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**TANGENTIAL EXTREMAL PRINCIPLES FOR FINITE  
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THEORY**

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# TANGENTIAL EXTREMAL PRINCIPLES FOR FINITE AND INFINITE SYSTEMS OF SETS, I: BASIC THEORY\*

BORIS S. MORDUKHOVICH<sup>†</sup> and HUNG M. PHAN<sup>‡</sup>

**Abstract.** In this paper we develop new extremal principles in variational analysis that deal with finite and infinite systems of convex and nonconvex sets. The results obtained, unified under the name of tangential extremal principles, combine primal and dual approaches to the study of variational systems being in fact first extremal principles applied to infinite systems of sets. The first part of the paper concerns the basic theory of tangential extremal principles while the second part presents applications to problems of semi-infinite programming and multiobjective optimization.

**Key words.** Variational analysis, extremal systems, extremal principles, tangent and normal cones, semi-infinite programming, multiobjective optimization

**AMS subject classifications.** Primary: 49J52, 49J53; Secondary: 90C30

## 1 Introduction

It has been well recognized that the *convex separation principle* plays a crucial role in many aspects of nonlinear analysis, optimization, and their applications. In particular, a conventional way to derive necessary optimality conditions in constrained optimization problems is to construct first an appropriate *tangential convex approximations* of the problem data around an optimal solution in *primal* spaces and then to apply a convex separation theorem to get supporting elements in dual spaces (Lagrange multipliers, adjoint arcs, shadow prices, etc.). For problems of nonsmooth optimization, this approach inevitably leads to the usage of *convex* sets of normals and subgradients whose calculi are also based on convex separation theorems and/or their equivalents.

Despite the well-developed technique of convex analysis, the convex separation approach has a number of serious limitations, especially concerning applications to problems of nonsmooth optimization and related topics; see, e.g., commentaries and discussions on pp. 132–140 of [5] and also on pp. 131–133 of [6]. To overcome some of these limitations, a *dual-space* approach revolving around *extremal principles* has been developed and largely applied in the frameworks of variational analysis, generalized differentiation, and optimization-related areas; see the two-volume monograph [5, 6] with their references. The extremal principles developed therein can be viewed as variational counterparts of convex separation theorems in nonconvex settings while providing normal cone descriptions of extremal points of finitely many closed sets in terms of the corresponding generalized *Euler equation*.

Note that the known extremal principles do not involve any tangential approximations of sets in primal spaces and do not employ convex separation. This dual-space approach exhibits a number of significant advantages in comparison with convex separation techniques and opens new perspectives in variational analysis, generalized differentiation, and their numerous applications. On the other hand, we are not familiar with any versions of extremal principles in the scope of [5, 6] for infinite systems of sets; it is not even clear how to formulate them appropriately in the lines of the developed methodology. Among primary motivations for considering infinite systems

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of sets we mention problems of semi-infinite programming, especially those concerning the most difficult case of countably many constraints vs. conventional ones with compact indexes; cf. [2].

The main purpose of this paper is to propose and justify extremal principles of a new type, which can be applied to infinite set systems while also provide independent results for finitely many nonconvex sets. To achieve this goal, we develop a novel approach that incorporates and unifies some ideas from both tangential approximations of sets in primal spaces and nonconvex normal cone approximations in dual spaces. The essence of this approach is as follows. Employing a variational technique, we first derive a new *conic extremal principle*, which concerns countable systems of general *nonconvex* cones in finite dimensions and describes their extremality at the origin via an appropriate countable version of the generalized Euler equation formulated in terms of the nonconvex limiting normal cone by Mordukhovich [4]. Then we introduce a notion of *tangential extremal points* for infinite (in particular, finite) systems of closed sets involving their tangential approximations. The corresponding *tangential extremal principles* are induced in this way by applying the conic extremal principle to the collection of selected tangential approximations. The major attention is paid in this paper to the case of tangential approximations generated by the (nonconvex) Bouligand-Severi *contingent cone*, which exhibits remarkable properties that are most appropriate for implementing the proposed scheme and subsequent applications. The contingent cone is replaced by its *weak* counterpart when the space in question is infinite-dimensional. Selected applications of the developed theory to problems of semi-infinite programming and multiobjective optimization are given in the second part of this study [7]

For the reader's convenience we briefly overview in Section 2 some basic constructions of tangent and normal cones in variational analysis widely used in what follows. Section 3 contains definitions of tangential extremal points of finite and infinite set systems as well as descriptions of the extremality conditions for them, which are at the heart of the tangential extremal principles established below. In this section we also compare the new notions of tangential extremality with the conventional notion of extremality previously known for finite systems of sets.

Section 4 is devoted to deriving the conic extremal principle for countable systems of arbitrary closed cones in finite-dimensional spaces. In Section 5 we apply this basic result to establishing several useful representations of Fréchet normals to countable intersections of cones at the origin.

Section 6 concerns the study of the weak contingent cone in infinite-dimensional spaces, which reduces to the classical Bouligand-Severi contingent cone in finite dimensions. We show that the weak contingent cone provides a remarkable tangential approximation for an arbitrary closed subset enjoying, in particular, the new tangential normal enclosedness and approximate normality properties in any reflexive Banach spaces. These properties are employed in Section 7 to derive contingent and weak contingent extremal principles for countable and finite systems of closed sets in finite and infinite dimensions. We also establish appropriate versions of the aforementioned results in a broader class of Asplund spaces.

Throughout the paper we use standard notation of variational analysis; see, e.g., [5, 8]. Unless otherwise stated, the space  $X$  in question is Banach with the norm  $\|\cdot\|$  and the canonical pairing  $\langle \cdot, \cdot \rangle$  between  $X$  and its topological dual  $X^*$  with  $IB \subset X$  and  $IB^* \subset X^*$  standing for the corresponding closed unit balls. The symbols  $\xrightarrow{w}$  and  $\xrightarrow{w^*}$  indicate the weak convergence in  $X$  and

the weak\* convergence in  $X^*$ , respectively. Given  $\emptyset \neq \Omega \subset X$ , denote by

$$\text{cone } \Omega := \bigcup_{\lambda \geq 0} \lambda \Omega = \bigcup_{\lambda \geq 0} \{ \lambda v \mid v \in \Omega \}$$

the *conic hull* of  $\Omega$  and by

$$\text{co } \Omega := \left\{ \sum_{i \in I} \lambda_i u_i \mid I \text{ finite, } \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1, u_i \in \Omega \right\}$$

the *convex hull* of this set. The notation  $x \xrightarrow{\Omega} \bar{x}$  means that  $x \rightarrow \bar{x}$  with  $x \in \Omega$ . Finally,  $\mathbb{N} := \{1, 2, \dots\}$  signifies the collection of all natural numbers.

## 2 Tangents and Normal to Nonconvex Sets

In this section we recall some basic notions of tangent and normal cones to nonempty sets closed around the reference points; see the books [1, 5, 8, 9] for more details and related material.

Given  $\Omega \subset X$  and  $\bar{x} \in \Omega$ , the closed (while often nonconvex) cone

$$T(\bar{x}; \Omega) := \{v \in X \mid \exists \text{ sequences } t_k \downarrow 0, v_k \rightarrow v \text{ with } \bar{x} + t_k v_k \in \Omega, \forall k \in \mathbb{N}\} \quad (2.1)$$

is the *Bouligand-Severi tangent/contingent cone* to  $\Omega$  at  $\bar{x}$ . We also use its weak counterpart

$$T_w(\bar{x}; \Omega) := \{v \in X \mid \exists \text{ sequences } t_k \downarrow 0, v_k \xrightarrow{w} v \text{ with } \bar{x} + t_k v_k \in \Omega, \forall k \in \mathbb{N}\} \quad (2.2)$$

known as the *weak contingent cone* to  $\Omega$  at this point. For any  $\varepsilon \geq 0$ , the collection

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\} \quad (2.3)$$

is called the set of  $\varepsilon$ -normals to  $\Omega$  at  $\bar{x}$ . In the case of  $\varepsilon = 0$  the set  $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$  is a cone known as the *Fréchet/regular normal cone* (or the prenormal cone) to  $\Omega$  at this point. Note that the Fréchet normal cone is always convex while it may be trivial (i.e., reduced to  $\{0\}$ ) at boundary points of simple nonconvex sets in finite dimensions as for  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -|x_1|\}$  at  $\bar{x} = (0, 0)$ . If the space  $X$  is reflexive, then

$$\widehat{N}(\bar{x}; \Omega) = T_w^*(\bar{x}; \Omega) := \{x^* \in X^* \mid \langle x^*, v \rangle \leq 0, \forall v \in T_w(\bar{x}; \Omega)\}. \quad (2.4)$$

The collection of sequential limiting normals

$$N(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \begin{array}{l} \exists \text{ sequences } \varepsilon_k \downarrow 0, x_k \xrightarrow{\Omega} \bar{x}, x_k^* \xrightarrow{w^*} x^* \text{ as } k \rightarrow \infty \\ \text{such that } x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega), \forall k \in \mathbb{N} \end{array} \right\} \quad (2.5)$$

is known as the *Mordukhovich/basic/limiting normal cone* to  $\Omega$  at  $\bar{x}$ . If the space  $X$  is Asplund, i.e., each of its separable subspaces has a separable dual (this is automatic, in particular, for any reflexive Banach space), then we can equivalently put  $\varepsilon_k = 0$  in (2.5); see [5] for more details. Observe also that for  $X = \mathbb{R}^n$  the normal cone (2.5) can be equivalently described in the form

$$N(\bar{x}; \Omega) = \left\{ x^* \in \mathbb{R}^n \mid \begin{array}{l} \exists \text{ sequences } x_k \rightarrow \bar{x}, w_k \in \Pi(x_k; \Omega), \alpha_k \geq 0 \\ \text{such that } \alpha_k(x_k - w_k) \rightarrow x^* \text{ as } k \rightarrow \infty \end{array} \right\}$$

via the *Euclidean projector*  $\Pi(x; \Omega) := \{w \in \Omega \mid \|x - w\| = \text{dist}(x; \Omega)\}$  of  $x \in \mathbb{R}^n$  onto  $\Omega$ .

It is worth mentioning that the limiting normal cone (2.5) is often nonconvex as, e.g., for the set  $\Omega \subset \mathbb{R}^2$  considered above, where  $N(0; \Omega) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2 = -|u_1|\}$ . It does not happen when  $\Omega$  is *normally regular* at  $\bar{x}$  in the sense that  $N(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega)$ . The latter class includes convex sets when both cones (2.3) as  $\varepsilon = 0$  and (2.5) reduce to the classical normal cone of convex analysis and also some other collections of “nice” sets of a certain locally convex type. At the same time it excludes a number of important settings that frequently appear in applications; see, e.g., the books [5, 6, 8] for precise results and discussions. Being nonconvex, the normal cone  $N(\bar{x}; \Omega)$  in (2.5) cannot be tangentially generated by duality of type (2.4), since the duality/polarity operation automatically implies convexity. Nevertheless, in contrast to Fréchet normals, this limiting normal cone enjoys *full calculus* in general Asplund spaces, which is mainly based on extremal principles of variational analysis and related variational techniques; see [5] for a comprehensive calculus account and further references.

The next simple observation is useful in what follows.

**Proposition 2.1 (generalized normals to cones).** *Let  $\Lambda \subset X$  be a cone, and let  $w \in \Lambda$ . Then we have the inclusion*

$$\widehat{N}(w; \Lambda) \subset N(0; \Lambda).$$

**Proof.** Pick any  $x^* \in \widehat{N}(w; \Lambda)$  and get by definition (2.3) of the Fréchet normal cone that

$$\limsup_{x \Delta w} \frac{\langle x^*, x - w \rangle}{\|x - w\|} \leq 0.$$

Fix  $x \in \Lambda$ ,  $t > 0$  and let  $u := x/t$ . Then  $(x/t) \in \Lambda$ ,  $tw \in \Lambda$ , and

$$\limsup_{x \Delta tw} \frac{\langle x^*, x - tw \rangle}{\|x - tw\|} = \limsup_{x \Delta w} \frac{t \langle x^*, (x/t) - w \rangle}{t \|(x/t) - w\|} = \limsup_{u \Delta w} \frac{\langle x^*, u - w \rangle}{\|u - w\|} \leq 0,$$

which gives  $x^* \in \widehat{N}(tw; \Lambda)$  by (2.3). Letting finally  $t \rightarrow 0$ , we get  $x^* \in N(0; \Lambda)$  and thus complete the proof of the proposition.  $\square$

### 3 Tangential Extremal Systems and Extremality Conditions

In this section we introduce the notions of conic and tangential extremal systems for finite and countable collections of sets and discuss extremality conditions, which are at the heart of the conic and tangential extremal principles justified in the subsequent sections. These new extremality concepts are compared with conventional notions of local extremality for set systems.

We start with the new definitions of extremal points and extremal systems of a countable or finite number of cones and general sets in normed spaces.

**Definition 3.1 (conic and tangential extremal systems).** *Let  $X$  be an arbitrary normed space. Then we say that:*

(a) *A countable system of cones  $\{\Lambda_i\}_{i \in \mathbb{N}} \subset X$  with  $0 \in \bigcap_{i=1}^{\infty} \Lambda_i$  is EXTREMAL AT THE ORIGIN, or simply is an EXTREMAL SYSTEM OF CONES, if there is a bounded sequence  $\{a_i\}_{i \in \mathbb{N}} \subset X$  with*

$$\bigcap_{i=1}^{\infty} (\Lambda_i - a_i) = \emptyset. \quad (3.1)$$

(b) Let  $\{\Omega_i\}_{i \in \mathbb{N}} \subset X$  be an countable system of sets with  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ , and let  $\Lambda := \{\Lambda_i(\bar{x})\}_{i \in \mathbb{N}}$  with  $0 \in \bigcap_{i=0}^{\infty} \Lambda_i(\bar{x}) \subset X$  be an approximating system of cones. Then  $\bar{x}$  is a  $\Lambda$ -TANGENTIAL LOCAL EXTREMAL POINT of  $\{\Omega_i\}_{i \in \mathbb{N}}$  if the system of cones  $\{\Lambda_i(\bar{x})\}_{i \in \mathbb{N}}$  is extremal at the origin. In this case the collection  $\{\Omega_i, \bar{x}\}_{i \in \mathbb{N}}$  is called a  $\Lambda$ -TANGENTIAL EXTREMAL SYSTEM.

(c) Suppose that  $\Lambda_i(\bar{x}) = T(\bar{x}; \Omega_i)$  are the contingent cones to  $\Omega_i$  at  $\bar{x}$  in (b). Then  $\{\Omega_i, \bar{x}\}_{i \in \mathbb{N}}$  is called a CONTINGENT EXTREMAL SYSTEM with the CONTINGENT LOCAL EXTREMAL POINT  $\bar{x}$ . We use the terminology of WEAK CONTINGENT EXTREMAL SYSTEM and WEAK CONTINGENT LOCAL EXTREMAL POINT if  $\Lambda_i(\bar{x}) = T_w(\bar{x}; \Omega_i)$  are the weak contingent cones to  $\Omega_i$  at  $\bar{x}$ .

Note that all the notions in Definition 3.1 obviously apply to the case of systems containing *finitely* many sets; indeed, in such a case the other sets reduce to the whole space  $X$ . Observe also that both parts in part (c) of this definition are equivalent in finite dimensions. Furthermore, they both reduce to (a) in the general case if all the sets  $\Omega_i$  are cones and  $\bar{x} = 0$ .

Let us now compare the new notions of Definition 3.1 with the conventional notion of locally extremal points for finitely many sets first formulated in [3]. Recall [5, Definition 2.1] that a point  $\bar{x} \in \bigcap_{i=1}^m \Omega_i$  is *locally extremal* for the system  $\{\Omega_1, \dots, \Omega_m\}$  if there are sequences  $\{a_{ik}\} \subset X$  with  $a_{ik} \rightarrow 0$  as  $k \rightarrow \infty$  for  $i = 1, \dots, m$  and a neighborhood  $U$  of  $\bar{x}$  such that

$$\bigcap_{i=1}^m (\Omega_i - a_{ik}) \cap U = \emptyset \text{ for all large } k \in \mathbb{N}. \quad (3.2)$$

We first observe that for finite systems of cones the local extremality of the origin in the sense of (3.2) is equivalent to the validity of condition (3.1) of Definition 3.1.

**Proposition 3.2 (equivalent description of cone extremality).** *The finite system of cones  $\{\Lambda_1, \dots, \Lambda_m\}$  is extremal at the origin in the sense of Definition 3.1(a) if and only if  $\bar{x} = 0$  is a local extremal point of  $\{\Lambda_1, \dots, \Lambda_m\}$  in the sense of (3.2).*

**Proof.** The “only if” part is obvious. To justify the “if” part, assume that there are elements  $a_1, \dots, a_m \in X$  such that

$$\bigcap_{i=1}^m (\Lambda_i - a_i) = \emptyset. \quad (3.3)$$

Now for any  $\eta > 0$  we have by (3.3) and the conic structure of  $\Lambda_i$  that

$$\emptyset = \bigcap_{i=1}^m \eta(\Lambda_i - a_i) = \bigcap_{i=1}^m (\eta\Lambda_i - \eta a_i) = \bigcap_{i=1}^m (\Lambda_i - \eta a_i).$$

Letting  $\eta \downarrow 0$  implies that the extremality condition (3.2) holds, i.e., the origin is a local extremal point of the cone system  $\{\Lambda_1, \dots, \Lambda_m\}$ .  $\square$

Next we show that the local extremality (3.2) and the contingent extremality from Definition 3.1(c) are independent notions even in the case of two sets in  $\mathbb{R}^2$ .

**Example 3.3 (contingent extremality versus local extremality).**

(i) Consider two closed subsets in  $\mathbb{R}^2$  defined by

$$\Omega_1 := \text{epi } \varphi \text{ with } \varphi(x) := x \sin(1/x) \text{ as } x \neq 0, \varphi(0) = 0 \text{ and } \Omega_2 := (\mathbb{R} \times \mathbb{R}_-) \setminus \text{int } \Omega_1.$$

Take the point  $\bar{x} = (0, 0) \in \Omega_1 \cap \Omega_2$  and observe that the contingent cones to  $\Omega_1$  and  $\Omega_2$  at  $\bar{x}$  are computed, respectively, by

$$T(\bar{x}; \Omega_1) = \text{epi}(-|\cdot|) \text{ and } T(\bar{x}; \Omega_2) = \mathbb{R} \times \mathbb{R}_-.$$

It is easy to see that  $\bar{x}$  is a local extremal point of  $\{\Omega_1, \Omega_2\}$  but not a contingent local extremal point of this set system.

(ii) Define two closed subsets of  $\mathbb{R}^2$  by

$$\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -x_1^2\} \text{ and } \Omega_2 := \mathbb{R} \times \mathbb{R}_-.$$

The contingent cones to  $\Omega_1$  and  $\Omega_2$  at  $\bar{x} = (0, 0)$  are computed by

$$T(\bar{x}; \Omega_1) = \mathbb{R} \times \mathbb{R}_+ \text{ and } T(\bar{x}; \Omega_2) = \mathbb{R} \times \mathbb{R}_-.$$

We can see that  $\{\Omega_1, \Omega_2, \bar{x}\}$  is a contingent extremal system but not an extremal system of sets.

Our further intention is to derive verifiable *extremality conditions* for tangentially extremal points of set systems in certain *countable* forms of the *generalized Euler equation* expressed via the limiting normal cone (2.5) at the points in question. Let us first formulate and discuss the desired conditions, which reflect the essence of the tangential extremal principles of this paper.

**Definition 3.4 (extremality conditions for countable systems).** *We say that:*

(a) *The system of cones  $\{\Lambda_i\}_{i \in \mathbb{N}}$  in  $X$  satisfies the CONIC EXTREMALITY CONDITIONS at the origin if there are normals  $x_i^* \in N(0; \Lambda_i)$  for  $i = 1, 2, \dots$  such that*

$$\sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 = 1. \quad (3.4)$$

(b) *Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  with  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  and  $\Lambda := \{\Lambda_i\}_{i \in \mathbb{N}}$  with  $0 \in \bigcap_{i=1}^{\infty} \Lambda_i$  be, respectively, systems of arbitrary sets and approximating cones in  $X$ . Then the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  satisfies the  $\Lambda$ -TANGENTIAL EXTREMALITY CONDITIONS at  $\bar{x}$  if the systems of cones  $\{\Lambda_i\}_{i \in \mathbb{N}}$  satisfies the conic extremality conditions at the origin. We specify the CONTINGENT EXTREMALITY CONDITIONS and the WEAK CONTINGENT EXTREMALITY CONDITIONS for  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x}$  if  $\Lambda = \{T(\bar{x}; \Omega_i)\}_{i \in \mathbb{N}}$  and  $\Lambda = \{T_w(\bar{x}; \Omega_i)\}_{i \in \mathbb{N}}$ , respectively.*

(c) *The system of sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  in  $X$  satisfies the LIMITING EXTREMALITY CONDITIONS at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  if there are limiting normals  $x_i^* \in N(\bar{x}; \Omega_i)$ ,  $i = 1, 2, \dots$ , satisfying (3.4).*

Let us briefly discuss the introduced extremality conditions.

**Remark 3.5 (discussions on extremality conditions).**

(i) All the conditions of Definition 3.4 can be obviously specified to the case of *finite systems* of sets by considering all the other sets as the whole space therein. Then the series in (3.4) become finite sums and the coefficients  $2^{-i}$  can be dropped by rescaling.

(ii) It easily follows from the constructions involved that the contingent, weak contingent, and limiting extremality conditions are *equivalent* to each other if all the sets  $\Omega_i$  are either *cones* with  $\bar{x} = 0$  or *convex* near  $\bar{x}$ .



(iii) As we show below, the weak contingent extremality conditions *imply* the limiting extremality conditions in any reflexive space  $X$  and also in Asplund spaces under a certain additional assumption, which is automatic under reflexivity. Thus the contingent extremality conditions imply the limiting ones in finite dimensions. The opposite implication does *not hold* even for two sets in  $\mathbb{R}^2$ . To illustrate it, consider the two sets from Example 3.3(i) for which  $\bar{x} = (0, 0)$  is a local extremal point in the usual sense, and hence the limiting extremality conditions hold due to [5, Theorem 2.8]. However, it is easy to see that the contingent extremality conditions are violated for this system.

Observe that for the case of *finitely many* sets  $\{\Omega_1, \dots, \Omega_m\}$  the limiting extremality conditions of Definition 3.4(c) correspond to the generalized Euler equation in the *exact extremal principle* of [5, Definition 2.5(iii)] applied to local extremal points of sets. A natural version of the “fuzzy” Euler equation in the *approximate extremal principle* of [5, Definition 2.5(ii)] for the case of a *countable* set system  $\{\Omega_i\}_{i \in \mathbb{N}}$  at  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  can be formulated as follows: for any  $\varepsilon > 0$  there are

$$x_i \in \Omega_i \cap (\bar{x} + \varepsilon B) \text{ and } x_i^* \in \widehat{N}(x_i; \Omega_i) + \frac{1}{2^i} \varepsilon B^*, \quad i \in \mathbb{N}, \quad (3.5)$$

such that the relationships in (3.4) is satisfied. It turns out that such a countable version of the approximate extremal principle always *holds trivially*, at least in Asplund spaces, for *any* system of closed sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  at *every* boundary point  $\bar{x}$  of infinitely many sets  $\Omega_i$ .

**Proposition 3.6 (triviality of the approximate extremality conditions for countable set systems).** *Let  $\{\Omega_i\}_{i \in \mathbb{N}}$  be a countable system of sets closed around some point  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$ , and let  $\varepsilon > 0$ . Assume that for infinitely many  $i \in \mathbb{N}$  there exist  $x_i \in \Omega_i \cap (\bar{x} + \varepsilon B)$  such that  $\widehat{N}(x_i; \Omega_i) \neq \{0\}$ ; this is the case when  $X$  is Asplund and  $\bar{x}$  belongs to the boundary of infinitely many sets  $\Omega_i$ . Then we always have  $\{x_i^*\}_{i \in \mathbb{N}}$  satisfying conditions (3.4) and (3.5).*

**Proof.** Observe first that the fulfillment of the assumption made in the proposition for the case of Asplund spaces follows from the density of Fréchet normals on boundaries of closed sets in such spaces; see, e.g., [5, Corollary 2.21]. To proceed further, fix  $\varepsilon > 0$  and find  $j \in \mathbb{N}$  so large that

$$\frac{\sqrt{2^j}}{2^{j-1}} \leq \frac{1}{2} \varepsilon \text{ and } \widehat{N}(x_j; \Omega_j) \neq \{0\} \text{ with } x_j \in \Omega_j \cap (\bar{x} + \varepsilon B).$$

This allows us to get  $0 \neq x_j^* \in \widehat{N}(x_j; \Omega_j)$  such that  $\|x_j^*\| = \sqrt{2^j}$  and then choose

$$x_1^* := -\frac{1}{2^{j-1}} x_j^* \in 0 + \frac{1}{2} \varepsilon B^* \subset \widehat{N}(x_1; \Omega_1) + \frac{1}{2} \varepsilon B^*, \quad x_j^* \in \widehat{N}(x_j; \Omega_j) + \frac{1}{2^j} \varepsilon B^*,$$

and  $x_i^* := 0 \in \widehat{N}(x_i; \Omega_i) + \frac{1}{2^i} \varepsilon B^*$  for all  $i \neq 1, j$ .

Thus we have the sequence  $\{x_i^*\}_{i \in \mathbb{N}}$  satisfying (3.5) and the relationships

$$\sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = \frac{1}{2} \left( -\frac{1}{2^{j-1}} x_j^* \right) + 0 + \dots + \frac{1}{2^j} x_j^* + \dots = 0, \quad \sum_{i=1}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 > 1,$$

which give (3.4) and complete the proof of the proposition.  $\square$

## 4 Conic Extremal Principle for Countable Systems of Sets

This section addresses the *conic extremal principle* for countable systems of cones in finite-dimensional spaces. This is the first extremal principle for infinite systems of sets, which ensures the fulfillment of the conic extremality conditions of Definition 3.4(a) for a conic extremal system at the origin under a natural nonoverlapping assumption. We present a number of examples illustrating the results obtained and the assumptions made.

To derive the main result of this section, we extend the *method of metric approximations* initiated in [4] to the case of countable systems of cones; cf. an essentially different realization of this method in the proof of the extremal principle for local extremal points of finitely many sets in  $\mathbb{R}^n$  given in [5, Theorem 2.8]. First observe an elementary fact needed in what follows.

**Lemma 4.1 (series differentiability).** *Let  $\|\cdot\|$  be the usual Euclidian norm in  $\mathbb{R}^n$ , and let  $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  be a bounded sequence. Then a function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  defined by*

$$\varphi(x) := \sum_{i=1}^{\infty} \frac{1}{2^i} \|x - z_i\|^2, \quad x \in \mathbb{R}^n,$$

*is continuously differentiable on  $\mathbb{R}^n$  with the derivative*

$$\nabla\varphi(x) = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} (x - z_i), \quad x \in \mathbb{R}^n.$$

**Proof.** It is easy to see that both series above converge for every  $x \in \mathbb{R}^n$ . Taking further any  $u, \xi \in \mathbb{R}^n$  with the norm  $\|\xi\|$  sufficiently small, we have

$$\|u + \xi\|^2 - \|u\|^2 - 2\langle u, \xi \rangle = \|u\|^2 + 2\langle u, \xi \rangle + \|\xi\|^2 - \|u\|^2 - 2\langle u, \xi \rangle = \|\xi\|^2 = o(\|\xi\|).$$

Thus it follows for any  $x \in \mathbb{R}^n$  and  $y$  close to  $x$  that

$$\begin{aligned} \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x \rangle &= \sum_{i=1}^{\infty} \frac{1}{2^i} \left[ \|y - z_i\|^2 - \|x - z_i\|^2 - 2\langle x - z_i, y - x \rangle \right] \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \|y - x\|^2 = o(\|y - x\|), \end{aligned}$$

which justifies that  $\nabla\varphi(x)$  is the derivative of  $\varphi$  at  $x$ , which is obviously continuous on  $\mathbb{R}^n$ .  $\square$

Here is the extremal principle for a countable systems of cones, which plays a crucial role in the subsequent applications of this paper and its continuation [7].

**Theorem 4.2 (conic extremal principle in finite dimensions).** *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be an extremal system of closed cones in  $X = \mathbb{R}^n$  satisfying the NONOVERLAPPING CONDITION*

$$\bigcap_{i=1}^{\infty} \Lambda_i = \{0\}. \quad (4.1)$$

*Then the conic extremal principle holds, i.e., there are  $x_i^* \in N(0; \Lambda_i)$  for  $i = 1, 2, \dots$  such that*

$$\sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 = 1.$$

*Moreover, one can find  $w_i \in \Lambda_i$  for which  $x_i^* \in \widehat{N}(w_i; \Lambda_i)$ ,  $i = 1, 2, \dots$*

**Proof.** Pick a bounded sequence  $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$  from Definition 3.1(a) satisfying

$$\bigcap_{i=1}^{\infty} (\Lambda_i - a_i) = \emptyset$$

and consider the unconstrained optimization problem:

$$\text{minimize } \varphi(x) := \left[ \sum_{i=1}^{\infty} \frac{1}{2^i} \text{dist}^2(x + a_i; \Lambda_i) \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^n. \quad (4.2)$$

Let us prove that problem (4.2) has an optimal solution. Since the function  $\varphi$  in (4.2) is continuous on  $\mathbb{R}^n$  due to the continuity of the distance function and the uniform convergence of the series therein, it suffices to show that there is  $\alpha > 0$  for which the nonempty level set  $\{x \in \mathbb{R}^n \mid \varphi(x) \leq \inf_x \varphi + \alpha\}$  is bounded and then to apply the classical Weierstrass theorem. Suppose by the contrary that the level sets are unbounded whenever  $\alpha > 0$ , for any  $k \in \mathbb{N}$  find  $x_k \in \mathbb{R}^n$  satisfying

$$\|x_k\| > k \quad \text{and} \quad \varphi(x_k) \leq \inf_x \varphi + \frac{1}{k}.$$

Setting  $u_k := x_k / \|x_k\|$  with  $\|u_k\| = 1$  and taking into account that all  $\Lambda_i$  are cones, we get

$$\frac{1}{\|x_k\|} \varphi(x_k) = \left[ \sum_{i=1}^{\infty} \frac{1}{2^i} \text{dist}^2\left(u_k + \frac{a_i}{\|x_k\|}; \Lambda_i\right) \right]^{\frac{1}{2}} \leq \frac{1}{\|x_k\|} \left( \inf_x \varphi + \frac{1}{k} \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.3)$$

Furthermore, there is  $M > 0$  such that for large  $k \in \mathbb{N}$  we have

$$\text{dist}\left(u_k + \frac{a_i}{\|x_k\|}; \Lambda_i\right) \leq \left\| u_k + \frac{a_i}{\|x_k\|} \right\| \leq M.$$

Without relabeling, assume  $u_k \rightarrow u$  as  $k \rightarrow \infty$  with some  $u \in \mathbb{R}^n$ . Passing now to the limit as  $k \rightarrow \infty$  in (4.3) and employing the uniform convergence of the series therein and the fact that  $a_i / \|x_k\| \rightarrow 0$  uniformly in  $i \in \mathbb{N}$  due to the boundedness of  $\{a_i\}_{i \in \mathbb{N}}$ , we have

$$\left[ \sum_{i=1}^{\infty} \frac{1}{2^i} \text{dist}^2(u; \Lambda_i) \right]^{\frac{1}{2}} = 0.$$

This implies by the closedness of the cones  $\Lambda_i$  and the *nonoverlapping condition* (4.1) of the theorem that  $u \in \bigcap_{i=1}^{\infty} \Lambda_i = \{0\}$ . The latter is impossible due to  $\|u\| = 1$ , which contradicts our intermediate assumption on the unboundedness of the level sets for  $\varphi$  and thus justifies the existence of an optimal solution  $\tilde{x}$  to problem (4.2).

Since the system of closed cones  $\{\Lambda_i\}_{i \in \mathbb{N}}$  is *extremal at the origin*, it follows from the construction of  $\varphi$  in (4.2) that  $\varphi(\tilde{x}) > 0$ . Taking into account the nonemptiness of the projection  $\Pi(x; \Lambda)$  of  $x \in \mathbb{R}^n$  onto an arbitrary closed set  $\Lambda \subset \mathbb{R}^n$ , pick any  $w_i \in \Pi(\tilde{x} + a_i; \Lambda_i)$  as  $i \in \mathbb{N}$  and observe from Proposition 2.1 above and the proof of [5, Theorem 1.6] that

$$\tilde{x} + a_i - w_i \in \Pi^{-1}(w_i; \Lambda_i) - w_i \subset \widehat{N}(w_i; \Lambda_i) \subset N(0; \Lambda_i). \quad (4.4)$$

Furthermore, the sequence  $\{a_i - w_i\}_{i \in \mathbb{N}}$  is bounded in  $\mathbb{R}^n$  due to

$$\|x + a_i - w_i\| = \text{dist}(x + a_i; \Lambda_i) \leq \|x + a_i\|.$$

Next we consider another unconstrained optimization problem:

$$\text{minimize } \psi(x) := \left[ \sum_{i=1}^{\infty} \frac{1}{2^i} \|x + a_i - w_i\|^2 \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}^n. \quad (4.5)$$

It follows from  $\psi(x) \geq \varphi(x) \geq \varphi(\tilde{x}) = \psi(\tilde{x})$  for all  $x \in \mathbb{R}^n$  that problem (4.5) has the same optimal solution  $\tilde{x}$  as (4.2). The main difference between these two problems is that the cost function  $\psi$  in (4.5) is *smooth* around  $\tilde{x}$  by Lemma 4.1, the smoothness of the function  $\sqrt{t}$  around nonzero points, and the fact that  $\psi(\tilde{x}) \neq 0$  due to the cone extremality. Applying now the *classical Fermat rule* to the smooth unconstrained minimization problem (4.5) and using the derivative calculation in Lemma 4.1, we arrive at the relationships

$$\nabla \psi(\tilde{x}) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = 0 \quad \text{with } x_i^* := \frac{1}{\psi(\tilde{x})} (\tilde{x} + a_i - w_i), \quad i \in \mathbb{N}. \quad (4.6)$$

The latter implies by (4.4) that  $x_i^* \in \widehat{N}(w_i; \Lambda_i) \subset N(0; \Lambda_i)$  for all  $i \in \mathbb{N}$ . Furthermore, it follows from the constructions of  $x_i^*$  in (4.6) and of  $\psi$  in (4.5) that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \|x_i^*\|^2 = 1,$$

which thus completes the proof of the theorem.  $\square$

In the remaining part of this section, we present three examples showing that all the assumptions made in Theorem 4.2 (nonoverlapping, finite dimension, and conic structure) are *essential* for the validity of this result.

**Example 4.3 (nonoverlapping condition is essential).** Let us show that the conic extremal principle may fail for countable systems of *convex* cones in  $\mathbb{R}^2$  if the nonoverlapping condition (4.1) is violated. Define the convex cones  $\Lambda_i \subset \mathbb{R}^2$  as  $i \in \mathbb{N}$  by

$$\Lambda_1 := \mathbb{R} \times \mathbb{R}_+ \quad \text{and} \quad \Lambda_i := \{(x, y) \in \mathbb{R}^2 \mid y \leq \frac{x}{i}\} \quad \text{for } i = 2, 3, \dots$$

Observe that for any  $\nu > 0$  we have

$$\left( \Lambda_1 + (0, \nu) \right) \bigcap_{k=2}^{\infty} \Lambda_k = \emptyset,$$

which means that the cone system  $\{\Lambda_i\}_{i \in \mathbb{N}}$  is extremal at the origin. On the other hand,

$$\bigcap_{i=1}^{\infty} \Lambda_i = \mathbb{R}_+ \times \{0\},$$

i.e., the nonoverlapping condition (4.1) is violated. Furthermore, we can easily compute the corresponding normal cones by

$$N(0; \Lambda_1) = \{\lambda(0, -1) \mid \lambda \geq 0\} \quad \text{and} \quad N(0; \Lambda_i) = \{\lambda(-1, i) \mid \lambda \geq 0\}, \quad i = 2, 3, \dots$$

Taking now any  $x_i^* \in N(0; \Lambda_i)$  as  $i \in \mathbb{N}$ , observe the equivalence

$$\left[ \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* = 0 \right] \iff \left[ \frac{\lambda_1}{2} (0, -1) + \sum_{i=2}^{\infty} \frac{\lambda_i}{2^i} (-1, i) = 0 \quad \text{with } \lambda_i \geq 0 \quad \text{as } i \in \mathbb{N} \right].$$

The latter implies that  $\lambda_i = 0$  and hence  $x_i^* = 0$  for all  $i \in \mathbb{N}$ . Thus the nontriviality condition in (3.4) is not satisfied, which shows that the conic extremal principle fails for this system.

**Example 4.4 (conic structure is essential).** If all the sets  $\Omega_i$  for  $i \in \mathbb{N}$  are *convex* but some of them are *not cones*, then the equivalent extremality conditions of Definition 3.4(b,c) are natural extensions of the conic extremality conditions in Theorem 4.2. We show nevertheless that the corresponding extension of the conic extremal principle under the nonoverlapping requirement

$$\bigcap_{i=1}^{\infty} \Omega_i = \{0\} \quad (4.7)$$

*fails* without imposing a conic structure on all the sets involved. Indeed, consider a countable system of closed and convex sets in  $\mathbb{R}^2$  defined by

$$\Omega_1 := \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\} \text{ and } \Omega_i := \{(x, y) \in \mathbb{R}^2 \mid y \leq \frac{x}{i}\} \text{ for } i = 2, 3, \dots$$

We can see that only the set  $\Omega_1$  is not a cone and that the nonoverlapping requirement (4.7) is satisfied. Furthermore, the system  $\{\Omega_i\}_{i \in \mathbb{N}}$  is extremal at the origin in the sense that (3.1) holds. However, the arguments similar to Example 4.3 show that the extremality conditions (3.4) with  $x_i^* \in N(0; \Omega_i)$  as  $i \in \mathbb{N}$  fail to fulfill. Note that, as shown in Section 7, both contingent and limiting extremal principles hold for countable systems of general nonconvex sets if nonoverlapping condition (4.7) is replaced by another one reflecting the *contingent extremality*.

**Example 4.5 (failure of the conic extremal principle in infinite dimensions).** The last example demonstrates that the conic extremal principle of Theorem 4.2 with the nonoverlapping condition (4.1) may fail for countable systems of *convex cones* (in fact, half-spaces) in an arbitrary infinite-dimensional *Hilbert space*. To proceed, consider a Hilbert space  $X$  with the orthonormal basis  $\{e_i \mid i \in \mathbb{N}\}$  and define a countable system of closed half-spaces by

$$\Lambda_1 := \{x \in X \mid \langle x, e_1 \rangle \leq 0\} \text{ and } \Lambda_i := \{x \in X \mid \langle x, e_i - e_{i-1} \rangle \leq 0\} \text{ for } i = 2, 3, \dots$$

It is easy to compute the corresponding normal cones to the above sets:

$$N(0; \Lambda_1) = \{\lambda e_1 \mid \lambda \geq 0\} \text{ and } N(0; \Lambda_i) = \{\lambda(e_i - e_{i-1}) \mid \lambda \geq 0\} \text{ for } i = 2, 3, \dots$$

Now let us check that the nonoverlapping condition (4.1) is satisfied. Indeed, picking any point

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \in \bigcap_{i=1}^{\infty} \Lambda_i,$$

we have  $\alpha_1 = \langle x, e_1 \rangle \leq 0$  and  $\alpha_i = \langle x, e_i \rangle \leq \langle x, e_{i-1} \rangle = \alpha_{i-1}$  for  $i = 2, 3, \dots$ . This clearly leads to  $\alpha_i = 0$  for all  $i \in \mathbb{N}$ , which yields  $x = 0$  and thus justifies (4.1). The same arguments show that

$$(\Lambda_1 - e_1) \cap \bigcap_{i=2}^{\infty} \Lambda_i = \emptyset,$$

i.e.,  $\{\Lambda_i\}_{i \in \mathbb{N}}$  is a *conic extremal system*. However, the conic extremality conditions of Definition 3.4(a) *fail* for this system. To check this, suppose that there exist  $x_i^* \in N(0; \Lambda_i)$  as  $i \in \mathbb{N}$  satisfying the relationships

$$\sum_{i=1}^{\infty} x_i^* = 0 \text{ and } \sum_{i=1}^{\infty} \|x_i^*\| > 0. \quad (4.8)$$

By the above structure of  $N(0; \Lambda_i)$  we have  $x_1^* = \lambda_1 e_1$  and  $x_i^* = \lambda_i(e_i - e_{i-1})$  as  $i = 2, 3, \dots$  for some  $\lambda_i \geq 0$  as  $i \in \mathbb{N}$ . Thus the first condition in (4.8) reduces to

$$\lambda_1 e_1 + \sum_{i=2}^{\infty} \lambda_i (e_i - e_{i-1}) = 0.$$

The latter is possible if either (a):  $\lambda_i = 1$  for all  $i \in \mathbb{N}$  or (b):  $\lambda_i = 0$  for all  $i \in \mathbb{N}$ . Case (a) surely contradicts the convergence of the series in the second condition of (4.8) while in case (b) the latter series converges to zero. Hence the conic extremal principle of Theorem 4.2 does not hold in this infinite-dimensional setting.

## 5 Fréchet Normals to Countable Intersections of Cones

In this section we present applications of the conic extremal principle established in Theorem 4.2 to deriving several representations, under appropriate assumptions, of Fréchet normals to *countable intersections* of cones in finite-dimensional spaces. These calculus results are certainly of their independent interest while they are largely employed in [7] to problems of semi-infinite programming and multiobjective optimization.

To begin with, we introduce the following qualification condition for countable systems of cones formulated in terms of limiting normals (2.5), which plays a significant role in deriving the results of this section as well as in the subsequent applications given in [7].

**Definition 5.1 (normal qualification condition for countable systems of cones).** *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be a countable system of closed cones in  $X$ . We say that it satisfies the NORMAL QUALIFICATION CONDITION at the origin if*

$$\left[ \sum_{i=1}^{\infty} x_i^* = 0, \quad x_i^* \in N(0; \Lambda_i) \right] \implies [x_i^* = 0, \quad i \in \mathbb{N}]. \quad (5.1)$$

This definition corresponds to the normal qualification condition of [5] for finite systems of sets; see the discussions and various applications of the latter condition therein. We refer the reader to [7] for a nonconic version of (5.1), its relationships with other qualification conditions for countable systems of sets, and sufficient conditions for its validity that equally apply to both conic and nonconic versions. In this section we use the normal qualification condition of Definition 5.1 to represent Fréchet normals to countable intersections of cones in terms of limiting normals to each of the sets involved. Let us start with the following “fuzzy” intersection rule at the origin.

**Theorem 5.2 (fuzzy intersection rule for Fréchet normals to countable intersections of cones).** *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be a countable system of arbitrary closed cones in  $\mathbb{R}^n$  satisfying the normal qualification condition (5.1). Then given a Fréchet normal  $x^* \in \widehat{N}(0; \bigcap_{i=1}^{\infty} \Lambda_i)$  and a number  $\varepsilon > 0$ , there are limiting normals  $x_i^* \in N(0; \Lambda_i)$  as  $i \in \mathbb{N}$  such that*

$$x^* \in \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* + \varepsilon B^*. \quad (5.2)$$

**Proof.** Fix  $x^* \in \widehat{N}(0; \bigcap_{i=1}^{\infty} \Lambda_i)$  and  $\varepsilon > 0$ . By definition (2.3) of Fréchet normals we have

$$\langle x^*, x \rangle - \varepsilon \|x\| < 0 \text{ whenever } x \in \bigcap_{i=1}^{\infty} \Lambda_i \setminus \{0\}. \quad (5.3)$$

Define a countable system of closed cones in  $\mathbb{R}^{n+1}$  by

$$O_1 := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Lambda_1, \alpha \leq \langle x^*, x \rangle - \varepsilon \|x\|\} \text{ and } O_i := \Lambda_i \times \mathbb{R}_+ \text{ for } i = 2, 3, \dots \quad (5.4)$$

Let us check that all the assumptions for the validity of the conic extremal principle in Theorem 4.2 are satisfied for the system  $\{O_i\}_{i \in \mathbb{N}}$ . Picking any  $(x, \alpha) \in \bigcap_{i=1}^{\infty} O_i$ , we have  $x \in \bigcap_{i=1}^{\infty} \Lambda_i$  and  $\alpha \geq 0$  from the construction of  $O_i$  as  $i \geq 2$ . This implies in fact that  $(x, \alpha) = (0, 0)$ . Indeed, supposing  $x \neq 0$  gives us by (5.3) that

$$0 \leq \alpha \leq \langle x^*, x \rangle - \varepsilon \|x\| < 0,$$

which is a contradiction. On the other hand, the inclusion  $(0, \alpha) \in O_1$  yields that  $\alpha \leq 0$  by the construction of  $O_1$ , i.e.,  $\alpha = 0$ . Thus the *nonoverlapping condition*

$$\bigcap_{i=1}^{\infty} O_i = \{(0, 0)\}$$

holds for  $\{O_i\}_{i \in \mathbb{N}}$ . Similarly we check that

$$(O_1 - (0, \gamma)) \cap \bigcap_{i=2}^{\infty} O_i = \emptyset \text{ for any fixed } \gamma > 0, \quad (5.5)$$

i.e.,  $\{O_i\}_{i \in \mathbb{N}}$  is a *conic extremal system* at the origin. Indeed, violating (5.5) means the existence of  $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}$  such that

$$(x, \alpha) \in [O_1 - (0, \gamma)] \cap \bigcap_{i=2}^{\infty} O_i,$$

which implies that  $x \in \bigcap_{i=1}^{\infty} O_i$  and  $\alpha \geq 0$ . Then by the construction of  $O_1$  in (5.4) we get

$$\gamma + \alpha \leq \langle x^*, x \rangle - \varepsilon \|x\| \leq 0,$$

a contradiction due the positivity of  $\gamma$  in (5.5).

Applying now the second conclusion of Theorem 4.2 to the system  $\{O_i\}_{i \in \mathbb{N}}$  gives us the pairs  $(w_i, \alpha_i) \in O_i$  and  $(x_i^*, \lambda_i) \in \widehat{N}((w_i, \alpha_i); O_i)$  as  $i \in \mathbb{N}$  satisfying the relationships

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \langle x_i^*, \lambda_i \rangle = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^i} \|(x_i^*, \lambda_i)\|^2 = 1. \quad (5.6)$$

It immediately follows from the constructions of  $O_i$  as  $i \geq 2$  in (5.4) that  $\lambda_i \leq 0$  and  $x_i^* \in \widehat{N}(w_i; \Lambda_i)$ ; thus  $x_i^* \in N(0; \Lambda_i)$  for  $i = 2, 3, \dots$  by Proposition 2.1. Furthermore, we get

$$\limsup_{(x, \alpha) \rightarrow (w_1, \alpha_1)} \frac{\langle x_1^*, x - w_1 \rangle + \lambda_1(\alpha - \alpha_1)}{\|x - w_1\| + |\alpha - \alpha_1|} \leq 0 \quad (5.7)$$

by the definition of Fréchet normals to  $O_1$  at  $(w_1, \alpha_1) \in O_1$  with  $\lambda_1 \geq 0$  and

$$\alpha_1 \leq \langle x^*, w_1 \rangle - \varepsilon \|w_1\| \quad (5.8)$$

by the construction of  $O_1$ . Examine next the two possible cases in (5.6):  $\lambda_1 = 0$  and  $\lambda_1 > 0$ .

**Case 1:**  $\lambda_1 = 0$ . If inequality (5.8) is strict in this case, find a neighborhood  $U$  of  $w_1$  such that

$$\alpha_1 < \langle x^*, \bar{x} \rangle - \varepsilon \|x\| \quad \text{for all } x \in U,$$

which ensures that  $(x, \alpha_1) \in O_1$  for all  $x \in \Lambda_1 \cap U$ . Substituting  $(x, \alpha_1)$  into (5.7) gives us

$$\limsup_{x \xrightarrow{\Lambda_1} w_1} \frac{\langle x_1^*, x - w_1 \rangle}{\|x - w_1\|} \leq 0,$$

which means that  $x_1^* \in \widehat{N}(w_1; \Lambda_1)$ . If (5.8) holds as equality, we put  $\alpha := \langle x^*, x \rangle - \varepsilon \|x\|$  and get

$$|\alpha - \alpha_1| = |\langle x^*, x - w_1 \rangle + \varepsilon(\|w_1\| - \|x\|)| \leq (\|x^*\| + \varepsilon) \|x - w_1\|.$$

Furthermore, it follows from (5.7) that

$$\limsup_{(x, \alpha) \xrightarrow{O_1} (w_1, \alpha_1)} \frac{\langle x_1^*, x - w_1 \rangle}{\|x - w_1\| + |\alpha - \alpha_1|} \leq 0.$$

Thus for any  $\nu > 0$  sufficiently small and  $\alpha$  chosen above, we have

$$\langle x_1^*, x - w_1 \rangle \leq \nu(\|x - w_1\| + |\alpha - \alpha_1|) \leq \nu(1 + \|x^*\| + \varepsilon) \|x - w_1\|$$

whenever  $x \in \Lambda_1$  is sufficiently closed to  $w_1$ . The latter yields that

$$\limsup_{x \xrightarrow{\Lambda_1} w_1} \frac{\langle x_1^*, x - w_1 \rangle}{\|x - w_1\|} \leq 0, \quad \text{i.e., } x_1^* \in \widehat{N}(w_1; \Lambda_1).$$

Thus in both cases of the strict inequality and equality in (5.8), we justify that  $x_1^* \in \widehat{N}(w_1; \Lambda_1)$  and thus  $x_1^* \in N(0; \Lambda_1)$  by Proposition 2.1. Summarizing the above discussions gives us

$$x_i^* \in N(0; \Lambda_i) \quad \text{and } \lambda_i = 0 \quad \text{for all } i \in \mathbb{N}$$

in Case 1 under consideration. Hence it follows from (5.6) that there are  $\tilde{x}_i^* := (1/2^i)x_i^* \in N(0; \Lambda_i)$  as  $i \in \mathbb{N}$ , not equal to zero simultaneously, satisfying

$$\sum_{i=1}^{\infty} \tilde{x}_i^* = 0.$$

This contradicts the normal qualification condition (5.1) and thus shows that the case of  $\lambda_1 = 0$  is actually *not possible* in (5.8).

**Case 2:**  $\lambda_1 > 0$ . If inequality (5.8) is strict, put  $x = w_1$  in (5.7) and get

$$\limsup_{\alpha \rightarrow \alpha_1} \frac{\lambda_1(\alpha - \alpha_1)}{|\alpha - \alpha_1|} \leq 0.$$

That yields  $\lambda_1 = 0$ , a contradiction. Hence it remains to consider the case when (5.8) holds as equality. To proceed, take  $(x, \alpha) \in O_1$  satisfying

$$x \in \Lambda_1 \setminus \{w_1\} \quad \text{and } \alpha = \langle x^*, x \rangle - \varepsilon \|x\|.$$



By the equality in (5.8) we have

$$\alpha - \alpha_1 = \langle x^*, x - w_1 \rangle + \varepsilon(\|w_1\| - \|x\|) \quad \text{and thus} \quad |\alpha - \alpha_1| \leq (\|x^*\| + \varepsilon)\|x - w_1\|.$$

On the other hand, it follows from (5.7) that for any  $\gamma > 0$  sufficiently small there exists a neighborhood  $V$  of  $w_1$  such that

$$\langle x_1^*, x - w_1 \rangle + \lambda_1(\alpha - \alpha_1) \leq \lambda_1\gamma\varepsilon(\|x - w_1\| + |\alpha - \alpha_1|) \quad (5.9)$$

whenever  $x \in \Lambda_1 \cap V$ . Substituting  $(x, \alpha)$  with  $x \in \Lambda_1 \cap V$  into (5.9) gives us

$$\begin{aligned} \langle x_1^*, x - w_1 \rangle + \lambda_1(\alpha - \alpha_1) &= \langle x_1^* + \lambda_1 x^*, x - w_1 \rangle + \lambda_1\varepsilon(\|w_1\| - \|x\|) \\ &\leq \lambda_1\gamma\varepsilon(\|x - w_1\| + |\alpha - \alpha_1|) \\ &\leq \lambda_1\gamma\varepsilon[\|x - w_1\| + (\|x^*\| + \varepsilon)\|x - w_1\|] \\ &= \lambda_1\gamma\varepsilon(1 + \|x^*\| + \varepsilon)\|x - w_1\|. \end{aligned}$$

It follows from the above that for small  $\gamma > 0$  we have

$$\langle x_1^* + \lambda_1 x^*, x - w_1 \rangle + \lambda_1\varepsilon(\|w_1\| - \|x\|) \leq \lambda_1\varepsilon\|x - w_1\|$$

and thus arrive at the estimates

$$\langle x_1^* + \lambda_1 x^*, x - w_1 \rangle \leq \lambda_1\varepsilon\|x - w_1\| + \lambda_1\varepsilon(\|x\| - \|w_1\|) \leq 2\lambda_1\varepsilon\|x - w_1\|$$

for all  $x \in \Lambda_1 \cap V$ . The latter implies by definition (2.3) of  $\varepsilon$ -normals that

$$x_1^* + \lambda_1 x^* \in \widehat{N}_{2\lambda_1\varepsilon}(w_1; \Lambda_1). \quad (5.10)$$

Furthermore, it is easy to observe from the above choice of  $\lambda_1$  and the structure of  $O_1$  in (5.4) that  $\lambda_1 \leq 2 + 2\varepsilon$ . Employing now the representation of  $\varepsilon$ -normals in (5.10) from [5, formula (2.51)] held in finite dimensions, we find  $v \in \Lambda_1 \cap (w_1 + 2\lambda_1\varepsilon B)$  such that

$$x_1^* + \lambda_1 x^* \in \widehat{N}(v; \Lambda_1) + 2\lambda_1\varepsilon B^* \subset N(0; \Lambda_1) + 2\lambda_1\varepsilon B^*. \quad (5.11)$$

Since  $\lambda_1 > 0$  in the case under consideration and by  $-x_1^* = 2 \sum_{i=2}^{\infty} \frac{1}{2^i} x_i^*$  due to the first equality in (5.6), it follows from (5.11) that

$$x^* \in N(0; \Lambda_1) + \frac{2}{\lambda_1} \sum_{i=2}^{\infty} \frac{1}{2^i} x_i^* + 2\varepsilon B^*,$$

and hence there exists  $\tilde{x}_1^* \in N(0; \Lambda_1)$  such that

$$x^* \in \sum_{i=1}^{\infty} \frac{1}{2^i} \tilde{x}_i^* + 2\varepsilon B^* \quad \text{with} \quad \tilde{x}_i^* := \frac{2x_i^*}{\lambda_1} \in N(0; \Lambda_i) \quad \text{for} \quad i = 2, 3, \dots$$

This justifies (5.2) and completes the proof of the theorem.  $\square$

Our next result shows that we can put  $\varepsilon = 0$  in representation (5.2) under an additional assumption on Fréchet normals to cone intersections.

**Theorem 5.3** (refined representation of Fréchet normals to countable intersections of cones). *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be a countable system of arbitrary closed cones in  $\mathbb{R}^n$  satisfying the normal qualification condition (5.1). Then for any Fréchet normal  $x^* \in \widehat{N}\left(0; \bigcap_{i=1}^{\infty} \Lambda_i\right)$  satisfying*

$$\langle x^*, x \rangle < 0 \text{ whenever } x \in \bigcap_{i=1}^{\infty} \Lambda_i \setminus \{0\} \quad (5.12)$$

*there are limiting normals  $x_i^* \in N(0; \Lambda_i)$ ,  $i = 1, 2, \dots$ , such that*

$$x^* = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^*. \quad (5.13)$$

**Proof.** Fix a Fréchet normal  $x^* \in \widehat{N}\left(0; \bigcap_{i=1}^{\infty} \Lambda_i\right)$  satisfying condition (5.12) and construct a countable system of closed cones in  $\mathbb{R}^n \times \mathbb{R}$  by

$$O_1 := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Lambda_1, \alpha \leq \langle x^*, x \rangle\} \text{ and } O_i := \Lambda_i \times \mathbb{R}_+ \text{ for } i = 2, 3, \dots \quad (5.14)$$

Similarly to the proof Theorem 5.2 with taking (5.12) into account, we can verify that all the assumptions of Theorem 4.2 hold. Applying the conic extremal principle from this theorem gives us pairs  $(w_i, \alpha_i) \in O_i$  and  $(x_i^*, \lambda_i) \in \widehat{N}((w_i, \alpha_i); O_i)$  such that the extremality conditions in (5.6) are satisfied. We obviously get  $\lambda_i \leq 0$  and  $x_i^* \in \widehat{N}(w_i; \Lambda_i)$  for  $i = 1, 2, \dots$ , which ensures that  $x_i^* \in N(0; \Lambda_i)$  as  $i \geq 2$  by Proposition 2.1. It follows furthermore that for  $i = 1$  the limiting inequality (5.7) holds. The latter implies by the structure of the set  $O_1$  in (5.14) that

$$\lambda_1 \geq 0 \text{ and } \alpha_1 \leq \langle x^*, w_1 \rangle. \quad (5.15)$$

Similarly to the proof of Theorem 5.2 we consider the two possible cases  $\lambda_1 = 0$  and  $\lambda_1 > 0$  in (5.15) and show that the first case contradicts the normal qualification condition (5.1). In the second case we arrive at representation (5.13) based on the extremality conditions in (5.6) and the structures of the sets  $O_i$  in (5.14).  $\square$

The next theorem in this section provides constructive upper estimates of the Fréchet normal cone to countable intersections of closed cones in finite dimensions and of its interior via limiting normals to the sets involved at the origin.

**Theorem 5.4** (Fréchet normal cone to countable intersections). *Let  $\{\Lambda_i\}_{i \in \mathbb{N}}$  be a countable system of arbitrary closed cones in  $\mathbb{R}^n$  satisfying the normal qualification condition (5.1), and let  $\Lambda := \bigcap_{i=1}^{\infty} \Lambda_i$ . Then we have the inclusions*

$$\text{int } \widehat{N}(0; \Lambda) \subset \left\{ \sum_{i=1}^{\infty} x_i^* \mid x_i^* \in N(0; \Lambda_i) \right\}, \quad (5.16)$$

$$\widehat{N}(0; \Lambda) \subset \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(0; \Lambda_i), I \in \mathcal{L} \right\}, \quad (5.17)$$

*where  $\mathcal{L}$  stands for the collection of all finite subsets of the natural series  $\mathbb{N}$ .*

**Proof.** First we justify inclusion (5.16) assuming without loss of generality that  $\text{int } N(0; \Lambda) \neq \emptyset$ . Pick any  $x^* \in \text{int } \widehat{N}(0; \Lambda)$  and also  $\gamma > 0$  such that  $x^* + 3\gamma B^* \subset \widehat{N}(0; \Lambda)$ . Then for any  $x \in \Lambda \setminus \{0\}$  find  $z^* \in \mathbb{R}^n$  satisfying the relationships

$$\|z^*\| = 2\gamma \text{ and } \langle z^*, x \rangle < -\gamma\|x\|.$$

Since  $x^* - z^* \in x^* + 3\gamma B^* \subset \widehat{N}(0; \Lambda)$ , we have  $\langle x^* - z^*, x \rangle \leq 0$  and hence

$$\langle x^*, x \rangle = \langle x^* - z^*, x \rangle + \langle z^*, x \rangle < -\gamma\|x\| < 0.$$

This allows us to employ Theorem 5.3 and thus justify the first inclusion (5.16).

To prove the remaining inclusion (5.17), pick  $x^* \in \widehat{N}(0; \Lambda)$  and for any fixed  $\varepsilon > 0$  apply Theorem 5.2. In this way we find  $x_i^* \in N(0; \Lambda_i)$ ,  $i \in \mathbb{N}$ , such that

$$x^* \in \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* + \varepsilon B^*.$$

Since  $\varepsilon > 0$  was chosen arbitrarily, it follows that

$$x^* \in A := \text{cl} \left\{ \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* \mid x_i^* \in N(0; \Lambda_i) \right\}.$$

Let us finally justify the inclusion

$$A \subset \text{cl } C \text{ with } C := \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(0; \Lambda_i), I \in \mathcal{L} \right\}.$$

To proceed, pick  $z^* \in A$  and for any fixed  $\varepsilon > 0$  find  $x_i^* \in N(0; \Lambda_i)$  satisfying

$$\left\| z^* - \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^* \right\| \leq \frac{\varepsilon}{2}.$$

Then choose a number  $k \in \mathbb{N}$  so large that

$$\left\| z^* - \sum_{i=1}^k \frac{1}{2^i} x_i^* \right\| \leq \varepsilon.$$

Since  $\sum_{i=1}^k \frac{1}{2^i} x_i^* \in C$ , we get  $(z^* + \varepsilon B^*) \cap C \neq \emptyset$ , which means that  $z^* \in \text{cl } C$ . This justifies (5.17) and completes the proof of the theorem.  $\square$

Finally in this section, we present a consequence of Theorem 5.4, which gives an exact computation of Fréchet normals to countable intersections of cones normally regular at the origin.

**Corollary 5.5 (countable intersections of normally regular cones).** *In addition to the assumptions of Theorem 5.4, suppose that all the cones  $\Lambda_i$ ,  $i \in \mathbb{N}$ , are normally regular at the origin. Then the Fréchet normal cone to the intersection  $\Lambda = \bigcap_{i=1}^{\infty} \Lambda_i$  is computed by*

$$\widehat{N}(0; \Lambda) = \text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in N(0; \Lambda_i), I \in \mathcal{L} \right\}. \quad (5.18)$$

**Proof.** It is easy to check that

$$\text{cl} \left\{ \sum_{i \in I} x_i^* \mid x_i^* \in \widehat{N}(0; \Lambda_i), I \in \mathcal{L} \right\} \subset \widehat{N}(0; \Lambda)$$

for arbitrary set systems. Combining this with inclusion (5.17) of Theorem 5.4 and the normal regularity of each cone  $\Lambda_i$  as  $i \in \mathbb{N}$ , gives us equality (5.18).  $\square$

## 6 Tangential Normal Enclosedness and Approximate Normality

In this section we introduce and study two important properties of tangents cones that are of their own interest while allow us make a bridge between the extremal principles for cones and the limiting extremality conditions for arbitrary closed sets at their tangential extremal points. The main attention is paid to the contingent and weak contingent cones, which are proved to enjoy these properties under natural assumptions.

Let us start with introducing a new property of sets that is formulated in terms of the limiting normal cone (2.5) and plays a crucial rôle of what follows.

**Definition 6.1 (tangential normal enclosedness).** *Given a nonempty subset  $\Omega \subset X$  and a subcone  $\Lambda \subset X$  of a Banach space  $X$ , we say that  $\Lambda$  is TANGENTIALLY NORMALLY ENCLOSED (TNE) into  $\Omega$  at a point  $\bar{x} \in \Omega$  if*

$$N(0; \Lambda) \subset N(\bar{x}; \Omega). \quad (6.1)$$

The word “tangential” in Definition 6.1 reflects the fact that this normal enclosedness property is applied to tangential approximations of sets at reference points. Observe that if the set  $\Omega$  is convex near  $\bar{x}$ , then its classical tangent cone at  $\bar{x}$  enjoys the TNE property; indeed, in this case inclusion (6.1) holds as equality. We establish below a remarkable fact on the validity of the TNE property for the weak contingent cone (2.2) to any closed subset of a reflexive Banach space.

To study this and related properties, fix  $\Omega \subset X$  with  $\bar{x} \in \Omega$  and denote by  $\Lambda_w := T_w(\bar{x}; \Omega)$  the weak contingent cone to  $\Omega$  at  $\bar{x}$  without indicating  $\Omega$  and  $\bar{x}$  for brevity. Given a direction  $d \in \Lambda_w$ , let  $T_d^w$  be the collection of all sequences  $\{x_k\} \subset \Omega$  such that

$$\frac{x_k - \bar{x}}{t_k} \xrightarrow{w} d \text{ for some } t_k \downarrow 0.$$

It follows from definition (2.2) of  $\Lambda_w = T(\bar{x}; \Omega)$  that  $T_d^w \neq \emptyset$  whenever  $d \in \Lambda_w$ .

**Definition 6.2 (tangential approximate normality).** *We say that  $\Omega \subset X$  has the TANGENTIAL APPROXIMATE NORMALITY (TAN) property at  $\bar{x} \in \Omega$  if whenever  $d \in \Lambda_w$  and  $x^* \in \widehat{N}(d; \Lambda_w)$  are chosen there is a sequence  $\{x_k\} \in T_d^w$  along which the following holds: for any  $\varepsilon > 0$  there exists  $\delta \in (0, \varepsilon)$  such that*

$$\limsup_{k \rightarrow \infty} \left[ \sup \left\{ \frac{\langle x^*, z - x_k \rangle}{t_k} \mid z \in \Omega \cap (x_k + t_k \delta B) \right\} \right] \leq 2\varepsilon \delta, \quad (6.2)$$

where  $t_k \downarrow 0$  is taken from the construction of  $T_d^w$ .

The meaning of this property that gives the name is as follows: any  $x^* \in \widehat{N}(d; \Lambda_w)$  for the tangential approximation of  $\Omega$  at  $\bar{x}$  behaves approximately like a true normal at appropriate points  $x_k$  near  $\bar{x}$ . It occurs that the TAN property holds for any closed subset of a reflexive Banach space. The next proposition provides even a stronger result.

**Proposition 6.3 (approximate tangential normality in reflexive spaces).** *Let  $\Omega$  be a subset of a reflexive space  $X$ , and let  $\bar{x} \in \Omega$ . Then given any  $d \in \Lambda_w = T(\bar{x}; \Omega)$  and  $x^* \in \widehat{N}(d; \Lambda_w)$ , we have (6.2) whenever sequences  $\{x_k\} \in T_d^w$  and  $t_k \downarrow 0$  are taken from the construction of  $T_d^w$ . In particular, the set  $\Omega$  enjoys the TAN property at  $\bar{x}$ .*

**Proof.** Assume that  $\bar{x} = 0$  for simplicity. Pick any  $\varepsilon > 0$  and by the definition of Fréchet normals find  $\delta \in (0, \varepsilon)$  such that

$$\langle x^*, v - d \rangle \leq \frac{\varepsilon}{2} \|v - d\| \text{ for all } v \in \Lambda_w \cap (d + \delta B). \quad (6.3)$$

Fix any sequences  $\{x_k\} \in \mathcal{T}_d^w$  and  $t_k \downarrow 0$  from the formulation of the proposition and show that property (6.2) holds with the numbers  $\varepsilon$  and  $\delta$  chosen above. Supposing the contrary, find  $\{x_k\} \in \mathcal{T}_d^w$  and the corresponding sequence  $t_k \downarrow 0$  such that

$$\lim_{k \rightarrow \infty} \left\{ \sup \frac{\langle x^*, z - x_k \rangle}{t_k} \mid z \in \Omega \cap (B(x_k + t_k \delta B)) \right\} > 2\varepsilon \delta$$

along some subsequence of  $k \in \mathbb{N}$ , with no relabeling here and in what follows. Hence there is a sequence of  $z_k \in \Omega \cap (B(x_k + t_k \delta B))$  along which

$$\frac{\langle x^*, z_k - x_k \rangle}{t_k} > \varepsilon \delta \text{ for } k \in \mathbb{N}.$$

Taking into account the relationships

$$\left\| \frac{z_k}{t_k} - \frac{x_k}{t_k} \right\| \leq \delta \text{ and } \frac{x_k}{t_k} \xrightarrow{w} d \text{ as } k \rightarrow \infty,$$

we get that the sequence  $\left\{ \frac{x_k}{t_k} \right\}$  is bounded in  $X$ , and so is  $\left\{ \frac{z_k}{t_k} \right\}$ . Since any bounded sequence in a reflexive Banach space contains a weakly convergent subsequence, we may assume with no loss of generality that the sequence  $\left\{ \frac{z_k}{t_k} \right\}$  weakly converges to some  $v \in X$  as  $k \rightarrow \infty$ . It follows from the weak convergence of this sequence that

$$\|v - d\| \leq \liminf_{k \rightarrow \infty} \left\| \frac{z_k}{t_k} - \frac{x_k}{t_k} \right\| \leq \delta.$$

This allows us to conclude that

$$\langle x^*, v - d \rangle \geq \varepsilon \delta > \frac{\varepsilon}{2} \delta \geq \frac{\varepsilon}{2} \|v - d\|,$$

which contradicts (6.3) and thus completes the proof of the proposition.  $\square$

The next theorem is the main result of this section showing that the TAN property of a closed set in an Asplund space implies the TNE property of the weak contingent cone to this set at the reference point. This unconditionally justifies the latter property in reflexive spaces.

**Theorem 6.4 (TNE property in Asplund spaces).** *Let  $\Omega$  be a closed subset of an Asplund space  $X$ , and let  $\bar{x} \in \Omega$ . Assume that  $\Omega$  has the tangential approximate normality property at  $\bar{x}$ . Then the weak contingent cone  $\Lambda_w = T(\bar{x}; \Omega)$  is tangentially normally enclosed into  $\Omega$  at this point. Furthermore, the latter TNE property holds for any closed subset of a reflexive space.*

**Proof.** We are going to show that the following holds in the Asplund space setting under the TAN property of  $\Omega$  at  $\bar{x}$ :

$$\widehat{N}(d; \Lambda_w) \subset N(\bar{x}; \Omega) \text{ for all } d \in \Lambda, \|d\| = 1, \quad (6.4)$$

which is obviously equivalent to  $N(0; \Lambda_w) \subset N(\bar{x}; \Omega)$ , the TNE property of the weak contingent cone  $\Lambda_w$ . Then the second conclusion of the theorem in reflexive spaces immediately follows from Proposition 6.3. Assume without loss of generality that  $\bar{x} = 0$ .

To justify (6.4), fix  $d \in \Lambda_w$  and  $x^* \in \widehat{N}(d; \Lambda_w)$  with  $\|d\| = 1$  and  $\|x^*\| = 1$ . Taking  $\{x_k\} \in \mathcal{T}_d^w$  from Definition 6.2, it follows that for any  $\varepsilon$  there is  $\delta < \varepsilon$  such that (6.2) holds with  $\bar{x} = 0$ . Hence

$$\langle x^*, z - x_k \rangle \leq 3t_k\varepsilon\delta \text{ whenever } z \in Q := \Omega \cap (x_k + t_k\delta B), \quad k \in \mathbb{N}. \quad (6.5)$$

Consider further the function

$$\varphi(z) := -\langle x^*, z - x_k \rangle, \quad z \in Q,$$

for which we have by (6.5) that

$$\varphi(x_k) = 0 \leq \inf_{z \in Q} \varphi(z) + 3t_k\varepsilon\delta.$$

Setting  $\lambda := \frac{t_k\delta}{3}$  and  $\tilde{\varepsilon} := 3t_k\varepsilon\delta$ , we apply the Ekeland variational principle (see, e.g., [5, Theorem 2.26]) with  $\lambda$  and  $\tilde{\varepsilon}$  to the function  $\varphi$  on  $Q$ . In this way we find  $\tilde{x} \in Q$  such that  $\|\tilde{x} - x_k\| \leq \lambda$  and  $\tilde{x}$  minimizes the perturbed function

$$\psi(z) := -\langle x^*, z - x_k \rangle + \frac{\tilde{\varepsilon}}{\lambda} \|z - \tilde{x}\| = -\langle x^*, z - x_k \rangle + 9\varepsilon \|z - \tilde{x}\|, \quad z \in Q.$$

Applying now the generalized Fermat rule to  $\psi$  at  $\tilde{x}_k$  and then the fuzzy sum rule in the Asplund space setting (see, e.g., [5, Lemma 2.32]) gives us

$$0 \in -x^* + (9\varepsilon + \lambda)B^* + \widehat{N}(\tilde{x}_k; Q) \quad (6.6)$$

with some  $\tilde{x}_k \in \Omega \cap (\tilde{x} + \lambda B)$ . The latter means that

$$\|\tilde{x}_k - x_k\| \leq \|\tilde{x}_k - \tilde{x}\| + \|\tilde{x} - x_k\| \leq 2\lambda < t_k\delta.$$

Hence  $\tilde{x}_k$  belongs to the interior of the ball centered at  $\tilde{x}$  with radius  $t_k\delta$ , which implies that  $\widehat{N}(\tilde{x}_k; Q) = \widehat{N}(\tilde{x}_k; \Omega)$ . Thus we get from (6.6) that

$$x^* \in \widehat{N}(\tilde{x}_k; \Omega) + (9\varepsilon + \lambda)B^*, \quad k \in \mathbb{N}.$$

Letting there  $k \rightarrow \infty$  and then  $\varepsilon \downarrow 0$  gives us  $\tilde{x}_k \rightarrow \bar{x}$  and  $x^* \in N(\bar{x}; \Omega)$ . This justifies (6.4) and completes the proof of the theorem.  $\square$

**Corollary 6.5 (TNE property of the contingent cone in finite dimensions).** *Let a set  $\Omega \subset \mathbb{R}^n$  be closed around  $\bar{x} \in \Omega$ . Then the contingent cone  $T(\bar{x}; \Omega)$  to  $\Omega$  at  $\bar{x}$  is tangentially normally enclosed into  $\Omega$  at this point, i.e., we have*

$$N(0; \Lambda) \subset N(\bar{x}; \Omega) \text{ with } \Lambda := T(\bar{x}; \Omega). \quad (6.7)$$

**Proof.** It follows from Theorem 6.4 due to  $T(\bar{x}; \Omega) = T_w(\bar{x}; \Omega)$  in  $\mathbb{R}^n$ .  $\square$

Note that another proof of inclusion (6.7) in  $\mathbb{R}^n$  can be found in [8, Theorem 6.27].

## 7 Contingent and Weak Contingent Extremal Principles for Countable and Finite Systems of Closed Sets

By *tangential extremal principles* we understand results justifying the validity of extremality conditions defined in Section 3 for countable and/or finite systems of closed sets at the corresponding *tangential extremal points*. Note that, given a system of  $\Lambda = \{\Lambda_i\}$ -approximating cones to a set system  $\{\Omega_i\}$  at  $\bar{x}$ , the results ensuring the fulfillment of the  $\Lambda$ -tangential extremality conditions at  $\Lambda$ -tangential local extremal points are directly induced by an appropriate conic extremal principle applied to the cone system  $\{\Lambda_i\}$  at the origin. It is remarkable, however, that for *tangentially normally enclosed* cones  $\{\Lambda_i\}$  we simultaneously ensure the fulfillment of the *limiting extremality conditions* of Definition 3.4(c) at the corresponding tangential extremal points. As shown in Section 6, this is the case of the contingent cone in finite dimensions and of the weak contingent cone in reflexive (and also in Asplund) spaces.

In this section we pay the main attention to deriving the *contingent* and *weak contingent extremal principle* involving the aforementioned extremality conditions for countable and finite systems of sets and finite-dimensional and infinite-dimensional spaces. Observe that in the case of countable collections of sets the results obtained are the first in the literature, while in the case of finite systems of sets they are independent of the those known before being applied to different notions of tangential extremal points; see the discussions in Section 3.

We begin with the contingent extremal principle for countable systems of arbitrary closed sets in finite-dimensional spaces.

**Theorem 7.1 (contingent extremal principle for countable sets systems in finite dimensions).** *Let  $\bar{x} \in \bigcap_{i=1}^{\infty} \Omega_i$  be a contingent local extremal point of a countable system of closed sets  $\{\Omega_i\}_{i \in \mathbb{N}}$  in  $\mathbb{R}^n$ . Assume that the contingent cones  $T(\bar{x}; \Omega_i)$  to  $\Omega_i$  at  $\bar{x}$  are nonoverlapping*

$$\bigcap_{i=1}^{\infty} \{T(\bar{x}; \Omega_i)\} = \{0\}.$$

*Then there are normal vectors*

$$x_i^* \in N(0; \Lambda_i) \subset N(\bar{x}; \Omega_i) \text{ for } \Lambda_i := T(\bar{x}; \Omega_i) \text{ as } i \in \mathbb{N}$$

*satisfying the extremality conditions in (3.4).*

**Proof.** This result follows from combining Theorem 4.2 and Corollary 6.5.  $\square$

Consider further systems of finitely many sets  $\{\Omega_1, \dots, \Omega_m\}$  in Asplund spaces and derive for them the weak contingent extremal principle. Recall that a set  $\Omega \subset X$  is *sequentially normally compact* (SNC) at  $\bar{x} \in \Omega$  if for any sequence  $\{(x_k, x_k^*)\}_{k \in \mathbb{N}} \subset \Omega \times X^*$  we have the implication

$$[x_k \rightarrow \bar{x}, x_k^* \xrightarrow{w} 0 \text{ with } x_k^* \in \widehat{N}(x_k; \Omega), k \in \mathbb{N}] \implies \|x_k^*\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

In [5, Subsection 1.1.4], the reader can find a number of efficient conditions ensuring the SNC property, which holds in rather broad infinite-dimensional settings. The next proposition shows that the SNC property of TAN sets is inherent by their weak contingent cones.

**Proposition 7.2 (SNC property of weak contingent cones).** *Let  $\Omega$  be a closed subset of an Asplund space  $X$  satisfying the tangential approximate normality property at  $\bar{x} \in \Omega$ . Then the weak contingent cone  $T_w(\bar{x}; \Omega)$  is SNC at the origin provided that  $\Omega$  is SNC at  $\bar{x}$ . In particular, in reflexive spaces the SNC property of a closed subset  $\Omega$  at  $\bar{x}$  unconditionally implies the SNC property of its weak contingent cone  $T_w(\bar{x}; \Omega)$  at the origin.*

**Proof.** To justify the SNC property of  $\Lambda_w := T_w(\bar{x}; \Omega)$  at the origin, take sequences  $d_k \rightarrow 0$  and  $x_k^* \in \widehat{N}(d_k; \Lambda_w)$  satisfying  $x_k^* \xrightarrow{w^*} 0$  as  $k \rightarrow \infty$ . Using the TAN property of  $\Omega$  at  $\bar{x}$  and following the proof of Theorem 6.4, we find sequences  $\varepsilon_k \downarrow 0$  and  $\tilde{x}_k \xrightarrow{\Omega} \bar{x}$  such that

$$x_k^* \in \widehat{N}(\tilde{x}_k; \Omega) + \varepsilon_k B^* \text{ for all } k \in \mathbb{N}.$$

Hence there are  $\tilde{x}_k^* \in \widehat{N}(\tilde{x}_k; \Omega)$  with  $\|\tilde{x}_k^* - x_k^*\| \leq \varepsilon_k$ , which implies that  $\tilde{x}_k^* \xrightarrow{w^*} 0$  as  $k \rightarrow \infty$ . By the SNC property of  $\Omega$  at  $\bar{x}$  we get that  $\|\tilde{x}_k^*\| \rightarrow 0$ , which yields in turn that  $\|x_k^*\| \rightarrow 0$  as  $k \rightarrow \infty$ . This justifies the SNC property of  $\Lambda_w$  at the origin. The second assertion of this proposition immediately follows from Proposition 6.3.  $\square$

Now we are ready to establish the weak contingent extremal principle for systems of finitely many closed subsets of Asplund spaces in both approximate and exact forms.

**Theorem 7.3 (weak contingent extremal principle for finite systems of sets in Asplund spaces).** *Let  $\bar{x} \in \bigcap_{i=1}^m \Omega_i$  be a weak contingent local extremal point of the system  $\{\Omega_1, \dots, \Omega_m\}$  of closed sets in an Asplund space  $X$ . Assume that all the sets  $\Omega_i$ ,  $i = 1, \dots, m$ , have the TAN property at  $\bar{x}$ , which is automatic in reflexive spaces. Then the following versions of the weak contingent extremal principle hold:*

(i) APPROXIMATE VERSION: *for any  $\varepsilon > 0$  there are  $x_i^* \in N(\bar{x}; \Omega_i)$  as  $i = 1, \dots, m$  satisfying*

$$\|x_1^* + \dots + x_m^*\| \leq \varepsilon \text{ and } \|x_1^*\| + \dots + \|x_m^*\| = 1. \quad (7.1)$$

(ii) EXACT VERSION: *if in addition all but one of the sets  $\Omega_i$  as  $i = 1, \dots, m$  are SNC at  $\bar{x}$ , then there exist  $x_i^* \in N(\bar{x}; \Omega_i)$  as  $i = 1, \dots, m$  satisfying*

$$x_1^* + \dots + x_m^* = 0 \text{ and } \|x_1^*\| + \dots + \|x_m^*\| = 1. \quad (7.2)$$

**Proof.** It follows from Proposition 3.2 that the cone system  $\{\Lambda_w^i = T_w(\bar{x}; \Omega_i)\}$  as  $i = 1, \dots, m$  is extremal at the origin in the conventional sense (3.2). Applying to it the approximate extremal principle from [5, Theorem 2.20], for any  $\varepsilon > 0$  we find  $x_i \in \Lambda_w^i$  and  $x_i^* \in \widehat{N}(x_i; \Lambda_w^i)$  as  $i = 1, \dots, m$  such that all the relationships in (7.1) hold. Then

$$x_i^* \in \widehat{N}(x_i; \Lambda_w^i) \subset N(0; \Lambda_w^i) \subset N(\bar{x}; \Omega_i), \quad i = 1, \dots, m,$$

by Proposition 2.1 and Theorem 6.4, which justifies assertion (i).

Now to justify (ii), observe that all but one of the cones  $\Lambda_w^i$  are SNC at the origin by Proposition 7.2. Thus the conclusion of (ii) follows from the exact extremal principle in [5, Theorem 2.22] and Theorem 6.4 established above.  $\square$



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