

11-1-2015

Structural Properties of Transmuted Weibull Distribution

Kaisar Ahmad

University of Kashmir, Srinagar, India, ahmadkaisar31@gmail.com


S. P. Ahmad

University of Kashmir, Srinagar, India, sprvz@yahoo.com

A. Ahmed

Aligarh Muslim University, Aligarh, India

Follow this and additional works at: <http://digitalcommons.wayne.edu/jmasm>

 Part of the [Applied Statistics Commons](#), [Social and Behavioral Sciences Commons](#), and the [Statistical Theory Commons](#)

Recommended Citation

Ahmad, Kaisar; Ahmad, S. P.; and Ahmed, A. (2015) "Structural Properties of Transmuted Weibull Distribution," *Journal of Modern Applied Statistical Methods*: Vol. 14 : Iss. 2 , Article 13.

DOI: 10.22237/jmasm/1446351120

Available at: <http://digitalcommons.wayne.edu/jmasm/vol14/iss2/13>

This Regular Article is brought to you for free and open access by the Open Access Journals at DigitalCommons@WayneState. It has been accepted for inclusion in Journal of Modern Applied Statistical Methods by an authorized editor of DigitalCommons@WayneState.

Structural Properties of Transmuted Weibull Distribution

Kaisar Ahmad
University of Kashmir
Srinagar, India

S. P. Ahmad
University of Kashmir
Srinagar, India

A. Ahmed
Aligarh Muslim University
Aligarh, India

The transmuted Weibull distribution, and a related special case, is introduced. Estimates of parameters are obtained by using a new method of moments.

Keywords: Transmuted Weibull distribution, moment generating function, sample coefficient of variation, Standard deviation, Skewness and kurtosis

Introduction

The Weibull distribution was introduced by the Swedish Physicist Waloddi Weibull in 1939. He applied this distribution to analyze the breaking strength of materials. This distribution has been extensively used in lifetime and reliability problem. The Weibull family is a generalization of the exponential family and can model data exhibiting monotone hazard rate behavior, i.e., it can accommodate three types of failure rates, namely increasing, decreasing and constant. Its application in connection with lifetimes of many types of manufactured items has been widely advocated (e.g., Weibull, 1951; Berrettoni, 1964), and it has been used as a model with diverse types of items such as vacuum tubes (Kao, 1959), ball bearings (Lieblein & Zelen, 1956), and electrical insulation. It is also widely used in biomedical applications.

A simple explanation of the Weibull distribution and its applications can be found in Franck (1988). A comprehensive review of this model is available in Johnson, Kotz, and Balakrishnan (1995). A generalization of the Weibull distribution with application to the analysis of survival data is given by Mudholkar, Srivastava, and Kollia (1996). Inferences from grouped data in the three-parameter Weibull models is introduced by Hirose and Lai (1997). Lawless

K. Ahmad and S. P. Ahmad are in the Department of Statistics. Email at ahmadkaisar31@gmail.com and sprvz@yahoo.com, respectively. A. Ahmed is in the Department of Statistics and O.R.

(2002) provided statistical models and methods for lifetime data. Al-Athari (2011) and Hossain and Zimmer (2003) did some comparative studies on the estimation of Weibull parameters using complete and censored samples. Nadarajah and Kotz (2005) presented a procedure on some recent modifications of Weibull distribution.

For deriving new moment estimators of three parameters transmuted Weibull distribution, a similar approach to that of Huang and Hwang (2006) was used. Nadarajah and Kotz (2005) discussed products and ratios of Weibull random variables. Gokarna and Tsokos (2009) proposed a method on the transmuted extreme value distribution with application. Ahmad and Ahmad (2013) presented a procedure of Bayesian analysis of Weibull distribution.

A random variable x is said to have a Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by

$$g(x) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \quad x \geq 0, \alpha > 0, \beta > 0$$

The cdf of Weibull distribution is given by

$$G(x) = \int_0^x g(x) dx$$

$$G(x) = \int_0^x \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) dx$$

$$\Rightarrow \quad G(x) = 1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \quad (1)$$

Transmuted Weibull distribution

In order to obtain the pdf of transmuted Weibull distribution, use the following cdf which is given by

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2 \quad (2)$$

STRUCTURAL PROPERTIES OF WEIBULL DISTRIBUTION

where $G(x)$ is the cdf of base distribution. If $\lambda = 0$, we have the distribution of base random variable.

Now using equation (1) in equation (2),

$$F(x) = (1 + \lambda) \left(1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \right) - \lambda \left(1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \right)^2$$

$$\Rightarrow F(x) = (1 + \lambda)k - \lambda k^2$$

where

$$k = 1 - \exp\left(-\frac{x^\beta}{\alpha}\right)$$

$$\Rightarrow F(x) = k(1 + \lambda - \lambda k)$$

$$\Rightarrow F(x) = k\{1 + \lambda(1 - k)\}$$

$$\Rightarrow F(x) = \left\{ 1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \right\} \left\{ 1 + \lambda \exp\left(-\frac{x^\beta}{\alpha}\right) \right\} \quad (3)$$

This is the required cdf of Transmuted Weibull distribution.

In order to find the pdf of Transmuted Weibull distribution, first differentiate equation (3) w.r.t. x which is given by

$$f(x) = \frac{d}{dx} \{F(x)\}$$

$$\Rightarrow f(x) = \frac{d}{dx} \left[\left\{ 1 - \exp\left(-\frac{x^\beta}{\alpha}\right) \right\} \left\{ 1 + \lambda \exp\left(-\frac{x^\beta}{\alpha}\right) \right\} \right]$$

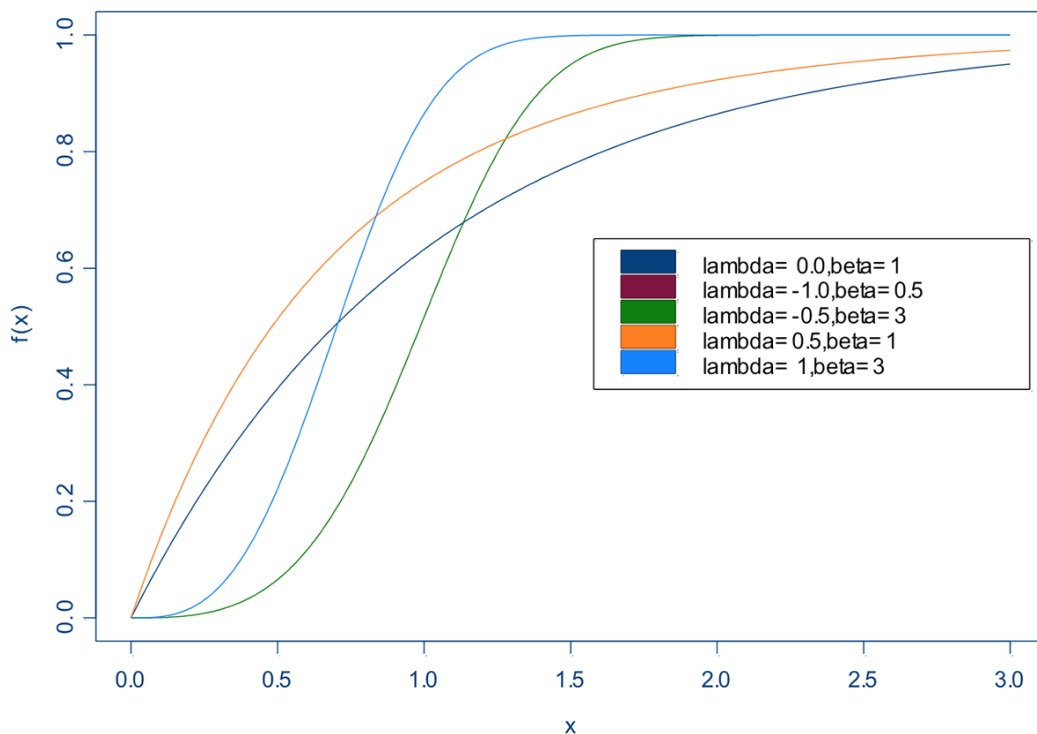


Figure 1. The cdfs of various transmuted Weibull distributions.

After differentiating the above equation w.r.t. x ,

$$f(x) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{ 1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right) \right\} \quad (4)$$

which is the required pdf of Transmuted Weibull distribution with parameters α , β and λ .

STRUCTURAL PROPERTIES OF WEIBULL DISTRIBUTION

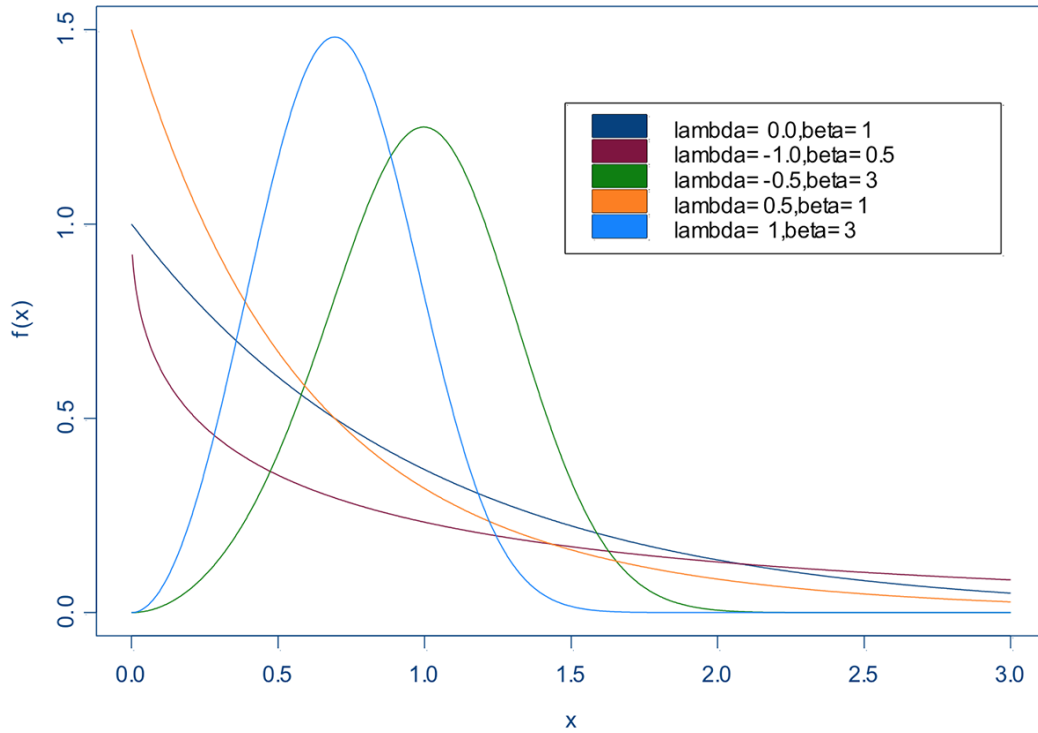


Figure 2. The pdfs of various Transmuted Weibull distributions.

Special cases

- 1) If $\lambda = 0$, then Transmuted Weibull distribution reduced to two parameter Weibull distribution with parameters α and β .

$$f(x; \alpha, \beta) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \quad x \geq 0, \alpha, \beta > 0$$

- 2) If $\lambda = 0$ and $\beta = 1$, then Transmuted Weibull distribution reduced to exponential distribution with parameter $\left(\frac{1}{\alpha}\right)$, i.e.

$$f(x) = \frac{1}{\alpha} \exp\left(-\frac{x}{\alpha}\right) \quad x > 0, \alpha > 0$$

- 3) If $\lambda = 0$ and $\alpha = \beta = 1$, then Transmuted Weibull distribution reduced to standard exponential distribution, i.e.

$$f(x) = \exp(-x) \quad x > 0$$

Moments of Transmuted Weibull distribution

Moments are the expected values of certain functions of a random variable. They serve to numerically describe the variable with respect to given characteristics for location, variation, skewness and kurtosis, to name a few. The expected value of x^r is termed as r^{th} moment about origin of the random variable x which is given by

$$\mu'_r = E(x)^r$$

Thus the r^{th} moment of Transmuted Weibull distribution is given by

$$\begin{aligned} \mu'_r &= \int_0^{\infty} x^r f(x; \alpha, \beta, \lambda) dx \\ \mu'_r &= \int_0^{\infty} x^r \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right)\right\} dx \end{aligned}$$

After solving the above equation,

$$\mu'_r = \alpha^{\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{r}{\beta}}\right) \quad (5)$$

Mean of the Transmuted Weibull distribution

Setting $r = 1$ in equation (5) leads to the mean of the Transmuted Weibull distribution, which is given by

$$\mu'_1 = \alpha^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right) \quad (6)$$

Second moment of the Transmuted Weibull distribution

Setting $r = 2$ in equation (5),

$$\mu'_2 = \alpha^{\frac{2}{\beta}} \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \quad (7)$$

Variance of Transmuted Weibull distribution

The variance of Transmuted Weibull distribution is given by

$$\mu_2 = \alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right\} \quad (8)$$

Third and fourth moments of Transmuted Weibull distribution

Setting $r = 3$ in equation (5),

$$\mu'_3 = \alpha^{\frac{3}{\beta}} \Gamma\left(\frac{3}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-3}{\beta}}\right)$$

and

$$\mu_3 = \alpha^{\frac{3}{\beta}} \left[\begin{array}{l} \Gamma\left(\frac{3}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-3}{\beta}}\right) \\ - \Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \left\{ \begin{array}{l} 3\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \\ - 2\Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \end{array} \right\} \end{array} \right] \quad (9)$$

If $r = 4$ in equation (5),

$$\mu'_4 = \alpha^{\frac{4}{\beta}} \Gamma\left(\frac{4}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-4}{\beta}}\right)$$

thus

$$\mu_4 = \alpha^{\frac{4}{\beta}} \left[\begin{array}{c} \Gamma\left(\frac{4}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{4}{\beta}}\right) \\ -\Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{1}{\beta}}\right) \left\{ \begin{array}{l} 4\Gamma\left(\frac{3}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{3}{\beta}}\right) \\ -6\Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{1}{\beta}}\right) \\ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{2}{\beta}}\right) \\ +3\Gamma^3\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{1}{\beta}}\right)^3 \end{array} \right\} \end{array} \right] \quad (10)$$

MGF of Transmuted Weibull distribution

The mgf of Transmuted Weibull distribution is given by

$$M_x(t) = \int_0^{\infty} e^{tx} f(x) dx$$

$$M_x(t) = \int_0^{\infty} \left\{ 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots + \frac{(tx)^n}{n!} + \dots \right\} f(x) dx$$

$$M_x(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} f(x) dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

Now by using the equation (5) in the above equation, we have

STRUCTURAL PROPERTIES OF WEIBULL DISTRIBUTION

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \alpha^{\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{r}{\beta}}\right) \quad (11)$$

This is the required mgf of Transmuted Weibull distribution.

Standard deviation of Transmuted Weibull distribution

The positive square root of the variance is called standard deviation. Symbolically, $\sigma = \sqrt{\sigma^2}$. From equation (8), the variance of Transmuted Weibull distribution is given as

$$\begin{aligned} \sigma^2 &= \alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)^2 \right\} \\ \Rightarrow \sigma &= \alpha^{\frac{1}{\beta}} \sqrt{\left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)^2 \right\}} \\ \Rightarrow \sigma &= \alpha^{\frac{1}{\beta}} \sqrt{\sigma_2 - \sigma_1^2} \end{aligned}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{k}{\beta}}\right) \quad (12)$$

Coefficient of variation of Transmuted Weibull distribution

This is the ratio of standard deviation and mean. Usually, it is denoted by C.V. and is given by

$$C.V. = \frac{\sigma}{\mu}$$

$$\Rightarrow C.V. = \frac{\alpha^{\frac{1}{\beta}} \sqrt{\left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right\}}}{\alpha^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)}$$

$$\Rightarrow C.V. = \frac{\sqrt{\sigma_2 - \sigma_1^2}}{\sigma_1} \quad (13)$$

where $\sigma_k = \Gamma\left(\frac{k}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-k}{\beta}}\right)$

Skewness and kurtosis of Transmuted Weibull distribution

The most popular way to measure the skewness and kurtosis of a distribution function rests upon ratios of moments. Lack of symmetry of tails (about mean) of frequency distribution curve is known as skewness. The formula for measure of skewness given by Karl Pearson in terms of moments of frequency distribution is given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

After using equation (8) and equation (9) in the above equation, we have

$$\beta_1 = \frac{\left[\Gamma\left(\frac{3}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-3}{\beta}}\right) - \Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \left\{ \begin{matrix} 3\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \\ - 2\Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \end{matrix} \right\} \right]^2}{\left[\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right]^3}$$

STRUCTURAL PROPERTIES OF WEIBULL DISTRIBUTION

$$\Rightarrow \beta_1 = \frac{\{\sigma_3 - \sigma_1(3\sigma_2 - \sigma_1^2)\}^2}{(\sigma_2 - \sigma_1^2)^3}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-k}{\beta}}\right)$$

Therefore

$$\begin{aligned} \gamma_1 &= \sqrt{\beta_1} \\ \Rightarrow \gamma_1 &= \frac{\{\sigma_3 - \sigma_1(3\sigma_2 - \sigma_1^2)\}}{(\sigma_2 - \sigma_1^2)^{\frac{3}{2}}} \end{aligned}$$

If $\gamma_1 < 0$, then the frequency curve is negatively skewed. If $\gamma_1 > 0$, then the frequency curve is positively skewed.

Kurtosis

The formula for measure of kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

After using equation (8) and equation (10) in the above equation,

$$\beta_2 = \frac{\left[\Gamma\left(\frac{4}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-4}{\beta}}\right) - \Gamma\left(\frac{1}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \left\{ 4\Gamma\left(\frac{3}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-3}{\beta}}\right) - 6\Gamma\left(\frac{1}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \Gamma\left(\frac{2}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \right\} + 3\Gamma^3\left(\frac{1}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^3 \right]}{\left[\Gamma\left(\frac{2}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \right]^2}$$

$$\Rightarrow \beta_2 = \frac{\{\sigma_4 - \sigma_1(4\sigma_3 - 6\sigma_1\sigma_2 + 3\sigma_1)\}^3}{(\sigma_2 - \sigma_1)^2}$$

where

$$\sigma_k = \Gamma\left(\frac{k}{\beta} + 1\right)\left(1 - \lambda + \lambda 2^{\frac{-k}{\beta}}\right)$$

and

$$\gamma_2 = \beta_2 - 3$$

$$\Rightarrow \gamma_2 = \frac{\{\sigma_4 - \sigma_1(4\sigma_3 - 6\sigma_1\sigma_2 + 3\sigma_1)\}^3}{(\sigma_2 - \sigma_1)^2} - 3$$

If $\gamma_2 > 0$, then the frequency curve is leptokurtic. If $\gamma_2 < 0$, then the frequency curve is platykurtic. If $\gamma_2 = 0$, then the frequency curve is mesokurtic, or we can say that there is no kurtosis.

Harmonic mean of Transmuted Weibull distribution

$$\frac{1}{H} = \int_0^{\infty} f(x; \alpha, \beta, \lambda) dx$$

$$\frac{1}{H} = \int_0^{\infty} \frac{1}{x} \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right)\right\} dx$$

$$\frac{1}{H} = (1 - \lambda) \int_0^{\infty} \frac{1}{x} \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) dx + 2\lambda \int_0^{\infty} \frac{1}{x} \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{2x^\beta}{\alpha}\right) dx$$

After substitution,

$$\frac{1}{H} = \frac{(1 - \lambda)}{\alpha} \int_0^{\infty} \frac{1}{z^{\frac{1}{\beta}}} \exp\left(-\frac{z}{\alpha}\right) dz + 2\frac{\lambda}{\alpha} \int_0^{\infty} \frac{1}{z^{\frac{1}{\beta}}} \exp\left(-\frac{2z}{\alpha}\right) dz$$

After solving the above equation

$$\begin{aligned} \frac{1}{H} &= \alpha^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right) \\ \Rightarrow H &= \frac{1}{\alpha^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)} \end{aligned} \tag{14}$$

New moment estimator of the Transmuted Weibull distribution

For deriving new moment estimators of three parameters transmuted Weibull distribution, we need the following theorem obtained by using the similar approach of Huang and Hwang (2006).

Theorem 1. Let $n \geq 3$ and let $X_1, X_2, X_3, \dots, X_n$ be n positive identical independently random variables having probability density function $f(x)$. Then the independence of the sample mean \bar{X}_n and the sample coefficient of variance

$V_n = \frac{S_n}{\bar{X}_n}$ is equivalent to that $f(x)$ is a Transmuted Weibull density where S_n is the sample standard deviation.

The next theorem requires the derivation of the expectation and the variance of $V_n^2 = \left(\frac{S_n}{\bar{X}_n}\right)^2$, where \bar{X}_n and S_n are respectively the sample mean and the sample standard deviation.

Theorem 2. Let $X_1, X_2, X_3, \dots, X_n$ be n positive identical independently distributed random samples drawn from a population having Transmuted Weibull density

$$f(x; \alpha, \beta, \lambda) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right)\right\}$$

then

$$E(S_n^2) = \alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{1}{\beta}}\right)^2 \right\}$$

Proof: Because the r^{th} moment of a random variable x about origin is given by

$$\begin{aligned} \mu'_r &= \int_0^\infty x^r f(x; \alpha, \beta, \lambda) dx \\ \mu'_r &= \int_0^\infty x^r \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right)\right\} dx \end{aligned}$$

After solving the above equation,

$$\mu'_r = \alpha^{\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{r}{\beta}}\right)$$

If $r = 1$ in the above equation,

STRUCTURAL PROPERTIES OF WEIBULL DISTRIBUTION

$$E(\bar{X}_n) = \alpha^{\frac{1}{\beta}} \Gamma\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)$$

Also if $r = 2$ in the above equation,

$$E(\bar{X}_n^2) = \frac{\alpha^{\frac{2}{\beta}} \left[\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{2}{\beta}}\right) + (n-1) \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)^2 \right]}{n} \quad (15)$$

and

$$V(\bar{X}_n) = \frac{\alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)^2 \right\}}{n} .$$

Thus

$$E(S_n^2) = nV(\bar{X}_n)$$

$$E(S_n^2) = \alpha^{\frac{2}{\beta}} \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{2}{\beta}}\right) - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{-\frac{1}{\beta}}\right)^2 \right\} \quad (16)$$

where \bar{X}_n and S_n^2 are respectively the sample mean and the sample variance.

Theorem 3. Let $X_1, X_2, X_3, \dots, X_n$ be n positive identical independently distributed random samples drawn from a population having Transmuted Weibull density

$$f(x; \alpha, \beta, \lambda) = \frac{\beta}{\alpha} x^{\beta-1} \exp\left(-\frac{x^\beta}{\alpha}\right) \left\{ 1 - \lambda + 2\lambda \exp\left(-\frac{x^\beta}{\alpha}\right) \right\}$$

then

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n \left\{ \begin{array}{l} \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \\ - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \end{array} \right\}}{\left[\begin{array}{l} \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \\ + (n-1) \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \end{array} \right]}$$

where \bar{X}_n and S_n^2 are respectively the sample mean and the sample variance.

Proof: By using the theorem (1), we have

$$\begin{aligned} E(S_n^2) &= E\left(\frac{S_n^2}{\bar{X}_n^2} \bar{X}_n^2\right) = E\left(\frac{S_n^2}{\bar{X}_n^2}\right) E(\bar{X}_n^2) \\ \Rightarrow E\left(\frac{S_n^2}{\bar{X}_n^2}\right) &= \frac{E(S_n^2)}{E(\bar{X}_n^2)} \end{aligned} \quad (17)$$

Now using equations (15) and (16) in equation (17), we have

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n \left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \right.}{\left[\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \right.} \\ \left. \left. - \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right\}}{\left[\Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \right.} \\ \left. + (n-1) \Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right]}$$

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) \rightarrow \frac{\left\{ \Gamma\left(\frac{2}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-2}{\beta}}\right) \right\}}{\left[\Gamma^2\left(\frac{1}{\beta} + 1\right) \left(1 - \lambda + \lambda 2^{\frac{-1}{\beta}}\right)^2 \right]} - 1$$

as $n \rightarrow \infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^2}{\bar{X}_n^2}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

References

- Ahmad, S. P. & Ahmad, K. (2013). Bayesian analysis of Weibull distribution using R software. *Australian Journal of Basic and Applied Sciences*, 7(9), 156-164. <http://ajbasweb.com/old/ajbas/2013/July/156-164.pdf>
- Al-Athari, F. M. (2011). Parameter estimation for the double-Pareto distribution. *Journal of Mathematics and Statistics*, 7(4), 289–294. doi:10.3844/jmssp.2011.289.294
- Berrettoni, J. N. (1964). Practical applications of the Weibull distribution, *Industrial Quality Control*, 21(2), 71-79.
- Franck, J. R. (1988). A simple explanation of the Weibull distribution and its applications. *Reliability Review*, 8(3), 93-116.
- Gokarna R. A. & Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods and Applications*, 71(12), e1401–e1407. doi:10.1016/j.na.2009.01.168

- Hirose, H. & Lai, T. L. (1997). Inference from grouped data in three-parameter Weibull models with applications to breakdown-voltage experiments, *Technometrics*, 39(2), 199-210. doi:10.1080/00401706.1997.10485085
- Hossain, A. & Zimmer, W. (2003). Comparison of estimation methods for Weibull parameters: complete and censored samples. *Journal of Statistical Computation and Simulation*, 73(2), 145–153. doi:10.1080/0094965021000033486
- Huang, P. H. & Hwang, T. Y. (2006). On new moment estimation of parameters of the generalized gamma distribution using its characterization. [sic] *Taiwanese Journal of Mathematics*, 10(4), 1083 -1093.
- Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions* (Vol. 2). New York: John Wiley & Sons.
- Kao, J. H. K. (1959). A graphical estimation of mixed Weibull parameters in life testing of electron tubes. *Technometrics*, 1(4), 389-407. doi:10.1080/00401706.1959.10489870
- Lawless, J. F. (2002). *Statistical models and methods for lifetime data* (2nd Ed.). New York: John Wiley & Sons.
- Lieblein, J. & Zelen, M. (1956). Statistical investigation of the fatigue life of deep groove ball bearing. *Journal of Research of the National Bureau of Standards*, 57(5), 273-316. doi:10.6028/jres.057.033
- Mudholkar, G. S., Srivastava, D. K. & Kollia, G. D. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of the American Statistical Association*, 91(436), 1575-1583. doi:10.1080/01621459.1996.10476725
- Nadarajah, S. & Kotz, S. (2005). On some recent modifications of Weibull distribution, *IEEE Transactions on Reliability*, 54(4), 561-562. doi:10.1109/TR.2005.858811
- Weibull, W. (1939). A statistical theory of strength of materials. *Ingeniörs Vetenskapsakademien Handlingar*, 151, 1-45.
- Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18, 293-297.