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Recently, the Bayesian analysis of the two-component mixture of lifetime models under singly type I censored samples was discussed. The Bayes estimation of the parameters of mixture of two Rayleigh distributions (MTRD) is developed under doubly censoring. Different informative priors, under squared error loss function and k-loss function, have been assumed for the posterior estimation. The performance of different estimators has been compared in terms of posterior risks by analyzing the simulated and real life data sets.

Keywords: Inverse transformation method, mixture model, doubly censoring, loss functions, Bayes estimator

Introduction

In survival analysis, data are subject to censoring. The most common type of censoring is right censoring, in which the survival time is larger than the observed right censoring time. In some cases, however, data are subject to left, as well as, right, censoring. When left censoring occurs, the only information available to an analyst is that the survival time is less than or equal to the observed left censoring time. A more complex censoring scheme is found when both initial and final times are interval-censored. This situation is referred as double censoring, or the data with both right and left censored observations are known as doubly censored data.

Analysis of doubly censored data for simple (single) distribution has been studied by many authors. Fernandez (2000) investigated maximum likelihood prediction based on type II doubly censored exponential data. Fernandez (2006) has discussed Bayesian estimation based on trimmed samples from Pareto populations. Khan et al. (2010) studied predictive inference from a two-parameter Rayleigh life

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model given a doubly censored sample. Kim and Song (2010) have discussed Bayesian estimation of the parameters of the generalized exponential distribution from doubly censored samples. Khan et al. (2011) studied sensitivity analysis of predictive modeling for responses from the three-parameter Weibull model with a follow-up doubly censored sample of cancer patients. Pak et al. (2013) has proposed the estimation of Rayleigh scale parameter under doubly type-II censoring from imprecise data.

In statistics, a mixture distribution is signified as a convex fusion of other probability distributions. It can be used to model a statistical population with subpopulations, where constituent of mixture probability densities are the densities of the subpopulations. Mixture distribution may appropriately be used for certain data set where the subsets of the whole data set possess different properties that can best be modeled separately. They can be more mathematically manageable, because the individual mixture components are dealt with more ease than the overall mixture density. The families of mixture distributions have a wider range of applications in different fields such as fisheries, agriculture, botany, economics, medicine, psychology, electrophoresis, finance, communication theory, geology and zoology.

Soliman (2006) derived estimators for the finite mixture of Rayleigh model based on progressively censored data. Sultan, et al. (2007) described the properties and estimation of mixture of two inverse Weibull distributions. Sultan, et al. (2007) have discussed some properties of the mixture of two inverse Weibull distributions. Saleem and Aslam (2008) presented a comparison of the Maximum Likelihood (ML) estimates with the Bayes estimates assuming the Uniform and the Jeffreys priors for the parameters of the Rayleigh mixture. Kundu and Howalder (2010) considered the Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. Saleem et al. (2010) considered the Bayesian analysis of the mixture of Power function distribution using the complete and the censored sample. Shi and Yan (2010) studied the case of the two parameter exponential distribution under type I censoring to get empirical Bayes estimates. Eluebaly and Bouguila (2011) have presented a Bayesian approach to analyze finite generalized Gaussian mixture models which incorporate several standard mixtures, widely used in signal and image processing applications, such as Laplace and Gaussian. Sultan and Al-Moisheer (2012) developed approximate Bayes estimation of the parameters and reliability function of mixture of two inverse Weibull distributions under Type-2 censoring.

The Proposed Mixture Model and the Likelihood Function

The probability density function (pdf) of the Rayleigh distribution with rate parameter λ_i is

$$f_i\left(x_{ij}\right)=2x_{ij}\lambda_i^{2}\exp\left(-x_{ij}^{2}\lambda_i^{2}\right),\ \ 0<x_{ij}<\infty,\ \ \lambda_i^{2}>0,\ \ i=1,2,\ \text{and}\ j=1,2,...,n_i \qquad (1)$$

The cumulative distribution function (CDF) of the distribution is

$$F_i\left(x_{ij}\right)=1-\exp\left(-\lambda_i^{2}x_{ij}^{2}\right),\ \ 0<x_{ij}<\infty,\ \ \lambda_i^{2}>0,\ \ i=1,2,\ \text{and}\ j=1,2,...,n_i \qquad (2)$$

A density function for mixture of two components densities with mixing weights (p_1, 1- p_1) is

$$f\left(x\right)=p_1f_1\left(x\right)+\left(1-p_1\right)f_2\left(x\right),\ \ 0<p_1<1. \qquad (3)$$

The cumulative distribution function for the mixture model is:

$$F\left(x\right)=p_1F_1\left(x\right)+\left(1-p_1\right)F_2\left(x\right) \qquad (4)$$

Consider a random sample of size 'n' from Rayleigh distribution, and let $x_r, x_{r+1},...,x_s$ be the ordered observations that can only be observed. The remaining '$r-1$' smallest observations and the '$n-s$' largest observations have been assumed to be censored. Now based on causes of failure, the failed items are assumed to come either from subpopulation 1 or from subpopulation 2; so the $x_{1r_1},...,x_{1s_1}$ and $x_{2r_2},...,x_{2s_2}$ failed items come from first and second subpopulations respectively. The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component. Where $x_s=\max\left(x_{1,s_1},x_{2,s_2}\right)$ and $x_r=\min\left(x_{1,r_1},x_{2,r_2}\right)$. Therefore, $m_1=s_1-r_1+1$ and $m_2=s_2-r_2+1$ number of failed items can be observed from first and second subpopulations respectively. The remaining $n-(s-r+2)$ items are assumed to be censored observations, and $s-r+2$ are the uncensored items. Where $r=r_1+r_2$,

$s = s_1 + s_2$ and $m = m_1 + m_2$. Then the likelihood function for the Type II doubly censored sample $\mathrm{x} = \left\{\left(x_{1r_1}, ..., x_{1s_1}\right), \left(x_{2r_2}, ..., x_{2s_2}\right)\right\}$, assuming the causes of the failure of the left censored items are identified, can be written as:

$$L\left(\lambda_1, \lambda_2, p_1 | \mathrm{x}\right) \propto p_1^{s_1} \left(1 - p_1\right)^{s_2} \left\{F_1\left(x_{(r_1)}, \lambda_1\right)\right\}^{r_1 - 1} \left\{F\left(x_{(r_2)}, \lambda_2\right)\right\}^{r_2 - 1}$$
$$\left\{1 - F\left(x_s, \lambda_1, \lambda_2\right)\right\}^{n-s} \left\{\prod_{i=r_1}^{s_1} f_1\left(x_{1(i)}, \lambda_1\right)\right\} \left\{\prod_{i=r_2}^{s_2} f_2\left(x_{2(i)}, \lambda_2\right)\right\} \quad (5)$$

$$L\left(\lambda_1, \lambda_2, p_1 | \mathrm{x}\right) \propto \sum_{k_1=0}^{r_1-1} \sum_{k_2=0}^{r_2-1} \sum_{k_3=0}^{n-s} (-1)^{k_1+k_2}$$
$$\binom{r_1 - 1}{k_1} \binom{r_2 - 1}{k_2} \binom{n - s}{k_3} p_1^{n-s-k_3+s_1} (1 - p_1)^{s_2+k_3} \quad (6)$$
$$\times \lambda_1^{2m_1} \lambda_2^{2m_2} \exp\left\{-\lambda_1^2\left(\Omega\left(x_{1j}\right)\right)\right\} \exp\left\{-\lambda_2^2\left(\Omega\left(x_{2j}\right)\right)\right\}$$

where $\Omega\left(x_{1j}\right) = \sum_{i=r_1}^{s_1} x^2_{1(i)} + \left(n - s - k_3\right) x^2_{(s)} + k x^2_{(r_1)}$, $\Omega\left(x_{2j}\right) = \sum_{i=r_2}^{s_2} x^2_{2(i)} + k_3 x^2_{(s)} + k x^2_{(r_2)}$,
$m_1 = s_1 - r_1 + 1$, and $m_2 = s_2 - r_2 + 1$

Bayes Estimation

For the Bayesian estimation, let us assume that the parameters λ_1 and p_1 $i = 1, 2$ are independent random variables, and then we consider the following priors for different parameters:

Bayesian Estimation using Nakagami Prior

The prior for the rate parameters λ_i for i = 1, 2, is assumed to be the Nakagami distribution, with the hyper-parameters a_i and b_i, given by

$$f_{\lambda_i}\left(\lambda_i\right)=\frac{2a_i^{a_i}}{\Gamma\left(a_i\right)b_i^{a_i}}\lambda_i^{2ai-1}\exp\left(\frac{-\lambda_i^2 a_i}{b_i}\right),\qquad a_i,b_i>0 \tag{7}$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_p\left(p_1\right)=\frac{\Gamma\left(c_1+d_1\right)}{\Gamma\left(c_1\right)\Gamma\left(d_1\right)}p_1^{c_1-1}\left(1-p_1\right)^{d_1-1},\qquad c_1,d_1>0 \tag{8}$$

From equation (7)-(8), propose the following joint prior density of the vector $\Theta=\left(\lambda_1,\lambda_2,p_1\right)$

$$g\left(\Theta\right)\propto\lambda_i^{2ai-1}\exp\left(\frac{-\lambda_i^2 a_i}{b_i}\right)p_1^{c_1-1}\left(1-p_1\right)^{d_1-1},$$
$$0<p_1<1,\ a_i>0,\ b_i>0,\ c_1>0,\ d_1>0 \tag{9}$$

By multiplying Equation (9) with Equation (6), the joint posterior density for the vector Θ given the data becomes

$$\pi\left(\Theta\,|\,\mathrm{x}\right)\propto\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\prod_{i=1}^{2}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{n-s-k_3+s_1+c_1-1}$$
$$\times(1-p_1)^{s_2+k_3+d_1-1}\lambda_i^{2(a_i+m_i)-1}\exp\left\{-\lambda_i^2\left(\frac{a_i}{b_i}+\Omega\left(x_{ij}\right)\right)\right\} \tag{10}$$

Marginal distributions of λ_i and p_1 $i=1,2$ can be obtained by integrating the nuisance parameters.

Bayesian Estimation using Chi Prior

The prior for the rate parameters λ_i for i=1, 2, is assumed to be the chi distribution, with the hyperparameter e_i, given by

$$f_{\lambda_i}(\lambda_i)=\frac{2^{1-\frac{e_i}{2}}}{\Gamma\left(\frac{e_i}{2}\right)}\lambda_i^{e_i-1}\exp\left(\frac{-\lambda_i^2}{2}\right),\qquad e_i>0 \tag{11}$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_p(p_1)=\frac{\Gamma(c_2+d_2)}{\Gamma(c_2)\Gamma(d_2)}p_1^{c_2-1}(1-p_1)^{d_2-1},\qquad c_2,d_2>0 \tag{12}$$

From equation (11)-(12), we propose the following joint prior density of the vector $\Theta=(\lambda_1,\lambda_2,p_1)$

$$g(\Theta)\propto\lambda_i^{ei-1}\exp\left(\frac{-\lambda_i^2}{2}\right)p_1^{c_2-1}(1-p_1)^{d_2-1},<0\,p_1<1,\ e_i>0,c_3>0,d_3>0 \tag{13}$$

By multiplying Equation (13) with Equation (6), the joint posterior density for the vector Θ given the data becomes

$$\begin{aligned}\pi(\Theta\,|\,\mathrm{x})\propto&\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\prod_{i=1}^{2}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{n-s-k_3+s_1+c_2-1}\\&\times(1-p_1)^{s_2+k_3+d_2-1}\lambda_i^{2m_i+e_i-1}\exp\left\{-\lambda_i^2\left(\frac{1}{2}+\Omega(x_{ij})\right)\right\}\end{aligned} \tag{14}$$

Bayesian Estimation using Rayleigh Prior

The prior for the rate parameters λ_i for i=1, 2, is assumed to be the Rayleigh distribution, with the hyperparameter v_i given by

$$f_{\lambda_i}(\lambda_i)=\frac{\lambda_i}{v_i^2}\exp\left(\frac{-\lambda_i^2}{2v_i^2}\right),\qquad v_i>0 \tag{15}$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_p\left(p_1\right)=\frac{\Gamma\left(c_3+d_3\right)}{\Gamma\left(c_3\right)\Gamma\left(d_3\right)}p_1^{c_3-1}\left(1-p_1\right)^{d_3-1},\quad c_3,d_3>0 \tag{16}$$

From equation (15)-(16), propose the following joint prior density of the vector $\Theta=\left(\lambda_1,\lambda_2,p_1\right)$:

$$g\left(\Theta\right)\propto\lambda_i\exp\left(\frac{-\lambda_i^2}{2v_i^2}\right)p_1^{c_3-1}\left(1-p_1\right)^{d_3-1},<0p_1<1,\ v_i,\ c_3>0,\ d_3>0 \tag{17}$$

By multiplying Equation (17) with Equation (6), the joint posterior density for the vector Θ given the data becomes

$$\begin{aligned}\pi\left(\Theta\,|\,\mathrm{x}\right)\propto\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\prod_{i=1}^{2}(-1)^{k_1+k_2}&\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{n-s-k_3+s_1+c_3-1}\\&\times(1-p_1)^{s_2+k_3+d_3-1}\lambda_i^{2(m_i)+1}\exp\left\{-\lambda_i^2\left(\frac{1}{2v_i^2}+\Omega\left(x_{ij}\right)\right)\right\}\end{aligned} \tag{18}$$

Marginal distributions of λ_1 and p_1 $i=1,2$ can be obtained by integrating the nuisance parameters.

Bayes Estimation of the Vector of Parameters Θ

The Bayesian point estimation is connected to a loss function in general, signifying the loss induced when the estimate $\hat{\theta}$ differ from true parameter θ. Because there is no specific rule that helps us to identify the appropriate loss function to be used, squared error loss is used in this article as it serve as standard loss. It is well known that under the squared error loss function, the Bayes estimator of a function of the parameters is the posterior mean of the function and risk is the posterior variance. It is defined as $l\left(\hat{\theta},\theta\right)=\left(\theta-\hat{\theta}\right)^2$.

It was originally used in estimation problems when the unbiased estimator of θ was being considered. Another reason for its popularity is due to its relationship to least squares theory. The use of SELF makes the calculations simpler.

The K-Loss function (KLF), defined as: $l\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2/\hat{\theta}\theta$, was proposed by Wasan (1970). It is well fitted for a measure of inaccuracy for an estimator of a scale parameter of a distribution defined on $R^+=\left(0,\infty\right)$. Under K-Loss function the Bayes estimates and posterior risks are defined as $\hat{\theta}=\sqrt{E(\theta\,|\,\mathbf{x})/E(\theta^{-1}\,|\,\mathbf{x})}$, and $\rho\left(\hat{\theta}\right)=2\left\{E(\theta\,|\,\mathrm{x})E(\theta^{-1}\,|\,\mathrm{x})-1\right\}$ respectively.

The respective marginal distribution of each parameter has been used to derive the Bayes estimators and posterior risks for λ_1,λ_2 and p_1 under the squared error loss function (SELF) and K- loss functions (KLF). The Bayes estimators and posterior risks of λ_1,λ_2 and p_1 under squared error loss function (SELF) assuming Nakagami prior are given as:

The Bayes estimators of λ_1,λ_2 and $p_1$are:

$$\hat{\lambda}_{1(SELF)}=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B\left(A_1,A_2\right)\Gamma\left(a_1+m_1+1/2\right)\Gamma\left(a_2+m_2\right)}{2\left\{a_1/b_1+\Omega\left(x_{1j}\right)\right\}^{\left(a_1+m_1+1/2\right)}2\left\{a_2/b_2+\Omega\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}$$

$$\hat{\lambda}_{2(SELF)}=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B\left(A_1,A_2\right)\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2+1/2\right)}{2\left\{a_1/b_1+\Omega\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}2\left\{a_2/b_2+\Omega\left(x_{2j}\right)\right\}^{\left(a_2+m_2+1/2\right)}}$$

$$\hat{p}_{1(SELF)}=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B\left(A_1+1,A_2\right)\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2\right)}{2\left\{a_1/b_1+\Omega\left(x_{1j}\right)\right\}^{\left(a_1+m_1\right)}2\left\{a_2/b_2+\Omega\left(x_{2j}\right)\right\}^{\left(a_2+m_2\right)}}$$

The Posterior risks of λ_1,λ_2 and p_1 are:

$$\rho\left(\hat{\lambda}_{1(SELF)}\right)=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1+1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1+1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}-\left(\hat{\lambda}_{1(SELF)}\right)^2$$

$$\rho\left(\hat{\lambda}_{2(SELF)}\right)=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2+1)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2+1)}}-\left(\hat{\lambda}_{2(SELF)}\right)^2$$

$$\rho\left(\hat{p}_{1(SELF)}\right)=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1+2,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}-\left(\hat{p}_{1(SELF)}\right)^2$$

The Bayes estimators of λ_1,λ_2 and p_1 under KLF are:

$$\hat{\lambda}_{1(KLF)}=\left(\frac{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1+1/2)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1+1/2)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}}{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1-1/2)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1-1/2)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}}\right)^{\frac{1}{2}}$$

$$\hat{\lambda}_{2(KLF)}=\left(\frac{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2+1/2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2+1/2)}}}{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2-1/2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2-1/2)}}}\right)^{\frac{1}{2}}$$

$$\hat{p}_{1(KLF)}=\left(\frac{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1+1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}}{\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1-1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2-1/2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}}\right)^{\frac{1}{2}}$$

The Posterior risks of λ_1, λ_2 and p_1 under KLF are:

$$\rho\left(\hat{\lambda}_{1(KLF)}\right)=2\left(\begin{array}{l} N^{-2}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1+1/2)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1+1/2)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}} \\ \times\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1-1/2)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1-1/2)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}-1 \end{array}\right)$$

$$\rho\left(\hat{\lambda}_{2\ (KLF)}\right)=2\left(\begin{array}{l} N^{-2}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2+1/2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2+1/2)}} \\ \times\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2-1/2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2-1/2)}}-1 \end{array}\right)$$

$$\rho\left(\hat{p}_{1(KLF)}\right)=2\left(\begin{array}{l} N^{-2}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1+1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}} \\ \times\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1-1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}-1 \end{array}\right)$$

Where N^{-1} is formulized as

$$N^{-1}=\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\left\{a_1/b_1+\Omega(x_{1j})\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega(x_{2j})\right\}^{(a_2+m_2)}}$$

$A_1=n-s-k_3+s_1+c_1$ and $A_2=s_2+k_3+d_1$

Similarly, expressions for Bayes estimators and their posterior risks under the rest of the priors can be obtained with little modifications.

Elicitation

In Bayesian analysis the elicitation of opinion is a crucial step. In statistical inference, the characteristics of a certain predictive distribution proposed by an expert determine the hyper-parameters of a prior distribution. Focus on a method of elicitation based on prior predictive distribution. The elicitation of hyper-parameter from the prior $p(\lambda)$ is a difficult task. The prior predictive distribution is used for the elicitation of the hyper-parameters which is compared with the experts' judgment about this distribution and then the hyper-parameters are chosen in such a way so as to make the judgment agree as closely as possible with the given distribution. See also Grimshaw et al. (2001), O'Hagan et al. (2006), Jenkinson (2005) and Leon et al. (2003). According to Aslam (2003), the method of elicitation is to compare the prior predictive distribution with experts' assessment about this distribution and then to choose the hyper-parameters that make the assessment agree closely with the member of the family. The prior predictive distributions under all the priors are derived using:

$$p(y) = \int_{\Theta} p(y \mid \Theta) p(\Theta) \, d\Theta$$

Elicitation under Nakagami distribution

The prior predictive distribution using Nakagami prior is:

$$\begin{aligned} p(y) &= 2\left(a_1 b_1^{-1}\right)^{a_1} \frac{y a_1 c_1}{\left(c_1 + d_1\right)\left(y^2 + a_1 b_1^{-1}\right)^{(a_1+1)}} \\ &\quad + 2\left(a_2 b_2^{-1}\right)^{a_2} \frac{y a_2 d_1}{\left(c_1 + d_1\right)\left(y^2 + a_2 b_2^{-1}\right)^{(a_2+1)}}, \ y > 0 \end{aligned} \tag{19}$$

For the elicitation of the six hyper-parameters, six different intervals are considered. From Equation (19), the experts' probabilities/assessments are supposed to be 0.10 for each case. The six integrals for equation (19) are considered with the following limits of the values of random variable 'Y': (0, 10), (10, 20), (20, 30), (30, 40), (40, 50) and (50, 60) respectively. For the elicitation of the hyper-parameters a_1, a_2, b_1, b_2, c_1, and d_1. These six equations are solved simultaneously through computer program developed in *SAS* package using the command of PROC SYSLIN. Thus the values of hyper-parameters obtained by applying this methodology are: 0.000231, 0.012109, 0.52114, 4.99325, 0.52130, and 0.14790 respectively.

Elicitation under Chi Prior

The prior predictive distribution using Chi prior is:

$$p(y)=\frac{(0.5)^{\frac{e_1}{2}}\, ye_1c_2}{(c_2+d_2)\left(y^2+0.50\right)^{(e_1/2+1)}}+\frac{(0.5)^{\frac{e_2}{2}}\, ye_2d_2}{(c_2+d_2)\left(y^2+0.50\right)^{(e_2/2+1)}},\ y>0$$

Now, elicit four hyper-parameters, so consider the four integrals. The expert probabilities are assumed to 0.15 for each integral with the following limits of the values of random variable ‘Y’: (0, 15), (15, 30), (30, 45) and (45, 60). Using the similar kind of program, as discussed above, we have the following values of the hyper-parameters e_1 = 20.1056, e_2 = 14.23569, c_2 = 0.09377 and d_2 = 0.08749.

Elicitation under Rayleigh Prior

The prior predictive distribution using Rayleigh prior is:

$$p(y)=\frac{(v_1)^{-2}\, yc_3}{(c_3+d_3)\left(y^2+2v_1^{-2}\right)^2}+\frac{(v_2)^{-2}\, yd_3}{(c_3+d_3)\left(y^2+2v_2^{-2}\right)^2},\ y>0$$

Again, elicit four hyper-parameters, so consider the four integrals. The expert probabilities are assumed to 0.15 for each integral with the following limits of the values of random variable ‘*Y*’: (0, 15), (15, 30), (30, 45) and (45, 60). Using the similar kind of program, as discussed above, we attained the following values of the hyper-parameters v_1 = 5.052104, v_2 = 5.03251, c_3 = 0.67213 and d_3 = 0.91035.

Simulation Study and Comparisons

A simulation study is carried out in order to investigate the performance of Bayes estimators under tenfold choice of the parametric values, different sample sizes, and the different values of the mixing proportion. Take random samples of sizes n = 20, 40, and 80 from the two component mixture of Rayleigh distributions with tenfold choice of parameters

$(\lambda_1, \lambda_2) \in \{(0.1,\ 0.12), (1,\ 1.2), (10,\ 12), (0.1,\ 12), (10,\ 0.12)\}$, $p_1 = 0.45$ and 0.6. To generate a mixture data we make use of probabilistic mixing with probabilities p_1 and (1- p_1). A uniform number u is generated n times and if $u < p_1$ the observation is taken randomly from F_1 (the Rayleigh distribution with parameter λ_1) otherwise from F_2 (from the Rayleigh distribution with parameter λ_2). The choice of the censoring time is made in such a way that the censoring rate in the resultant sample is approximately 20%. To implement censored samplings, we considered that the $x_{1r_1}, ..., x_{1s_1}$ and $x_{2r_2}, ..., x_{2s_2}$ failed items come from first and second subpopulations respectively. The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component. Where $x_s = \max\left(x_{1,s_1}, x_{2,s_2}\right)$ and $x_r = \min\left(x_{1,r_1}, x_{2,r_2}\right)$. The simulated data sets have been obtained using following steps:

Step 1: Draw samples of size 'n' from the mixture model

Step 2: Generate a uniform random no. u for each observation

Step 3: If $u \le \pi$, the take the observation from first subpopulation otherwise from the second subpopulation

Step 4: Determine the test termination points on left and right, that is, determine the values of x_r and x_s

Step 5: The observations which are less than x_r and greater than x_s have been considered to be censored from each component

Step 6: Use the remaining observations from each component for the analysis

To avoid an extreme sample, we simulate 10, 000 data sets each of size *n*. The Bayes estimates and posterior risks (in parenthesis) are computed using Mathematica 8.0. The average of these estimates and corresponding risks are reported in tables 1- 15. The abbreviations used in the tables are: B.Es: Bayes estimators; P.Rs: Posterior risks; NP: Nakagami prior; CP: Chi prior; RP: Rayleigh prior.

Table 1: B.Es and P.Rs under NP using $(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$ and $(0.1, 0.12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.104076	0.127713	0.498425	0.099628	0.135951	0.665779
	(0.000479)	(0.000558)	(0.013229)	(0.000285)	(0.000904)	(0.011667)
40	0.099427	0.12652	0.48622	0.094406	0.131125	0.659375
	(0.000223)	(0.000306)	(0.007231)	(0.000127)	(0.000431)	(0.006346)
80	0.099036	0.125807	0.478841	0.092618	0.13063	0.61648
	(0.000114)	(0.000161)	(0.003865)	(0.000057)	(0.000230)	(0.003218)
k-loss function						
20	0.101884	0.123181	0.480102	0.095373	0.136428	0.655312
	(0.086648)	(0.069120)	(0.129905)	(0.056005)	(0.104087)	(0.061058)
40	0.101669	0.123008	0.471869	0.096164	0.131118	0.645074
	(0.045012)	(0.037679)	(0.070063)	(0.027334)	(0.053150)	(0.031642)
80	0.090768	0.121778	0.470942	0.097312	0.125869	0.640866
	(0.021446)	(0.019760)	(0.034345)	(0.014017)	(0.028207)	(0.016573)

Table 2: B.Es and P.Rs under NP using $(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.45)$ and $(1, 1.2, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.03790	1.26375	0.498181	0.978002	1.35085	0.665995
	(0.046897)	(0.054602)	(0.013210)	(0.027554)	(0.087625)	(0.011594)
40	1.00642	1.25934	0.482283	0.979230	1.31290	0.657711
	(0.022787)	(0.028565)	(0.007234)	(0.013363)	(0.045048)	(0.005650)
80	0.996073	1.25518	0.478649	0.989340	1.307418	0.616586
	(0.011180)	(0.015747)	(0.003863)	(0.006855)	(0.023781)	(0.003222)
k-loss function						
20	1.02547	1.297250	0.484873	10.11040	12.737600	0.481255
	(0.085012)	(0.069721)	(0.126936)	(0.083902)	(0.068839)	(0.129790)
40	0.972684	1.24985	0.477066	9.85076	12.45580	0.474996
	(0.043184)	(0.037322)	(0.068486)	(0.043278)	(0.037003)	(0.069297)
80	0.994972	1.22715	0.469378	9.91883	12.11990	0.468498
	(0.024387)	(0.021374)	(0.037459)	(0.022678)	(0.019891)	(0.036650)

Table 3: B.Es and P.Rs under NP using $\left(\lambda_1, \lambda_2, p_1\right) = (10, 12, 0.45)$ and $(10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	10.82959 (4.256710)	12.9605 (4.8388)	0.497084 (0.013244)	9.58255 (2.66321)	13.61630 (7.85146)	0.663355 (0.01170)
40	10.12890 (2.39442)	12.88640 (2.71684)	0.479166 (0.007305)	9.77352 (1.40887)	13.49536 (4.29918)	0.656693 (0.006435)
80	9.61493 (1.05203)	12.67810 (1.58376)	0.462094 (0.003820)	9.88275 (0.58717)	13.45520 (2.30094)	0.616606 (0.003233)
k-loss function						
20	10.11040 (0.083902)	12.73760 (0.068839)	0.481255 (0.12979)	9.84880 (0.056169)	12.80650 (0.104837)	0.653839 (0.061849)
40	9.85076 (0.043278)	12.45580 (0.037003)	0.474996 (0.069297)	9.94419 (0.027645)	12.63070 (0.054150)	0.651788 (0.031611)
80	9.91883 (0.022678)	12.11990 (0.019891)	0.468498 (0.036650)	9.95821 (0.013426)	12.58110 (0.028228)	0.640724 (0.015684)

Table 4: B.Es and P.Rs under NP using $\left(\lambda_1, \lambda_2, p_1\right) = (0.10, 12, 0.45)$ and $(0.10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.095619 (0.000301)	13.67980 (4.868910)	0.534912 (0.012036)	0.092487 (0.000201)	13.81620 (7.189420)	0.687435 (0.010396)
40	0.090655 (0.000131)	13.54530 (2.343030)	0.51823 (0.006457)	0.091992 (0.000096)	13.63860 (3.41382)	0.677511 (0.005650)
80	0.090905 (0.000065)	13.48370 (1.148460)	0.509322 (0.003346)	0.091260 (0.000048)	13.5980 (1.767790)	0.672210 (0.002953)
k-loss function						
20	0.0914225 (0.067737)	12.87110 (0.053268)	0.522308 (0.097694)	0.093144 (0.048171)	13.86210 (0.078296)	0.679012 (0.049925)
40	0.092511 (0.032515)	12.72830 (0.025796)	0.511618 (0.052023)	0.0951075 (0.023389)	13.70450 (0.037372)	0.673099 (0.026303)
80	0.09455 (0.015937)	12.37891 (0.012696)	0.505934 (0.026871)	0.096113 (0.011527)	12.96763 (0.018268)	0.669953 (0.013511)

Table 5: B.Es and P.Rs under NP using $(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.45)$ and $(10, 0.12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	11.92930 (4.278770)	0.112064 (0.000325)	0.403231 (0.011879)	11.53870 (3.16833)	0.113048 (0.000483)	0.585750 (0.011740)
40	11.5720 (2.13578)	0.114125 (0.000155)	0.412042 (0.006265)	11.46580 (1.51987)	0.11513 (0.000219)	0.586132 (0.006334)
80	11.17240 (1.08183)	0.118567 (0.000075)	0.42970 (0.003216)	11.40931 (0.756977)	0.116862 (0.000103)	0.589618 (0.003294)
k-loss function						
20	11.64350 (0.067737)	0.10762 (0.053254)	0.417776 (0.15071)	10.98420 (0.048171)	0.106840 (0.078242)	0.574552 (0.078744)
40	11.44770 (0.032519)	0.118174 (0.025790)	0.423946 (0.080978)	10.87421 (0.023391)	0.108845 (0.037356)	0.586549 (0.041779)
80	11.15780 (0.015944)	0.119735 (0.012698)	0.438659 (0.042023)	10.74670 (0.011528)	0.109872 (0.018256)	0.596061 (0.021540)

Table 6: B.Es and P.Rs under CP using $(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$ and $(0.1, 0.12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.160528 (0.000519)	0.169018 (0.000592)	0.479278 (0.013082)	0.134094 (0.000277)	0.195013 (0.000893)	0.664322 (0.0116345)
40	0.133714 (0.000286)	0.147046 (0.000329)	0.468172 (0.007333)	0.11004 (0.000116)	0.172682 (0.000448)	0.663936 (0.006115)
80	0.111222 (0.000147)	0.139219 (0.000196)	0.448705 (0.004006)	0.103352 (0.000058)	0.151533 (0.000223)	0.66273 (0.003196)
k-loss function						
20	0.161522 (0.04053)	0.158551 (0.043568)	0.465088 (0.145568)	0.13577 (0.029991)	0.198858 (0.049799)	0.653999 (0.061370)
40	0.122886 (0.030112)	0.14806 (0.031199)	0.464529 (0.072191)	0.112829 (0.019077)	0.167765 (0.034061)	0.652043 (0.030841)
80	0.104790 (0.02132)	0.132506 (0.020869)	0.463772 (0.038239)	0.101234 (0.010604)	0.143011 (0.020665)	0.650817 (0.015416)

Table 7: B.Es and P.Rs under CP using $(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.45)$ and $(1, 1.2, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.52970 (0.046080)	1.57752 (0.051958)	0.476725 (0.013179)	1.35361 (0.027818)	1.88630 (0.080777)	0.663527 (0.011650)
40	1.26130 (0.026077)	1.41822 (0.031378)	0.469789 (0.007406)	1.16501 (0.013935)	1.66429 (0.044256)	0.659758 (0.006292)
80	1.14492 (0.014751)	1.25617 (0.015752)	0.45906 (0.004146)	1.05707 (0.006667)	1.42491 (0.021669)	0.656866 (0.003355)
k-loss function						
20	1.53236 (0.041669)	1.49543 (0.045185)	0.454182 (0.147876)	1.32818 (0.028990)	1.84131 (0.048999)	0.654349 (0.0615537)
40	1.25475 (0.028827)	1.43835 (0.028914)	0.452576 (0.073808)	1.13137 (0.019386)	1.63107 (0.034167)	0.654126 (0.031362)
80	1.06717 (0.018469)	1.33081 (0.018471)	0.450778 (0.036839)	1.04654 (0.0119729)	1.43309 (0.024099)	0.653583 (0.016246)

Table 8: B.Es and P.Rs under CP using $(\lambda_1, \lambda_2, p_1) = (10, 12, 0.45)$ and $(10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	5.53623 (0.433161)	5.39552 (0.43394)	0.470341 (0.013715)	5.58208 (0.415167)	5.00538 (0.452878)	0.640696 (0.012735)
40	6.27422 (0.39050)	6.44220 (0.396637)	0.456349 (0.0072919)	6.62719 (0.356135)	5.85830 (0.422084)	0.629771 (0.006889)
80	7.34364 (0.339835)	7.76371 (0.337258)	0.45011 (0.003809)	7.60293 (0.287163)	7.02488 (0.384135)	0.625043 (0.003610)
k-loss function						
20	5.52495 (0.029004)	5.37683 (0.030637)	0.453655 (0.14971)	5.8203 (0.024786)	4.95789 (0.037429)	0.629618 (0.072137)
40	6.24029 (0.020289)	6.40839 (0.019526)	0.447779 (0.077333)	6.59809 (0.016483)	5.99867 (0.025214)	0.623712 (0.037649)
80	7.05432 (0.012209)	7.72934 (0.010219)	0.448146 (0.040194)	7.46259 (0.010121)	6.96302 (0.015946)	0.622423 (0.019236)

Table 9: B.Es and P.Rs under CP using $(\lambda_1, \lambda_2, p_1) = (0.10, 12, 0.45)$ and $(0.10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.135859 (0.000265)	5.51759 (0.447943)	0.526231 (0.012354)	0.13051 (0.000208)	5.08452 (0.461858)	0.682633 (0.010735)
40	0.115993 (0.000132)	6.65447 (0.40904)	0.513532 (0.006542)	0.112149 (0.000099)	6.04774 (0.435674)	0.674904 (0.005747)
80	0.099126 (0.000059)	8.88344 (0.346268)	0.506878 (0.003368)	0.102298 (0.000048)	7.33437 (0.384372)	0.670852 (0.002976)
k-loss function						
20	0.142139 (0.028685)	5.48345 (0.030311)	0.513036 (0.104196)	0.129112 (0.024474)	5.04152 (0.037048)	0.673849 (0.052484)
40	0.115279 (0.019664)	6.60253 (0.018872)	0.506765 (0.053772)	0.110234 (0.015909)	6.371890 (0.024397)	0.670396 (0.026986)
80	0.105346 (0.012065)	8.09129 (0.010753)	0.503448 (0.027325)	0.104518 (0.009303)	7.380910 (0.015178)	0.668463 (0.0141179)

Table 10: B.Es and P.Rs under CP using $(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.45)$ and $(10, 0.12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	5.62845 (0.441878)	0.146896 (0.000328)	0.421965 (0.012086)	5.96024 (0.424476)	0.151952 (0.000431)	0.578365 (0.012083)
40	6.44104 (0.400232)	0.127722 (0.000153)	0.43595 (0.006316)	6.85842 (0.368546)	0.136448 (0.000225)	0.578932 (0.006429)
80	7.58808 (0.343598)	0.117307 (0.000074)	0.45587 (0.0032287)	8.06977 (0.302239)	0.120046 (0.000104)	0.5886153 (0.003318)
k-loss function						
20	5.58490 (0.028685)	0.14275 (0.030323)	0.405755 (0.162971)	5.89857 (0.024474)	0.159564 (0.037102)	0.566651 (0.083544)
40	6.40615 (0.019662)	0.126893 (0.018875)	0.417657 (0.084299)	6.84141 (0.015908)	0.132547 (0.024410)	0.561311 (0.043066)
80	7.44424 (0.01207)	0.117069 (0.010756)	0.439330 (0.042888)	7.71344 (0.009358)	0.121066 (0.014498)	0.558488 (0.021872)

Table 11: B.Es and P.Rs under RP using $(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$ and $(0.1, 0.12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.105782	0.135123	0.434012	0.103079	0.138375	0.645401
	(0.000427)	(0.000581)	(0.012329)	(0.000282)	(0.000938)	(0.003282)
40	0.10453	0.131766	0.474736	0.096193	0.138033	0.641649
	(0.000233)	(0.000296)	(0.007004)	(0.000128)	(0.000453)	(0.006242)
80	0.096183	0.129324	0.468102	0.093818	0.130254	0.64095
	(0.000103)	(0.0001600)	(0.003756)	(0.000058)	(0.000234)	(0.003206)
k-loss function						
20	0.101651	0.123051	0.485036	0.093688	0.114119	0.630198
	(0.0687089)	(0.066684)	(0.119068)	(0.049411)	(0.088853)	(0.06578)
40	0.101989	0.112941	0.467163	0.0975664	0.129533	0.624327
	(0.042361)	(0.036825)	(0.069873)	(0.026666)	(0.049849)	(0.031978)
80	0.100056	0.118725	0.46499	0.0978534	0.124279	0.623173
	(0.022779)	(0.020242)	(0.037457)	(0.0121946)	(0.023696)	(0.014910)

Table 12: B.Es and P.Rs under RP using $(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.45)$ and $(1, 1.2, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.05922	1.33314	0.433188	1.03293	1.51575	0.644157
	(0.042667)	(0.056026)	(0.012338)	(0.027869)	(0.098507)	(0.011345)
40	1.00664	1.29054	0.476983	0.970745	1.39351	0.642962
	(0.021771)	(0.029663)	(0.007093)	(0.012762)	(0.045143)	(0.006225)
80	0.983895	1.289237	0.475851	0.983541	1.37069	0.62145
	(0.011111)	(0.016032)	(0.003774)	(0.011043)	(0.019325)	(0.003245)
k-loss function						
20	1.16454	1.28713	0.468544	0.996425	1.09914	0.62148
	(0.072345)	(0.063822)	(0.131557)	(0.064813)	(0.119828)	(0.070912)
40	1.00749	1.12108	0.466946	0.985194	1.35608	0.620150
	(0.041506)	(0.035513)	(0.069635)	(0.027093)	(0.050711)	(0.032581)
80	0.954177	1.11020	0.457543	0.995916	1.34535	0.618873
	(0.020998)	(0.019217)	(0.034475)	(0.013199)	(0.025414)	(0.015959)

Table 13: B.Es and P.Rs under RP using $(\lambda_1, \lambda_2, p_1) = (10, 12, 0.45)$ and $(10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	9.58324	11.43070	0.427949	9.58708	10.9565	0.634954
	(3.169330)	(3.83663)	(0.012469)	(2.322100)	(5.00616)	(0.011817)
40	9.67622	11.78300	0.47243	9.599110	11.9233	0.630872
	(1.924840)	(2.34378)	(0.007076)	(1.29289)	(3.42684)	(0.006453)
80	9.68654	12.34210	0.470593	9.73014	12.86730	0.615151
	(0.947672)	(1.416400)	(0.003749)	(0.586285)	(2.08719)	(0.003283)
k-loss function						
20	8.12011	11.32660	0.464824	10.09989	12.1777	0.629444
	(0.068480)	(0.058682)	(0.133263)	(0.0493049)	(0.088018)	(0.066076)
40	9.37871	11.84870	0.461711	9.41742	11.96030	0.616346
	(0.038674)	(0.034311)	(0.069343)	(0.027143)	(0.051405)	(0.034086)
80	9.42150	11.9648	0.460929	9.571406	12.45630	0.610353
	(0.021294)	(0.019479)	(0.036338)	(0.012793)	(0.025410)	(0.015892)

Table 14: B.Es and P.Rs under RP using $(\lambda_1, \lambda_2, p_1) = (0.10, 12, 0.45)$ and $(0.10, 12, 0.60)$

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.094884	11.89990	0.469921	0.098341	11.96510	0.664261
	(0.000263)	(3.293650)	(0.011542)	(0.000209)	(4.61595)	(0.010333)
40	0.095518	12.63033	0.509872	0.098953	12.78527	0.656383
	(0.000137)	(1.93028)	(0.006313)	(0.000098)	(2.78527)	(0.005624)
80	0.091143	12.39790	0.495107	0.099681	12.49280	0.646003
	(0.000063)	(1.10531)	(0.003307)	(0.000047)	(1.58923)	(0.002943)
k-loss function						
20	0.114775	12.94880	0.506062	0.107666	13.3107	0.655632
	(0.059662)	(0.048205)	(0.099563)	(0.043940)	(0.067830)	(0.052989)
40	0.094401	12.89850	0.503309	0.091838	12.75870	0.646092
	(0.030531)	(0.024543)	(0.052498)	(0.022345)	(0.034796)	(0.027125)
80	0.091618	12.72390	0.501733	0.0904158	12.41740	0.636729
	(0.015444)	(0.012387)	(0.026989)	(0.011268)	(0.017632)	(0.013727)

Table 15: B.Es and P.Rs under RP using $(\lambda_1, \lambda_2, p_1)$ = (10, 0.12, 0.45) and (10, 0.12, 0.60)

n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	10.7884 (3.33464)	0.11846 (0.000331)	0.372751 (0.010833)	10.50830 (2.35803)	0.119567 (0.000474)	0.567091 (0.0113719)
40	10.3168 (1.92637)	0.11824 (0.000153)	0.406198 (0.006094)	10.41890 (1.30849)	0.116756 (0.000221)	0.567909 (0.006219)
80	10.30871 (0.992713)	0.119613 (0.000074)	0.427844 (0.003169)	10.39070 (0.686735)	0.1198576 (0.0001029)	0.585739 (0.003262)
k-loss function						
20	9.81163 (0.059662)	0.152163 (0.048174)	0.406292 (0.150848)	9.22044 (0.043940)	0.127570 (0.067755)	0.555927 (0.081129)
40	10.94770 (0.030533)	0.111752 (0.024540)	0.419822 (0.080943)	10.64808 (0.022345)	0.12357 (0.034789)	0.55487 (0.042404)
80	10.57770 (0.015446)	0.112851 (0.012385)	0.429373 (0.042003)	10.41660 (0.011269)	0.123356 (0.017625)	0.550732 (0.021699)

Numerical results of the simulation study, presented in tables 1-15, reveal interesting properties of the proposed Bayes estimators. The estimated values of the parameters converge to the true values, and amounts of posterior risks tend to decrease for lager choice of sample size. Another interesting point concerning the posterior risks of the estimates of λ_1, λ_2 is that increasing (decreasing) the proportion of the component in mixture reduces (increases) the amount of the posterior risk for the estimates of λ_1. In addition, when SELF is assumed and values of λ_i are relatively smaller i.e. for (λ_1, λ_2) = (0.1, 0.12) and (1, 1.2), the Bayes estimates assuming Rayleigh prior are more precise than the rest of the informative priors, as the averaged posterior risks of the mixture components are smaller as compared to those under other priors. On the other hand, for quite larger values of parameters, i.e. for (λ_1, λ_2) = (10, 12), and for significantly different values of the parameters, i.e. for (λ_1, λ_2) = (0.1, 12), the estimates under chi prior (with few exceptions) perform better than those under Nakagami and Rayleigh priors. However, the estimates for the mixing parameter (p_1), under Rayleigh prior, are associated with the minimum amounts of posterior risks irrespective of choice of true parametric values. When KLF is assumed, the estimates under chi prior are found to be the most efficient for all combinations of the values of the parameters, with an exception in case of (λ_1, λ_2) = (0.1, 0.12), where the estimates under the assumption of Nakagami prior are better than those under other priors. However, the estimates for the mixing parameter (p_1) are having mixed behavior, as for various choices of the true parametric values indicate the preference of different priors.

The Bayes estimates of the lifetime parameters are over/under-estimated but the size of over/under-estimation is greater under squared error loss function. On the other hand, estimates of the mixing proportion parameter have mixed behavior sometimes over-estimated and sometimes under-estimated, but the Bayes estimates under Rayleigh prior are much closer to the true parametric value. In comparison of loss functions it has been assessed that the magnitudes of posterior risks under squared error loss function are smaller than those under k-loss function for smaller choice of true parametric values i.e. for $(\lambda_1, \lambda_2) = (0.1, 0.12)$ and $(1, 1.2)$. On the other hand, for quite larger values of parameters, i.e. for $(\lambda_1, \lambda_2) = (10, 12)$, and for significantly different values of the parameters, i.e. for $(\lambda_1, \lambda_2) = (0.1, 12)$ and $(10, 0.12)$, the k-loss function produces the better results. It should also be mentioned here that the squared error loss function produces better convergence than k-loss function.

Real Data Analysis

In this section, real datasets are analyzed to illustrate the methodology discussed in the previous sections. In order to show the usefulness of the proposed mixture model, we applied the findings of the paper to the survival times (in years) of a group of patients given chemotherapy treatment. The data has been reported by Bekker et al. (2000). We have used the Kolmogorov-Smirnov and chi square tests to see whether the data follow the Rayleigh distribution. These tests say that the data follow the Rayleigh distribution at 5% level of significance with p-values 0.2170 and 0.2681 respectively. The data consisting of 46 survival times (in years) for 46 patients are:

Table 16: Survival times (in years) of patients given chemotherapy treatment

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395,
0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841,
0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343,
2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Consider the case when the data are doubly type II censored. Data are randomly grouped into two sets using probabilistic mixing for $p_1 = 0.60$.

Table 17: Doubly censored mixture real life data regarding survival times of patients given chemotherapy treatment

Population-I	**Population-II**
0.197, 0.534, 0.115, 0.296, 0.121, 0.466, 0.529, 1.447, 0.863, 0.132, 0.395, 0.696, 2.825, 3.658, 3.978, 3.743, 2.343, 2.178, 0.540, 4.003, 1.553, 1.485, 2.83, 2.416	0.260, 1.099, 0.501, 0.458, 0.641, 0.334, 0.570, 0.164, 0.203, 0.282, 0.047, 1.271, 1.589, 1.326, 0.841, 2.444

The following characteristics are extracted from censored data for the analysis of mixture model:

$p_1 = 0.6$, $n = 40$, $r = 5$, $r_1 = 2$, $r_2 = 3$, $n - r = 9$, $s = 36$, $s_1 = 22$, $s_2 = 14$, $n_1 = 24$, $n_2 = 16$, $x_{r_1} = 0.121,\ x_{s_1} = 3.978,\ x_{r_2} = 0.203,$ and $x_{s_2} = 2.444.$ $\sum_{i=r_1}^{s_1} x^2{}_{1(i)} = 84.6037$ and $\sum_{i=r_2}^{s_2} x^2{}_{2(i)} = 15.2833.$

The similar methodology has been employed when $p_1 = 0.45$.

$p_1 = 0.45$, $n = 40$, $r = 5$, $r_1 = 2$, $r_2 = 3$, $n - r = 9$, $s = 36$, $s_1 = 16$, $s_2 = 20$, $n_1 = 18$, $n_2 = 22$, $x_{r_1} = 0.121,\ x_{s_1} = 3.658,\ x_{r_2} = 0.164,$ and $x_{s_2} = 3.978,$ $\sum_{i=r_1}^{s_1} x^2{}_{1(i)} = 48.704$ and $\sum_{i=r_2}^{s_2} x^2{}_{2(i)} = 37.1999.$

Bayes estimates are obtained assuming informative priors under squared error loss function, and k-loss function.

Table 18: B.Es and P.Rs in parentheses under squared error loss function, and k-loss function for real data set.

Priors	squared error loss function			k-loss function		
p1 = 0.6	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Nakagami Prior	0.383514 (0.001682)	0.947312 (0.016257)	0.677477 (0.005652)	0.38129 (0.023399)	0.938511 (0.037685)	0.673064 (0.026312)
Chi Prior	0.462925 (0.001678)	1.148390 (0.015708)	0.674903 (0.005747)	0.461094 (0.015909)	1.141450 (0.024405)	0.670395 (0.026986)
Rayleigh Prior	0.392190 (0.001681)	0.980312 (0.016213)	0.665364 (0.005626)	0.390017 (0.022350)	0.971852 (0.034973)	0.6608996 (0.027130)
***p*1 = 0.45**	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Nakagami Prior	0.377204 (0.0024201)	0.722023 (0.007572)	0.516536 (0.006537)	0.373998 (0.034429)	0.716491 (0.030999)	0.509802 (0.053186)
Chi Prior	0.482646 (0.002974)	0.837202 (0.008188)	0.511665 (0.006693)	0.479778 (0.023980)	0.831775 (0.026185)	0.504667 (0.055851)
Rayleigh Prior	0.388672 (0.002412)	0.740606 (0.007504)	0.508438 (0.006381)	0.385574 (0.032264)	0.735272 (0.029121)	0.50177 (0.053512)

The findings from the real life analysis are in close accordance with those of simulation study. It can be assessed that the chi prior produces better results for parameters $\lambda_1$and λ_2, while in case of mixing parameter the Rayleigh prior provides comparatively better results than other priors. It should further be noted that the estimates under squared error loss function are associated with smaller amounts of posterior risks.

Conclusion

The Bayesian inference of the mixture of Rayleigh model under doubly type II censoring has been considered assuming informative priors. The simulation study has displayed some interesting properties of the Bayes estimates. It is noted in each case that the posterior risks of estimates of lifetime parameters are reduced as the sample size increases. The results indicated that by using SELF and relatively smaller values of λ_i i.e. for (λ_1, λ_2) = (0.1, 0.12) and (1, 1.2), the Bayes estimates assuming Rayleigh prior are more precise than the rest of the informative priors. While, for quite larger values of parameters, i.e. for (λ_1, λ_2) = (10, 12), and for significantly different values of the parameters, i.e. for (λ_1, λ_2) = (0.1, 12) and (10, 0.12), the estimates under chi prior perform better than other priors. Similarly, when KLF is considered, the estimates under chi prior are found to be the most efficient for most of the combinations of the values of the parameters. The performance of the squared error loss function is better than k-loss function for $(\lambda_1, \lambda_2) = (0.1, 0.12)$ and (1, 1.2). However, for $(\lambda_1, \lambda_2) = (10, 12)$, (0.1, 12) and (10, 0.12), the k-loss function produces the better results. It should also be mentioned here that the squared error loss function produces better convergence than k-loss function for almost all the cases. The real life example further strengthened the findings from the simulation study. The study can further be extended by considering some other censoring techniques, and using some more flexible probability distribution.

References

Aslam, M. (2003). An application of prior predictive distribution to elicit the prior density. *Journal of Statistical Theory and Applications*, *2*(1), 70-83.

Bekker, A., Roux, J., & Mostert, P. (2000). A generalization of the compound Rayleigh distribution: using a Bayesian method on cancer survival times. *Communications in Statistics, Theory and Methods*, *29*(7), 1419-1433.

Blischke, W. R., & Murthy, D. N. P. (2000). *Reliability*. New York: John Wiley and Sons.

Bolstad, W. M. (2004). *Introduction to Bayesian Statistics*. New York: John Wiley and Sons.

Eluebaly, T., & Bougila, N. (2011). Bayesian learning of finite generalized Gaussian mixture models on images. *Signal Processing*, *91*(4), 801-820.

Fernandez, A. J. (2000). On maximum likelihood prediction based on type II doubly censored exponential data. *Metrika*, *50*(3), 211-220.

Fernandez, A. J. (2006). Bayesian estimation based on trimmed samples from Pareto populations. *Computational statistics & data analysis*, *51*(2), 1119-1130.

Ghosh, S. K., & Ebrahimi, N. (2001). *Bayesian analysis of the mixing function in a mixture of two exponential distributions*. Tech. Rep. 2531, Institute of Statistics Mimeographs, North Carolina State University, North Carolina State University.

Grimshaw, S D., Collings, B. J., Larsen, W. A., & Hurt, C. R. (2001). Eliciting Factor Importance in a Designed Experiment. *Technometrics*, *43*(2), 133-146.

Jenkinson, D. (2005). *The elicitation of probabilities: A review of the statistical literature*. Department of Probability and Statistics, University of Sheffield.

Khan, H. M., Albatineh, A., Alshahrani, S., Jenkins, N., & Ahmed, N. U. (2011). Sensitivity analysis of predictive modeling for responses from the three-parameter Weibull model with a follow-up doubly censored sample of cancer patients. *Computational Statistics & Data Analysis*, *55*(12), 3093-3103.

Khan, H. M., Provost, S. B., & Singh, A. (2010). Predictive inference from a two-parameter Rayleigh life model given a doubly censored sample. *Communications in Statistics—Theory and Methods*, *39*(7), 1237-1246.

Kim, C., & Song, S. (2010). Bayesian estimation of the parameters of the generalized exponential distribution from doubly censored samples. *Statistical Papers*, *51*(3), 583-597.

Kundu, D., & Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. *Computational Statistics & Data Analysis*, *54*(6), 1547-1558.

Leon, J. C., Vazquez-Polo, J. F., & Gonzalez, L. R. (2003). Environmental and resource economics. *Kluwer Academic Publishers*, *26*, 199-210.

Mendenhall, W. & Hadar, R. J. (1958). Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. *Biometrika*, *45*, 504-520.

Murthy, D. N. P., Bulmer, M. & Eccleston, J. A. (2004). Weibull model selection for reliability modelling. *Reliability Engineering & System Safety*, *86*, 257-267.

Noor, F. & Aslam, M. (2008). Bayesian inference of the inverse Weibull mixture distribution using type-I censoring. *Journal of Applied Statistics*, *40*(5), 1076-1089.

O'Hagan, A., Buck, C. E., Daneshkhah, A., Eiser, J. E., Garthwaite, P. H., Jenkinson, D. J., … Rakow, T. (2006). *Uncertain Judgements: Eliciting expert probabilities*. New York: John Wiley and Sons.

Pak, A., Parham, G. A. & Saraj, M. (2013). On Estimation of Rayleigh Scale Parameter under Doubly Type-II Censoring from Imprecise Data. *Journal of Data Science*, *11*, 305-322.

Saleem, M., & Aslam, M. (2008). Bayesian analysis of the two component mixture of the Rayleigh distribution assuming the Uniform and the Jeffreys prior from censored data. *Journal of Applied Statistical Science*, *16*(4), 105-113.

Saleem, M., Aslam, M., & Economou, P. (2010). On the Bayesian analysis of the mixture of Power function distribution using the complete and the censored sample. *Journal of Applied Statistics*, *37*(1), 25-40.

Shi, Y., & Yan, W. (2010). The EB estimation of scale parameter for two parameter exponential distribution under the type-I censoring life test. *Journal of Physical Science*, *4*, 25-30.

Singh, S. K., Singh, U., & Sharma, V. K. (2013). Bayesian analysis for Type-II hybrid censored sample from inverse Weibull distribution. *International Journal of System Assurance Engineering*, *1-8*.

Soliman, A. A. (2006). Estimators for the finite mixture of Rayleigh model based on progressively censored data. *Communications in Statistics—Theory and Methods*, *35*, 803-820.

Sultan, K. S., & Al-Moisheer, A. S. (2012). Approximate Bayes estimation of the parameters and reliability function of a mixture of two inverse Weibull distributions under type-2 censoring. *Journal of Statistical Computation and Simulation*, *1-15*.

Sultan, K. S., Ismail, M. A., & Al-Moisheer, A. S. (2007). Mixture of two inverse Weibull distributions: Properties and estimation. *Computational Statistics & Data Analysis*, *51*(11), 5377-5387.

Wasan, M. T. (1970). *Parametric estimation*. New York: McGraw-Hill.