

12-3-2010

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Recommended Citation

Cánovas, M.J.; Lopez, M.A.; Mordukhovich, Boris S.; and Parra, J, "Quantitative Stability and Optimality Conditions in Convex Semi-Infinite and Infinite Programming" (2010). *Mathematics Research Reports*. Paper 81.
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**QUANTITATIVE STABILITY AND OPTIMALITY
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Detroit, MI 48202

**Department of Mathematics
Research Report**

**2010 Series
#14**

This research was partly supported by the US National Science Foundation

QUANTITATIVE STABILITY AND OPTIMALITY CONDITIONS IN CONVEX SEMI-INFINITE AND INFINITE PROGRAMMING¹

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Abstract. This paper concerns parameterized convex infinite (or semi-infinite) inequality systems whose decision variables run over general infinite-dimensional Banach (resp. finite-dimensional) spaces and that are indexed by an arbitrary fixed set T . Parameter perturbations on the right-hand side of the inequalities are measurable and bounded, and thus the natural parameter space is $l_\infty(T)$. Based on advanced variational analysis, we derive a precise formula for computing the exact Lipschitzian bound of the feasible solution map, which involves only the system data, and then show that this exact bound agrees with the coderivative norm of the aforementioned mapping. On one hand, in this way we extend to the convex setting the results of [4] developed in the linear framework under the boundedness assumption on the system coefficients. On the other hand, in the case when the decision space is reflexive, we succeed to remove this boundedness assumption in the general convex case, establishing therefore results new even for linear infinite and semi-infinite systems. The last part of the paper provides verifiable necessary optimality conditions for infinite and semi-infinite programs with convex inequality constraints and general nonsmooth and nonconvex objectives. In this way we extend the corresponding results of [5] obtained for programs with linear infinite inequality constraints.

Key words. semi-infinite and infinite programming, parametric optimization, variational analysis, convex infinite inequality systems, quantitative stability, Lipschitzian bounds, generalized differentiation, coderivatives

AMS subject classification. 90C34, 90C25, 49J52, 49J53, 65F22

1 Introduction

Many optimization problems are formulated in the form:

$$(P) \quad \begin{array}{ll} \inf & \varphi(x) \\ \text{s.t.} & f_t(x) \leq 0, \quad t \in T, \end{array}$$

¹This research was partially supported by grants MTM2008-06695-C03 (01-02) from MICINN (Spain).

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where T is an arbitrary *index set*, where $x \in X$ is a *decision variable* selected from a general Banach space X with its topological dual denoted by X^* , and where $f_t : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $t \in T$, are proper lower semicontinuous (lsc) convex functions; these are our *standing assumptions*. In this paper we analyze *quantitative stability* of the feasible set of (P) under small perturbations on the right-hand side of the constraints. In more detail, the paper is focused on characterizing *Lipschitzian behavior* of the feasible solution map, with computing the *exact bound* of Lipschitzian moduli by using appropriate tools of advanced variational analysis and generalized differentiation particularly based on *coderivatives*; see below.

In what follows we consider the *parametric convex inequality system*

$$\sigma(p) := \{f_t(x) \leq p_t, t \in T\}, \quad (1)$$

where the functional parameter p is a measurable and essentially bounded function $p : T \rightarrow \mathbb{R}$, i.e., p belongs to the Banach space $l_\infty(T)$; we use the notation p_t for $p(t)$, $t \in T$. The zero function $\bar{p} = 0$ is regarded as the *nominal parameter*. This assumption does not entail any loss of generality.

Recall that the *parameter space* $l_\infty(T)$ is a Banach space with the norm

$$\|p\| := \sup_{t \in T} |p_t|.$$

If no confusion arises, we also use the same notation $\|\cdot\|$ for the given norm in X and for the corresponding dual norm in X^* defined by

$$\|x^*\| := \sup_{\|x\| \leq 1} \langle x^*, x \rangle \text{ for any } x^* \in X^*,$$

where $\langle x^*, x \rangle$ stands for the standard canonical pairing. Our main attention is focused on the *feasible solution map* $\mathcal{F} : l_\infty(T) \rightrightarrows X$ defined by

$$\mathcal{F}(p) := \{x \in X \mid x \text{ is a solution to } \sigma(p)\}. \quad (2)$$

The convex system $\sigma(p)$ with $p \in l_\infty(T)$ can be *linearized* by using the *Fenchel-Legendre conjugate* $f_t^* : X^* \rightarrow \overline{\mathbb{R}}$ for each function f_t given by

$$f_t^*(u^*) := \sup \{ \langle u^*, x \rangle - f_t(x) \mid x \in X \} = \sup \{ \langle u^*, x \rangle - f_t(x) \mid x \in \text{dom} f_t \},$$

where $\text{dom} f_t := \{x \in X \mid f_t(x) < \infty\}$ is the effective domain of f_t . Specifically, under the current assumptions on each f_t its conjugate f_t^* is also a proper lsc convex function such that

$$f_t^{**} = f_t \text{ on } X \text{ with } f_t^{**} := (f_t^*)^*.$$

In this way, for each $t \in T$, the inequality $f_t(x) \leq p_t$ turns out to be equivalent to the linear system

$$\{ \langle u^*, x \rangle - f_t^*(u^*) \leq p_t, u^* \in \text{dom} f_t^* \}$$

in the sense that they have the same solution sets. Then we consider the following parametric family of linear systems:

$$\tilde{\sigma}(\rho) := \{ \langle u^*, x \rangle \leq f_t^*(u^*) + \rho(t, u^*), (t, u^*) \in \tilde{T} \}, \quad (3)$$

where $\tilde{T} := \{(t, u^*) \in T \times X^* \mid u^* \in \text{dom} f_t^*\}$, and the associated feasible set mapping $\tilde{\mathcal{F}} : l_\infty(\tilde{T}) \rightrightarrows X$ given by

$$\tilde{\mathcal{F}}(\rho) := \{x \in X \mid x \text{ is a solution to } \tilde{\sigma}(\rho)\}. \quad (4)$$

Thus our initial family $\{\mathcal{F}(p), p \in l_\infty(T)\}$ can be straightforwardly embedded into the family $\{\tilde{\mathcal{F}}(\rho), \rho \in l_\infty(\tilde{T})\}$ through the relation

$$\mathcal{F}(p) = \tilde{\mathcal{F}}(\rho_p) \text{ for } p \in l_\infty(T), \quad (5)$$

where $\rho_p \in l_\infty(\tilde{T})$ is defined by

$$\rho_p(t, u^*) := p_t \text{ for } (t, u^*) \in \tilde{T}.$$

We consider the supremum norm in $l_\infty(\tilde{T})$, which for the sake of simplicity is also denoted by $\|\cdot\|$. Note that

$$\|p\| = \sup_{t \in T} |p_t| = \sup_{(t, u) \in \tilde{T}} |\rho_p(t, u)| = \|\rho_p\|.$$

Since the f_t 's are fixed functions, the structure of $\tilde{\sigma}(\rho)$, $\rho \in l_\infty(\tilde{T})$, fits into the context analyzed in [4], and some results of the present paper take advantage of this fact. The implementation of this idea requires establishing precise relationships between Lipschitzian behavior of \mathcal{F} at the nominal parameter $\bar{p} = 0$ and that of $\tilde{\mathcal{F}}$ at $\rho_{\bar{p}} = 0$, which is done in what follows. This approach allows us to derive *characterizations of quantitative/Lipschitzian stability* of parameterized sets of feasible solutions described by infinite systems of *convex* inequalities, with computing the exact bound of Lipschitzian moduli, from those obtained in [4] for their linear counterparts in general Banach spaces.

Furthermore, in the case of *reflexive* spaces of decision variables we manage to *remove* the *boundedness* requirement on coefficients of linear systems imposed in [4] and thus establish in this way complete characterizations of quantitative stability of general convex systems of infinite inequalities under the most natural assumptions on the initial data.

Our approach to the study of quantitative stability of infinite convex systems is mainly based on *coderivative analysis* of set-valued mappings of type (2). As a by-product of this approach, we derive verifiable *necessary optimality conditions* for semi-infinite programs with convex inequality constraints and general (nonsmooth and nonconvex) objective functions.

The rest of the paper is organized as follows. Section 2 presents some basic definitions and key results from variational analysis and generalized differentiation needed in what follows.

In Section 3 we derive auxiliary results for infinite systems of convex inequalities used in the proofs of the main results of the paper.

Section 4 is devoted to the quantitative stability analysis of parameterized infinite systems of convex inequalities by means of coderivatives in arbitrary Banach spaces of decision variables. Based on this variational technique, we establish verifiable characterizations of the Lipschitz-like property of the perturbed feasible solution map (2) with precise computing the exact Lipschitzian bound in terms of the initial data of (1). This is done by reducing (1) to the linearized system (3) in the way discussed above.

In Section 5 we show how to remove, in the case of reflexive decision spaces, the boundedness assumption on coefficients of linear infinite systems and hence for the general convex infinite systems (1) via the linearization procedure (3) in the above quantitative stability analysis and characterizations.

Finally, Section 6 is devoted to deriving subdifferential optimality conditions for semi-infinite and infinite programs of type (P) with convex infinite constraints and nondifferentiable (generally nonconvex) objectives.

Our notation is basically standard in the areas of variational analysis and semi-infinite/infinite programming; see, e.g., [10, 17]. Unless otherwise stated, all the spaces under consideration are Banach. The symbol w^* signifies the weak* topology of a dual space, and thus the weak* topological limit corresponds to the weak* convergence of nets. Some particular notation will be recalled, if necessary, in the places where they are introduced.

2 Preliminaries from Variational Analysis

Given a set-valued mapping $F: Z \rightrightarrows Y$ between Banach spaces Z and Y , we say the F is *Lipschitz-like around* $(\bar{z}, \bar{y}) \in \text{gph } F$ with *modulus* $\ell \geq 0$ if there are neighborhoods U of \bar{z} and V of \bar{y} such that

$$F(z) \cap V \subset F(u) + \ell \|z - u\| \mathbb{B} \text{ for any } z, u \in U, \quad (6)$$

where \mathbb{B} stands for the closed unit ball in the space in question. The infimum of moduli $\{\ell\}$ over all the combinations of $\{\ell, U, V\}$ satisfying (6) is called the *exact Lipschitzian bound* of F around (\bar{z}, \bar{y}) and is labeled as $\text{lip } F(\bar{z}, \bar{y})$.

If $V = Y$ in (6), this relationship signifies the classical (Hausdorff) *local Lipschitzian* property of F around \bar{z} with the *exact Lipschitzian bound* denoted by $\text{lip } F(\bar{z})$ in this case.

It is worth mentioning that the Lipschitz-like property (also known as the Aubin or pseudo-Lipschitz property) of an arbitrary mapping $F: Z \rightrightarrows Y$ between Banach spaces is equivalent to other two fundamental properties in nonlinear analysis while defined for the inverse mapping $F^{-1}: Y \rightrightarrows X$; namely, to the *metric regularity* of F^{-1} and to the *linear openness* of F^{-1} around (\bar{y}, \bar{z}) , with the corresponding relationships between their exact bounds (see, e.g. [12, 17, 19]). From these relationships we can easily observe the following

representation for the exact Lipschitzian bound:

$$\text{lip } F(\bar{z}, \bar{y}) = \limsup_{(z,y) \rightarrow (\bar{z}, \bar{y})} \frac{\text{dist}(y; F(z))}{\text{dist}(z; F^{-1}(y))}, \quad (7)$$

where $\inf \emptyset := \infty$ (and hence $\text{dist}(x; \emptyset) = \infty$) as usual, and where $0/0 := 0$. We have accordingly that $\text{lip } F(\bar{z}, \bar{y}) = \infty$ if F is not Lipschitz-like around (\bar{z}, \bar{y}) .

A remarkable fact consists of the possibility to characterize pointwisely the (derivative-free) Lipschitz-like property of F around (\bar{z}, \bar{y}) —and hence its local Lipschitzian, metric regularity, and linear openness counterparts—in terms of a dual-space construction of generalized differentiation called the *coderivative* of F at $(\bar{z}, \bar{y}) \in \text{gph } F$. The latter is a positively homogeneous multifunction $D^*F(\bar{z}, \bar{y}): Y^* \rightrightarrows Z^*$ defined by

$$D^*F(\bar{z}, \bar{y})(y^*) := \{z^* \in Z^* \mid (z^*, -y^*) \in N((\bar{z}, \bar{y}); \text{gph } F)\}, \quad y^* \in Y^*, \quad (8)$$

where $N(\cdot; \Omega)$ stands for the collection of generalized normals to a set at a given point known as the *basic*, or *limiting*, or *Mordukhovich normal cone*; see, e.g. [14, 17, 19, 20] and references therein. When both Z and Y are finite-dimensional, it is proved in [15] (cf. also [19, Theorem 9.40]) that a closed-graph mapping $F: Z \rightrightarrows Y$ is Lipschitz-like around $(\bar{z}, \bar{y}) \in \text{gph } F$ if and only if

$$D^*F(\bar{z}, \bar{y})(0) = \{0\}, \quad (9)$$

and the exact Lipschitzian bound of moduli $\{\ell\}$ in (6) is computed by

$$\text{lip } F(\bar{z}, \bar{y}) = \|D^*F(\bar{z}, \bar{y})\| := \sup \{\|z^*\| \mid z^* \in D^*F(\bar{z}, \bar{y})(y^*), \|y^*\| \leq 1\}. \quad (10)$$

There is an extension [17, Theorem 4.10] of the coderivative criterion (9), via the so-called mixed coderivative of F at (\bar{z}, \bar{y}) , to the case when both spaces Z and Y are Asplund (i.e., their separable subspaces have separable duals) under some additional “partial normal compactness” assumption that is automatic in finite dimensions. Also the aforementioned theorem contains an extension of the exact bound formula (10) provided that Y is Asplund while Z is finite-dimensional. Unfortunately, none of these results is applied in our setting (2).

Indeed, the underlying set-valued mapping (2) considered in this paper is $\mathcal{F}: l_\infty(T) \rightrightarrows X$ defined by the infinite system of convex inequalities (1). The graph $\text{gph } \mathcal{F}$ of this mapping is obviously convex, and we can easily verify that it is also closed with respect to the product topology. If the index set T is infinite, $l_\infty(T)$ is an infinite-dimensional Banach space, which is *never Asplund*. There exists an isometric isomorphism between the topological dual $l_\infty(T)^*$ and the space $ba(T)$ of additive and bounded measures $\mu: T \rightrightarrows \mathbb{R}$ such that

$$\langle \mu, y \rangle = \int_T y_t \mu(dt).$$

The *dual norm* $\|\mu\|$ is the total variation of μ on T , i.e.,

$$\|\mu\| = \sup_{A \subset T} \mu(A) - \inf_{B \subset T} \mu(B).$$

All these topological facts are classical and can be found, e.g., in [8].

3 Auxiliary Results for Infinite Convex Systems

Given a subset S of a normed space, the notation $\text{co}S$ and $\text{cone}S$ stand for the convex hull and the conic convex hull of S , respectively. The symbol \mathbb{R}_+ signifies the interval $[0, \infty)$, and by $\mathbb{R}_+^{(T)}$ we denote the collection of all the functions $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^T$ such that $\lambda_t > 0$ for only *finitely many* $t \in T$. As usual, cl^*S stands for the weak* (w^* in brief) topological closure of S .

The *indicator function* $\delta_S = \delta(\cdot; S)$ of the set S is defined by $\delta_S(x) := 0$ if $x \in S$ and $\delta_S(x) := \infty$ if $x \notin S$. It is easy to see that S is a nonempty closed convex set if and only if δ_S is a proper lsc convex function. For a function $h : X \rightarrow \overline{\mathbb{R}}$ the *epigraph* of h is given by

$$\text{epi } h := \{(x, \gamma) \in X \times \mathbb{R} \mid x \in \text{dom } h, h(x) \leq \gamma\}.$$

The following *extended Farkas' Lemma* is a key tool in our analysis.

Lemma 1 (cf. [6, Theorem 4.1]) *For $p \in \text{dom}(\mathcal{F})$ and $(v, \alpha) \in X^* \times \mathbb{R}$, the following statements are equivalent:*

- (i) $v(x) \leq \alpha$ is a consequence of $\sigma(p)$; i.e., $v(x) \leq \alpha$ for all $x \in \mathcal{F}(p)$.
- (ii) $(v, \alpha) \in \text{cl}^*(\text{cone} \bigcup_{t \in T} \text{epi}(f_t - p_t)^*)$.

Proof. Theorem 4.1 in [6] yields the equivalence between (i) and the inclusion

$$(v, \alpha) \in \text{cl}^* \left(\text{cone} \bigcup_{t \in T} \text{epi}(f_t - p_t)^* + \mathbb{R}_+(0, 1) \right).$$

Thus it suffices to observe that $(0, 1) \in \text{cl}^*(\text{cone} \bigcup_{t \in T} \text{epi}(f_t - p_t)^*)$. To do this, pick any $(w, \beta) \in \text{epi}(f_{t_0} - p_{t_0})^*$ for some $t_0 \in T$ and note that

$$(0, 1) = \lim_{r \rightarrow \infty} \frac{1}{r} (w, \beta + r),$$

where the limit is taken with respect to the strong topology. ■

Remark 2 As an application of the previous lemma, together with the Brøndsted-Rockafellar theorem (which yields, for each $t \in T$, that

$$\text{rge}(\partial f_t) \subset \text{dom}(f_t^*) \subset \text{cl}^*(\text{rge}(\partial f_t));$$

see, e.g. [21, Theorem 3.1.2]), we get the representation

$$\mathcal{F}(p) = \{x \in X \mid \langle u^*, x \rangle - f_t^*(u^*) \leq p_t, t \in T, u^* \in \text{rge}(\partial f_t)\}$$

providing an alternative way of linearizing our convex system (1).

Let us now define, for $p \in l_\infty(T)$, the sets

$$H(p) := \text{co} \left(\bigcup_{t \in T} \text{epi}(f_t - p_t)^* \right) \subset X^* \times \mathbb{R}, \quad (11)$$

$$C(p) := \text{co} \left(\bigcup_{t \in T} \text{gph}(f_t - p_t)^* \right) \subset X^* \times \mathbb{R}. \quad (12)$$

Note that in the case of linear constraints of the type " \geq " the set $H(p)$ in (11) coincides with what was called *hypographical set* in [3].

We say that the system $\sigma(0)$ satisfies the *strong Slater condition* (SSC) if there exists a point $\hat{x} \in X$ such that

$$\sup_{t \in T} f_t(\hat{x}) < 0.$$

In this case \hat{x} is called a *strong Slater point* for $\sigma(0)$. Note that \hat{x} is a strong Slater point for $\sigma(0)$ if and only if \hat{x} is a strong Slater point for the linear system $\bar{\sigma}(0)$, i.e., $\sup_{(t, u^*) \in \bar{T}} \{ \langle u^*, \hat{x} \rangle - f_t^*(u^*) \} < 0$.

Lemma 3 *Assume that $0 \in \text{dom } \mathcal{F}$. The following statements are equivalent:*

- (i) $\sigma(0)$ satisfies the SSC.
- (ii) $0 \in \text{int}(\text{dom } \mathcal{F})$.
- (iii) \mathcal{F} is Lipschitz-like around $(0, x)$ for all $x \in \mathcal{F}(0)$
- (iv) $(0, 0) \notin \text{cl}^* H(0)$.
- (v) $(0, 0) \notin \text{cl}^* C(0)$.

Proof. The equivalence between (i), (ii), and (iv) are established in Theorem 5.1 of [7]. The equivalence between (ii) and (iii) follows from the classical Robinson-Ursescu theorem. Implication (iv) \Rightarrow (v) is obvious by the inclusion $C(0) \subset H(0)$ due to (11) and (12).

Let us now check that the inclusion $(0, 0) \in \text{cl}^* H(0)$ implies the one in $(0, 0) \in \text{cl}^* C(0)$, which thus yields (v) \Rightarrow (iv). To proceed, assume that $(0, 0) \in \text{cl}^* H(0)$ and write

$$(0, 0) = w^* \text{-} \lim_{\nu \in \mathcal{N}} \left\{ \sum_{t \in T} \alpha_{t\nu} (v_{t\nu}^*, f_t^*(v_{t\nu}^*) + \beta_{t\nu}) \right\} \quad (13)$$

for some net indexed by a certain directed set \mathcal{N} and satisfying the conditions

$$\begin{aligned} \sum_{t \in T} \alpha_{t\nu} &= 1 \text{ for all } \nu \in \mathcal{N}, \\ (v_{t\nu}^*, f_t^*(v_{t\nu}^*) + \beta_{t\nu}) &\in \text{epi } f_t^* \text{ for all } t \text{ and all } \nu \in \mathcal{N} \end{aligned}$$

with $\alpha_\nu = (\alpha_{t\nu})_{t \in T}$ and $\beta_\nu = (\beta_{t\nu})_{t \in T}$ belonging to $\mathbb{R}_+^{(T)}$. Take then any $\bar{x} \in \mathcal{F}(0)$ and observe from (13) the relationships

$$\begin{aligned} 0 &= \lim_{\nu \in \mathcal{N}} \left\{ \sum_{t \in T} \alpha_{t\nu} (\langle v_{t\nu}^*, \bar{x} \rangle - f_t^*(v_{t\nu}^*) - \beta_{t\nu}) \right\} \\ &\leq \lim_{\nu \in \mathcal{N}} \left\{ \sum_{t \in T} \alpha_{t\nu} (f_t(\bar{x}) - \beta_{t\nu}) \right\} \leq 0 \end{aligned}$$

held due to the feasibility of \bar{x} and the fact that $\beta_{t\nu} \geq 0$ for all t and all ν . Hence we arrive at the equality

$$\lim_{\nu \in \mathcal{N}} \left\{ \sum_{t \in T} \alpha_{t\nu} \beta_{t\nu} \right\} = 0$$

yielding in turn that

$$(0, 0) = w^* \text{-} \lim_{\nu \in \mathcal{N}} \left\{ \sum_{t \in T} \alpha_{t\nu} (v_{t\nu}^*, f_t^*(v_{t\nu}^*)) \right\} \in \text{cl}^* \left(\text{co} \left(\bigcup_{t \in T} \text{gph } f_t^* \right) \right),$$

which thus completes the proof of the lemma. ■

The following two technical statements are of their own interest while playing an essential role in proving the main results presented in the subsequent sections. We keep the convention $0/0 := 0$.

Proposition 4 *Suppose that X is a Banach space and that $g : X \rightarrow \overline{\mathbb{R}}$ is a proper convex function such that there exists $\hat{x} \in X$ with $g(\hat{x}) < 0$. If*

$$S := \{y \in X \mid g(y) \leq 0\}, \quad (14)$$

then for all $x \in X$ we have the equality

$$\text{dist}(x; S) = \sup_{(x^*, \alpha) \in \text{epi } g^*} \frac{[(x^*, x) - \alpha]_+}{\|x^*\|}. \quad (15)$$

Proof. Observe that the nonemptiness of the set S defined in (14) ensures that $\alpha \geq 0$ whenever $(x^*, \alpha) \in \text{epi } g^*$, and so the possibility of $x^* = 0$ is not an obstacle in (15) under our convention that $0/0 = 0$. Note also that $\text{dist}(x; S)$ is nothing else but the optimal value in the convex optimization problem

$$\inf \|y - x\| \quad \text{s.t. } g(y) \leq 0.$$

Since for this problem the classical Slater condition is satisfied, the *strong Lagrange duality* holds (see, e.g., [21, Theorem 2.9.3]); namely,

$$\begin{aligned} \text{dist}(x; S) &= \max_{\lambda \geq 0} \inf_{y \in X} \{ \|y - x\| + \lambda g(y) \} \\ &= \max \left\{ \sup_{\lambda > 0} \inf_{y \in X} \{ \|y - x\| + \lambda g(y) \}, \inf_{y \in X} \|y - x\| \right\} \\ &= \max \left\{ \sup_{\lambda > 0} \inf_{y \in X} \{ \|y - x\| + \lambda g(y) \}, 0 \right\}. \end{aligned}$$

Applying now the Fenchel duality theorem to the inner infimum problem for every fixed $\lambda > 0$, which is possible due to the obvious fulfillment of the Rockafellar regularity condition, we get

$$\inf_{y \in X} \{ \|y - x\| + \lambda g(y) \} = \max_{y^* \in X^*} \{ -\|\cdot - x\|^*(-y^*) - (\lambda g)^*(y^*) \}.$$

By employing next the well-known formula

$$\|\cdot - x\|^* (-y^*) = \begin{cases} \langle -y^*, x \rangle & \text{if } \|y^*\| \leq 1, \\ \infty & \text{otherwise,} \end{cases}$$

we arrive at the relationships

$$\begin{aligned} \inf_{y \in X} \{ \|y - x\| + \lambda g(y) \} &= \max_{\|y^*\| \leq 1} \{ \langle y^*, x \rangle - (\lambda g)^*(y^*) \} \\ &= \max_{\substack{y^* \in X^*, \rho \in \mathbb{R}, \\ \|y^*\| \leq 1, (\lambda g)^*(y^*) \leq \rho}} \{ \langle y^*, x \rangle - \rho \} \\ &= \max_{\substack{y^* \in X^*, \rho \in \mathbb{R}, \\ \|y^*\| \leq 1, \lambda g^*(y^*/\lambda) \leq \rho}} \{ \langle y^*, x \rangle - \rho \} \\ &= \max_{\substack{y^* \in X^*, \rho \in \mathbb{R}, \\ \|y^*\| \leq 1, (1/\lambda)(y^*, \rho) \in \text{epi } g^*}} \{ \langle y^*, x \rangle - \rho \}. \end{aligned}$$

Thus defining $x^* := (1/\lambda)y^*$ and $\alpha := (1/\lambda)\rho$ gives us

$$\inf_{y \in X} \{ \|y - x\| + \lambda g(y) \} = \max_{\substack{x^* \in X^*, \alpha \in \mathbb{R}, \\ \|x^*\| \leq 1/\lambda, (x^*, \alpha) \in \text{epi } g^*}} \lambda \{ \langle x^*, x \rangle - \alpha \} \quad \text{and}$$

$$\begin{aligned} \text{dist}(x; S) &= \max \left\{ \sup_{\substack{\lambda > 0, x^* \in X^*, \alpha \in \mathbb{R}, \\ \|x^*\| \leq 1/\lambda, (x^*, \alpha) \in \text{epi } g^*}} \lambda \{ \langle x^*, x \rangle - \alpha \}, 0 \right\} \quad (16) \\ &= \sup_{\substack{\lambda > 0, x^* \in X^*, \alpha \in \mathbb{R}, \\ \|x^*\| \leq 1/\lambda, (x^*, \alpha) \in \text{epi } g^*}} \lambda \{ \langle x^*, x \rangle - \alpha \}_+. \end{aligned}$$

Again with $\lambda > 0$ fixed, for $x^* = 0$ we observe that

$$\begin{aligned} \max_{(0, \alpha) \in \text{epi } g^*} \lambda \{ \langle 0, x \rangle - \alpha \} &= \max_{g^*(0) \leq \alpha} \lambda \{ \langle 0, x \rangle - \alpha \} \\ &= \lambda \{ -g^*(0) \} \\ &\leq \lambda \inf_{x \in X} g(x) \\ &\leq \lambda g(\hat{x}) < 0. \end{aligned}$$

According to this, the second representation in (16) implies the equalities

$$\begin{aligned} \text{dist}(x; S) &= \sup_{\substack{\lambda > 0, \|x^*\| \leq 1/\lambda, \\ (x^*, \alpha) \in \text{epi } g^*}} \lambda \{ \langle x^*, x \rangle - \alpha \}_+ \\ &= \max_{(x^*, \alpha) \in \text{epi } g^*} \sup_{\lambda > 0, \|x^*\| \leq 1/\lambda} \lambda \{ \langle x^*, x \rangle - \alpha \}_+ \\ &= \sup_{(x^*, \alpha) \in \text{epi } g^*} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}, \end{aligned}$$

which complete the proof of the proposition. ■

Lemma 5 Assume that SSC is satisfied for the system $\sigma(p)$ in (1). Then for any $x \in X$ and any $p \in l_\infty(T)$ we have the representation

$$\text{dist}(x; \mathcal{F}(p)) = \sup_{(x^*, \alpha) \in \text{cl}^* C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}. \quad (17)$$

If furthermore the space X is reflexive, then

$$\text{dist}(x; \mathcal{F}(p)) = \sup_{(x^*, \alpha) \in C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}. \quad (18)$$

Proof. We can obviously write

$$\mathcal{F}(p) = \{x \in X \mid g(x) \leq 0\} \quad \text{with } g := \sup_{t \in T} (f_t - p_t),$$

where the SSC is equivalent to the existence of $\hat{x} \in X$ such that $g(\hat{x}) < 0$.

Employing further [9, formula (2.3)] gives us

$$\text{epi } g^* = \text{epi} \left\{ \sup_{t \in T} (f_t - p_t) \right\}^* = \text{cl}^* \text{co} \left(\bigcup_{t \in T} \text{epi} (f_t - p_t)^* \right) = \text{cl}^* H(p),$$

and thus (17) comes straightforwardly from (15) together with the fact that $\text{cl}^* H(p) = [\text{cl}^* C(p)] + \mathbb{R}_+(0, 1)$ with $0 \in X^*$.

Consider now the case when the space X is reflexive. Arguing by contradiction, assume that (18) does not hold and then find a scalar β such that

$$\sup_{(x^*, \alpha) \in \text{cl}^* C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|} > \beta > \sup_{(x^*, \alpha) \in C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}. \quad (19)$$

Thus there exists a pair $(\bar{x}^*, \bar{\alpha}) \in \text{cl}^* C(p)$, with $\bar{x}^* \in X^* \setminus \{0\}$ and $\bar{\alpha} \in \mathbb{R}$, satisfying the strict inequality

$$\frac{[\langle \bar{x}^*, x \rangle - \bar{\alpha}]_+}{\|\bar{x}^*\|} > \beta.$$

Since X is reflexive and the set $C(p)$ is convex, the classical Mazur theorem allows us to replace the weak* closure of C by its norm closure. Hence there is a sequence $(x_k^*, \alpha_k) \in C(p)$, $k = 1, 2, \dots$, converging in norm to $(\bar{x}^*, \bar{\alpha})$ with

$$\lim_{k \rightarrow \infty} \frac{[\langle x_k^*, x \rangle - \alpha_k]_+}{\|x_k^*\|} = \frac{[\langle \bar{x}^*, x \rangle - \bar{\alpha}]_+}{\|\bar{x}^*\|} > \beta.$$

Therefore we find a natural number k_0 for which

$$\frac{[\langle x_{k_0}^*, x \rangle - \alpha_{k_0}]_+}{\|x_{k_0}^*\|} > \beta.$$

This clearly contradicts (19) and thus completes the proof of the lemma. ■

4 Qualitative Stability via Coderivatives

In this section we consider the parametric convex system (1) in the general framework of Banach decision spaces X . The main goals of this section are to establish necessary and sufficient conditions for the Lipschitz-like property of the solution map (2) to (1) and to compute the exact Lipschitzian bound of (2) in the general Banach space setting. As mentioned in Section 1, our approach to these quantitative stability issues relies on reducing the convex infinite system $\sigma(p)$ in (1) to its linearization $\tilde{\sigma}(\rho_p)$ in (3) and then employing the corresponding results of [4] derived for linear infinite systems. This is done on the base of coderivative analysis.

We start with deriving an upper estimate of the exact Lipschitzian bound for the solution map (2) by using the aforementioned approach.

Lemma 6 *For any $x \in X$ and any $p \in l_\infty(T)$ the following holds:*

$$\text{dist}(p; \mathcal{F}^{-1}(x)) \geq \text{dist}(\rho_p; \tilde{\mathcal{F}}^{-1}(x)).$$

Proof. First observe that $\tilde{\mathcal{F}}^{-1}(x) = \emptyset$ yields $\mathcal{F}^{-1}(x) = \emptyset$. Consider further the nontrivial case when both sets $\tilde{\mathcal{F}}^{-1}(x)$ and $\mathcal{F}^{-1}(x)$ are nonempty. Thus we get for any sequence $\{p_r\}_{r \in \mathbb{N}} \subset l_\infty(T)$ that

$$\text{dist}(p; \mathcal{F}^{-1}(x)) = \lim_{r \in \mathbb{N}} \|p - p_r\| = \lim_{r \in \mathbb{N}} \|\rho_p - \rho_{p_r}\| \geq \text{dist}(\rho_p; \tilde{\mathcal{F}}^{-1}(x)).$$

To complete the proof, recall that $p_r \in \mathcal{F}^{-1}(x)$ if and only if $\rho_{p_r} \in \tilde{\mathcal{F}}^{-1}(x)$. ■

From now on we consider the nominal parameter $\bar{p} = 0$, i.e, the zero function from T to \mathbb{R} ; the corresponding function $\rho_{\bar{p}}$ is also the zero function from \tilde{T} to \mathbb{R} . Both zero functions will be denoted simply by 0.

Lemma 7 *Let $\bar{x} \in \mathcal{F}(0)$. Then we have the upper estimate*

$$\text{lip } \mathcal{F}(0, \bar{x}) \leq \text{lip } \tilde{\mathcal{F}}(0, \bar{x}).$$

Proof. The aimed inequality comes straightforwardly from the exact Lipschitzian bound representation (7) combined with the linearized relationship (5) and the previous lemma. ■

The latter lemma and the results of [4] for linear infinite systems lead us to a constructive upper modulus estimate for the original convex system.

Theorem 8 *Let $\bar{x} \in \mathcal{F}(0)$. Assume that the SSC is satisfied for $\sigma(0)$ and that the set $\bigcup_{t \in T} \text{dom } f_t^*$ is bounded in X^* . The following assertions hold:*

- (i) *If \bar{x} is a strong Slater point of $\sigma(0)$, then $\text{lip } \mathcal{F}(0, \bar{x}) = 0$.*
- (ii) *If \bar{x} is not a strong Slater point of $\sigma(0)$, then*

$$\text{lip } \mathcal{F}(0, \bar{x}) \leq \text{lip } \tilde{\mathcal{F}}(0, \bar{x}) = \max \{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl}^* C(0) \}. \quad (20)$$

Proof. First of all, recall from Section 1 that the SSC property for the convex system $\sigma(p)$ is equivalent to the SSC condition for the linear one $\tilde{\sigma}(\rho_p)$. Thus the equality in (20) follows from [4, Theorem 4.6] under the boundedness assumptions made in the theorem. The upper estimate in (20) is the content of Lemma 7, and thus the proof of the theorem is complete. ■

In what follows we show that the upper estimate in (20) holds in fact as *equality* under the assumptions of Theorem 8. Furthermore, the boundedness assumption of this theorem (which may be violated even in simple examples) can be avoided in the case of reflexive decision spaces X .

To justify the equality in (20), we proceed by using *coderivative analysis*. For each $t \in T$, consider a convex function $h_t : l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$ defined by

$$h_t(p, x) := \langle -\delta_t, p \rangle + f_t(x), \quad (21)$$

where δ_t denotes the classical *Dirac measure* at $t \in T$, i.e.,

$$\langle \delta_t, p \rangle := p_t \text{ for every } p = (p_t)_{t \in T} \in l_\infty(T).$$

It is easy to see that

$$\text{dom } h_t^* = \{-\delta_t\} \times \text{dom } f_t^* \quad \text{and} \quad \text{gph } h_t^* = \{-\delta_t\} \times \text{gph } f_t^*. \quad (22)$$

The next result computes the coderivative of the solution map (2) to the original infinite convex system (1) in terms of its initial data. It is important for the subsequent qualitative stability analysis conducted in this section as well as for deriving optimality conditions in Section 6.

Proposition 9 *Let $\bar{x} \in \mathcal{F}(0)$ for the solution map (2) to the convex system (1). Then $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$ if and only if*

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left(\bigcup_{t \in T} [\{-\delta_t\} \times \text{gph } f_t^*] \right). \quad (23)$$

Proof. Due to the obvious convexity of the graphical set $\text{gph } \mathcal{F}$ for (2), the cone $N((0, \bar{x}); \text{gph } \mathcal{F})$ reduces to the classical normal cone of convex analysis. Thus we have that $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$ if and only if $\langle p^*, p \rangle - \langle x^*, x \rangle \leq -\langle x^*, \bar{x} \rangle$ by considering the convex system

$$\{h_t(p, x) \leq 0, t \in T\}$$

with h_t defined in (21). It now follows from the extended Farkas Lemma formulated in Lemma 1 that

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^* \left(\text{cone} \bigcup_{t \in T} \text{epi } h_t^* \right). \quad (24)$$

It is easy to see by applying both sides of (24) on $(0, \bar{x}, -1)$ that the epigraph in (24) can be replaced by the graph of h_t^* therein. Thus representation (23) follows from that in (24) and the expression of the graph of h_t^* given in (22). ■

The next important result provides a complete computation of the coderivative norm, defined in (10), via the characteristic set $C(0)$ from (12).

Theorem 10 Let $\bar{x} \in \mathcal{F}(0)$. Assume that the SSC is satisfied for $\sigma(0)$ and that the set $\bigcup_{t \in T} \text{dom } f_t^*$ is bounded in X^* . The following assertions hold:

- (i) If \bar{x} is a strong Slater point of $\sigma(0)$, then $\|D^*\mathcal{F}(0, \bar{x})\| = 0$.
- (ii) If \bar{x} is not a strong Slater point of $\sigma(0)$, then

$$\|D^*\mathcal{F}(0, \bar{x})\| = \max \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl}^*C(0) \right\} > 0.$$

Proof. It follows the lines in the proof of [4, Theorem 3.5] with using the equivalent descriptions of the strong Slater condition for the convex inequality system (1) via the characteristic set $C(0)$ obtained in Lemma 3. ■

Now we are ready to establish the main result of this section containing the coderivative characterization of the Lipschitz-like property of the solution map (2) with the precise computation of the exact Lipschitzian bound.

Theorem 11 Let $\bar{x} \in \mathcal{F}(0)$ for the solution map (2) to the convex inequality system $\sigma(p)$ in (1) with an arbitrary Banach decision space X . Then \mathcal{F} is Lipschitz-like around $(0, \bar{x})$ if and only if

$$D^*\mathcal{F}(0, \bar{x})(0) = \{0\}. \quad (25)$$

If furthermore the SSC is satisfied for $\sigma(0)$ and the set $\bigcup_{t \in T} \text{dom } f_t^*$ is bounded in X^* , then the following hold:

- (i) $\text{lip } \mathcal{F}(0, \bar{x}) = 0$ provided that \bar{x} is a strong Slater point of $\sigma(0)$;
- (ii) otherwise we have

$$\text{lip } \mathcal{F}(0, \bar{x}) = \max \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl}^*C(0) \right\} > 0. \quad (26)$$

Proof. The “only if” part in the coderivative criterion (25) is a consequence of [17, Theorem 1.44] established for general set-valued mappings of closed graph between Banach spaces. The proof of the “if” part in (25) follows the lines in the proof of [4, Theorem 4.1].

The equality $\text{lip } \mathcal{F}(0, \bar{x}) = 0$ for the exact Lipschitzian bound in case (i) can be checked directly from the definitions while it also follows by combining assertion (i) of Theorem 8 and assertion (i) of Theorem 10.

It remains to justify equality (26) in the case when \bar{x} is not a strong Slater point of $\sigma(0)$. Indeed, the upper estimate for $\text{lip } \mathcal{F}(0, \bar{x})$ follows from assertion (ii) of Theorem 8 and computing the coderivative norm in assertion (ii) of Theorem 10 under the assumptions made. The lower bound estimate

$$\text{lip } \mathcal{F}(0, \bar{x}) \geq \|D^*\mathcal{F}(0, \bar{x})\|$$

is proved in [17, Theorem 1.44] for general set-valued mappings between Banach spaces. This completes the proof of the theorem. ■

Remark 12 For the Lipschitzian modulus results obtained in Theorem 8 and Theorem 11 we imposed the *boundedness* assumption on the set $\bigcup_{t \in T} \text{dom } f_t^*$

in the convex infinite system (1). This corresponds to the boundedness on the coefficient set $\{a_t^* \mid t \in T\}$ in the case of parametric linear infinite systems $\{\langle a_t^*, x \rangle \leq b_t + p_t\}$. While the latter assumption does not look restrictive in the linear framework, it may be too strong in the convex setting under consideration, being violated even in some simple examples as in the case of the following single constraint involving one-dimensional decision and parameter variables:

$$x^2 \leq p \text{ for } x, p \in \mathbb{R}. \quad (27)$$

Note that the linearized system (3) associated with (27) reads as follows:

$$ux \leq p + \frac{u^2}{4}, \quad u \in \mathbb{R}.$$

In the next section we show that the aforementioned coefficient boundedness assumption for linear systems and the corresponding boundedness assumption on the set $\bigcup_{t \in T} \text{dom } f_t^*$ in the convex framework can be dropped in the case of reflexive Banach spaces X of decision variables.

Remark 13 After the publication of [4], Alex Ioffe drew our attention to the possible connections of some of the results therein with those obtained in [13] for general set-valued mappings of convex graph. Examining this approach, we were able to check, in particular, that [4, Corollary 4.7] on the computing the exact Lipschitzian bound of linear infinite systems via the coderivative norm under the coefficient boundedness can be obtained by applying Theorem 3 and Proposition 5 from [13]. However, our proofs are far from being straightforward.

5 Enhanced Stability Results in Reflexive Spaces

In this section we primarily deal with the linear infinite system

$$\sigma(p) := \{ \langle a_t^*, x \rangle \leq b_t + p_t, t \in T \}, \quad (28)$$

where $a_t^* \in X^*$ and $b_t \in \mathbb{R}$ are fixed for each t from an arbitrary index sets T . Due to the linearization approach developed above, the results obtained below for linear systems can be translated to convex infinite systems of type (1).

Note that in the linear case (28) the characteristic set (12) takes the form

$$C(p) = \text{co} \{ (a_t^*, b_t + p_t) \mid t \in T \}. \quad (29)$$

This is our setting in [4], where the coefficient set $\{a_t^* \mid t \in T\} \subset X^*$ is assumed to be bounded while computing the coderivative norm $\|D^* \mathcal{F}(0, \bar{x})\|$ in [4, Theorem 3.5] and the exact Lipschitzian bound $\text{lip } \mathcal{F}(0, \bar{x})$ in [4, Theorem 4.5] for the solution map \mathcal{F} to (28). In the case when X is reflexive, we are going to remove now the coefficient boundedness assumption from both referred theorems, which implies that the boundedness of the set $\bigcup_{t \in T} \text{dom } f_t^*$ can also be removed as an assumption throughout Section 4 when X is reflexive.

First we observe that the boundedness of the coefficients $\{a_t^* \mid t \in T\}$ yields that only ε -active indices are relevant in (29) with respect to the set of elements in the form $(u^*, \langle u^*, \bar{x} \rangle)$ belonging to $\text{cl}^* C(0)$, which from now on is written as $\{(\bar{x}, -1)\}^\perp \cap \text{cl}^* C(0)$. Given $\bar{x} \in \mathcal{F}(0)$ and $\varepsilon \geq 0$, we use the notation

$$T_\varepsilon(\bar{x}) := \{t \in T \mid \langle a_t^*, \bar{x} \rangle \geq b_t - \varepsilon\}$$

for the set of ε -active indices. Let us make the above statement precise.

Proposition 14 *Assume that the coefficient set $\{a_t^* \mid t \in T\}$ is bounded in X^* . Then given $\bar{x} \in \mathcal{F}(0)$, we have the representation*

$$\{(\bar{x}, -1)\}^\perp \cap \text{cl}^* C(0) = \bigcap_{\varepsilon > 0} \text{cl}^* \text{co} \{ (a_t^*, b_t) \mid t \in T_\varepsilon(\bar{x}) \}. \quad (30)$$

Proof. It follows the lines of justifying Step 1 in the proof of [2, Theorem 1]. Note that both sets in (30) are nonempty if and only if \bar{x} is not a strong Slater point for $\sigma(0)$. Note also that the fulfillment of the SSC for $\sigma(0)$ in (28) is not required for the fulfillment of (30). ■

Observe that in the continuous case considered in [1] (where T is assumed to be a compact Hausdorff space, $X = \mathbb{R}^n$, and the mapping $t \mapsto (a_t^*, b_t)$ is continuous on T) representation (30) reads as

$$\{(\bar{x}, -1)\}^\perp \cap C(0) = \text{co} \{ (a_t^*, b_t) \mid t \in T_0(\bar{x}) \}.$$

The following example shows that the statement of Proposition 14 is *no longer valid* without the boundedness assumption on $\{a_t^* \mid t \in T\}$ and that in (26) the set $\text{cl}^* C(0)$ cannot be replaced by $\text{cl}^* \text{co} \{ (a_t^*, b_t) \mid t \in T_\varepsilon(\bar{x}) \}$ for some *small* $\varepsilon > 0$; i.e., it is not sufficient to consider just ε -active constraints. However, the exact bound formula (26) *remains true* in this example, with the replacement of “max” by “sup” therein.

Example 15 Consider the countable linear system in \mathbb{R}^2 :

$$\sigma(p) = \left\{ \begin{array}{ll} (-1)^t t x_1 \leq 1 + p_t, & t = 1, 2, \dots, \\ x_1 + x_2 \leq 0 + p_0, & t = 0 \end{array} \right\}.$$

The reader can easily check that

$$\text{co} \{ (a_t^*, b_t) \mid t \in T_\varepsilon(\bar{x}) \} = \{ (1, 1, 0) \} \text{ and}$$

$$\{(\bar{x}, -1)\}^\perp \cap \text{cl}^* C(0) = \{ (\alpha, 1, 0), \alpha \in \mathbb{R} \}$$

for $\bar{x} = 0_2$ and $0 \leq \varepsilon < 1$. Moreover

$$\mathcal{F}(p) = \{0\} \times (-\infty, p_0] \text{ whenever } \|p\| \leq 1,$$

which easily implies that $\text{lip } \mathcal{F}(0, \bar{x}) = 1$, and hence the exact bound formula (26) holds in this example. Observe however that for $0 < \varepsilon < 1$ we have

$$\max \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl}^* \text{co} \{ (a_t^*, b_t) \mid t \in T_\varepsilon(\bar{x}) \} \right\} = \frac{1}{\sqrt{2}},$$

which shows that $T(\bar{x})$ cannot be replaced by $T_\varepsilon(\bar{x})$ in (26).

As we mentioned above, it is clear that $\{(\bar{x}, -1)\}^\perp \cap \text{cl}^*C(0) = \emptyset$ when \bar{x} is a strong Slater point for $\sigma(0)$. The following example (where the SSC is satisfied for $\sigma(0)$) shows that the set $\{(\bar{x}, -1)\}^\perp \cap \text{cl}^*C(0)$ may be empty when $\bar{x} \in \mathcal{F}(0)$ is not a strong Slater point for $\sigma(0)$. According to Theorem 11, this cannot be the case when the coefficient set $\{a_t^* \mid t \in T\}$ is bounded. Observe however that in this example we have $\text{lip } \mathcal{F}(0, \bar{x}) = 0$, and thus (26) still holds under the convention that $\text{sup } \emptyset := 0$.

Example 16 Consider the infinite linear system in \mathbb{R} :

$$\sigma(p) = \left\{ tx \leq \frac{1}{t} + p_t, \quad t \in [1, \infty) \right\}$$

and take $\bar{x} = 0$. It is easy to see that $\{(\bar{x}, -1)\}^\perp \cap \text{cl}^*C(0) = \emptyset$. Let us now check that $\text{lip } \mathcal{F}(0, \bar{x}) = 0$. Indeed, representation (7) yields

$$\begin{aligned} \text{lip } \mathcal{F}(0, \bar{x}) &= \limsup_{(p,x) \rightarrow (0,0)} \frac{\text{dist}(x; \mathcal{F}(p))}{\text{dist}(p; \mathcal{F}^{-1}(x))} = \limsup_{(p,x) \rightarrow (0,0)} \frac{[x - \inf_{t \geq 1} (\frac{1}{t^2} + \frac{p_t}{t})]_+}{\sup_{t \geq 1} [tx - \frac{1}{t} - p_t]_+} \\ &= \limsup_{(p,x) \rightarrow (0,0)} \frac{\sup_{t \geq 1} [x - \frac{1}{t^2} - \frac{p_t}{t}]_+}{\sup_{t \geq 1} [tx - \frac{1}{t} - p_t]_+}. \end{aligned}$$

Taking into account that $\sup_{t \geq 1} [tx - \frac{1}{t} - p_t]_+ = \infty$ if $x > 0$ for every $p \in l_\infty([1, \infty))$ and that for any $(p, x) \in \varepsilon B_{l_\infty([1, \infty))} \times [-\varepsilon, 0]$ with $0 < \varepsilon \leq 1$ we have $x - \frac{1}{t^2} - \frac{p_t}{t} \leq 0$, it follows that

$$\text{lip } \mathcal{F}(0, \bar{x}) = \limsup_{(p,x) \rightarrow (0,0)} \frac{\sup_{t \geq 1/\varepsilon} [x - \frac{1}{t^2} - \frac{p_t}{t}]_+}{\sup_{t \geq 1/\varepsilon} [tx - \frac{1}{t} - p_t]_+} \leq \varepsilon.$$

Since this holds for any $\varepsilon \in (0, 1]$, we get $\text{lip } \mathcal{F}(0, \bar{x}) = 0$ and thus conclude our consideration in this example.

Now we are ready to establish our major result in the case of reflexive decision spaces X in (28). Recall that in this case the weak* closure cl^*S and the norm closure $\text{cl } S$ in X^* agree for convex subsets $S \subset X^*$.

Theorem 17 *Assume that X is reflexive and let $\bar{x} \in \mathcal{F}(0)$. If the SSC is satisfied for $\sigma(0)$ in (28), then we have*

$$\text{lip } \mathcal{F}(0, \bar{x}) = \|D^*\mathcal{F}(0, \bar{x})\| = \sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0) \right\} \quad (31)$$

with $C(0)$ defined in (29), under the convention that $\text{sup } \emptyset := 0$.

Proof. As mentioned above, the inequality $\text{lip } \mathcal{F}(0, \bar{x}) \geq \|D^*\mathcal{F}(0, \bar{x})\|$ holds for general set-valued mappings due to [17, Theorem 1.44]. Let us next consider the nontrivial case $\{(\bar{x}, -1)\}^\perp \cap \text{cl } C(0) \neq \emptyset$ and show that

$$\|D^*\mathcal{F}(0, \bar{x})\| \geq \sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0) \right\}. \quad (32)$$

To proceed, take $u^* \in X^*$ such that $(u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0)$. The fulfillment of the SSC for $\sigma(0)$ in (28) ensures that $u^* \neq 0$ according to Lemma 3. By the latter inclusion, find a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\lambda_k = (\lambda_{tk})_{t \in T} \in \mathbb{R}_+^{(T)}$ and $\sum_{t \in T} \lambda_{tk} = 1$ as $k \in \mathbb{N}$ satisfying

$$(u^*, \langle u^*, \bar{x} \rangle) = \lim_{k \rightarrow \infty} \sum_{t \in T} \lambda_{tk} (a_t^*, b_t). \quad (33)$$

Since the sequence $\left\{ \|u^*\|^{-1} \sum_{t \in T} \lambda_{tk} (-\delta_t) \right\}_{k \in \mathbb{N}}$ is contained in $\|u^*\|^{-1} B_{l_\infty(T)}$, the classical Alaoglu-Bourbaki theorem ensures that a certain subnet of this sequence (indexed by $\nu \in \mathcal{N}$) weak* converges to some $p^* \in l_\infty(T)^*$ with $\|p^*\| \leq \|u^*\|^{-1}$. Denoting by $e \in l_\infty(T)$ the function whose coordinates are identically one, we get

$$\langle p^*, -e \rangle = \lim_{\nu \in \mathcal{N}} \|u^*\|^{-1} \sum_{t \in T} \lambda_{t\nu} = \|u^*\|^{-1},$$

and hence $\|p^*\| = \|u^*\|^{-1}$. Appealing now to (33) gives us, for the subnet under consideration, the equality

$$\left(p^*, \|u^*\|^{-1} u^*, \left\langle \|u^*\|^{-1} u^*, \bar{x} \right\rangle \right) = w^* \text{-} \lim_{\nu \in \mathcal{N}} \|u^*\|^{-1} \sum_{t \in T} \lambda_{t\nu} (-\delta_t, a_t^*, b_t).$$

Employing further the coderivative description from Proposition 9 yields

$$p^* \in D^* \mathcal{F}(0, \bar{x}) \left(-\|u^*\|^{-1} u^* \right),$$

which implies by definition (10) of the coderivative norm that

$$\|D^* \mathcal{F}(0, \bar{x})\| \geq \|p^*\| = \|u^*\|^{-1}.$$

Since u^* was arbitrarily chosen from those satisfying $(u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0)$, we arrive at the lower estimate (32) for the coderivative norm.

Now let us prove the upper estimate for the exact Lipschitzian bound

$$\text{lip } \mathcal{F}(0, \bar{x}) \leq \sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0) \right\}, \quad (34)$$

which ensures, together with the lower estimates above, the fulfillments of both equalities in (31). Arguing by contradiction, find $\alpha > 0$ such that

$$\text{lip } \mathcal{F}(0, \bar{x}) > \alpha > \sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0) \right\}. \quad (35)$$

According to the first inequality of (35), there are sequences $p_r = (p_{tr})_{t \in T} \rightarrow 0$ and $x_r \rightarrow \bar{x}$ such that

$$\text{dist}(x_r; \mathcal{F}(p_r)) > \alpha \text{dist}(p_r; \mathcal{F}^{-1}(x_r)) \quad \text{for all } r \in \mathbb{N}. \quad (36)$$

By the SSC for $\sigma(0)$ we have that $\mathcal{F}(p_r) \neq \emptyset$ for r sufficiently large (say for all r without loss of generality). This SSC is equivalent to the Lipschitz-like property of the corresponding solution map \mathcal{F} around $(0, \bar{x})$ and also to the inner/lower semicontinuity of \mathcal{F} around \bar{x} by [7, Theorem 5.1], which entails that

$$\lim_{r \rightarrow \infty} \text{dist}(x_r; \mathcal{F}(p_r)) = 0. \quad (37)$$

Moreover, it follows from (36) that the quantity

$$\begin{aligned} \text{dist}(p_r; \mathcal{F}^{-1}(x_r)) &= \sup_{t \in T} [\langle a_t^*, x_r \rangle - b_t - p_{tr}]_+ \\ &= \sup_{(x^*, \alpha) \in C(p_r)} [\langle x^*, x_r \rangle - \alpha]_+ \end{aligned} \quad (38)$$

is finite. It follows from Lemma 5 while $\|p_r\| \leq \eta$, $r = 1, 2, \dots$, that

$$\text{dist}(x_r; \mathcal{F}(p_r)) = \sup_{\substack{x^* \in X^* \setminus \{0\}, \alpha \in \mathbb{R}, \\ (x^*, \alpha) \in C(p_r)}} \frac{[\langle x^*, x_r \rangle - \alpha]_+}{\|x^*\|}, \quad r = 1, 2, \dots$$

This allows us to find $(x_r^*, \alpha_r) \in C(p_r) \setminus \{0\}$ as $r = 1, 2, \dots$ satisfying

$$0 \leq \text{dist}(x_r; \mathcal{F}(p_r)) - \frac{\langle x_r^*, x_r \rangle - \alpha_r}{\|x_r^*\|} < \frac{\text{dist}(p_r; \mathcal{F}^{-1}(x_r))}{r}. \quad (39)$$

Furthermore, by (36) and (38) we can choose (x_r^*, α_r) in such a way that

$$\alpha \text{dist}(p_r; \mathcal{F}^{-1}(x_r)) < \frac{\langle x_r^*, x_r \rangle - \alpha_r}{\|x_r^*\|} + \frac{\text{dist}(p_r; \mathcal{F}^{-1}(x_r))}{r} \leq \frac{\text{dist}(p_r; \mathcal{F}^{-1}(x_r))}{\|x_r^*\|}. \quad (40)$$

Since $\text{dist}(p_r; \mathcal{F}^{-1}(x_r)) > 0$, we deduce from (40) that

$$\alpha < \frac{1}{\|x_r^*\|} + \frac{1}{r} \quad \text{and} \quad 0 < \|x_r^*\| < \frac{1}{\alpha - r^{-1}} \quad \text{for all } r = 1, 2, \dots,$$

and thus, by the weak* sequential compactness of the unit ball in dual to reflexive spaces, select a subsequence $\{x_{r_k}^*\}_{k \in \mathbb{N}}$ which weak* converges to some $x^* \in X^*$ satisfying $\|x^*\| \leq 1/\alpha$. Then we get from (37) and (39) that

$$\lim_{k \in \mathbb{N}} \frac{\langle x_{r_k}^*, x_{r_k} \rangle - \alpha_{r_k}}{\|x_{r_k}^*\|} = 0,$$

which implies in turn that

$$\lim_{k \in \mathbb{N}} (\langle x_{r_k}^*, x_{r_k} \rangle - \alpha_{r_k}) = 0.$$

Since the sequence $\{x_{r_k}\}_{k \in \mathbb{N}}$ converges (in norm) to \bar{x} , the latter implies that

$$\lim_{k \in \mathbb{N}} \alpha_{r_k} = \lim_{k \in \mathbb{N}} \langle x_{r_k}^*, x_{r_k} \rangle = \langle x^*, \bar{x} \rangle.$$

Taking into account for each $k \in \mathbb{N}$ we have $(x_{r_k}^*, \alpha_{r_k}) \in C(p_{r_k})$, there exist $\lambda_{r_k} = (\lambda_{tr_k})_{t \in T}$ such that $\lambda_{tr_k} \geq 0$, only finitely many of them are positive,

$$\sum_{t \in T} \lambda_{tr_k} = 1, \text{ and } (x_{r_k}^*, \alpha_{r_k}) = \sum_{t \in T} \lambda_{tr_k} (a_t^*, b_t + p_{tr_k}), \quad k \in \mathbb{N}.$$

Combining all the above gives us the relationships

$$\begin{aligned} (x^*, \langle x^*, \bar{x} \rangle) &= w^* \text{-} \lim_{k \in \mathbb{N}} (x_{r_k}^*, \langle x_{r_k}^*, x_{r_k} \rangle) \\ &= w^* \text{-} \lim_{k \in \mathbb{N}} (x_{r_k}^*, \alpha_{r_k}) \\ &= w^* \text{-} \lim_{k \in \mathbb{N}} \sum_{t \in T} \lambda_{tr_k} (a_t^*, b_t + p_{tr_k}) \\ &= w^* \text{-} \lim_{k \in \mathbb{N}} \sum_{t \in T} \lambda_{tr_k} (a_t^*, b_t) \in \text{cl } C(0). \end{aligned}$$

Observe finally that $x^* \neq 0$ because, by Lemma 3, the linear infinite system $\sigma(0)$ satisfies the SSC. This allows us to conclude that

$$\sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl } C(0) \right\} \geq \|x^*\|^{-1} \geq \alpha,$$

which contradicts (35) and thus completes the proof of the theorem. ■

As mentioned above, the results obtained in this section for linear infinite systems make it possible to drop the major boundedness assumptions imposed in the corresponding results of Section 4 for convex infinite systems in reflexive spaces. Let us present the improved “reflexive” version of Theorem 11, the main result of the preceding section.

Theorem 18 *Let $\bar{x} \in \mathcal{F}(0)$ for the solution map (2) to the convex inequality system $\sigma(p)$ in (1) with a reflexive Banach decision space X . If the SSC is satisfied for $\sigma(0)$, then the following hold:*

- (i) $\text{lip } \mathcal{F}(0, \bar{x}) = 0$ provided that \bar{x} is a strong Slater point of $\sigma(0)$;
- (ii) otherwise we have

$$\text{lip } \mathcal{F}(0, \bar{x}) = \sup \left\{ \|u^*\|^{-1} \mid (u^*, \langle u^*, \bar{x} \rangle) \in \text{cl}^* C(0) \right\},$$

where the characteristic set $C(0)$ is defined in (12).

Proof. Follows the lines in the proof of Theorem 11 with the usage of Theorem 17 instead of [4, Theorem 4.6] therein. ■

6 Optimality Conditions for Infinite Programs

In this section we consider an infinite (or semi-infinite) optimization problem of type (P) written in the form:

$$\inf \varphi(p, x) \text{ s.t. } x \in \mathcal{F}(p), \quad (41)$$

where $x \in X$, $p = (p_t)_{t \in T} \in l_\infty(T)$ with an arbitrary index set T , and where the set of feasible solutions

$$\mathcal{F}(p) := \{x \in X \mid f_t(x) \leq p_t, t \in T\} \quad (42)$$

is defined by the parameterized infinite system of convex inequalities (1) over a general Banach space X satisfying the standing assumptions formulated in Section 1. We refer the reader to [5] for the justification and valuable examples of such a two-variable version of the infinite/semi-infinite program (P) in the particular case of linear infinite inequality systems (28).

The main goal of the section is to derive *necessary optimality conditions* for optimal solutions to (41) under general requirements on *nonconvex* and *nonsmooth* cost functions $\varphi: l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$. Involving the coderivative analysis of Section 4, we obtain optimality conditions in the general *asymptotic form* developed in [5] for linear infinite systems; see also the discussions and references therein on the comparison with other kinds of optimality conditions in semi-definite and infinite optimization. Furthermore, we establish results of two independent types: *lower subdifferential* and *upper subdifferential* depending on the type of subgradients used for cost functions; see below.

Let us start with *lower* subdifferential conditions, which are of the conventional type in nonsmooth *minimization*. Since our infinite/semi-infinite setup is given intrinsically in general Banach spaces by the structure of (42) with $p \in l_\infty(T)$ independently of the dimension of X , we cannot employ the well-developed Asplund space theory from [17, 18]. The most appropriate subdifferential construction in our framework is the so-called approximate subdifferential by Ioffe [11, 12], which is a general (while more complicated, topological) Banach space extension of the (sequential) basic/limiting subdifferential by Morukhovich [14, 17] that may be larger than the latter even for locally Lipschitzian functions on nonseparable Asplund spaces while it is always smaller than the Clarke subdifferential; see [17, Subsection 3.2.3] for more details.

The approximate subdifferential constructions in Banach spaces are defined by the following multistep procedure. Given a function $\varphi: Z \rightarrow \overline{\mathbb{R}}$ finite at \bar{z} , we first consider its *lower Dini* (or Dini-Hadamard) *directional derivative*

$$d^- \varphi(\bar{z}; v) := \liminf_{u \rightarrow v, t \downarrow 0} \frac{\varphi(\bar{z} + tu) - \varphi(\bar{z})}{t}, \quad v \in Z,$$

and then define the *Dini ε -subdifferential* of φ at \bar{z} by

$$\partial_\varepsilon^- \varphi(\bar{z}) := \{z^* \in Z^* \mid \langle z^*, v \rangle \leq d^- \varphi(\bar{z}; v) + \varepsilon \|v\| \text{ for all } v \in Z\}, \quad \varepsilon \geq 0,$$

putting $\partial_\varepsilon^- \varphi(\bar{z}) := \emptyset$ if $\varphi(\bar{z}) = \infty$. The *A-subdifferential* of φ at \bar{z} is defined via topological limits involving finite-dimensional reductions of ε -subgradients by

$$\partial_A \varphi(\bar{z}) := \bigcap_{L \in \mathcal{L}, \varepsilon > 0} \overline{\text{Lim sup}}_{z \rightarrow^\varphi \bar{z}} \partial_\varepsilon^-(\varphi + \delta_L)(z),$$

where \mathcal{L} signifies the collection of all the finite-dimensional subspaces of Z , where $z \rightarrow^\varphi \bar{z}$ means that $z \rightarrow \bar{z}$ with $\varphi(z) \rightarrow \varphi(\bar{z})$, and where $\overline{\text{Lim sup}}$ stands for the

topological Painlevé-Kuratowski upper/outer limit of a mapping $F: Z \rightrightarrows Z^*$ as $z \rightarrow \bar{z}$ defined by

$$\overline{\text{Lim sup}}_{z \rightarrow \bar{z}} F(z) := \left\{ z^* \in Z^* \mid \exists \text{ net } (z_\nu, z_\nu^*)_{\nu \in \mathcal{N}} \subset Z \times Z^* \text{ s.t. } z_\nu^* \in F(z_\nu), \right. \\ \left. (z_\nu, z_\nu^*) \xrightarrow{\|\cdot\| \times w^*} (\bar{z}, z^*) \right\}.$$

Then the *approximate G -subdifferential* of φ at \bar{z} (the main construction here called the “nucleus of the G -subdifferential” in [11]) is defined by

$$\partial_G \varphi(\bar{z}) := \left\{ z^* \in X^* \mid (z^*, -1) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{z}, \varphi(\bar{z})); \text{epi } \varphi) \right\}, \quad (43)$$

where $\text{epi } \varphi := \{(z, \mu) \in Z \times \mathbb{R} \mid \mu \geq \varphi(z)\}$. This construction, in any Banach space Z , reduces to the classical derivative in the case of smooth functions and to the classical subdifferential of convex analysis if φ is convex. In what follows we also need the *singular G -subdifferential* of φ at \bar{x} defined by

$$\partial_G^\infty \varphi(\bar{x}) := \left\{ z^* \in X^* \mid (z^*, 0) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi) \right\}. \quad (44)$$

Note that $\partial_G^\infty \varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around \bar{x} .

Now we are ready to derive the lower subdifferential necessary optimality conditions for problem (41) with the convex infinite constraints (42) and a general nonsmooth cost function φ . These conditions and the subsequent results of this section address an arbitrary local minimizer $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$ to the problem under consideration. Following our convention in the previous sections, we suppose without loss of generality that $\bar{p} = 0$.

Theorem 19 *Let $(0, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (41) with the constraint system (42) given by the infinite convex inequalities $\sigma(p)$ in a Banach space X . Assume that the cost function $\varphi: l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$ is lower semicontinuous around $(0, \bar{x})$ with $\varphi(0, \bar{x}) < \infty$. Suppose furthermore that:*

- (a) *either φ is locally Lipschitzian around $(0, \bar{x})$;*
- (b) *or $\text{int}(\text{gph } \mathcal{F}) \neq \emptyset$ (which holds, in particular, when the SSC holds for $\sigma(0)$ and the set $\cup_{t \in T} \text{dom } f_t^*$ is bounded) and the system*

$$(p^*, x^*) \in \partial_G^\infty \varphi(0, \bar{x}), \quad -(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left(\bigcup_{t \in T} [\{-\delta_t\} \times \text{gph } f_t^*] \right) \quad (45)$$

admits only the trivial solution $(p^, x^*) = (0, 0)$.*

Then there is a G -subgradient pair $(p^, x^*) \in \partial_G \varphi(0, \bar{x})$ such that*

$$-(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone} \left(\bigcup_{t \in T} [\{-\delta_t\} \times \text{gph } f_t^*] \right). \quad (46)$$

Proof. The original problem (41) can be rewritten as a mathematical program with geometric constraints:

$$\text{minimize } \varphi(p, x) \text{ subject to } (p, x) \in \text{gph } \mathcal{F}, \quad (47)$$

which can be equivalently described in the form of by unconstrained minimization with “infinite penalties”:

$$\text{minimize } \varphi(p, x) + \delta((p, x); \text{gph } \mathcal{F}).$$

By the G -generalized Fermat stationary rule for the latter problem, we have

$$(0, 0) \in \partial_G[\varphi + \delta(\cdot; \text{gph } \mathcal{F})](0, \bar{x}). \quad (48)$$

Employing the G -subdifferential sum rule to (48), formulated in [11, Theorem 7.4] for the “nuclei”, gives us

$$(0, 0) \in \partial_G \varphi(0, \bar{x}) + N((0, \bar{x}); \text{gph } \mathcal{F}) \quad (49)$$

provided that either φ is locally Lipschitzian around $(0, \bar{x})$, or the interior of $\text{gph } \mathcal{F}$ is nonempty and the qualification condition

$$\partial_G^\infty \varphi(0, \bar{x}) \cap [-N((0, \bar{x}); \text{gph } \mathcal{F})] = \{0, 0\} \quad (50)$$

is satisfied. It is not hard to check (cf. [4, Remark 2.4]) that the strong Slater condition for $\sigma(0)$ and the boundedness of the set $\cup_{t \in T} \text{dom } f_t^*$ imply that the interior of $\text{gph } \mathcal{F}$ is not empty. Observe further that, due to the coderivative definition (8), the optimality condition (49) can be equivalently written as

$$\text{there is } (p^*, x^*) \in \partial_G \varphi(0, \bar{x}) \text{ with } -p^* \in D^* \mathcal{F}(0, \bar{x})(x^*). \quad (51)$$

Employing now in (51) the coderivative calculation from Proposition 9, we arrive at (46). Similar arguments show that the qualification condition (50) can be expressed in the explicit form (45), and thus the proof is complete. ■

The result of Theorem 19 can be represented in a much simpler form for smooth cost functions in (41); it also seems to be new for infinite programming under consideration. Recall that a function $\varphi: Z \rightarrow \mathbb{R}$ is *strictly differentiable* at \bar{z} , with its gradient at this point denoted by $\nabla \varphi(\bar{z}) \in Z^*$, if

$$\lim_{z, u \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(u) - \langle \nabla \varphi(\bar{z}), z - u \rangle}{\|z - u\|} = 0,$$

which surely holds if φ is continuously differentiable around \bar{z} . Since we have

$$\partial_G \varphi(0, \bar{x}) = \{(\nabla_p \varphi(0, \bar{x}), \nabla_x \varphi(0, \bar{x}))\}$$

provided that φ in (41) is strictly differentiable at $(0, \bar{x})$ (and hence locally Lipschitzian around this point), then condition (46) reduces in this case to

$$-\langle \nabla_p \varphi(0, \bar{x}), \nabla_x \varphi(0, \bar{x}), \langle \nabla_x \varphi(0, \bar{x}), \bar{x} \rangle \rangle \in \text{cl}^* \text{cone} \left(\bigcup_{t \in T} [\{-\delta_t\} \times \text{gph } f_t^*] \right). \quad (52)$$

Next we derive qualified asymptotic necessary optimality condition of a new *upper subdifferential* type, initiated in [16] for other classes of optimization problems with finitely many constraints; see also [5, Section 4] for infinite programs with linear constraints. The upper subdifferential optimality conditions presented below are generally independent of Theorem 19 for problems with non-smooth objectives; see the discussion below. The main characteristic feature of upper subdifferential conditions is that they apply to *minimization* problems but not to the expected framework of maximization.

To proceed, we recall the notion of the *Fréchet upper subdifferential* (known also as the Fréchet or viscosity superdifferential) of $\varphi: X \rightarrow \mathbb{R}$ at \bar{z} defined by

$$\widehat{\partial}^+ \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \limsup_{z \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leq 0 \right\}, \quad (53)$$

which reduces to the classical gradient $\nabla \varphi(\bar{z})$ if φ is Fréchet differentiable at \bar{z} (may not be strictly) and to the (upper) subdifferential of *concave* functions in the framework of convex analysis.

Theorem 20 *Let $(0, \bar{x}) \in \text{gph } \mathcal{F}$ be a local minimizer for problem (41) with the convex infinite constraint system (42) in Banach spaces. Then every upper subgradient $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$ satisfies inclusion (46) in Theorem 19.*

Proof. It follows the proof of [5, Theorem 4.1] based on the variational description of Fréchet subgradients in [17, Theorem 1.88(i)] and computing the coderivative of the feasible solution map (42) given in Proposition 9. ■

As a consequence of Theorem 20, we get the simplified necessary optimality condition (52) for the infinite program whose objective φ is merely *Fréchet differentiable* at the optimal point $(0, \bar{x})$.

Note also that, in contrast to Theorem 19, we impose no additional assumptions on φ and \mathcal{F} in Theorem 20. Furthermore, the resulting inclusion (46) is proved to hold for *every* Fréchet upper subgradient $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$ in Theorem 20 instead of *some* G -subgradient $(p^*, x^*) \in \partial_G \varphi(0, \bar{x})$ in Theorem 19. On the other hand, it occurs that $\widehat{\partial}^+ \varphi(0, \bar{x}) = \emptyset$ in many important situations (e.g., for convex objectives) while $\partial_G \varphi(0, \bar{x}) \neq \emptyset$ for every local Lipschitzian function on a Banach space. We refer the reader to [5, Remark 4.5] and [18, Commentary 5.5.4] for extended comments on various classes of functions admitting upper Fréchet subgradients and additional regularity properties ensuring strong advantages of upper subdifferential optimality conditions in comparison with their lower subdifferential counterparts.

Acknowledgements. The authors are indebted to Radu Boţ who suggested Proposition 4 and the current version of Lemma 5 allowing us to remove a certain boundedness assumption therein, which was present in the previous version of the paper. We also very grateful to Alex Ioffe who brought our attention to the recent paper [13] and its connections with some results of [4]; see more discussions in Remark 13.

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