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APPLICATIONS OF VARIATIONAL ANALYSIS
TO A GENERALIZED HERON PROBLEM

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Abstract. This paper is a continuation of our ongoing efforts to solve a number of geometric problems and their extensions by using advanced tools of variational analysis and generalized differentiation. Here we propose and study, from both qualitative and numerical viewpoints, the following optimal location problem as well as its further extensions: on a given nonempty subset of a Banach space, find a point such that the sum of the distances from it to n given nonempty subsets of this space is minimal. This is a generalized version of the classical Heron problem: on a given straight line, find a point C such that the sum of the distances from C to the given points A and B is minimal. We show that the advanced variational techniques allow us to completely solve optimal location problems of this type in some important settings.

Key words. Heron problem and its extensions, variational analysis and optimization, generalized differentiation, minimal time function, convex and nonconvex sets.

AMS subject classifications. 49J52, 49J53, 90C31.

1 Introduction and Problem Formulation

In this paper we propose and largely investigate various extensions of the Heron problem, which seem to be mathematically interesting and important for applications. In particular, the one of this type is to replace two given points in the classical Heron problem by finitely many nonempty closed subsets of a Banach space and to replace the straight line therein by another nonempty closed subset of this space. The reader are referred to our paper [14] for partial results concerning a convex version of this problem in the Euclidean space $\mathbb{R}^n$.

Recall that the classical Heron problem was posted by Heron from Alexandria (10–75 AS) in his *Catroptica* as follows: find a point on a straight line in the plane such that the sum of the distances from it to two given points is minimal; see [4, 6] for more discussions. We formulate the *distance function version* of the *generalized Heron problem* as follows:

$$\text{minimize } D(x) := \sum_{i=1}^{n} d(x;\Omega_i) \text{ subject to } x \in \Omega, \tag{1.1}$$

where Ω and Ω_i, $i = 1, \ldots, n, n \geq 2$, are given nonempty closed subsets of a Banach space X endowed with the norm $\|\cdot\|$, and where

$$d(x;Q) := \inf\{\|x-y\| \mid y \in Q\}. \tag{1.2}$$

is the usual distance from $x \in X$ to a set Q. Observe that in this new formulation the generalized Heron problem (1.1) is an extension of the *generalized Fermat-Torricelli* problem proposed and studied in [13]. The difference is that the latter problem in unconstrained, i.e., $\Omega = X$ in (1.1) while the presence of the *geometric constraint* in the generalized Heron version (1.1) makes it more mathematically complicated and more realistic for applications. Among the most natural areas of applications we mention constrained problems arising in location science, optimal networks, wireless communications, etc. We refer the reader to the corresponding discussions and results in [13] and the bibliographies therein concerning unconstrained Fermat-Torricelli-Steiner-Weber versions. Needless

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to say that the presence of geometric (generally nonconvex) constraints in (1.1) essentially changes these versions while referring us to the original Heron geometric problem.

In fact, we are able to investigate a more general version of problem (1.1), where the distance function (1.2) is replaced by the so-called *minimal time function*

$$T_Q^F(x) := \inf\{t \geq 0 \big| \, Q \cap (x + tF) \neq \emptyset\} \tag{1.3}$$

with the *constant dynamics* $\dot{x} \in F \subset X$ and the *target* set $Q \subset X$ in a Banach space X; see [12] and the references therein for more discussions and results on this class of functions important for various aspects of optimization theory and its numerous applications.

The main problem under consideration in this paper, called below the *generalized Heron problem*, is formulated as follows:

$$\text{minimize } \; T(x) := \sum_{i=1}^{n} T_{\Omega_i}^F(x) \; \text{ subject to } \; x \in \Omega, \tag{1.4}$$

where F is a closed, bounded, and convex set containing the origin as an interior point, and where Ω and Ω_i for $i = 1, \ldots, n$ are nonempty closed subsets of a Banach space X; these are the *standing assumptions* of the paper.

When $F = I\!B$ in (1.4), this problem reduces to the one in (1.1). Note that involving the minimal time function (1.3) into (1.4) instead of the distance function in (1.1) allows us to cover some important location models that cannot be encompassed by formalism (1.1); cf. [15] for the case of convex unconstrained problems of type (1.4) and [13] for the generalized Fermat-Torricelli problem corresponding to (1.4) with $\Omega = X$.

A characteristic feature of the generalized Heron problem (1.4) and its distance function specification (1.1) is that they are *intrinsically nonsmooth*, since the functions (1.2) and (1.3) are nondifferentiable. These problems are generally nonconvex while the convexity of both cost functions in (1.1) and (1.4) follows from the convexity the sets Ω_i. This makes it natural to apply advanced methods and tools of variational analysis and generalized differentiation to study these problems. To proceed in this direction, we largely employ the recent results from [12] on generalized differentiation of the minimal time function (1.3) in convex and nonconvex settings as well as comprehensive rules of generalized differential calculus. As can be seen from the solutions below, the constraint nature of the Heron problem and its extensions leads to new structural phenomena in comparison with the corresponding Fermat-Torricelli counterparts. Note that a number of the results obtained in this paper are new even for the unconstrained setting of the generalized Fermat-Torricelli problem.

The rest of the paper is organized as follows. In Section 2, we present some basic constructions and properties from variational analysis that are widely used in the sequel. Section 3 concerns deriving necessary optimality conditions for solutions to the generalized Heron problem in the case of arbitrary closed sets Ω and Ω_i, $i = 1, \ldots, n$, in (1.4) and its specification (1.1). The results obtained are expressed in terms of the limiting normal cone to closed sets in the sense of Mordukhovich [9]. We pay a special attention to the Hilbert space setting, which allows us to establish necessary (in some cases necessary and sufficient) optimality conditions in the most efficient forms. Some examples are given to illustrate applications of general results in particular situations. In Section 4 we develop a numerical algorithm to solve some versions of the generalized Heron problem in finite dimensions while the concluding Section 5 is devoted to the implementation of this algorithm and its specifications in various settings of their own interest.

Our notation is basically standard in the area of variational analysis and generalized differentiation; see [9, 16]. We recall some of them in the places they appear.

2 Tools of Generalized Differentiation

This section contains basic constructions and results of the generalized differentiation theory in variational analysis employed in what follows. The reader can find all the proofs, discussions, and additional material in the books [2, 9, 10, 16, 17] and the references therein.

Given an extended-real-valued function $\varphi\colon X \to \overline{I\!R} := (-\infty,\infty]$ with $\bar{x}$ from the domain $\operatorname{dom}\varphi := \{x \in X \mid \varphi(x) < \infty\}$ and given $\varepsilon \ge 0$, define first the *ε-subdifferential* of φ at $\bar{x}$ by

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) := \Big\{x^* \in X^* \Big|\ \liminf_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \ge -\varepsilon\Big\}. \tag{2.1}$$

For $\varepsilon = 0$ the set $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0\varphi(\bar{x})$ is known as *Fréchet/regular subdifferential* of φ at $\bar{x}$. It follows from definition (2.1) that regular subgradients are described as follows: $x^* \in \widehat{\partial}_\varepsilon\varphi(\bar{x})$ if and only if for any $\eta > 0$ there is $\gamma > 0$ such that

$$\langle x^*, x-\bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}) + (\varepsilon+\eta)\|x-\bar{x}\| \ \text{ whenever } \ x \in \bar{x} + \gamma I\!B$$

with $I\!B$ standing for the closed unit ball of the space in question. When φ is Fréchet differentiable at $\bar{x}$, its regular subdifferential $\widehat{\partial}\varphi(\bar{x})$ reduces to the classical gradient $\{\nabla\varphi(\bar{x})\}$. Despite the simple definition (2.1) closely related to the classical derivative, the regular subdifferential and its ε-enlargements in general do not happen to be appropriate for applications to the generalized Heron problem under consideration due to the serious lack of calculus rules.

To get a better construction, we need to employ a certain robust limiting procedure, which lies at the heart of variational analysis. Recall that, given a set-valued mapping $G\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, the *sequential Painlevé-Kuratowski outer limit* of G as $x \to \bar{x}$ is defined by

$$\begin{aligned}\operatorname*{Lim\,sup}_{x\to\bar{x}} G(x) := \Big\{x^* \in X^* \Big|\ & \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ as } k \to \infty \\ & \text{such that } x_k^* \in G(x_k) \text{ for all } k \in I\!N := \{1,2,\ldots\}\Big\},\end{aligned} \tag{2.2}$$

where w^* signifies the weak* topology of X^*. Applying the limiting operation (2.2) to the set-valued mapping $(x,\varepsilon) \rightrightarrows \widehat{\partial}_\varepsilon\varphi(x)$ in (2.1) and using the notation $x \xrightarrow{\varphi} \bar{x} := x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ give us the subgradient set

$$\partial\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon\varphi(x) \tag{2.3}$$

known as the *Mordukhovich/limiting subdifferential* of φ at $\bar{x}$. We can equivalently put $\varepsilon = 0$ in (2.3) if φ is lower semicontinuous around $\bar{x}$ and if X is *Asplund*, i.e., each of its separable subspaces has a separable dual; the latter is automatics, e.g., when X is reflexive. Recall that φ is *subdifferentially regular* at $\bar{x}$ if $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x})$.

Note that every convex function φ is subdifferentially regular at any point $\bar{x} \in \operatorname{dom}\varphi$ with the classical subdifferential representation

$$\partial\varphi(\bar{x}) = \big\{x^* \in X^* \ \big|\ \langle x^*, x-\bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}) \ \text{ for all } \ x \in X\big\}. \tag{2.4}$$

However, the latter property often fails in nonconvex setting, where $\widehat{\partial}\varphi(\bar{x})$ may be empty (as for $\varphi(x) = -|x|$ at $\bar{x} = 0$) with a poor calculus, while the limiting subdifferential (2.3) enjoys a *full calculus* (at least in Asplund spaces) due to *variational/extremal principles* of variational analysis. We following calculus results are most useful in this paper.

Theorem 2.1 (subdifferential sum rules). *Let $\varphi_i\colon X \to \overline{I\!R}$, $i = 1,\ldots,n$, be lower semicontinuous functions on a Banach space X. Suppose that all but one of them are locally Lipschitzian around $\bar{x} \in \cap_{i=1}^n \operatorname{dom}\varphi_i$. Then:*

(i) *We have the inclusion*

$$\partial\Big(\sum_{i=1}^n \varphi_i\Big)(\bar{x}) \subset \sum_{i=1}^n \partial\varphi_i(\bar{x}) \tag{2.5}$$

provided that X is Asplund. Furthermore, inclusion (2.5) becomes an equality if all the functions φ_i are subdifferentially regular at $\bar{x}$.

(ii) *When all the functions* φ_i *are convex, the equality*

$$\partial\Big(\sum_{i=1}^{n}\varphi_i\Big)(\bar{x})=\sum_{i=1}^{n}\partial\varphi_i(\bar{x}) \tag{2.6}$$

holds with no Asplund space requirement.

Note that assertion (ii) of Theorem 2.1, which is the classical Moreau-Rockafellar theorem, is a consequence of assertion (i) in the case of Asplund spaces; see [9, Theorem 3.36].

Finally in this section, recall that the corresponding *normal cones* to a set Ω at $\bar{x}\in\Omega$ can be defined via the subdifferentials (2.1) and (2.3) of the indicator function by

$$\widehat{N}(\bar{x};\Omega):=\widehat{\partial}\delta(\bar{x};\Omega)\ \text{ and }\ N(\bar{x};\Omega):=\partial\delta(\bar{x};\Omega), \tag{2.7}$$

where $\delta(x;\Omega)=0$ if $x\in\Omega$ and $\delta(x;\Omega)=\infty$ otherwise.

3 Optimality Conditions for the Generalized Heron Problem

The main results of this section give necessary optimality conditions for the generalized Heron problem under consideration, which occur to be necessary and sufficient for optimality in the case of convex data. To begin with, we would like make sure that problem (1.4) admits an optimal solution under natural assumptions.

Proposition 3.1 (existence of optimal solutions to the generalized Heron problem). *The generalized Heron problem* (1.4) *admits an optimal solution in each of the following three cases:*
(i) X *is a Banach space, and the constraint set* Ω *is compact.*
(ii) X *is finite-dimensional, and one of the sets* Ω *and* Ω_i *as* $i=1,\ldots,n$ *is bounded.*
(iii) X *is reflexive, the sets* Ω *and* Ω_i *as* $i=1,\ldots,n$ *are convex and one of them is bounded.*

Proof. It follows from [11, Proposition 2.2] that the minimal time function (1.3) and hence the function T in (1.4) are Lipschitz continuous. Thus the conclusion in the case (i) follows from the classical Weierstrass theorem.

Consider the infimum value

$$\gamma:=\inf_{x\in\Omega}T(x)<\infty$$

in problem (1.4) and take a minimizing sequence $\{x_k\}$ with $T(x_k)\to\gamma$ as $k\to\infty$ and $x_k\in\Omega$ for all $k\in I\!N$. Now assume that X is finite dimensional and Ω_1 is bounded. When k is sufficiently large, one has

$$T^F_{\Omega_1}(x_k)\le T(x_k)<\gamma+1.$$

Thus there exist $0\le t_k<\gamma+1$, $f_k\in F$, and $w_k\in\Omega_1$ such that

$$x_k+t_kf_k=w_k.$$

Since both F and Ω_1 are bounded, (x_k) is a bounded sequence, and hence it has subsequence that converges to $\bar{x}\in\Omega$. Then $\bar{x}$ is a solution of the problem under (ii). The proof in case (iii) is similar to that given in [14, Proposition 4.1]. △

To proceed with deriving optimality conditions for the generalized Heron problem (1.4) and its specification (1.1), we need more notation. Define the *support level set*

$$C^*:=\big\{x^*\in X^*\big|\ \sigma_F(-x^*)\le 1\big\}$$

via the *support function* of the constant dynamics

$$\sigma_F(x^*):=\sup_{x\in F}\langle x^*,x\rangle,\quad x^*\in X^*.$$

The *generalized projection* to the target set Q via the minimal time function (1.3) is a set-valued mapping $\Pi_Q^F\colon X \rightrightarrows X$ defined by

$$\Pi_Q^F(x) := Q \cap \big(x + T_Q^F(x)F\big), \quad x \in X. \tag{3.1}$$

Considering further the *Minkowski gauge*

$$\rho_F(x) := \inf\big\{t \ge 0\big|\, x \in tF\big\}, \quad x \in X, \tag{3.2}$$

and involving the limiting normal cone from (2.7), we define the sets

$$A_i(x) := \begin{cases} \displaystyle\bigcup_{\omega \in \Pi_{\Omega_i}^F(x)} \big[-\partial\rho_F(\omega - x) \cap N(\omega;\Omega_i)\big] & \text{for } x \notin \Omega_i,\ \Pi_{\Omega_i}^F(x) \ne \emptyset, \\ N(x;\Omega_i) \cap C^* & \text{for } x \in \Omega_i \text{ as } i = 1,\ldots,n. \end{cases} \tag{3.3}$$

We say that the minimal time function $T_Q^F(\cdot)$ is *well posed* at $\bar{x}$ if for every sequence $\{x_k\}$ converging to $\bar{x}$ there is a sequence $\{w_k\}$ such that $w_k \in \Pi_Q^F(x_k)$ and $\{w_k\}$ contains a convergent subsequence. The reader is referred to [12, Proposition 6.2] for a number of verifiable conditions ensuring such a well-posedness of the minimal time function.

Our first theorem establishes necessary as well as necessary and sufficient conditions for optimality in (1.4) via the sets $A_i(x)$ from (3.3) in general infinite-dimensional settings.

Theorem 3.2 (necessary and sufficient optimality conditions for the generalized Heron problem in Banach and Asplund spaces). *Given $\bar{x} \in \Omega$, suppose in the setting of* (1.4) *that the minimal time function $T_{\Omega_i}^F$ is well posed at $\bar{x}$ for each $i \in \{1,\ldots,n\}$ such that $\bar{x} \notin \Omega_i$. The following assertions hold:*

(i) *Let $\bar{x}$ be a local optimal solution to* (1.4), *and let X be Asplund. Then we have*

$$0 \in \sum_{i=1}^n A_i(\bar{x}) + N(\bar{x};\Omega), \tag{3.4}$$

where the sets $A_i(\bar{x})$ are defined in (3.3).

(ii) *Let X be a general Banach space, and let all the sets Ω and Ω_i as $i = 1,\ldots,n$ be convex. Given $\bar{x} \in \Omega$, assume that $\Pi_{\Omega_i}^F(\bar{x}) \ne \emptyset$ for $i = 1,\ldots,n$ with $\bar{x} \notin \Omega_i$, select any $\omega \in \Pi_{\Omega_i}^F(\bar{x})$, and construct $A_i(\bar{x})$ by*

$$A_i(\bar{x}) := N(\bar{\omega};\Omega_i) \cap \big[-\partial\rho_F(\bar{\omega} - \bar{x})\big] \quad \textit{for } \bar{x} \notin \Omega_i \tag{3.5}$$

and by the second formula in (3.3) *otherwise. Then $\bar{x}$ is an optimal solution to* (1.4) *if and only if inclusion* (3.4) *is satisfied.*

Proof. Observe first that problem (1.4) can be equivalently written in the form

$$\text{minimize } \ T(x) + \delta(x;\Omega). \tag{3.6}$$

It easily follows from definitions (2.1) and (2.3) of regular and limiting subgradients and their description (2.4) for convex functions that the generalized Fermat rule

$$0 \in \widehat{\partial} f(\bar{x}) \subset \partial f(\bar{x}) \tag{3.7}$$

is a necessary condition for a local minimizer $\bar{x}$ of any function $f\colon X \to \overline{I\!R}$ being also sufficient for this if f is convex. To justify now assertion (i), we apply (3.7) via $\partial f(\bar{x})$ to the cost function $f(x) := T(x) + \delta(x;\Omega)$ in (3.6) and then use the subdifferential sum rule for limiting subgradients from Theorem 2.1(i) in Asplund spaces by taking into account that the functions $T^F_{\Omega_i}$ are Lipschitz continuous. It follows in this way that

$$\begin{aligned} 0 &\in \partial\big(T+\delta(\cdot;\Omega)\big)(\bar x)\subset \partial T(\bar x)+N(\bar x;\Omega)\\ &\subset \sum_{i=1}^n \partial T^F_{\Omega_i}(\bar x)+N(\bar x;\Omega). \end{aligned} \tag{3.8}$$

Employing further the subdifferential formulas for the minimal time function from [13, Theorem 3.1 and Theorem 3.2] gives us

$$\partial T^F_{\Omega_i}(\bar x)\subset A_i(\bar x),\quad i=1,\ldots,n. \tag{3.9}$$

Substituting the latter into (3.8) justifies inclusion (3.4) in assertion (i) of the theorem.

To justify assertion (ii), we apply Theorem 2.1(ii) for convex functions on Banach spaces and conclude in this way that both inclusions "$\subset$" in (3.8) hold as equalities and provide necessary and sufficient optimality conditions for optimality of $\bar x$ in (1.4). Employing finally [12, Theorem 7.1 and 7.3] gives us the equalities in (3.9), where the sets $A_i(\bar x)$ are calculated by (3.5) when $\bar x\notin\Omega_i$. This completes the proof of the theorem. △

It is not hard to check under our standing assumptions that the requirement $\Pi^F_{\Omega_i}(\bar x)\neq\emptyset$ in Theorem 3.2(ii) is automatically satisfied when the space X is reflexive.

The next theorem allows us to significantly simplify the calculation of the sets $A_i(\bar x)$ in Theorem 3.2 for the case of Hilbert spaces and thus to ease the implementation of the optimality conditions obtained therein. Besides this, it leads us to an improvement of optimality under some additional assumptions. Namely, we can replace the limiting normal cone in (3.4) by the smaller regular one for an arbitrary closed constraint set Ω. Define the index sets

$$I(x):=\big\{i\in\{1,\ldots,n\}\;\big|\;x\in\Omega_i\big\}\ \text{ and }\ J(x)=\big\{i\in\{1,\ldots,n\}\;\big|\;x\notin\Omega_i\big\},\quad x\in X. \tag{3.10}$$

We obviously have $I(x)\cup J(x)=\{1,\ldots,n\}$ and $I(x)\cap J(x)=\emptyset$ for all $x\in X$.

Theorem 3.3 (improved optimality conditions in Hilbert spaces). *Consider version (1.1) of the generalized Heron problem with a Hilbert space X in the assumptions of Theorem 3.2. The following assertions hold:*

(i) *Let $\bar x\in\Omega$ be a local optimal solution to (1.1), and let $\Pi(\bar x;\Omega_i)\neq\emptyset$ whenever $i\in J(\bar x)$. Then for any $a_i(\bar x)\in A_i(\bar x)$ as $i\in J(\bar x)$ we have*

$$-\sum_{i\in J(\bar x)} a_i(\bar x)\in\sum_{i\in I(\bar x)} A_i(\bar x)+N(\bar x;\Omega), \tag{3.11}$$

where each set $A_i(\bar x)$ is computed by

$$A_i(\bar x)=\begin{cases} \dfrac{\bar x-\Pi(\bar x;\Omega_i)}{d(\bar x;\Omega_i)} & \text{for } \bar x\notin\Omega_i,\\[2ex] N(\bar x;\Omega_i)\cap I\!B & \text{for } \bar x\in\Omega_i \end{cases} \tag{3.12}$$

whenever $i=1,\ldots,n$. If in addition $I(\bar x)=\emptyset$, then

$$-\sum_{i=1}^n a_i(\bar x)\in\widehat N(\bar x;\Omega). \tag{3.13}$$

(ii) *If all the sets Ω and Ω_i as $i=1,\ldots,n$ are convex, then each set $A_i(\bar x)$ as $i\in J(\bar x)$ in (3.12) is a singleton $\{a_i(\bar x)\}$ and condition (3.11) is necessary and sufficient for the global optimality of $\bar x\in\Omega$ in problem (1.1).*

Proof. To justify assertion (i), pick $\bar{\omega}_i \in \Pi(\bar{x};\Omega_i)$ for all $i \in J(\bar{x})$ such that $a_i(\bar{x}) = \dfrac{\bar{x}-\bar{\omega}_i}{d(\bar{x};\Omega_i)}$ and get the relationships

$$\sum_{i\in J(\bar{x})} \|\bar{x}-\bar{\omega}_i\| + \sum_{i\in I(\bar{x})} d(\bar{x};\Omega_i) = \sum_{i=1}^{n} d(\bar{x};\Omega_i) \le \sum_{i=1}^{n} d(x;\Omega_i) \le \sum_{i\in J(\bar{x})} \|x-\bar{\omega}_i\| + \sum_{i\in I(\bar{x})} d(x;\Omega_i)$$

for all $x \in \Omega$ around $\bar{x}$. This shows that $\bar{x}$ is a local optimal solution to the problem

$$\text{minimize } p(x) := \sum_{i\in J(\bar{x})} \|x-\bar{\omega}_i\| + \sum_{i\in I(\bar{x})} d(x;\Omega_i) \text{ subject to } x \in \Omega. \tag{3.14}$$

Since the norm function on a Hilbert space is Fréchet differentiable in any nonzero point, we conclude that each $p_i(x) := \|x-\bar{\omega}_i\|$ as $i \in J(\bar{x})$ is Fréchet differentiable at $\bar{x}$ with

$$\nabla p_i(\bar{x}) = \frac{\bar{x}-\bar{\omega}_i}{\|\bar{x}-\bar{\omega}_i\|} = \frac{\bar{x}-\bar{\omega}_i}{d(\bar{x};\Omega_i)} = a_i(\bar{x}).$$

Applying to (3.14) the first inclusion in the generalized Fermat rule (3.7) and then using the subdifferential sum rules from [9, Proposition 1.107(i)] for regular subgradients and from Theorem 2.1(i) for limiting ones, we get

$$\begin{aligned}
0 \in \widehat{\partial}\big[p+\delta(\cdot;\Omega)\big](\bar{x}) &= \sum_{i\in J(\bar{x})} \nabla p_i(\bar{x}) + \widehat{\partial}\Big[\sum_{i\in I(\bar{x})} d(\cdot;\Omega_i) + \delta(\cdot;\Omega)\Big](\bar{x}) \\
&\subset \sum_{i\in J(\bar{x})} a_i(\bar{x}) + \partial\Big[\sum_{i\in I(\bar{x})} d(\cdot;\Omega_i) + \delta(\cdot;\Omega)\Big](\bar{x}) \\
&\subset \sum_{i\in J(\bar{x})} a_i(\bar{x}) + \sum_{i\in I(\bar{x})} \partial d(\bar{x};\Omega_i) + N(\bar{x};\Omega) \\
&\subset \sum_{i\in J(\bar{x})} a_i(\bar{x}) + \sum_{i\in I(\bar{x})} \big[N(\bar{x};\Omega_i)\cap I\!B\big] + N(\bar{x};\Omega) \\
&= \sum_{i\in J(\bar{x})} a_i(\bar{x}) + \sum_{i\in I(\bar{x})} A_i(\bar{x}) + N(\bar{x};\Omega),
\end{aligned}$$

where the last three relationships hold since $\bar{x} \in \Omega_i$ for each $i \in I(\bar{x})$. This justifies inclusion (3.11). In the case of $I(\bar{x}) = \emptyset$, we arrive at inclusion (3.13) by the first row of the above relationships and the normal cone definition (2.7).

Assertion (ii) is justified similarly to the proof of Theorem 3.2(ii) by using the results of assertion (i) and the well-known fact that the projection operator for a closed and convex set in a Hilbert space is single-valued. $\triangle$

Observe that in Theorem 3.3, in contrast to Theorem 3.2, we do not impose the well-posedness requirement. In fact, under the assumptions of Theorem 3.3(ii) it holds automatically; see [9, Corollary 1.106]. Note also that in finite-dimensional spaces X we always have the Fréchet differentiability of the distance function at out-of-set points with unique projections (see, e.g., [16, Exercise 8.53]), and so we can deal in the proof of Theorem 3.3(i) directly with the cost function in the generalized Heron problem (1.1), without considering the auxiliary problem (3.14). However, in Hilbert spaces this approach requires additional and unavoidable assumptions on the projection continuity; see [5, Corollary 3.5]. In finite dimensions the projection continuity and Fréchet differentiability of the distance functions actually follows from the projection uniqueness, while it is not the case in Hilbert spaces as shown in [5, Example 5.2]. Observe to this end that neither uniqueness nor continuity of projections is required in Theorem 3.3.

On the other hand, the next result shows that for the unconstrained version of (1.1), i.e., for the generalized Fermat-Torricelli problem [13] with disjoint sets Ω_i, the projection nonemptiness at a local optimal solution automatically implies the projection uniqueness in arbitrary Hilbert spaces.

Proposition 3.4 (projection uniqueness at optimal solutions). *Let $\bar{x}$ be a local optimal solution to problem* (1.4) *in a Hilbert space X with $\Omega = X$ and $\cap_{i=1}^{n}\Omega_i = \emptyset$. Assume that $\bar{x} \notin \Omega_i$ as $i = 1, \ldots, n$. Then the fulfillment of the condition $\Pi(\bar{x}; \Omega_i) \neq \emptyset$ for all $i = 1, \ldots, n$ implies that the projection set $\Pi(\bar{x}; \Omega_i)$ is a singleton whenever $i \in \{1, \ldots, n\}$.*

Proof. Since $I(\bar{x}) = \emptyset$ for the first index set in (3.10), it follows from the proof of Theorem 3.3(i) with $\Omega = X$ that for every $\omega_i \in \Pi(\bar{x}; \Omega_i)$ as $i = 1, \ldots, n$ we have the equality

$$0 = \sum_{i=1}^{n} \frac{\bar{x} - \omega_i}{d(\bar{x}; \Omega_i)}. \tag{3.15}$$

Picking any Ω_i, say Ω_1, let us check that the set $\Pi(\bar{x}; \Omega_1)$ is singleton. Indeed, take two projections $\bar{\omega}_{1,1}, \bar{\omega}_{1,2} \in \Pi(\bar{x}; \Omega_1)$ and fix arbitrary projections $\bar{\omega}_i \in \Pi(\bar{x}; \Omega_i)$ for $i = 2, \ldots, n$. Then from (3.15) we get the relationships

$$0 = \frac{\bar{x} - \bar{\omega}_{1,1}}{d(\bar{x}; \Omega_1)} + \sum_{i=2}^{n} \frac{\bar{x} - \bar{\omega}_i}{d(\bar{x}; \Omega_i)} = \frac{\bar{x} - \bar{\omega}_{1,2}}{d(\bar{x}; \Omega_1)} + \sum_{i=2}^{n} \frac{\bar{x} - \bar{\omega}_i}{d(\bar{x}; \Omega_i)},$$

which imply that $\bar{\omega}_{1,1} = \bar{\omega}_{1,2}$ and thus complete the proof of the proposition. △

Observe that if $\bar{x}$ belongs to one of the sets Ω_i as $i = 1, \ldots, n$, the conclusion of Proposition 3.4 does not generally hold even in finite dimensions as it is demonstrated by the following example.

Example 3.5 (nonuniqueness of projections at solution points). Let $X = \mathbb{R}^2$ in the setting of Proposition 3.4, let Ω_1 be the unit circle of $\mathbb{R}^2$, and let $\Omega_2 = \{(0,0)\}$. Then $\bar{x} = \{(0,0)\}$ is a solution of the Fermat-Torricelli problem generated by Ω_1 and Ω_2, but the projection $\Pi(\bar{x}; \Omega_1)$ is the whole unit circle. It is also clear that any point inside of the unit circle other than $(0,0)$ is also a solution to this problem, and $\Pi(\bar{x}; \Omega_i)$ is a singleton for both $i = 1, 2$, which is consistent with the result of Proposition 3.4.

The observation made in Proposition 3.4 allows us to improve the optimality conditions obtained in [13, Corollary 4.1] for the generalized Fermat-Torricelli problem.

Corollary 3.6 (improved optimality conditions for the generalized Fermat-Torricelli problem with three nonconvex sets in Hilbert spaces). *Let $n = 3$ in the framework of Theorem 3.3, where Ω_1, Ω_2, and Ω_3 are pairwisely disjoint subsets of X and $\Omega = X$. The following alternative holds for a local optimal solution $\bar{x} \in X$ with the sets $A_i(\bar{x})$ defined by (3.12):*

(i) *The point $\bar{x}$ belongs to one of the sets Ω_i, say Ω_1. Then for any $a_i \in A_i(\bar{x})$ as $i = 2, 3$ we have the relationships*

$$\langle a_2, a_3 \rangle \leq -1/2 \quad \text{and} \quad -a_2 - a_3 \in \widehat{N}(\bar{x}; \Omega_1).$$

(ii) *The point $\bar{x}$ does not belong to all the three sets Ω_1, Ω_2, and Ω_3. Then $A_i(\bar{x}) = \{a_i\}$ for all $i = 1, 2, 3$ and we have*

$$\langle a_i, a_j \rangle = -1/2 \quad \text{for} \quad i \neq j \quad \text{as} \quad i, j \in \{1, 2, 3\}.$$

Conversely, suppose that the sets Ω_i, $i = 1, 2, 3$, are convex and that $\bar{x}$ satisfies either (i) *or* (ii). *Then it is a global optimal solution to the problem under consideration.*

Proof. In case (i) for any $a_i \in A_i(\bar{x})$ as $i = 2, 3$ take $\bar{\omega}_i \in \Pi(\bar{x}; \Omega_i)$ such that

$$a_i = \frac{\bar{x} - \bar{\omega}_i}{d(\bar{x}; \Omega_i)}, \quad i = 2, 3.$$

Since $\bar{x} \in \Omega_1$, we have the relationships

$$\|\bar{x} - \bar{\omega}_1\| + \|\bar{x} - \omega_2\| = \sum_{i=1}^{3} d(\bar{x}; \Omega_i) \leq \sum_{i=1}^{3} d(x; \Omega_i) \leq d(x; \Omega_1) + \|x - \bar{\omega}_2\| + \|x - \bar{\omega}_3\|$$

whenever x is near $\bar{x}$. Thus $\bar{x}$ is a local optimal solution to the problem

$$\text{minimize} \;\; q(x) := d(x;\Omega_1) + \|x - \bar{\omega}_2\| + \|x - \bar{\omega}_3\|. \tag{3.16}$$

Employing the generalized Fermat rule in (3.16) and then the aforementioned sum rule for regular subgradients gives us by using the well-known formula for the regular subdifferential of the distance function (see, e.g., [9, Corollary 1.96]) that

$$0 \in \widehat{\partial} q(\bar{x}) = \widehat{\partial} d(\bar{x};\Omega_1) + a_2 + a_3 = \widehat{N}(\bar{x};\Omega_1) \cap I\!B + a_2 + a_3.$$

The latter implies therefore that

$$-a_2 - a_3 \in \widehat{N}(\bar{x};\Omega_1) \;\; \text{with} \;\; \|a_2 + a_3\| \le 1.$$

The rest of the proof follows the lines of that in [13, Corollary 4.1]. Assertion (ii) and the converse statement are derived similarly from Proposition 3.4 and the proof of [13, Corollary 4.1] by the same procedure, which thus allows us to fully justify the corollary. △

From now on in this section we concentrate on the distance function version (1.1) of the generalized Heron problem while paying the main attention to deriving efficient forms of optimality conditions for (1.1) under additional structural assumptions on the constraint set Ω. In what follows in this section we impose the *nonintersection condition*

$$\Omega \cap \Omega_i = \emptyset \;\; \text{for all} \;\; i = 1,\ldots,n \tag{3.17}$$

on the sets Ω and Ω_i in (1.1), which is specific for the (constrained) generalized Heron problem. In this case we obviously have $I(\bar{x}) = \emptyset$ for the first index set in (3.10) whenever $\bar{x} \in \Omega$, and so the sets $A_i(\bar{x})$ are calculated by

$$A_i(\bar{x}) = \frac{\bar{x} - \Pi(\bar{x};\Omega_i)}{d(\bar{x};\Omega_i)}, \quad i = 1,\ldots,n, \tag{3.18}$$

in the Hilbert space setting under consideration.

To proceed, for any nonzero vectors $u, v \in X$ define the quantity

$$\cos(u,v) := \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$$

and, given a linear subspace L of X, recall that

$$L^{\perp} := \big\{x^* \in X \big|\, \langle x^*, v\rangle = 0 \;\; \text{for all} \;\; v \in L\big\}.$$

We say that $\Omega \subset X$ has a *tangent space* $L = L(\bar{x})$ at $\bar{x}$ if $L^{\perp} = \widehat{N}(\bar{x};\Omega)$. Note that for any affine subspace $\Omega \subset X$ parallel to a linear subspace L the tangent space to Ω at every $\bar{x} \in \Omega$ is L.

Next we derive verifiable necessary and sufficient conditions for optimal solutions to (1.1) in Hilbert spaces provided that the constraint set admits a tangent space at the reference point.

Proposition 3.7 (optimality conditions for the case of constraint sets with tangent spaces). *Consider the generalized Heron problem* (1.1) *under condition* (3.17) *in Hilbert spaces. The following assertions hold:*

(i) *Let $\bar{x} \in \Omega$ be a local optimal solution to* (1.1), *let $A_i(\bar{x})$ be computed in* (3.18) *where $\Pi(\bar{x};\Omega_i) \neq \emptyset$ for $i = 1,\ldots,n$, and let Ω admit a tangent space $L(\bar{x})$ at $\bar{x}$. Then for any $a_i(\bar{x}) \in A_i(\bar{x})$, one has*

$$\sum_{i=1}^{n} \cos\big(a_i(\bar{x}), v\big) = 0 \;\; \textit{for every} \;\; v \in L(\bar{x}) \setminus \{0\}. \tag{3.19}$$

(ii) *Let all the sets Ω_i, $i = 1,\ldots,n$, be convex. Then $A_i(\bar{x}) = \{a_i(\bar{x})\}$ and condition* (3.19) *with the tangent space $L(\bar{x})$ for Ω is necessary and sufficient for the global optimality of $\bar{x}$ in* (1.1).

Proof. To justify (i), observe by the assumptions made and the definition of the tangent space $L(\bar{x})$ to Ω at $\bar{x}$ that

$$\widehat{N}(\bar{x};\Omega) = L^{\perp} = \big\{v \in X \big| \, \langle v, x\rangle = 0 \ \text{ for all } \ x \in L(\bar{x})\big\}.$$

By Theorem 3.3 for any $a_i(\bar{x}) \in A_i(\bar{x})$, one has

$$0 \in \sum_{i=1}^{n} a_i(\bar{x}) + L^{\perp}(\bar{x}),$$

which implies in turn that

$$\Big\langle \sum_{i=1}^{n} a_i(\bar{x}), v\Big\rangle = 0 \ \text{ for all } \ v \in L(\bar{x}).$$

Since $\bar{x} \notin \Omega_i$ by (3.17), we have due to (3.18) that $\|a_i(\bar{x})\| = 1$ for $i = 1, \ldots, n$, and hence

$$\sum_{i=1}^{n} \frac{\langle a_i(\bar{x}), v\rangle}{\|a_i(\bar{x})\| \cdot \|v\|} = 0 \ \text{ whenever } \ v \in L(\bar{x}) \setminus \{0\}.$$

Thus we arrive at the the necessary optimality condition (3.19).

To justify (ii), observe that the implication "$\Longrightarrow$" follows directly from assertion (i) of the theorem, since the sets $A_i(\bar{x})$ are singletons for $i = 1, \ldots, n$ in this case. The oppositive implication "$\Longleftarrow$" follows from Theorem 3.3(ii) by taking into account the special structure of the normal cone $\widehat{N}(\bar{x};\Omega) = L^{\perp}(\bar{x})$. This completes the proof of the proposition. $\triangle$

We have the following specification of optimality conditions in Proposition 3.7 when the tangent space therein is finitely generated.

Corollary 3.8 (optimality conditions for the case of finitely generated tangent spaces). *Let* $L(\bar{x}) = \text{span}\{v_1, \ldots, v_s\}$ *with* $v_j \neq 0$ *as* $j = 1, \ldots, s$ *in the setting of Proposition* 3.7. *Then condition* (3.19) *in all of its conclusions is equivalent to*

$$\sum_{i=1}^{n} \cos(a_i, v_j) = 0 \ \textit{ for all } \ j = 1, \ldots, s. \tag{3.20}$$

Proof. We obviously have that (3.19)$\Longrightarrow$(3.20). To justify the converse implication, set $a := \sum_{i=1}^{n} a_i$ and observe by $v_j \neq 0$ as $j = 1, \ldots, s$ and $\|a_i\| = 1$ as $i = 1, \ldots, n$ that (3.20) yields $\langle a, v_j\rangle = 0$ for all $j = 1, \ldots, s$. Picking further an arbitrary vector $v \in L(\bar{x}) \setminus \{0\}$, we arrive at the representation

$$v = \sum_{j=1}^{s} \lambda_j v_j$$

with some $\lambda_j \in I\!R$. It gives by linearity that $\langle a, v\rangle = \sum_{j=1}^{n} \lambda_j \langle a, v_j\rangle = 0$, which yields (3.19) and completes the proof of the proposition. $\triangle$

The next result concerns the generalized Heron problem for two nonconvex sets in Hilbert spaces with a one-dimensional structure of the regular normal cone to the constraint.

Proposition 3.9 (necessary conditions for the generalized Heron problem with two nonconvex sets in Hilbert spaces). *Consider problem* (1.1) *for two sets* ($n = 2$) *in Hilbert spaces under the nonintersection condition* (3.17). *Let* $\bar{x} \in \Omega$ *be a local optimal solution to* (1.1) *such that* $\widehat{N}(\bar{x};\Omega) = \text{span}\{v\}$ *with some* $v \neq 0$ *and that* $\Pi(\bar{x};\Omega_i) \neq \emptyset$ *for* $i = 1, 2$. *Then for any* $a_i(\bar{x}) \in A_i(\bar{x})$ *as* $i = 1, 2$ *we have the conditions:*

$$\textit{either } \ a_1(\bar{x}) + a_2(\bar{x}) = 0 \ \textit{ or } \ \cos\big(a_1(\bar{x}), v\big) = \cos\big(a_2(\bar{x}), v\big). \tag{3.21}$$

Proof. It follows from Theorem 3.3(i) in this setting that

$$-a_1(\bar{x}) - a_2(\bar{x}) \in \widehat{N}(\bar{x};\Omega) \text{ for any } a_i(\bar{x}) \in A_i(\bar{x}),\ i = 1,2. \tag{3.22}$$

Denoting for simplicity $a_i := a_i(\bar{x})$ as $i = 1,2$ and taking into account the assumed structure of the regular normal cone to Ω, we get that (3.22) is equivalent to the following:

$$\text{either } a_1 + a_2 = 0 \text{ or } a_1 + a_2 = \lambda v \text{ with some } \lambda \neq 0.$$

Let us show that the latter condition implies that $\cos(a_1, v) = \cos(a_2, v)$. Indeed, in this case we have $\|a_1\| = \|a_1\| = 1$, which gives by the Euclidean norm on X that

$$\lambda^2\|v\|^2 = \|a_1 + a_2\|^2 = \|a_1\|^2 + \|a_2\|^2 + 2\langle a_1, a_2\rangle = 2 + 2\langle a_1, a_2\rangle.$$

This implies in turn the relationships

$$\begin{aligned}
\langle a_1, \lambda v\rangle &= \langle \lambda v - a_2, \lambda v\rangle \\
&= \lambda^2\|v\|^2 - \lambda\langle a_2, v\rangle \\
&= 2 + 2\langle a_1, a_2\rangle - \lambda\langle a_2, v\rangle \\
&= 2\langle a_2, a_2\rangle + 2\langle a_1, a_2\rangle - \lambda\langle a_2, v\rangle \\
&= 2\langle a_2 + a_1, a_2\rangle - \lambda\langle a_2, v\rangle \\
&= 2\langle \lambda v, a_2\rangle - \lambda\langle a_2, v\rangle = \langle a_2, \lambda v\rangle,
\end{aligned}$$

which yield that $\langle a_1, v\rangle = \langle a_2, v\rangle$ since $\lambda \neq 0$. By taking into account that $\|a_1\| = \|a_2\| = 1$ and $v \neq 0$, we conclude that $\cos(a_1, v) = \cos(a_2, v)$ and thus complete the proof. $\triangle$

Observe that sufficient optimality conditions in the form of Proposition 3.9 do not hold even in convex settings. The next result provides slightly modified conditions, which are sufficient for optimality in the case of the convex generalized Heron problem on the plane.

Proposition 3.10 (characterizing optimal solutions for the generalized Heron problem with two convex sets). *Let the sets Ω_1 and Ω_2 be convex in the setting of Proposition* 3.9, *and let $a_i := a_i(\bar{x})$ as $i = 1,2$. Then the modification*

$$\text{either } a_1 + a_2 = 0 \text{ or } \big[a_1 \neq a_2 \text{ and } \cos(a_1, v) = \cos(a_2, v)\big], \tag{3.23}$$

of the necessary condition (3.21) *is sufficient for the global optimality of $\bar{x} \in \Omega$ in* (1.1) *when $X = \mathbb{R}^2$.*

Proof. To justify the sufficiency of conditions (3.23) for the optimality of $\bar{x}$ in (1.1), we need to show—by taking into account Theorem 3.3(ii) and the assumed structure of the regular normal cone to Ω—that the relationships in (3.23) imply the fulfillment of

$$-a_1 - a_2 \in \widehat{N}(\bar{x};\Omega) = \text{span}\{v\}. \tag{3.24}$$

When $-a_1 - a_2 = 0$, inclusion (3.24) is obviously satisfied. Consider the alternative in (3.23) when $a_1 \neq a_2$ and $\cos(a_1, v) = \cos(a_2, v)$. Since we are in $\mathbb{R}^2$, represent $a_1 = (x_1, y_1)$, $a_2 = (x_2, y_2)$, and $v = (x, y)$ with two real coordinates. Then the equality $\cos(a_1, v) = \cos(a_2, v)$ can be written as

$$x_1 x + y_1 y = x_2 x + y_2 y, \text{ i.e., } (x_1 - x_2)x = (y_2 - y_1)y. \tag{3.25}$$

Since $v \neq 0$, assume without loss of generality that $y \neq 0$. By the equivalence

$$\|a_1\|^2 = \|a_2\|^2 \Longleftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

we have the equality $(x_1 - x_2)(x_1 + x_2) = (y_2 - y_1)(y_2 + y_1)$, which implies by (3.25) that

$$y(x_1 - x_2)(x_1 + x_2) = x(x_1 - x_2)(y_2 + y_1). \tag{3.26}$$

Note that $x_1 \neq x_2$, since otherwise we have from (3.25) that $y_1 = y_2$, which contradicts the condition $a_1 \neq a_2$ in (3.23). Dividing both sides of (3.26) by $x_1 - x_2$, we get

$$y(x_1 + x_2) = x(y_2 + y_1),$$

which implies in turn that

$$y(a_1 + a_2) = y(x_1 + x_2, y_1 + y_2) = \big(x(y_1 + y_2), y(y_1 + y_2)\big) = (y_1 + y_2)v.$$

In this way we arrive at the representation

$$a_1 + a_2 = \frac{y_1 + y_2}{y} v$$

showing that inclusion (3.24) is satisfied. This ensures the optimality of $\bar{x}$ in (1.1) and thus completes the proof of the proposition. △

We conclude this section by a simple example showing how the results obtained allow us to completely solve a direct generalization of the classical Heron problem in $\mathbb{R}^2$, where the constraint straight line is replaced by a convex set.

Example 3.11 (complete solution of a a convex set extension of the Heron problem on the plane). Consider problem (1.1), where Ω is the epigraph of the nonsmooth convex function $y = |x|$ in $\mathbb{R}^2$, and where Ω_1 and Ω_2 are two points (x_1, y_1) and (x_2, y_2) that do not lie on Ω. This problem admits optimal solutions due to Proposition 3.1(ii). To solve it, we are going to employ appropriate necessary optimality conditions obtained above. Observe first that the normal cone to Ω at $(0,0)$ is given by

$$N\big((0,0);\Omega\big) = \big\{(x,y) \in \mathbb{R}^2 \big|\ y \leq -|x|\big\}$$

while the classical normals at other points of Ω are calculated trivially. Using this, we can easily check that if the points (x_1, y_1) and (x_2, y_2) belong to the region

$$\big\{(x,y) \in \mathbb{R}^2 \big|\ y \leq -|x|\big\},$$

then the origin $\bar{x} = (0,0)$ is the only point that satisfies the necessary optimality condition from Theorem 3.3(i) written now as:

$$-a_1 - a_2 \in N(\bar{x};\Omega) \ \text{ with } \ a_i = \frac{(x_i, y_i)}{\|(x_i, y_i)\|} \ \text{ as } \ i = 1, 2.$$

If the points (x_1, y_1) and (x_2, y_2) belong to another region

$$\big\{(x,y) \in \mathbb{R}^2 \big|\ x > |y|\big\},$$

then the problem also has a unique optimal solution constructed by connecting the reflection point of (x_1, y_1) through the line $y = x$ and (x_2, y_2).

4 Subgradient Algorithm in the Generalized Heron Problem

In this section we develop a subgradient algorithm for the numerical solution of the generalized Heron problem (1.4) for finitely many convex sets and convex constraints in the finite-dimensional Euclidean space $\mathbb{R}^m$. These are our standing assumptions for the rest of the paper. Recall that $\Pi(x;\Omega)$ denotes the (unique) Euclidean projection of x to Ω while $\Pi^F_{\Omega_i}(x)$ stands for the generalized/minimal time projection (3.1) of this point to the target sets Ω_i in (1.4). Here is the algorithm whose various implementations are presented in the next section.

Theorem 4.1 (subgradient algorithm for the generalized Heron problem). *Let $S \neq \emptyset$ be the set of optimal solutions to problem (1.4). Picking a sequence $\{\alpha_k\}_{k\in I\!N}$ of positive numbers and a starting point $x_1 \in \Omega$, consider the algorithm*

$$x_{k+1} = \Pi\Big(x_k - \alpha_k \sum_{i=1}^{n} q_{ik}; \Omega\Big), \quad k = 1, 2, \ldots, \tag{4.1}$$

with an arbitrary choice of vectors

$$q_{ik} \in -\partial\rho_F(\omega_{ik} - x_k) \cap N(\omega_{ik}; \Omega_i) \ \text{ for some } \ \omega_{ik} \in \Pi^F_{\Omega_i}(x_k) \ \text{ if } \ x_k \notin \Omega_i \tag{4.2}$$

via the Minkowski gauge (3.2) and with $q_{ik} := 0$ otherwise. Assume that

$$\sum_{k=1}^{\infty} \alpha_k = \infty \ \text{ and } \ \ell^2 := \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \tag{4.3}$$

Then the iterative sequence $\{x_k\}$ in (4.1) converges to an optimal solution of problem (1.4) and the numerical value sequence

$$V_k := \min\big\{T(x_j)\big|\, j = 1, \ldots, k\big\} \tag{4.4}$$

converges to the optimal value $\widehat{V}$ in this problem. Furthermore, we have the estimate

$$V_k - \widehat{V} \le \frac{d(x_1; S)^2 + L^2 \sum_{i=1}^{k} \alpha_i^2}{2\sum_{i=1}^{k} \alpha_i},$$

where $0 \le L < \infty$ is a Lipschitz constant of the function $T(\cdot)$ from (1.4) on R^m.

Proof. We know that the value function $T(\cdot)$ in (1.4) is convex and globally Lipschitzian on $I\!R^m$. Employing [12, Theorems 7.1 and 7.3], the convex subdifferential of the minimal time functions (1.3) at any point x_k is computed by

$$\partial T^F_{\Omega_i}(x_k) = \begin{cases} N(x_k; \Omega_i) \cap \big\{v \in X \big|\, \sigma_F(-v) \le 1\big\} & \text{if } x_k \in \Omega_i, \\ N(\omega_{ik}; \Omega_i) \cap \big[-\partial\rho_F(\omega_{ik} - x_k)\big] & \text{if } x_k \notin \Omega_i, \end{cases} \tag{4.5}$$

where $\omega_{ik} \in \Pi^F_{\Omega_i}(x_k)$ is an arbitrary generalized projection vector for $i \in \{1, \ldots, n\}$ and $k \in I\!N$. Recalling now the subgradient algorithm for minimizing the convex function $T(\cdot)$ in (1.4) subject to $x \in \Omega$, we construct the iteration sequence by

$$x_{k+1} = \Pi\Big(x_k - \alpha_k v_k; \Omega\Big) \ \text{ with } \ v_k \in \partial T(x_k), \quad k = 1, 2, \ldots. \tag{4.6}$$

It follows from the convex subdifferential sum rule of Theorem 2.1(ii) that

$$v_k = \sum_{i=1}^{n} q_{ik} \ \text{ with } \ q_{ik} \in \partial T^F_{\Omega_i}(x_k)$$

for the subgradients v_k in (4.6). Substituting the latter into (4.6) gives us algorithm (4.1) with q_{ik} satisfying (4.2). Then all the conclusions of the theorem are derived from the so-called "square summable but not summable case" of the subgradient method for constrained convex functions under the conditions in (4.3); see [1, 3] for more details. $\triangle$

In the case of $F = I\!B$, the closed unit ball in $I\!R^m$, we are able to provide a more explicit algorithm to solve the distance function version (1.1) of the generalized Heron problem with now uniquely defined vectors q_{ik} in (4.1).

Corollary 4.2 (explicit subgradient algorithm for the distance version of the generalized Heron problem). *Consider the distance function specification* (1.1) *of the generalized Heron problem under the assumptions of Theorem* 4.1. *Then all the conclusions of this theorem hold with* q_{ik} *in* (4.1) *calculated by*

$$q_{ik} = \begin{cases} 0 & \text{if } x_k \in \Omega_i, \\ \dfrac{x_k - \Pi(x_k;\Omega_i)}{d(x_k;\Omega_i)} & \text{if } x_k \notin \Omega_i. \end{cases} \tag{4.7}$$

Proof. As follows from the proof of Theorem 3.3, in the case of problem (1.1) the vectors q_{ik} from (4.2) are uniquely determined and reduce to (4.7). △

The next corollary specifies algorithm (4.1) in the case of balls for the distance function version (1.1) of the generalized Heron problem.

Corollary 4.3 (subgradient algorithm in the case of multidimensional balls). *Consider problem* (1.1) *with* $\Omega_i = B(c_i, r_i) \subset I\!R^m$ *as* $i = 1, \ldots, n$. *Then the quantities* q_{ik} *in Theorem* 4.1 *are uniquely calculated by*

$$q_{ik} = \begin{cases} 0 & \text{if } \|x_k - c_i\| \le r_i, \\ \dfrac{x_k - c_i}{\|x_k - c_i\|} & \text{if } \|x_k - c_i\| > r_i \end{cases} \tag{4.8}$$

and the corresponding values V_k *are evaluated by formula* (4.4) *with*

$$T(x_j) = \sum_{i=1,\, x_j \notin \Omega_i}^{n} \Big(\|x_j - c_i\| - r_i \Big). \tag{4.9}$$

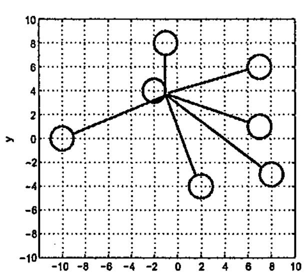
Proof. Formula (4.8) directly follows from (4.7) due to the projection representation

$$\Pi(x_k;\Omega_i) = c_i + r_i \frac{x_k - c_i}{\|x_k - c_i\|}$$

in the case under consideration. It is easy to see furthermore that the value function in (4.4) reduces to (4.9) in this case. △

5 Implementation of the Subgradient Algorithm

The final section of the paper is devoted to implementations of the subgradient algorithm from Theorem 4.1 and its specifications to solve the generalized Heron problem in a number of underlying examples of their own interest. Let us start with a two-dimensional problem involving a ball constraint in the setting of Corollary 4.3.



MATLAB RESULTS		
k	x_k	V_k
1	(-1,4)	44.58483
10	(-1.07737,3.61433)	44.36969
100	(-1.07779,3.61332)	44.36969
1000	(-1.07779,3.61331)	44.36969
10,000	(-1.07779,3.61331)	44.36969

Figure 1: A Generalized Heron Problem for Balls with a Ball Constraint.

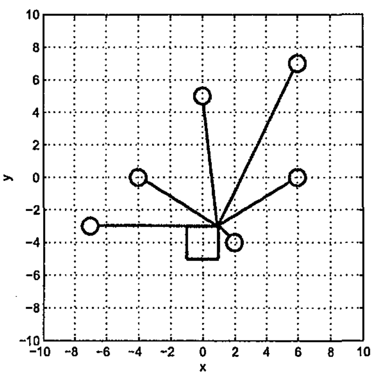
Example 5.1 (two-dimensional Heron problem for balls with ball constraints). Consider the generalized Heron problem (1.1) for balls in $\mathbb{R}^2$ subject to a given ball constraint. Let $c_i = (a_i, b_i)$ and r_i as $i = 1, \ldots, n$ be the centers and the radii of the balls Ω_i under consideration, and let $c = (x_0, y_0)$ and r be the center and radius for the given ball constraint Ω. The subgradient algorithm is given by (4.1), where the projection $P(x,y) := \Pi((x,y);\Omega)$ is computed by

$$P(x,y) = (v_x + x_0, v_y + y_0) \text{ with } v_x = \frac{r(x-x_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}},\ v_y = \frac{r(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}},$$

and where the quantities q_{ik} and V_k are calculated in Corollary 4.3.

To specify the calculations, take the ball constraint Ω with center $(-2,4)$ and radius 1. The sets Ω_i, $i = 1, \ldots, 6$, are the balls with centers $(-10,0)$, $(-1,8)$, $(2,-4)$, $(7,6)$, $(7,1)$, and $(8,-3)$ and with the same radius $r = 1$. The MATLAB calculations performed by algorithm (4.1) with the sequence $\alpha_k = 1/k$ satisfying (4.3) and the starting point $x_1 = (-1,4)$ are presented in Figure 1. Observe that the numerical results indicate points on the ball constraint with the optimal solution $\bar{x} \approx (-1.07779, 3.61331)$ and the optimal value $\widehat{V} \approx 44.36969$.

The next example concerns the generalized Heron problem with square constraints.



MATLAB RESULTS		
k	x_k	V_k
1	(-1,-4)	41.23881
50	(0.89884,-3)	37.32496
100	(0.95169,-3)	37.32091
150	(0.97352,-3)	37.31974
200	(0.98595,-3)	37.31920
250	(0.99413,-3)	37.31890
300	(1.00000,-3)	37.31872
350	(1.00000,-3)	37.31872

Figure 2: A Generalized Heron Problem for Balls with a Square Constraint.

Example 5.2 (generalized Heron problem with square constraints). Consider the implementation of algorithm (4.1) for problem (1.1) using a MATLAB program with the square constraint Ω of center $(a,b) = (0,-4)$ and short radius $r = 1$ and with the balls Ω_i as $i = 1, \ldots, 6$ centered at (-7,-3), (0,5), (-4,0), (2,-4), (6,0), and (6,7) with the same radius 0.5. Note that the projection $P(x,y) = \Pi((x,y);\Omega)$ is calculated by

$$P(x,y) = \begin{cases} (a+r, b+r) & \text{if } x-a > r,\ y-b > r, \\ (x, b+r) & \text{if } |x-a| \le r,\ y-b > r, \\ (a-r, b+r) & \text{if } x-a < -r,\ y-b > r, \\ (a-r, y) & \text{if } x-a < -r,\ |y-b| \le r, \\ (a-r, b-r) & \text{if } x-a < -r,\ y-b < -r, \\ (x, b-r) & \text{if } |x-a| \le r,\ y-b < -r, \\ (a+r, b-r) & \text{if } x-a > r,\ y-b < -r, \\ (a+r, b) & \text{if } x-a > r,\ |y-b| \le r, \\ (x,y) & \text{if } (x,y) \in \Omega. \end{cases}$$

The quantities q_{ik} and V_k are given by Corollary 4.3. In Figure 2 we present the results of calculations performed by the subgradient algorithm (4.1) for the sequence $\alpha_k = 1/k$ and the starting point $x_1 = (-1, -4)$. Observe that the computed optimal solution is $\bar{x} \approx (1.00000, -3.00000)$ and the optimal value is $\widehat{V} \approx 37.31872$.

Prior to the calculations in two next examples concerning the generalized Heron problem (1.1) for squares in $I\!R^2$ we formulate a specification of Theorem 4.1 in a general setting of such a type. Recall that a square in $I\!R^2$ is of *right position* if the sides of this square are parallel to the x-axis and the y-axis, respectively.

Corollary 5.3 (subgradient algorithm for the generalized Heron problem squares targets). *Consider problem (1.1) in $I\!R^2$, where each target set Ω_i is a square of right position with center $c_i = (a_i, b_i)$ and short radius r_i as $i = 1, \ldots, n$, and where the constraint Ω is an arbitrary closed and convex set. Denote the vertices of the i^{th} square by $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, $v_{4i} = (a_i + r_i, b_i - r_i)$, and let $x_k = (x_{1k}, x_{2k})$. Then the quantities q_{ik} in Theorem 4.1 are computed by*

$$q_{ik} = \begin{cases} 0 & \text{if } |x_{1k} - a_i| \le r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ \dfrac{x_k - v_{1i}}{\|x_k - v_{1i}\|} & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i > r_i, \\ \dfrac{x_k - v_{2i}}{\|x_k - v_{2i}\|} & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i > r_i, \\ \dfrac{x_k - v_{3i}}{\|x_k - v_{3i}\|} & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i < -r_i, \\ \dfrac{x_k - v_{4i}}{\|x_k - v_{4i}\|} & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i < -r_i, \\ (0,1) & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i > r_i, \\ (0,-1) & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i < -r_i, \\ (1,0) & \text{if } x_{1k} - a_i > r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ (-1,0) & \text{if } x_{1k} - a_i < -r_i \text{ and } |x_{2k} - b_i| \le r_i \end{cases}$$

for all $i = 1, \ldots, n$ and $k \in N$ with the corresponding quantities V_k defined by (4.4).

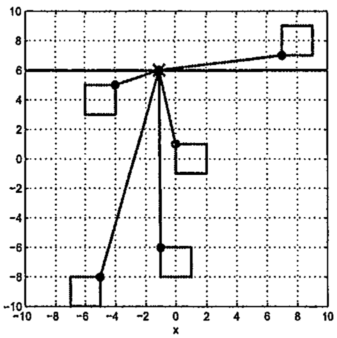
Proof. This statement follows from Corollary 4.2 by a direct calculation of the projection from an out-of-set point to each square Ω_i in formula (4.7). $\triangle$

Now we present the results of MATLAB calculations in the case of straight line constraints in the setting of Corollary 5.3.

Example 5.4 (generalized Heron problem for squares with line constraints). Consider the generalized Heron problem (1.1) for squares of right position in $I\!R^2$ subject to a straight line constraint Ω. Let $c_i = (a_i, b_i)$ and r_i as $i = 1, \ldots, n$ be the centers and short radius of the squares Ω_i under consideration. Denote by $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, $v_{4i} = (a_i + r_i, b_i - r_i)$ the vertices of the i^{th} square, and let $v = [s, h]$ and $p = (x_0, y_0)$, be the direction and point vectors of the given line Ω. Then the projection $P(x, y) = \Pi((x, y); \Omega)$ in the the subgradient algorithm (4.1) is calculated by

$$P(x, y) = (x_0 + st, y_0 + ht) \text{ and } t = \frac{s(x - x_0) + h(y - y_0)}{s^2 + h^2}$$

while the quantities q_{ik} and V_k for all $i = 1, \ldots, n$ and $k \in N$ are given by Corollary 5.3.

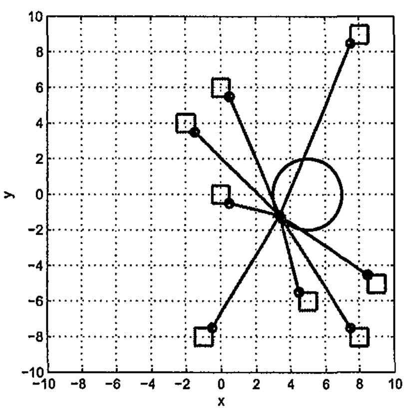


MATLAB RESULTS		
k	x_k	V_k
1	(-1,6)	42.8838
100	(-1.0826,6)	42.8821
1000	(-1.0896,6)	42.8821
100,000	(-1.0938,6)	42.8821
1,000,000	(-1.0944,6)	42.8821
5,000,000	(-1.0946,6)	42.8821
10,000,000	(-1.0946,6)	42.8821

Figure 3: A Generalized Heron Problem for Squares with a Line Constraint.

In Figure 3 we present the results of calculations by algorithm (4.1) with $\alpha_k = 1/k$ and the starting point $x_1 = (-1, 6)$ for the case above with the line constraint defined by $v = [1, 0]$ and $p = (1, 6)$ and the squares Ω_i as $i = 1, \ldots, 5$ centered at $(-6, -9)$, $(-5, 4)$, $(0, -7)$, $(1, 0)$, and $(8, 8)$ with the same short radius r=1. Observe that the calculated optimal solution is $\bar{x} \approx (-1.0946, 6)$ and the optimal value is $\widehat{V} \approx 42.8821$.

The next example concerns the generalized Heron problem (1.1) for squares in right position with a ball constraint on the plane.



MATLAB RESULTS		
k	x_k	V_k
1	(5,-2)	54.41891
10	(3.51379,-1.33835)	53.05740
100	(3.41230,-1.21623)	53.04403
1000	(3.39607,-1.19475)	53.04364
100,000	(3.39279,-1.19033)	53.04363
600,000	(3.39271,-1.19022)	53.04363
1,000,000	(3.39271,-1.19021)	53.04363
1,200,000	(3.39270,-1.19021)	53.04363
1,400,000	(3.39270,-1.19021)	53.04363

Figure 4: A Generalized Heron Problem for Squares with a Ball Constraint.

Example 5.5 (generalized Heron problem for squares with ball constraints). By taking into account the previous formulas for algorithm (4.1), we provide the following calculations concerning the generalized Heron problem (1.1) with the ball constraint Ω centered at $(5, 0)$ and radius 2 and the squares Ω_i, $i = 1, \ldots, 8$ of right position with the centers $(-2, 4)$, $(-1, -8)$, $(0, 0)$, $(0, 6)$, $(5, -6)$, $(8, -8)$, $(8, 9)$, and $(9, -5)$ and the same short radius $r = 0.5$. Figure 4 presents the results of calculations for algorithm (4.1) with the sequence $\alpha_k = 1/k$ and the starting point $x_1 = (5, -2)$. Observe that the obtained numerical results give us the optimal solution $\bar{x} \approx (3.39270, -1.19021)$ and the optimal value $\widehat{V} \approx 53.04363$.

Now let us illustrate applications of the subgradient algorithm from Theorem 4.1 to solving the generalized Heron problem (1.4) formulated via the minimal time function with dynamics sets F different from the ball. First we consider the dynamics F described by the *closed unit diamond*

$$F := \{(x_1, x_2) \in I\!R^2 \big|\ |x_1| + |x_2| \le 1\}. \tag{5.10}$$

In this case the corresponding Minkowski gauge (3.2) is given by the formula

$$\rho_F(x_1, x_2) = |x_1| + |x_2|. \tag{5.11}$$

The following proposition provides an explicit calculation of a subgradient of the minimal time function (1.3) generated by the diamond dynamics (5.10) and a square target in $\mathbb{R}^2$. We further use this calculation in implementing algorithm (4.1) with the corresponding selection of q_{ik} in (4.2).

Proposition 5.6 (subgradients of the minimal time function with diamond dynamics). *Let F be the closed unit diamond in $\mathbb{R}^2$, and let Ω be the square of right position centered at $c = (a, b)$ with short radius $r > 0$. Then we can calculate a subgradient $v(\bar{x}_1, \bar{x}_2) \in \partial T_\Omega^F(\bar{x}_1, \bar{x}_2)$ of the minimal time function T_Ω^F at $(\bar{x}_1, \bar{x}_2) \notin \Omega$ by*

$$v(\bar{x}_1, \bar{x}_2) = \begin{cases} (1,0) & \text{if } |\bar{x}_2 - b| \le r,\ \bar{x}_1 > a + r, \\ (-1,0) & \text{if } |\bar{x}_2 - b| \le r,\ \bar{x}_1 < a - r, \\ (0,1) & \text{if } |\bar{x}_1 - a| \le r,\ \bar{x}_2 > b + r, \\ (0,-1) & \text{if } |\bar{x}_1 - a| \le r,\ \bar{x}_2 < b - r, \\ (1,1) & \text{if } \bar{x}_1 > a + r,\ \bar{x}_2 > b + r, \\ (-1,1) & \text{if } \bar{x}_1 < a - r,\ \bar{x}_2 > b + r, \\ (-1,-1) & \text{if } \bar{x}_1 < a - r,\ \bar{x}_2 < b - r, \\ (1,-1) & \text{if } \bar{x}_1 > a + r,\ \bar{x}_2 < b - r, \\ 0 & \text{if } (\bar{x}_1, \bar{x}_2) \in \Omega. \end{cases} \tag{5.12}$$

Proof. By [12, Theorem 7.3] we have the relationship

$$\partial T_\Omega^F(\bar{x}) = N(\bar{\omega}; \Omega) \cap \big[-\partial \rho_F(\bar{\omega} - \bar{x}) \big] \quad \text{for any } \bar{\omega} \in \Pi_\Omega^F(\bar{x}) \tag{5.13}$$

between the subdifferentials of the minimal time function at $\bar{x} \notin \Omega$ and the corresponding Minkowski gauge. In the setting under consideration it is easy to find the minimal time projection $\Pi_\Omega^F(\bar{x}_1, \bar{x}_2)$ of a given vector $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$ to the square Ω. Furthermore, the convex subdifferential of (5.11) at (x_1, x_2) is computed by

$$\partial \rho_F(\bar{x}_1, \bar{x}_2) = \begin{cases} [-1,1] \times [-1,1] & \text{if } (\bar{x}_1, \bar{x}_2) = (0,0), \\ [-1,1] \times \{1\} & \text{if } \bar{x}_1 = 0,\ \bar{x}_2 > 0, \\ [-1,1] \times \{-1\} & \text{if } \bar{x}_1 = 0,\ \bar{x}_2 < 0, \\ \{1\} \times [-1,1] & \text{if } \bar{x}_1 > 0,\ \bar{x}_2 = 0, \\ \{-1\} \times [-1,1] & \text{if } \bar{x}_1 < 0,\ \bar{x}_2 = 0, \\ \{1\} \times \{1\} & \text{if } \bar{x}_1 > 0,\ \bar{x}_2 > 0, \\ \{1\} \times \{-1\} & \text{if } \bar{x}_1 > 0,\ \bar{x}_2 < 0, \\ \{-1\} \times \{1\} & \text{if } \bar{x}_1 < 0,\ \bar{x}_2 > 0, \\ \{-1\} \times \{-1\} & \text{if } \bar{x}_1 < 0,\ \bar{x}_2 < 0. \end{cases}$$

The rest of the proof is a direct verification that the vector $v(\bar{x}_1, \bar{x}_2)$ from (5.12) belongs to the set on the right-hand side of (5.13) and hence to $\partial T^F_\Omega(\bar{x}_1, \bar{x}_2)$. $\triangle$

Proposition 5.6 and the previous considerations lead us to the following realization of the subgradient algorithm (4.1).

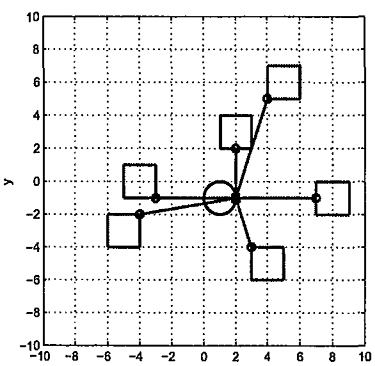
Corollary 5.7 (subgradient algorithm for finitely many squares and diamond dynamics in the generalized Heron problem). *Consider problem* (1.4) *generated by the diamond dynamics* (5.10) *and* n *squares* Ω_i *of right position in* $\mathbb{R}^2$. *Let* $c_i = (a_i, b_i)$ *and* r_i *as* $i = 1, \ldots, n$ *be the centers and the short radii of the squares under consideration, and let* $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, *and* $v_{4i} = (a_i + r_i, b_i - r_i)$ *be the vertices of the* i^{th} *square. Denoting* $x_k = (x_{1k}, x_{2k})$ *in algorithm* (4.1), *we compute the quantities* q_{ik} *as follows:*

$$q_{ik} = \begin{cases} 0 & \text{if } |x_{1k} - a_i| \le r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ (1,1) & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i > r_i, \\ (-1,1) & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i > r_i, \\ (-1,-1) & \text{if } x_{1k} - a_i < -r_i \text{ and } x_{2k} - b_i < -r_i, \\ (1,-1) & \text{if } x_{1k} - a_i > r_i \text{ and } x_{2k} - b_i < -r_i, \\ (0,1) & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i > r_i, \\ (0,-1) & \text{if } |x_{1k} - a_i| \le r_i \text{ and } x_{2k} - b_i < -r_i, \\ (1,0) & \text{if } x_{1k} - a_i > r_i \text{ and } |x_{2k} - b_i| \le r_i, \\ (-1,0) & \text{if } x_{1k} - a_i < -r_i \text{ and } |x_{2k} - b_i| \le r_i \end{cases}$$

for all $i \in \{1, \ldots, n\}$ *and* $k \in \mathbb{N}$.

Proof. It follows from Proposition 5.6, comparison between the right-hand side of (4.2) and formula (5.13), and the square calculations of Corollary 5.3. $\triangle$

Now we implement the results of Corollary 5.7 to solve the generalized Heron problem of the above type with ball constraints.

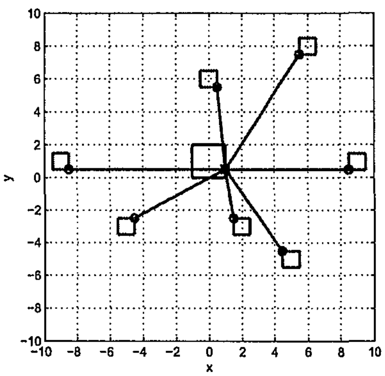


MATLAB RESULT		
k	x_k	V_k
1	(1,-2)	34
10	(1.98703,-0.83947)	32.01297
100	(1.99987,-0.98385)	32.00013
1,000	(2.00000,-0.99838)	32.00000
10,000	(2.00000,-0.99984)	32.00000
50,000	(2.00000,-0.99997)	32.00000
100,000	(2.00000,-0.99998)	32.00000
150,000	(2.00000,-0.99999)	32.00000
200,000	(2.00000,-0.99999)	32.00000

Figure 5: A Generalized Heron Problem for Squares with a Ball Constraint with Respect to "Sum" Distances.

Example 5.8 (generalized Heron problems with diamond dynamics for squares and ball constraints). Consider problem (1.4) with the diamond dynamics (5.10) for squares Ω_i as $i = 1, \ldots, 6$ of right position in $\mathbb{R}^2$ with the centers at (-5,-3), (-4,0), (2,3), (4,-5), (5,6), and (8,-1) and the same short radius 1 subject to the ball constraint Ω centered at $(-1,1)$ and radius 1. The results of calculations by the subgradient algorithm (4.1) with $\alpha_k = 1/k$ and the starting point $x_1 = (1,-2)$ are presented in Figure 5. Observe that the obtained optimal solution is the point $\bar{x} \approx (2.00000, -0.99999)$ on the ball constraint with the optimal value $\widehat{V} \approx 32.00000$.

The following example is a modification of the previous one for the case of square constraints.



MATLAB RESULT		
k	x_k	V_k
1	(-1,2)	61
10	(1,0.57897)	54.5
100	(1,0.48990)	54.5
1000	(1,0.50100)	54.5
1500	(1,0.49933)	54.5
2000	(1,0.49950)	54.5
2500	(1,0.49960)	54.5
3000	(1,0.50000)	54.5
3500	(1,0.50000)	54.5

Figure 6: A Generalized Heron Problem for Squares with a Square Constraint with Respect to "Sum" Distances.

Example 5.9 (generalized Heron problems for squares with diamond dynamics and square constraints). Consider the generalized Heron problem (1.4) with the diamond dynamics (5.10) for the squares $\Omega_i \in I\!R^2$ as $i = 1,\ldots,7$ of right position centered at $(-5,-3)$, $(-9,1)$, $(0,6)$, $(2,-3)$, $(6,8)$, $(5,-5)$, and $(9,1)$ with the same short radius 1 subject to the square constraint Ω of right position centered at (0,1) with the short radius 0.5. The calculations presented in Figure 6 are performed for the sequence $\alpha_k = 1/k$ in (4.1) and the starting point $x_1 = (-1,2)$. The obtained optimal solution is the point $\bar{x} \approx (1, 0.50000)$ on the square and the optimal value is $\widehat{V} \approx 54.50000$.

Next we consider the generalized Heron problem (1.4) with the *square dynamics* $F = [-1,1] \times [-1,1]$ on the plane. The corresponding Minkowski gauge is now given by

$$\rho_F(x_1,x_2) = \max\{|x_1|,|x_2|\}.$$

First we calculate a subgradient $v(\bar{x}_1,\bar{x}_2) \in \partial T^F_\Omega(\bar{x}_1,\bar{x}_2)$ of the cost function in (1.4) at any $(\bar{x}_1,\bar{x}_2)$, which is further used for a specification of algorithm (4.1) in this setting.

Proposition 5.10 (subgradients of minimal time functions with square dynamics and square targets). *Let $F = [-1,1]\times[-1,1]$, and let Ω be the square of right position in $I\!R^2$ centered at $c = (a,b)$ with short radius $r > 0$. Then a subgradient $v(\bar{x}_1,\bar{x}_2) \in \partial T^F_\Omega(\bar{x}_1,\bar{x}_2)$ of the minimal time function T^F_Ω at $(\bar{x}_1,\bar{x}_2)$ is computed by*

$$v(\bar{x}_1,\bar{x}_2) = \begin{cases} (1,0) & \text{if } |\bar{x}_2 - b| \le \bar{x}_1 - a,\ \bar{x}_1 > a + r,\\ (-1,0) & \text{if } |\bar{x}_2 - b| \le a - \bar{x}_1,\ \bar{x}_1 < a - r,\\ (0,1) & \text{if } |\bar{x}_1 - a| \le \bar{x}_2 - b,\ \bar{x}_2 > b + r,\\ (0,-1) & \text{if } |\bar{x}_1 - a| \le b - \bar{x}_2,\ \bar{x}_2 < b - r,\\ 0 & \text{if } (\bar{x}_1,\bar{x}_2) \in \Omega. \end{cases} \tag{5.14}$$

Proof. It is given in [13, Proposition 5.1]. △

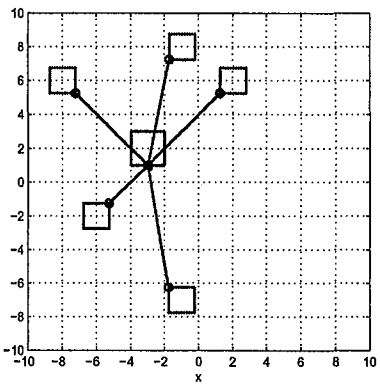
As a consequence of the proposition above, we calculate the quantities q_{ik} in algorithm (4.1) for the corresponding version of the generalized Heron problem.

Corollary 5.11 (subgradient algorithm for the generalized Heron problem with square dynamics). *Consider problem (1.4) for the square dynamics $F = [-1,1] \times [-1,1]$ and the square targets Ω_i as $i = 1,\ldots,n$ of right position in $\mathbb{R}^2$. Denote by $c_i = (a_i, b_i)$ and r_i the centers and the short radii of the squares Ω_i under consideration, and let the vertices of the i^{th} square be $v_{1i} = (a_i + r_i, b_i + r_i)$, $v_{2i} = (a_i - r_i, b_i + r_i)$, $v_{3i} = (a_i - r_i, b_i - r_i)$, and $v_{4i} = (a_i + r_i, b_i - r_i)$. Then the quantities q_{ik} in algorithm (4.1) of Theorem 4.1 in this setting along the iterative sequence $x_k = (x_{1k}, x_{2k})$ are calculated for all $i \in \{1,\ldots,n\}$ and $k \in \mathbb{N}$ by*

$$q_{ik} = \begin{cases} (1,0) & \text{if } |x_{2k} - b_i| \le x_{1k} - a_i \text{ and } x_{1k} > a_i + r_i, \\ (-1,0) & \text{if } |x_{2k} - b_i| \le a_i - x_{1k} \text{ and } x_{1k} < a_i - r_i, \\ (0,1) & \text{if } |x_{1k} - a_i| \le x_{2k} - b_i \text{ and } x_{2k} > b_i + r_i, \\ (0,-1) & \text{if } |x_{1k} - a_i| \le b_i - x_{2k} \text{ and } x_{2k} < b_i - r_i, \\ (0,0) & \text{otherwise.} \end{cases}$$

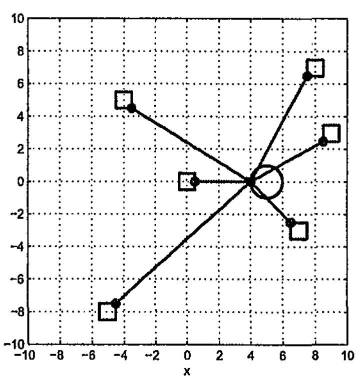
Proof. It follows from Proposition 5.11, comparison between the right-hand side of (4.2) and formula (5.13), and the square calculations of Corollary 5.3. $\triangle$

The following two examples present implementations of the subgradient algorithm realization from Corollary 5.11 in the generalized Heron problem under consideration with square and ball



MATLAB RESULT		
k	x_k	V_k
1	(-4,3)	26.25000
10	(-3.12500,1.04603)	24.37500
100	(-2.99136,1.00070)	24.25068
1,000	(-3.00133,1.00133)	24.25000
10,000	(-2.99996,1.00001)	24.25000
15,000	(-3.00013,1.00007)	24.25000
20,000	(-3.00000,1.00000)	24.25000
25,000	(-3.00001,1.00001)	24.25000
30,000	(-3.00001,1.00001)	24.25000

Figure 7: A Generalized Heron Problem for Squares with a Ball Constraint with Respect to "Max" Distances.



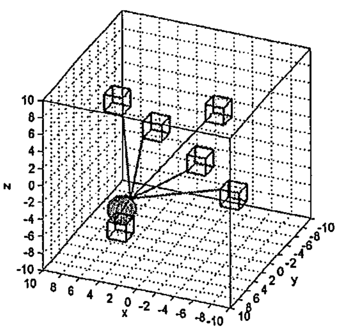
MATLAB RESULT		
k	x_k	V_k
1	(5,0)	35
10	(4.00062,0.03519)	33
100	(4.00000,0.00038)	33
200	(4.00000,0.00010)	33
400	(4.00000,0.00002)	33
600	(4.00000,0.00001)	33
800	(4.00000,0.00001)	33
1,000	(4.00000,0.00000)	33
1,200	(4.00000,0.00000)	33

Figure 8: A Generalized Heron Problem for Squares with Ball Constraint with Respect to "Max" Distances.

Example 5.12 (generalized Heron problem with square dynamics, targets, and constraints). Consider the implementation of the algorithm from Corollary 5.11 in problem (1.4) with the square constraint Ω of center (-3,2) and short radius 1 and the target square sets Ω_i as $i = 1, \ldots, 5$ of centers (-8,6), (-6,-2), (-1,8), (-1,-7), and (2,6) with the same short radius $r = 0.75$. In Figure 7 we present the results of calculations by (4.1) with $\alpha_k = 1/k$ and the starting point $x_1 = (-4, 3)$. The optimal solution here is $\bar{x} \approx (-3.00001, 1.00001)$ and the optimal value is $\widehat{V} \approx 24.25000$.

Example 5.13 (generalized Heron problem with square dynamics and targets and with ball constraints). Consider the implementation of the subgradient algorithm from Corollary 5.11 in problem (1.4) with the square dynamics, the square targets Ω_i as $i = 1, \ldots, 6$ of centers (-5,-8), (-4,5), (0,0), (8,7), (9,3), and (7,-3) with the same short radius $r = 0.5$, and with the ball constraint Ω of center (5,0) and radius 1. The presented calculations are performed by (4.1) with $\alpha_k = 1/k$ and the starting point $x_1 = (5, 0)$; see Figure 8. The obtained optimal solution is $\bar{x} \approx (4.00000, 0.00000)$ with the optimal value $\widehat{V} \approx 33.00000$.

Our last example concerns a three-dimensional distance version of the generalized Heron problem (1.1) for *cubes* of right position in $I\!R^3$ subject to a ball constraint.



MATLAB RESULT		
k	x_k	V_k
1	(5,0.5,-6)	51.58786
10	(4.23949,1.52680,-4.79680)	47.19028
100	(4.23948,1.53023,-4.79546)	47.19026
1,000	(4.23948,1.53024,-4.79546)	47.19026
10,000	(4.23948,1.53024,-4.79546)	47.19026
100,000	(4.23948,1.53024,-4.79546)	47.19026
1,000,000	(4.23948,1.53024,-4.79546)	47.19026

Figure 9: A Generalized Heron Problem for Cubes with Ball Constraint in Three Dimensions.

Example 5.14 (generalized Heron problem for cubes with ball constraints). Consider problem (1.1) for cubes Ω_i as $i = 1, \ldots, 6$ of right position in $I\!R^3$ with the centers $(8, -4, 3)$, $(-2, -6, 3)$, $(3, -2, 2)$, $(-4, -5, -6)$, $(-3, 1, 1)$, and $(3, 7, -5)$ and the same short radius 1 subject to the ball constraint Ω of center (5,2,-6) and radius 1.5. The projection $P((x, y, z); \Omega)$ and quantities q_{ik} in algorithm (4.1) are calculated similarly to Example 5.1. Figure 9 presents the implementation of the subgradient algorithm (4.1) with $\alpha_k = 1/k$ and the starting point $x_1 = (5, 5, -6)$. As we see, the optimal solution calculated here up to five significant digits is $\bar{x} \approx (4.23948, 1.53024, -4.79546)$ and the optimal value is $\widehat{V} \approx 47.19026$.

We conclude the paper by the following three observations.

Remark 5.15 (extensions and other location problems).

(i) Note that the approach and results of this paper can be easily extended to the *weighted version* of the generalized Heron problem (1.4):

$$\text{minimize } T(x) := \sum_{i=1}^{n} \mu_i T^F_{\Omega_i}(x), \text{ subject to } x \in \Omega, \tag{5.15}$$

where $\mu_i \geq 0$ as $i = 1, \ldots, n$ are given weights. Since we have

$$\partial\big(\mu_i T^F_{\Omega_i}\big)(\bar{x}) = \mu_i \partial T^F_{\Omega_i}(\bar{x})$$

for both convex and nonconvex subdifferentials used in this paper, it is straightforward to derive counterparts of the qualitative and numerical results obtained above for the case of the weighted generalized Heron problem (5.15). For example, the equation

$$\sum_{i=1}^{n} \mu_i \cos\big(a_i(\bar{x}), v\big) = 0 \text{ for every } v \in L(\bar{x}) \setminus \{0\}$$

replaces the one in (3.19) for all the corresponding results.

(ii) Our variational approach can be used to solve a variety of other facility location problems. In particular, the following *smallest intersecting ball problem* can be naturally formulated and investigated by using the above tools of variational analysis and generalized differentiation: given n nonempty closed subsets $\Omega_i \subset X$, $i = 1, \ldots, n$, find a point $\bar{x}$ on a given set Ω and the smallest number $r > 0$ such that the ball with center at $\bar{x}$ and radius r has nonempty intersection with all the sets Ω_i as $i = 1, \ldots, n$. This problem is modeled as follows:

$$\text{minimize } \mathrm{M}(x) := \max\big\{d(x;\Omega_i)\big|\, i = 1, \ldots, n\big\} \text{ subject to } x \in \Omega.$$

We intend to address this and other facility location problems in our future research.

(iii) For some results in the Hilbert space setting of Section 3, it is possible to use *the proximal normal cone* instead of the Fréchet normal cone. However, we use the Fréchet normal cone consistently for the simplicity of presentation.

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