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Lipschitzian Stability of Parametric Variational Inequalities over Generalized Polyhedra in Banach Spaces

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Abstract

This paper concerns the study of solution maps to parameterized variational inequalities over generalized polyhedra in reflexive Banach spaces. It has been recognized that generalized polyhedral sets are significantly different from the usual convex polyhedra in infinite dimensions and play an important role in various applications to optimization, particularly to generalized linear programming. Our main goal is to fully characterize robust Lipschitzian stability of the aforementioned solutions maps entirely via their initial data. This is done on the base of the coderivative criterion in variational analysis via efficient calculations of the coderivative and related objects for the systems under consideration. The case of generalized polyhedra is essentially more involved in comparison with usual convex polyhedral sets and requires developing elaborated techniques and new proofs of variational analysis.

Keywords: Variational analysis, reflexive Banach spaces, generalized polyhedral sets, parametric variational inequalities, robust Lipschitzian stability, generalized differentiation, coderivatives

1 Introduction

Parametric variational inequalities are among the most important objects in optimization theory and variational analysis; see, e.g., the books [2, 8, 19, 20, 22, 25] and the references therein. A breakthrough in their study and applications goes back to the seminal work by Robinson [23, 24] who treated them as parametric "generalized equations"

$$0 \in f(p,x) + N(x;\Theta) \quad \text{for all} \quad x \in \Theta, \tag{1.1}$$

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where $x \in X$ is the decision variable and $p \in Z$ is the parameter taking values in the corresponding Banach spaces. The "base" mapping $f\colon Z \times X \to X^*$ in (1.1) takes values in the dual space X^* while the set-valued "field" part $N\colon X \rightrightarrows X^*$ is the normal cone mapping to a convex set $\Theta \subset X$. By the classical definition of the normal cone in convex analysis with $N(x;\Omega) := \emptyset$ if $x \notin \Theta$, the generalized equation form (1.1) is equivalent to the standard form of variational inequalities: for each $p \in Z$ find $x \in \Theta$ such that

$$\langle f(p,x), x-u\rangle \le 0 \quad \text{whenever} \quad u \in \Theta.$$

It has been well recognized that the generalized equation formalism (1.1) is a convenient model to describe parametric complementarity problems, moving sets of optimal solutions to various optimization and equilibrium problems, KKT systems, and the like; see the references above with the bibliographies therein.

Consider the *solution map* $S\colon Z \rightrightarrows X$ to the parametric variational inequality/generalized equation (1.1) defined by

$$S(p) := \big\{x \in X\big|\ 0 \in f(p,x) + N(x;\Theta)\big\}. \tag{1.2}$$

The dependence of (1.2) on the parameter variable $p \in Z$ is one of the major issues from the viewpoints of sensitivity and stability analysis of the variational systems under consideration and their applications to parametric and hierarchical optimization, mathematical programs with equilibrium constraints, etc. *Robust Lipschitzian behavior* (i.e., stable with respect to perturbations of the initial data) of the solution map (1.2) and its *quantitative* characteristics are among the most important goals to achieve.

Advanced variational analysis and generalized differentiation offer verifiable pointwise characterizations of such behavior around reference points with computing the exact Lipschitzian moduli via the so-called *coderivatives* of general set-valued mappings; see [19, 20, 25] and Section 2 for more details. However, implementations of these criteria and their realizations in terms of the initial data of variational systems of type (1.2) is definitely not an easy job in both finite and infinite dimensions, where the latter case creates additional serious complications due to the lack of compactness.

A remarkable class of convex sets is described by *convex polyhedra*

$$\Theta := \big\{x \in X\big|\ \langle x_i^*, x\rangle \le c_i \ \text{ for } \ i = 1,\ldots,m\big\}, \tag{1.3}$$

where $x_i^* \in X^*$ are fixed elements. Significant progress in the study and applications of Lipschitzian stability for parametric variational inequalities (1.1) over polyhedral convex sets (1.3) has been achieved on the base of coderivative characterizations mainly in finite dimensions [6, 12, 13, 28, 29] and quite recently in reflexive Banach spaces [11, 21].

The major attention of this paper is paid to robust Lipschitzian stability of parametric variational inequalities over the so-called *generalized polyhedral sets* defined by

$$\Theta := \big\{x \in X\big|\ Ax = b \ \text{ and } \ \langle x_i^*, x\rangle \le c_i, \ \text{ for } \ i = 1,\ldots,m\big\} \tag{1.4}$$

and formed by fixed elements $x_i^* \in X^*$, $b \in Y$, $c_i \in I\!R$ and a linear bounded operator $A\colon X \to Y$ from X to another Banach space Y.

In contrast to the case of finite-dimensional spaces, the generalized polyhedra (1.4) do not reduce to the usual ones (1.3) in infinite dimensions. The "generalized polyhedral" terminology has been coined in [2], where systems (1.4) were largely investigated from the viewpoint of applications to the generalized linear programs

$$\text{minimize } \langle a, x\rangle \text{ subject to } Ax = b \text{ and } \langle x_i^*, x\rangle \le c_i \text{ for } i = 1, \ldots, m\} \tag{1.5}$$

as well as to problems of concave minimization under generalized polyhedral constraints (1.4). We refer the reader to the book [1] for the study of (generalized) linear programs in infinite dimensions and their applications to problems in approximation theory, mass-transfer, optimal control, dynamic network, and semi-infinite and infinite programming. The book [10] is particularly devoted to linear semi-infinite programming, while the recent papers [4, 5] concern robust stability issues and optimality conditions for semi-infinite and infinite programs with linear inequality constraints.

Among important classes of infinite-dimensional problems that can be written in the general polyhedral form (1.5) but not with merely polyhedral constraints (1.3) we mention discrete-time Markov decision processes with discounted cost, deterministic continuous-time control problems and those of singular stochastic control, problems related to Mather's variational principle, etc.; see, e.g., [7, 14, 15, 16, 27] for more details and references.

The main goal of this paper is to obtain complete characterizations of the *Lipschitz-like property* (known also as the Aubin property) of solution maps (1.2) to parametric variational inequalities (1.1) over generalized polyhedral sets Θ from (1.4) entirely in terms of the initial data f, A, b, x_i^*, and c_i. The Lipschitz-like/Aubin property has been well recognized in nonlinear analysis as the most natural extension of the classical local Lipschitz continuity to the case of set-valued mappings, with a localization around the reference point of the graph. This property usually accumulates the amount of robust stability needed for the analysis of constraint and variational systems; see [19, 20, 25] and the references therein.

Similarly to [11] the approach of this paper is based on implementing the coderivative characterizations [19] of the Lipschitz-like property for general set-valued mappings between infinite-dimensional spaces to the case of the solution map (1.2) generated by Θ from (1.4) instead of that from (1.3) as in [11]. It occurs however that the case of generalized polyhedra is significantly more involved and requires essential elaborations, which are done below. Furthermore, some of the results obtained in this paper are new even in the case of usual convex polyhedra in finite and infinite dimensions.

The rest of the paper is organized as follows. Section 2 collects preliminaries from variational analysis and generalized differentiations widely used in the sequel.

Sections 3 and 4 are devoted to technical issues of generalized differentiation of undoubted independent interest, which is crucial for employing the coderivative characterizations of robust stability. Namely, in Section 3 we compute the so-called precoderivative of the normal cone mapping $N(\cdot;\Theta)$ over the generalized polyhedron Θ from (1.4) in reflexive Banach spaces. This serves as a building block for computing the coderivative of the mapping $N(\cdot;\Theta)$ by a limiting procedure.

Section 5 contains the main results of the paper on complete characterizing the Lipschitz-like property of the solution map (1.2) to the underlying variational inequality (1.1) over the generalized polyhedron (1.4) in reflexive Banach spaces. We not only derive necessary and sufficient conditions for this property but also compute the exact bound of Lipschitzian moduli, which provides the most important qualitative characteristics of robust stability entirely in terms of the initial data of the variational system (1.2), (1.4) under consideration.

2 Preliminaries from Variational Analysis

Let us start with basic definitions, notation, and terminology conventional in variational analysis and generalized differentiation; see, e.g., [19, 25, 26]. Unless otherwise stated, every Banach spaces in question is assumed to be *reflexive* with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between the space X and its topological dual X^*. Note that a number of the results below hold (as follows from their proofs) in arbitrary Banach spaces or for the class of Asplund spaces, which contains reflexive ones. But it is more convenient for us to keep the reflexivity assumption overall for definiteness.

As usual, $B(X)$ stands for the closed unit balls of X, and the symbol $x_k^* \xrightarrow{w} x^*$ with $k \in I\!N := \{1, 2, \ldots\}$ indicates the *weak* convergence of a sequence in X^*. By

$$K^* := \big\{x^* \in X^* \big|\ \langle x^*, x\rangle \le 0 \ \text{ for all } \ x \in K\big\}$$

we denote the *polar* to a cone $K \subset X$ and by

$$\ker\big\{v_j^* \big|\ j \in J\big\} := \big\{x \in X \big|\ \langle v_j^*, x\rangle = 0 \ \text{ as } \ j \in J\big\}$$

the *kernel/orthogonality* subspace generated by the elements $v_j^* \in X^*$ as $j \in J$. This notation is in agreement with the *kernel* $\ker A := \{x \in X|\ Ax = 0\}$ of a linear operator $A\colon X \to Y$. In the case of just one generating element $v^* \in X^*$ we also use the notation

$$\{v^*\}^\perp := \big\{x \in X \big|\ \langle v^*, x\rangle = 0\big\}.$$

This is in agreement with the *orthogonality* notation for a liner subspace $L \subset X$, that is, $L^\perp := \{x^* \in X^*|\ \langle x^*, x\rangle = 0 \ \text{ for all } \ x \in L\}$.

Recall further that $\operatorname{span}\Omega$ stands for the *span* of a nonempty subset $\Omega \subset X$, i.e., the smallest linear subspace containing Ω and that $\operatorname{cone}\Omega$ signifies the *convex conic hull* of Ω; by convention we put $\operatorname{span}\emptyset := \{0\}$ and $\operatorname{cone}\emptyset := \{0\}$. For convenience, denote $\operatorname{cone}\{x_1, \ldots, x_m\} := \operatorname{pos}\{x_1, \ldots, x_m\}$. As usual, $\operatorname{cl}\Omega$ stands for the *closure* of the set Ω.

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping between two Banach spaces with the *domain* $\operatorname{dom} F := \{x \in X \big|\ F(x) \ne \emptyset\}$ and the *graph* $\operatorname{gph} F := \{(x, y) \in X \times Y|\ y \in F(x)\}$. The (sequential) *Painlevú-Kuratowski upper/outer limit* of F as $x \to \bar{x}$ is

$$\mathop{\rm Lim\,sup}_{x \to \bar{x}} F(x) := \Big\{y \in Y \Big|\ \begin{array}{l} \exists \text{ sequences } \ x_k \to \bar{x},\ y_k \to y \ \text{ as } \ k \to \infty \\ \qquad\qquad \text{such that } \ y_k \in F(x_k) \ \text{ for all } \ k \in I\!N \end{array}\Big\}. \tag{2.1}$$

In this way the *Bouligand-Severi contingent cone* to a set $\Omega \subset X$ at $\bar{x} \in \Omega$ is defined by

$$T(\bar{x}; \Omega) := \mathop{\rm Lim\,sup}_{\lambda \downarrow 0} \frac{\Omega - \bar{x}}{\lambda}. \tag{2.2}$$

When Ω is convex, the contingent cone (2.2) reduces to

$$T(x;\Omega) = \mathrm{cl}\big[\mathrm{cone}\,(\Omega - x)\big].$$

In what follows we often consider set-valued mappings $F\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*. In this case the sequential Painlevé-Kuratowski outer limit (2.1) is always understood in the sense of the *weak* topology on $Y = X^*$.

Let us next define the notions of generalized normals to a nonempty set $\Omega \subset X$ used in the paper. Given $\bar{x} \in \Omega$ and putting $x \xrightarrow{\Omega} \bar{x}$ when $x \to \bar{x}$ with $x \in \Omega$, we say that

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\Big\} \tag{2.3}$$

is the *prenormal cone* (known also as the regular or Fréchet normal cone) to Ω at $\bar{x}$. Note that the set $\widehat{N}(\bar{x};\Omega)$ is convex and weakly closed in X^* and is contained in the polar $T^*(\bar{x};\Omega)$ to the the contingent cone (2.2). Furthermore, $\widehat{N}(\bar{x};\Omega) = T^*(\bar{x};\Omega)$ if either $X = I\!R^n$ or Ω is convex. In the latter case the cone $\widehat{N}(\bar{x};\Omega)$ reduces to the normal cone of convex analysis. However, for nonconvex sets Ω the prenormal cone (2.3) does not possess natural properties of normal cones even in simple finite-dimensional settings. In particular, we often have $\widehat{N}(\bar{x};\Omega) = \{0\}$ for boundary points of sets (e.g., for $\Omega = \{(u,v) \in I\!R^2 |\ v \ge -|u|\}$ at the origin), and the cone (2.3) does not satisfy required calculus rules.

The situation dramatically changes if we consider the sequential regularization of the mapping $\widehat{N}(\cdot;\Omega)\colon X \rightrightarrows X^*$ by using the outer limit (2.1) in the weak topology of X^*:

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(\bar{x};\Omega) \tag{2.4}$$

and arrive at the construction known as the (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x} \in \Omega$; see [19, 20] and also [3, 25, 26] for more details and references in both finite and infinite dimensions. In spite of the intrinsic nonconvexity of the set of limiting normals (2.4), the normal cone $N(\bar{x};\Omega)$ and related subdifferential and coderivative constructions for functions and mappings satisfy comprehensive calculus rules and other required properties in the reflexive Banach space setting under consideration (as well as in more generality), which are mainly based on the variational/extremal principles of variational analysis.

Given now a set-valued mapping $F\colon X \rightrightarrows Y$ and following the pattern initiated in [17], we define two "adjoint derivative–coderivative" constructions via generalized normals to the graph of F. The *precoderivative* (known also as the Fréchet coderivative) of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$ is a positively homogeneous mapping $\widehat{D}^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ with the values

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}\big((\bar{x},\bar{y});\mathrm{gph}\,F\big)\big\}, \quad y^* \in Y^*, \tag{2.5}$$

while the (normal, limiting, Mordukhovich) *coderivative* of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$ is given by

$$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N\big((\bar{x},\bar{y});\mathrm{gph}\,F\big)\big\}, \quad y^* \in Y^*. \tag{2.6}$$

If $F = f\colon X \to Y$ is single-valued and *strictly differentiable* at $\bar{x}$ with the derivative $\nabla f(\bar{x})\colon X \to Y$ in the sense that

$$\lim_{x,u \to \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - \bar{x}\rangle}{\|x - u\|} = 0$$

(this is automatic when f is C^1 around $\bar{x}$), then we have

$$\widehat{D}^*f(\bar{x})(y^*) = D^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^*y^*\} \quad \text{for all} \quad y^* \in Y^*, \tag{2.7}$$

where $\bar{y} = f(\bar{x})$ is omitted in the coderivative notation for single-valued mappings. The coderivative representations in (2.7) show that both constructions (2.5) and (2.6) reduce to the *adjoint* derivative operator in the classical setting.

It follows from the above definitions that the coderivative (2.6) admits the representation

$$D^*F(\bar{x},\bar{y})(y^*) = \mathop{\mathrm{Lim\,sup}}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ z^*\xrightarrow{w}y^*}} \widehat{D}^*F(x,y)(z^*), \quad y^* \in Y^*, \tag{2.8}$$

via the outer limit (2.1) with respect to the weak topology in both dual spaces X^* and Y^*. As in [19], we say that F is (strongly) *coderivatively normal* at $(\bar{x},\bar{y})$ if

$$D^*F(\bar{x},\bar{y})(y^*) = \mathop{\mathrm{Lim\,sup}}_{\substack{(x,y)\to(\bar{x},\bar{y})\\ \|z^*-y^*\|\to 0}} \widehat{D}^*F(x,y)(z^*), \tag{2.9}$$

which means that the coderivative construction (2.6) does not change if we replace the weak convergence $z^* \xrightarrow{w} y^*$ in (2.8) by the norm one $z^* \to y^*$ in (2.9), while the convergence on X^* in (2.9) stays weak by (2.1).

Another definition needed in what follows is due to [9]: a set $\Omega \subset X$ is *dually norm-stable* at $\bar{x} \in \Omega$ if the basic normal cone (2.4) admits the representation

$$N(\bar{x};\Omega) = \{x^* \in X^* \mid \exists x_k \to \bar{x},\ x_k^* \in \widehat{N}(x_k;\Omega) \ \text{with} \ \|x_k^* - x^*\| \to 0 \ \text{as} \ k \to \infty\}. \tag{2.10}$$

Observe that the latter property obviously holds if either $X = I\!R^n$ or $N(\bar{x};\Omega) = \widehat{N}(\bar{x};\Omega)$, which is automatic when Ω is convex. Being applied to graphical sets, the dual norm-stability (2.10) surely yields the coderivative normality (2.9) of set-valued mappings.

Recall further a certain "normal compactness" property of set-valued mappings that is needed for characterizing robust Lipschitzian stability in infinite dimensions. A mapping $F\colon X \rightrightarrows Y$ is *partially sequentially normally compact* (PSNC) at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if for any sequence $\{(x_k,y_k,x_k^*,y_k^*)\} \subset X \times Y \times X^* \times Y^*$ satisfying $(x_k^*,y_k^*) \in \widehat{N}((x_k,y_k);\operatorname{gph} F)$ for all $k \in I\!N$ we have the implication

$$\big[(x_k,y_k) \to (\bar{x},\bar{y}),\ x_k^* \xrightarrow{w} 0,\ \|y_k^*\| \to 0\big] \Longrightarrow \|x_k^*\| \to 0 \text{ as } k \to \infty. \tag{2.11}$$

The PSNC property obviously holds if the domain space X is finite-dimensional. It is important to mention that F is PSNC at $(\bar{x},\bar{y})$ if it is *Lipschitz-like* around this point, i.e., there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + l\|x-u\|B(Y) \quad \text{whenever} \quad x,u \in U \tag{2.12}$$

with some constant/modulus $l \geq 0$. The infimum of all moduli $\{l\}$ in (2.12) is called the *exact Lipschitzian bound* of F around $(\bar{x},\bar{y})$ and is denoted by $\operatorname{lip}F(\bar{x},\bar{y})$. Note that property (2.12) is also known as Aubin's "pseudo-Lipschitzian" property and reduces to the Hausdorff one around $\bar{x}$ for $V = Y$ in (2.12). Furthermore, the Lipschitz-like property of an arbitrary mapping F between Banach spaces around $(\bar{x}, \bar{y})$ is equivalent to the fundamental properties of metric regularity and linear openness of the inverse mapping F^{-1} around $(\bar{y}, \bar{x})$; see [3, 19, 25] for more details, discussions, and references.

The following *coderivative characterization* of the Lipschitz-like property as well as a lower estimate and precise formula for computing the *exact bound* of Lipschitzian moduli are consequences of [19, Theorem 4.10]. In finite dimensions Theorem 2.1 reduced to [18, Theorem 5.7] and [25, Theorem 9.40] named in the latter as the Mordukhovich criterion.

Theorem 2.1 (coderivative characterization and exact bound formula for Lipschitz-like mappings). *Let $F\colon X \rightrightarrows Y$ be closed-graph around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and coderivatively normal at this point. Then F is Lipschitz-like around $(\bar{x}, \bar{y})$ if and only if*

$$D^*F(\bar{x}, \bar{y})(0) = \{0\} \tag{2.13}$$

and F is PSNC at $(\bar{x}, \bar{y})$. Furthermore, we have the estimate

$$\operatorname{lip} F(\bar{x}, \bar{y}) \geq \|D^*F(\bar{x}, \bar{y})\| := \sup\big\{\|x^*\| \;\big|\; x^* \in D^*F(\bar{x}, \bar{y})(y^*),\ \|y^*\| \leq 1\big\}, \tag{2.14}$$

which holds as equality if $\dim X < \infty$.

Finally in this section, we present a generalized Farkas lemma, which is taken from [2, Proposition 1.201] and widely employed in the paper being different from that used in [11].

Theorem 2.2 (generalized Farkas lemma). *Let X and Y be Banach spaces, let $a_i \in X^*$ for $i = 1, \ldots, p$, and let $A\colon X \to Y$ be a linear continuous operator of closed range. Then the polar to the cone*

$$K := \big\{x \in X \big|\ Ax = 0,\ \langle a_i, x\rangle \leq 0 \ \text{ for }\ i = 1, \ldots, p\big\}$$

can be equivalently written in the form

$$K^* = A^*(Y^*) + \Big\{\sum_{i=1}^{p} \lambda_i a_i \Big|\ \lambda_i \geq 0,\ i = 1, \ldots, p\Big\}.$$

3 Precoderivatives of Normal Cone Mappings to Generalized Polyhedra in Infinite Dimensions

In this section we start studying the *normal cone mapping* $\mathcal{F}\colon X \rightrightarrows X^*$ defined by

$$\mathcal{F}(x) := N(x; \Theta), \quad x \in X, \tag{3.1}$$

which is a significant component of describing the underlying solution map (1.2) to the parametric variational inequality (1.1). Indeed, we have

$$\operatorname{gph} S = \big\{(p, x) \in Z \times \Theta \big|\ -f(p, x) \in N(x; \Theta)\big\},$$

which can be equivalently written in the forms

$$\operatorname{gph} S = \big\{(p, x) \in Z \times \Theta \big|\ g(p, x) \in \operatorname{gph} \mathcal{F}\big\} = g^{-1}(\operatorname{gph} \mathcal{F}) \tag{3.2}$$

via the mapping $g\colon Z \times X \to X \times X^*$ defined by

$$g(p,x) := \big(x, -f(p,x)\big) \quad \text{for} \quad p \in Z \quad \text{and} \quad x \in X. \tag{3.3}$$

In what follows we consider the normal cone mapping (3.1) when Θ is the generalized polyhedron (1.4), which is now written for convenience as

$$\Theta = \big\{x \in X \big|\, Ax = b \quad \text{and} \quad \langle x_i^*, x\rangle \le c_i \quad \text{for} \quad i \in I\big\}, \tag{3.4}$$

where I is any given finite index set. We always assume that the range of A in (3.4) is *closed*, although a number of the results below hold without this assumption; see their proofs.

The major goal of this section is to efficiently compute the prenormal cone (2.3) to the graph of (3.1) with the generalized polyhedron (3.4) therein and hence the precoderivative (2.5) of this normal cone mapping entirely in terms of the initial data of Θ. The results obtained below significantly extend those from [11] derived for convex polyhedra (1.3) by developing an advanced technique of its own interest, which is new even for standard convex polyhedra in both finite and infinite dimensions.

To proceed, consider for fixed $\bar{x} \in \Theta$ the collection of *active constraint indices*

$$I(\bar{x}) := \big\{i \in I \big|\, \langle x_i^*, \bar{x}\rangle = c_i\big\}. \tag{3.5}$$

Recall that a *face* of a convex set $C \subset X$ is a convex subset M of C such that: if x_1 and x_2 belong to C and $\lambda x_1 + (1-\lambda)x_2 \in M$ for some $\lambda \in (0,1)$, then x_1 and x_2 actually belong to M. Both the empty set $\emptyset$ and C itself belong to the collection of faces of C. We denote the collection of all nonempty faces of C by $\mathcal{M}(C)$.

For any nonempty face M of Θ given in (3.4) there is a *maximal index subset* I_M of I such that $\langle x_i^*, x\rangle = c_i$ for every point $x \in M$ and each $i \in I_M$. We call this I_M the *active index set* associated with the face M of Θ and denote all the active index sets associated with nonempty faces of Θ by $\mathfrak{I}(\Theta) := \{I_M \mid M \in \mathcal{M}(\Theta)\}$. Note that I_M depends not only on the set Θ and the face M, but also on the representation of Θ because of the possible multi-representability of the set Θ. With no further mentioning we always refer to the active index set for a face of a generalized polyhedron in its given representation. For distinct nonempty faces of Θ their active index sets are distinct as well, while there is a one-to-one correspondence between $\mathcal{M}(\Theta)$ and $\mathfrak{I}(\Theta)$.

It is not hard to observe that a nonempty face M of the generalized polyhedron Θ in (3.4) admits the representation

$$M = \big\{x \in X \big|\, Ax = b,\ \langle x_i^*, x\rangle \le c_i \quad \text{for} \quad i \in I \setminus I_M \quad \text{and} \quad \langle x_j^*, x\rangle = c_j \quad \text{for} \quad j \in I_M\big\}. \tag{3.6}$$

Furthermore, it follows from the definition of active index set associated with M that for each $i \in I\setminus I_M$ there is a point $\bar{x} \in M$ such that $\langle x_i^*, \bar{x}\rangle < c_i$. We also have the representation

$$\operatorname{aff} M = \big\{x \in X \big|\, Ax = b,\ \langle x_i^*, x\rangle = c_i \quad \text{for} \quad i \in I_M\big\}$$

of the *affine hull* of M and the one

$$\operatorname{ri} M = \big\{x \in X \big|\, Ax = b,\ \langle x_i^*, x\rangle < c_i \quad \text{for} \quad i \in I \setminus I_M \quad \text{and} \quad \langle x_j^*, x\rangle = c_j \quad \text{for} \quad j \in I_M\big\} \tag{3.7}$$

of the *relative interior* of M. Taking into account definition (3.5) of the active constraint indices, observe that $I(x) = I_M$ whenever $x \in \mathrm{ri}\, M$.

It follows from the above that any nonempty face M of the generalized polyhedron Θ in (3.4) is a generalized polyhedral set itself with $\mathrm{ri}\, M \neq \emptyset$. Given $\bar{x} \in \Theta$, the set

$$M = \{x \in X \big| \, Ax = b, \ \langle x_i^*, x\rangle \le c_i \ \text{ for } \ i \in I \setminus I(\bar{x}) \ \text{ and } \ \langle x_j^*, x\rangle = c_j \ \text{ for } \ j \in I(\bar{x})\}$$

is a nonempty face of Θ, with the active index set $I_M = I(\bar{x})$ for $\bar{x} \in \mathrm{ri}\, M$. Therefore, the generalized polyhedron (3.4) is the union of the relative interiors of its nonempty faces, i.e.,

$$\Theta = \bigcup_{M \in \mathcal{M}(\Theta)} M = \bigcup_{M \in \mathcal{M}(\Theta)} \mathrm{ri}\, M. \tag{3.8}$$

Note also that distinct nonempty faces of Θ have no intersection of their relative interiors. Thus all the relative interiors of nonempty faces of Θ form a partition of Θ.

Next we present simple while useful in what follows relationships for contingent (2.2) and prenormal (2.3) cones to finite unions of sets valid in general Banach spaces X.

Proposition 3.1 (contingent and prenormal cones to set unions). *Let $\Lambda \subset X$ be the union of finitely many closed sets $\Lambda_i \neq \emptyset$ as $i \in I$. Given any $\bar{x} \in \Lambda$, define the index set $\widehat{I}(\bar{x}) := \{i \in I \mid \bar{x} \in \Lambda_i\}$. Then we have the relationships*

$$T(\bar{x}; \Lambda) = \bigcup_{i \in \widehat{I}(\bar{x})} T(\bar{x}; \Lambda_i), \tag{3.9}$$

$$\widehat{N}(\bar{x}; \Lambda) \supset \bigcap_{i \in \widehat{I}(\bar{x})} \widehat{N}(\bar{x}; \Lambda_i). \tag{3.10}$$

Furthermore, (3.10) *holds as equality if all the sets Λ_i are convex.*

Proof. The inclusion "$\supset$" in (3.9) obvious follows from definition (2.2) of the contingent cone. To justify the opposite inclusion in (3.9), pick any $h \in T(\bar{x}; \Lambda)$ and get from (2.2) sequences $t_k \downarrow 0$ and $h_k \to h$ as $k \to \infty$ such that

$$\bar{x} + t_k h_k \in \bigcup_{i \in I} \Lambda_i \ \text{ for all } \ k \in I\!N.$$

Taking into account that I is a finite index set, we can suppose by passing to a subsequence if necessary that $\bar{x} + t_k h_k \in \Lambda_i$ for some $i \in I$ and all $k \in I\!N$. Furthermore, $\bar{x} \in \Lambda_i$ due to closedness of all Λ_i. This implies that $i \in \widehat{I}(\bar{x})$ and $h \in T(\bar{x}; \Lambda_i) \subset \bigcup_{i \in \widehat{I}(\bar{x})} T(\bar{x}; \Lambda_i)$, which completes the proof of equality (3.9).

To justify inclusion (3.10), assume the contrary and by definition (2.3) of the prenormal cone find a dual element $x^* \in \bigcap_{i \in \widehat{I}(\bar{x})} \widehat{N}(\bar{x}; \Lambda_i)$, a positive number γ, and a sequence $x_k \to \bar{x}$ as $k \to \infty$ with $x_k \in \Lambda$ such that

$$\frac{\langle x^*, x_k - \bar{x}\rangle}{\|x_k - \bar{x}\|} > \gamma \tag{3.11}$$

for all $k \in I\!N$ sufficiently large. Since I is a finite index set, we can assume by passing to a subsequence if necessary that $x_k \in \Lambda_{i_0}$ for some $i_0 \in I$ and for all $k \in I\!N$. The closedness of Λ_{i_0} yields $\bar{x} \in \Lambda_{i_0}$. By (3.11) the latter implies that $x^* \notin \widehat{N}(\bar{x}; \Lambda_{i_0})$, a contradiction ensuring the fulfillment of inclusion (3.10).

It remains to justify the opposite inclusion "$\subset$" in (3.10) provided that all the sets Λ_i are convex. We have in general that $\widehat{N}(\bar{x}; \Lambda) \subset T(\bar{x}; \Lambda)^*$ and that the polar to the union of sets is the intersection of the polars to all of the sets. Thus

$$\widehat{N}(\bar{x}; \Lambda) \subset \bigcap_{i \in \widehat{I}(\bar{x})} T(\bar{x}; \Lambda_i)^*.$$

Combining the latter with the relationship $T(\bar{x}; \Lambda_i)^* = \widehat{N}(\bar{x}; \Lambda_i)$ in the convex case, we arrive at the equality in (3.10) and complete the proof of the proposition. $\triangle$

The next proposition gives a convenient representation of the normal cone mapping $N(\cdot; \Theta)$ on the relative interior of a given face of Θ.

Proposition 3.2 (normal cone mappings on faces of generalized polyhedra). *Let Θ be the generalized polyhedron defined in (3.4), and let $M \in \mathcal{M}(\Theta)$ be its face with the active index subset I_M. Then the normal cone mapping $N(\cdot; \Theta)$ is constant on the relative interior of M. Denoting the latter value by $N(\mathrm{ri}\, M; \Theta)$, we have the representation*

$$N(\mathrm{ri}\, M; \Theta) = A^*(Y^*) + \mathrm{pos}\big\{x_i^* \big|\, i \in I_M\big\}. \tag{3.12}$$

Proof. Let us show first the contingent cone (2.2) to Θ admits the representation

$$T(\bar{x}; \Theta) = \big\{x \in X \big|\, Ax = 0,\ \langle x_i^*, x\rangle \le 0 \ \text{ as } \ i \in I(\bar{x})\big\} \ \text{ for any } \ \bar{x} \in \Theta. \tag{3.13}$$

Note that the convexity of Θ yields $T(\bar{x}; \Theta) = \mathrm{cl}\,[\mathrm{cone}(\Theta - \bar{x})]$. Piking now any $h \in X$ from the right-hand side of (3.13) gives us

$$Ah = 0 \ \text{ and } \ \langle x_i^*, h\rangle \le 0 \ \text{ whenever } \ i \in I(\bar{x}).$$

Since $\bar{x} \in \Theta$, we have by (3.4) and definition (3.5) of the active constraint indices that

$$A\bar{x} = b,\ \langle x_i^*, \bar{x}\rangle = c_i \ \text{ for } \ i \in I(\bar{x}), \ \text{ and } \ \langle x_i^*, \bar{x}\rangle < c_i \ \text{ for } \ i \in I \setminus I(\bar{x}).$$

This implies, for all $\lambda > 0$ sufficiently small, that

$$A(\bar{x} + \lambda h) = b,\ \langle x_i^*, \bar{x} + \lambda h\rangle \le c_i \ \text{ as } \ i \in I(\bar{x}), \ \text{ and } \ \langle x_i^*, \bar{x} + \lambda h\rangle < c_i \ \text{ as } \ i \in I \setminus I(\bar{x}),$$

which gives $\bar{x} + \lambda h \in \Theta$ and thus justifies the inclusion "$\supset$" in (3.13).

To prove the opposite inclusion "$\subset$" in (3.13), take any $h \in T(\bar{x}; \Theta)$ and find by definition (2.2) of the contingent cone sequences $t_k \downarrow 0$ and $h_k \to h$ such that $\bar{x} + t_k h_k \in \Theta$ whenever $k \in I\!N$, which means that

$$A(\bar{x} + t_k h_k) = b \ \text{ and } \ \langle x_i^*, \bar{x} + t_k h_k\rangle \le c_i \ \text{ for } \ i \in I.$$

The latter implies the relationships

$$A(t_k h_k) = 0 \quad \text{and} \quad \langle x_i^*, t_k h_k \rangle \le 0 \quad \text{for} \quad i \in I(\bar{x}), \qquad k \in I\!N.$$

Thus we have by passing to the limit as $k \to \infty$ that $Ah = 0$ and $\langle x_i^*, h\rangle \le 0$ for all $i \in I(\bar{x})$, which justifies the equality in (3.13).

The polarity correspondence $N(\bar{x};\Theta) = T(\bar{x};\Theta)^*$ held due to the convexity of Θ and the established contingent cone representation (3.13) allows us to conclude by the generalized Farkas lemma of Theorem 2.2 that

$$\begin{aligned} N(\bar{x};\Theta) &= A^*(Y^*) + \Big\{ \sum_{i \in I(\bar{x})} \lambda_i x_i^* \Big| \, \lambda_i \ge 0 \Big\} \\ &= A^*(Y^*) + \operatorname{pos}\big\{ x_i^* \big| \, i \in I(\bar{x}) \big\}. \end{aligned}$$

Noting finally that $I(\bar{x}) = I_M$ for each $\bar{x} \in \operatorname{ri} M$ as shown above, we arrive at (3.12) and thus complete the proof of the proposition. $\triangle$

Let us present a useful consequence of Proposition 3.2 employed in what follows.

Corollary 3.3 (monotonicity relationships for faces of generalized polyhedra). *Let $I_1, I_2 \in \mathfrak{I}(\Theta)$ for the generalized polyhedron (3.4), and let M_1 and M_2 be nonempty faces of Θ. Denoting $N_i := N(\operatorname{ri} M_i;\Theta)$ for $i = 1,2$, we have*

$$M_1 \supset M_2 \Longleftrightarrow I_1 \subset I_2 \Longleftrightarrow N_1 \subset N_2.$$

Proof. Directly follows from the explicit normal cone representation (3.12). $\triangle$

Note also that for any face $M \subset \Theta$ we have the relationships

$$N(\bar{x};\Theta) = N(\operatorname{ri} M;\Theta) \subset N(x;\Theta) \quad \text{whenever} \quad \bar{x} \in \operatorname{ri} M \quad \text{and} \quad x \in M. \tag{3.14}$$

Indeed, the equality in (3.14) is proved in Proposition 3.2. To check the inclusion therein, take any $x \in M \setminus \operatorname{ri} M$ and find $\widetilde{M} \in \mathcal{M}(\Theta)$ such that $\widetilde{M} \subset M$ with $x \in \operatorname{ri} \widetilde{M}$. Then we get from Corollary 3.3 and Proposition 3.2 again that $I_M \subset I_{\widetilde{M}} = I(x)$ and that

$$N(\operatorname{ri} M;\Theta) \subset N(\operatorname{ri} \widetilde{M};\Theta) = N(x;\Theta),$$

which justifies the second relationship in (3.14).

Next we derive an exact representation of the normal cone to a face of the generalized polyhedron under consideration in terms of its initial data.

Proposition 3.4 (representation of normals to faces of generalized polyhedra). *Let $M \subset \Theta$ be an nonempty face of the generalized polyhedron (3.4), and let $\bar{x} \in M$. Then we have the normal cone representation*

$$N(\bar{x}; M) = A^*(Y^*) + \operatorname{pos}\big\{ x_i^* \big| \, i \in I(\bar{x}) \setminus I_M \big\} + \operatorname{span}\big\{ x_j^* \big| \, j \in I_M \big\}. \tag{3.15}$$

Proof. As shown above, the given face M of the generalized polyhedron (3.4) is a generalized polyhedron itself, which admits representation (3.6). Denote

$$\widetilde{A} := \begin{pmatrix} A \\ B \end{pmatrix}, \ \widetilde{b} := \begin{pmatrix} b \\ c \end{pmatrix}, \quad \text{and} \quad \widetilde{I} := I \setminus I_M,$$

where B is a matrix with the rows x_j^* for $j \in I_M$, and where c is a vector with the components c_j for $j \in I_M$. Then the aforementioned representation of M can be rewritten as

$$M = \big\{x \in X \big|\ \widetilde{A}x = \widetilde{b} \ \text{ and } \ \langle x_i^*, x\rangle \le c_i \ \text{ as } \ i \in \widetilde{I}\big\}.$$

Denote further $\widetilde{I}(\bar{x}) := \{i \in \widetilde{I} \mid \langle x_i^*, \bar{x}\rangle = c_i\} = I(\bar{x}) \setminus I_M$ and consider the set

$$\widetilde{M} := \big\{x \in X \big|\ \widetilde{A}x = \widetilde{b}, \ \langle x_i^*, x\rangle \le c_i \ \text{ as } \ i \in \widetilde{I} \setminus \widetilde{I}(\bar{x}), \ \text{ and } \ \langle x_j^*, x\rangle = c_j \ \text{ as } \ j \in \widetilde{I}(\bar{x})\big\}.$$

Then we have from (3.6) and (3.7) that $\widetilde{M}$ is a face of M with $\bar{x} \in \operatorname{ri} \widetilde{M}$. Applying now Proposition 3.2 to the generalized polyhedron M and its face $\widetilde{M}$, we get

$$\begin{aligned} N(\bar{x}; M) &= \widetilde{A}^*(Y^*) + \operatorname{pos}\big\{x_i^* \big|\ i \in \widetilde{I}(\bar{x})\big\} \\ &= A^*(Y^*) + \operatorname{pos}\big\{x_i^* \big|\ i \in I(\bar{x}) \setminus I_M\big\} + \operatorname{span}\big\{x_j^* \big|\ j \in I_M\big\} \end{aligned}$$

and thus complete the proof of the proposition. $\triangle$

It makes sense to illustrate the usage of the precise formulas of Propositions 3.2 and 3.4 in particular settings. For simplicity we consider a convex polyhedron in $I\!R^3$.

Example 3.5 (computing normals to faces of convex polyhedra). Consider a convex polyhedral set Θ with the generating vectors

$$x_1^* = (1,1,1), \quad x_2^* = (-1,0,0), \quad x_3^* = (0,-1,0), \quad x_4^* = (0,0,-1)$$

and the numbers $c_1 = 1$, $c_2 = c_3 = c_4 = 0$ in (1.3). Thus

$$\Theta = \big\{x \in I\!R^3 \big|\ \langle x_i^*, x\rangle \le c_i \ \text{ with } \ i \in I = \{1,2,3,4\}\big\}. \tag{3.16}$$

Then all the active index sets are given by

$$\begin{aligned} \mathfrak{I}(\Theta) = \big\{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \\ &\{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}\big\} \end{aligned}$$

and the corresponding collection of nonempty faces is

$$\begin{aligned} \mathcal{M}(\Theta) = \big\{&\Theta, \triangle ABC, \triangle BOC, \triangle AOC, \triangle AOB, \overline{BC}, \overline{AC}, \overline{AB}, \overline{OC}, \\ &\overline{OB}, \overline{OA}, C, A, B, O\big\}. \end{aligned}$$

Here $A = (1,0,0), B = (0,1,0), C = (0,0,1), O = (0,0,0)$, the symbol $\triangle ABC$ stands for the triangle with the zeniths A, B, C while $\overline{AB}$ denotes the line segment between A and B.

Consider now the face

$$M = \triangle ABC = \big\{x \in I\!R^3 \big|\ \langle x_i^*, x\rangle \le c_i \ \text{ for } \ i \in \{2,3,4\} \ \text{ and } \ \langle x_1^*, x\rangle = c_1\big\}$$

with the corresponding index set $I_M = \{1\}$ and the relative interior

$$\mathrm{ri}\, M = \big\{x \in I\!R^3 \big|\ \langle x_i^*, x\rangle < c_i \ \text{ for } \ i \in \{2,3,4\} \ \text{ and } \ \langle x_1^*, x\rangle = c_1\big\}.$$

Select $\bar{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \mathrm{ri}\, M$ and observe that the active constraint index set at $\bar{x}$ is $I(\bar{x}) = \{1\}$ while the normal cones to Θ and M at $\bar{x}$ are, respectively,

$$N(\bar{x};\Theta) = \mathrm{pos}\big\{x_i^* \big|\ i \in I(\bar{x})\big\} = \mathrm{pos}\big\{x_1^*\big\},$$

$$N(\bar{x};M) = \mathrm{pos}\big\{x_i^* \big|\ i \in I(\bar{x}) \setminus I_M\big\} + \mathrm{span}\big\{x_j^* \big|\ j \in I_M\big\} = \mathrm{span}\big\{x_1^*\big\}.$$

For the point $\bar{z} = (0, \frac{1}{2}, \frac{1}{2})$ the active constraint index set is $I(\bar{z}) = \{1,2\}$ and the normal cones to Θ and M at $\bar{z}$ are, respectively,

$$N(\bar{z};\Theta) = \mathrm{pos}\big\{x_i^* \big|\ i \in I(\bar{z})\big\} = \mathrm{pos}\big\{x_1^*, x_2^*\big\},$$

$$N(\bar{z};M) = \mathrm{pos}\big\{x_i^* \big|\ i \in I(\bar{z}) \setminus I_M\big\} + \mathrm{span}\big\{x_j^* \big|\ j \in I_M\big\} = \mathrm{pos}\big\{x_2^*\big\} + \mathrm{span}\big\{x_1^*\big\}.$$

All this is in accordance with the results of Propositions 3.2 and 3.4.

Let us now employ the above "face" relationships to the study of the prenormal cone (2.3) to the normal cone mapping (3.1) induced by the generalized polyhedron (3.4). Fixing an arbitrary pair $(\bar{x}, \bar{x}^*) \in \mathrm{gph}\, \mathcal{F}$ and using the normal cone representation

$$N(\bar{x};\Theta) = A^*(Y^*) + \mathrm{pos}\big\{x_i^* \big|\ i \in I(\bar{x})\big\} \tag{3.17}$$

that follows from Theorem 2.2, for any $\bar{x}^* \in N(\bar{x};\Theta)$ we get

$$\bar{x}^* = A^*(\bar{y}^*) + \sum_{\substack{i \in I(\bar{x}) \\ \lambda_i \ge 0}} \lambda_i x_i^* \ \text{ with some } \ \bar{y}^* \in Y^*. \tag{3.18}$$

Denote the index set of the corresponding "positive multipliers " by

$$\mathcal{J}(\bar{x}, \bar{x}^*) := \big\{i \in I(\bar{x}) \big|\ \lambda_i > 0 \ \text{ in } (3.18)\big\} \tag{3.19}$$

and observe that the multipliers λ_i in representation (3.18) may not uniquely defined. However, it is not hard to check that all the subsequent constructions and results involving the index collection $\mathcal{J}(\bar{x}; \bar{x}^*)$ are invariant with respect to any choice of positive multipliers; see, e.g., the proof of Theorem 3.6.

To proceed with deriving a constructive representation of the prenormal cone to the graph of $\mathcal{F}$ in (3.1) entirely via the initial data of the generalized polyhedron (3.4), define similarly to [11] the following sets depending on indices $P \subset Q \subset I$ by

$$A_{Q,P} := \mathrm{pos}\big\{x_i^* \big|\ i \in Q \setminus P\big\} + \mathrm{span}\big\{x_j^* \big|\ j \in P\big\}, \tag{3.20}$$

$$B_{Q,P} := \big\{x \in X \big|\ \langle x_i^*, x\rangle \le 0 \ \text{ as } \ i \in Q \setminus P \ \text{ and } \ \langle x_j^*, x\rangle = 0 \ \text{ as } \ j \in P\big\}. \tag{3.21}$$

Now we are ready to establish the main result of this section.

Theorem 3.6 (computing the prenormal cone to graphs of normal cone mappings over generalized polyhedra). *Let* $(\bar{x},\bar{x}^*)\in\operatorname{gph}\mathcal{F}$ *for the normal cone mapping* (3.1) *over the generalized polyhedron* Θ *from* (3.4), *and let* $Q=I(\bar{x})$ *and* $P=\mathcal{J}(\bar{x},\bar{x}^*)$ *be the corresponding index sets. Then we have the prenormal cone representation*

$$\widehat{N}\big((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F}\big)=\big[A_{Q,P}+A^*(Y^*)\big]\times\big[B_{Q,P}\cap(\ker A)\big] \tag{3.22}$$

via the sets $A_{Q,P}$ *and* $B_{Q,P}$ *computed in* (3.20) *and* (3.21), *respectively.*

Proof. It follows from (3.8) that the graph of the normal cone mapping (3.1) generated by (3.4) admits the face representation:

$$\begin{aligned}\operatorname{gph}\mathcal{F}&=\bigcup\big\{\{x\}\times N(x;\Theta)\big|\;x\in\Theta\big\}\\&=\bigcup\big\{\operatorname{ri}M\times N(\operatorname{ri}M;\Theta)\big|\;M\in\mathcal{M}(\Theta)\big\}\\&=\bigcup\big\{M\times N(\operatorname{ri}M;\Theta)\big|\;M\in\mathcal{M}(\Theta)\big\}.\end{aligned} \tag{3.23}$$

Then we have from the normal cone equality in Proposition 3.1 and the convexity of the above sets M and $N(\operatorname{ri}M;\Theta)$ that

$$\widehat{N}\big((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F}\big)=\bigcap_{\substack{(\bar{x},\bar{x}^*)\in M\times N(\operatorname{ri}M;\Theta)\\ M\in\mathcal{M}(\Theta)}}N\big((\bar{x},\bar{x}^*);M\times N(\operatorname{ri}M;\Theta)\big). \tag{3.24}$$

Applying the product formula for normals in the right-hand side of (3.24) gives

$$\widehat{N}\big((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F}\big)=\Big[\bigcap_{\substack{(\bar{x},\bar{x}^*)\in M\times N(\operatorname{ri}M;\Theta)\\ M\in\mathcal{M}(\Theta)}}N(\bar{x};M)\Big]\times\Big[\bigcap_{\substack{(\bar{x},\bar{x}^*)\in M\times N(\operatorname{ri}M;\Theta)\\ M\in\mathcal{M}(\Theta)}}N\big(\bar{x}^*;N(\operatorname{ri}M;\Theta)\big)\Big].$$

By Proposition 3.2 and Theorem 2.2 we have

$$\begin{aligned}N(\operatorname{ri}M;\Theta)&=A^*(Y^*)+\operatorname{pos}\big\{x_i^*\big|\;i\in I_M\big\}\\&=\big\{x\in X\big|\;Ax=0\;\text{ and }\;\langle x_i^*,x\rangle\le 0\;\text{ as }\;i\in I_M\big\}^*,\end{aligned}$$

which is a closed and convex cone. This yields the representation

$$N\big(\bar{x}^*;N(\operatorname{ri}M;\Theta)\big)=\big\{x\in X\big|\;Ax=0\;\text{ and }\;\langle x_i^*,x\rangle\le 0\;\text{ as }\;i\in I_M\big\}\cap\big\{\bar{x}^*\big\}^\perp. \tag{3.25}$$

Consider further the two cases: **(a)** $\bar{x}\in\operatorname{ri}M$ and **(b)** $\bar{x}\notin\operatorname{ri}M$. Starting with (a), we have $I_M=I(\bar{x})$ and $\mathcal{J}(\bar{x},\bar{x}^*)\subset I_M$. Since $\bar{x}^*\in N(\bar{x};\Theta)$, it gives by (3.18) and (3.19) that

$$\bar{x}^*=A^*(y^*)+\sum_{i\in\mathcal{J}(\bar{x},\bar{x}^*)}\lambda_i x_i^*\;\text{ for some }\;y^*\in Y^*.$$

This allows us to deduce from (3.25) and definition (3.21) that

$$\begin{aligned}&N\big(\bar{x}^*;N(\operatorname{ri}M;\Theta)\big)\\=\;&\big\{x\in X\big|\;Ax=0,\;\langle x_i^*,x\rangle\le 0\;\text{ as }\;i\in I(\bar{x})\setminus\mathcal{J}(\bar{x},\bar{x}^*),\;\text{ and }\;\langle x_j^*,x\rangle=0\;\text{ as }\;j\in\mathcal{J}(\bar{x},\bar{x}^*)\big\}\\=\;&B_{I(\bar{x}),\mathcal{J}(\bar{x},\bar{x}^*)}\cap(\ker A).\end{aligned}$$

In case (b) we find a face $\widetilde{M} \in \mathcal{M}(\Theta)$ such that $\bar{x} \in \operatorname{ri} \widetilde{M} \subset M$ and $I_M \subset I_{\widetilde{M}} = I(\bar{x})$. This implies by Corollary 3.3 that $\bar{x}^* \in N(\operatorname{ri} M; \Theta) \subset N(\operatorname{ri} \widetilde{M}; \Theta)$, and hence

$$N\big(\bar{x}^*; N(\operatorname{ri} \widetilde{M}; \Theta)\big) \subset N\big(\bar{x}^*; N(\operatorname{ri} M; \Theta)\big).$$

Combining the relationships above, we arrive at the equality

$$\bigcap_{\substack{(\bar{x},\bar{x}^*) \in M \times N(\operatorname{ri} M;\Theta) \\ M \in \mathcal{M}(\Theta)}} N\big(\bar{x}^*; N(\operatorname{ri} M; \Theta)\big) = \bigcap_{\substack{(\bar{x},\bar{x}^*) \in M \times N(\operatorname{ri} M;\Theta) \\ \bar{x} \in \operatorname{ri} M,\ M \in \mathcal{M}(\Theta)}} \big[B_{I(\bar{x}),\mathcal{J}(\bar{x},\bar{x}^*)} \cap (\ker A)\big]. \tag{3.26}$$

Let us show next that for different representations of $\bar{x}^*$ in (3.18)—i.e., for different collections of positive multipliers $\mathcal{J}_1(\bar{x},\bar{x}^*)$ and $\mathcal{J}_2(\bar{x},\bar{x}^*)$ in (3.19)—the following equality

$$B_{I(\bar{x}),\mathcal{J}_1(\bar{x},\bar{x}^*)} \cap (\ker A) = B_{I(\bar{x}),\mathcal{J}_2(\bar{x},\bar{x}^*)} \cap (\ker A) \tag{3.27}$$

holds. Indeed, picking any $\mathcal{J}_1(\bar{x},\bar{x}^*)$, $\mathcal{J}_2(\bar{x},\bar{x}^*)$, and $x \in B_{I(\bar{x}),\mathcal{J}_1(\bar{x},\bar{x}^*)} \cap (\ker A)$, we get by definitions (3.19) and (3.21) that

$$\bar{x}^* = A^* y_2^* + \sum_{j \in \mathcal{J}_2(\bar{x},\bar{x}^*)} \mu_j x_j^* = A^* y_1^* + \sum_{i \in \mathcal{J}_1(\bar{x},\bar{x}^*)} \lambda_i x_i^*$$

for some $y_1^*, y_2^* \in Y^*$ and that

$$\langle \bar{x}^*, x \rangle = \langle y_2^*, Ax \rangle + \sum_{j \in \mathcal{J}_2(\bar{x},\bar{x}^*)} \mu_j \langle x_j^*, x \rangle = \langle y_1^*, Ax \rangle + \sum_{i \in \mathcal{J}_1(\bar{x},\bar{x}^*)} \lambda_i \langle x_i^*, x \rangle = 0.$$

The latter implies in turn that

$$\sum_{j \in \mathcal{J}_2(\bar{x},\bar{x}^*)} \mu_j \langle x_j^*, x \rangle = 0.$$

This yields that $\langle x_j^*, x \rangle = 0$ for all $j \in \mathcal{J}_2(\bar{x},\bar{x}^*)$, since $\langle x_j^*, x \rangle \le 0$ and $\mu_j > 0$ whenever $j \in \mathcal{J}_2(\bar{x},\bar{x}^*) \subset I(\bar{x})$. Hence

$$B_{I(\bar{x}),\mathcal{J}_1(\bar{x},\bar{x}^*)} \cap (\ker A) \subset B_{I(\bar{x}),\mathcal{J}_2(\bar{x},\bar{x}^*)} \cap (\ker A),$$

which ensures the equality in (3.27) due to the arbitrary choice of $\mathcal{J}_1(\bar{x},\bar{x}^*)$ and $\mathcal{J}_2(\bar{x},\bar{x}^*)$.

It follows thus from (3.26) and (3.27) that

$$\bigcap_{\substack{(\bar{x},\bar{x}^*) \in M \times N(\operatorname{ri} M;\Theta) \\ M \in \mathcal{M}(\Theta)}} N\big(\bar{x}^*; N(\operatorname{ri} M; \Theta)\big) = B_{I(\bar{x}),\mathcal{J}(\bar{x},\bar{x}^*)} \cap (\ker A). \tag{3.28}$$

Applying now Propositions 3.2 and 3.4, for every $M \in \mathcal{M}(\Theta)$ with $(\bar{x},\bar{x}^*) \in M \times N(\operatorname{ri} M; \Theta)$ we have the relationships

$$\begin{aligned} N(\bar{x}; M) &= A^*(Y^*) + \operatorname{pos}\{x_i^* \mid i \in I(\bar{x}) \setminus I_M\} + \operatorname{span}\{x_j^* \mid j \in I_M\} \\ &\supset N(\bar{x}; \Theta) + \operatorname{span}\{\bar{x}^*\}, \end{aligned}$$

which imply therefore the inclusion

$$\bigcap_{\substack{(\bar{x},\bar{x}^*) \in M \times N(\operatorname{ri} M;\Theta) \\ M \in \mathcal{M}(\Theta)}} N(\bar{x}; M) \supset N(\bar{x}; \Theta) + \operatorname{span}\{\bar{x}^*\}. \tag{3.29}$$

Taking into account the duality relationship

$$N(\bar{x};\Theta)+\text{span}\{\bar{x}^*\}=\big[T(\bar{x},\Theta)\cap\{\bar{x}^*\}^\perp\big]^*,$$

the inclusion opposite to (3.29) follows from the implication

$$\Big[x^*\in\bigcap_{\substack{(\bar{x},\bar{x}^*)\in M\times N(\text{ri}\,M;\Theta)\\ M\in\mathcal{M}(\Theta)}}N(\bar{x};M)\Big]\Longrightarrow\Big[\langle x^*,v\rangle\le 0\ \text{ for any }\ v\in T(\bar{x};\Theta)\cap\{\bar{x}^*\}^\perp\Big].\quad(3.30)$$

To proceed with the proof of (3.30), pick any $v\in T(\bar{x};\Theta)\cap\{\bar{x}^*\}^\perp$ and let $x_k:=\bar{x}+\frac{1}{k}v$. Then it follows from the proof of Proposition 3.2 that $x_k\in\Theta$ for all $k\in I\!N$ sufficiently large. Taking into account the second representation of the generalized polyhedron Θ in (3.8) and that Θ has finitely many facies, suppose by passing to a subsequence if necessary that $x_k\in\text{ri}\,M_0$ for some $M_0\in\mathcal{M}(\Theta)$ and all $k\in I\!N$, and hence $\bar{x}\in M_0$ by passing to the limit as $k\to\infty$. Since $\bar{x}^*\in N(\bar{x};\Theta)$ and $v\in\{\bar{x}^*\}^\perp$, we have

$$\langle\bar{x}^*,x-x_k\rangle=\langle\bar{x}^*,x-\bar{x}\rangle-\frac{1}{k}\langle\bar{x}^*,v\rangle\le 0\ \text{ for all }\ x\in\Theta\ \text{ and }\ k\in I\!N.$$

The latter implies that $\bar{x}^*\in N(x_k;\Theta)$, and consequently $\bar{x}^*\in N(\text{ri}\,M_0;\Theta)$ by Proposition 3.2. Hence $x^*\in N(\bar{x};M_0)$, which yields that $\langle x^*,v\rangle\le 0$ due to $x_k=\bar{x}+\frac{1}{k}v\in M_0$ for all $k\in I\!N$. Thus we get the equality

$$\bigcap_{\substack{(\bar{x},\bar{x}^*)\in M\times N(\text{ri}\,M;\Theta)\\ M\in\mathcal{M}(\Theta)}}N(\bar{x};M)=N(\bar{x};\Theta)+\text{span}\{\bar{x}^*\}.$$

To complete the proof of the theorem, it remains to show that

$$N(\bar{x};\Theta)+\text{span}\{\bar{x}^*\}=A_{I(\bar{x}),\mathcal{J}(\bar{x},\bar{x}^*)}+A^*(Y^*).$$

The latter clearly follows from the definitions of $\mathcal{J}(\bar{x},\bar{x}^*)$ in (3.19) and $A_{Q,P}$ in (3.20) by representation (3.17) of the normal cone $N(\bar{x};\Theta)$. Thus we are done. $\triangle$

As an immediate consequence of Theorem 3.6, we obtain the following precise representation of the precoderivative (2.5) of the normal cone mapping (3.1), which allows us to compute it entirely in terms of the initial data of the generalized polyhedron (3.4).

Corollary 3.7 (computing the precoderivative of normal cone mappings over generalized polyhedra). *In the notation of Theorem 3.6 we have*

$$\widehat{D}^*\mathcal{F}(\bar{x},\bar{x}^*)(u)=\begin{cases}\text{pos}\{x_i^*\,|\,i\in Q\setminus P\}+\text{span}\{x_j^*\,|\,j\in P\}+A^*(Y^*)\\ \textit{if}\ \langle x_j^*,u\rangle=0\ \textit{ for }\ j\in P,\ Au=0,\ \textit{ and }\ \langle x_i^*,u\rangle\ge 0\ \textit{ for }\ i\in Q\setminus P;\\ \emptyset\qquad\textit{for all other}\ \ u\in X.\end{cases}$$

Proof. Follows directly from precoderivative definition and the results of Theorem 3.6 on computing the prenormal cone to the graph of the normal cone mapping $\mathcal{F}$. $\triangle$

4 Coderivatives of Normal Cone Mappings to Generalized Polyhedral Sets

In this section we compute the basic coderivative (2.6) of the normal cone mapping (3.1) entirely via the initial data of the generalized polyhedron (3.4) in infinite dimensions. The results obtained extend those established in [11] for usual polyhedra (1.3). Similarly to the latter paper, the proofs here are mainly based on passing to the limit from the corresponding results of Section 3 for prenormals and precoderivatives with some significant modifications in comparison with [11] due to the nature of generalized polyhedra.

Given $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \mathcal{F}$, consider the family of indices

$$\mathcal{I}(\bar{x}, \bar{x}^*) := \big\{P \subset I(\bar{x}) \big|\, \bar{x}^* \in A^*(Y^*) + \operatorname{pos}\{x_i^* |\, i \in P\}\big\}. \tag{4.1}$$

The next theorem represents the basic normal cone (2.4) to the graph of the normal cone mapping (3.1) via the indexed sets defined in (3.20) and (3.21) and the collection of faces of the generalized polyhedron (3.4).

Theorem 4.1 (face representation of basic normals to graphs of normal cone mappings). *Let* $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \mathcal{F}$ *for the normal cone mapping* (3.1) *built upon the generalized polyhedron* (3.4), *and let* $I(\bar{x})$ *and* $\mathcal{I} = \mathcal{I}(\bar{x}, \bar{x}^*)$ *be defined in* (3.5) *and* (4.1), *respectively. Then we have the representation*

$$N\big((\bar{x}, \bar{x}^*); \operatorname{gph} \mathcal{F}\big) = \bigcup_{P \subset I_M \subset I(\bar{x}),\, P \in \mathcal{I},\, M \in \mathcal{M}(\Theta)} \big[A^*(Y^*) + A_{I_M, P}\big] \times \big[B_{I_M, P} \cap (\ker A)\big]. \tag{4.2}$$

Furthermore, the graphical set $\operatorname{gph} \mathcal{F} \subset X \times X^*$ *is dually norm-stable at* $(\bar{x}, \bar{x}^*)$.

Proof. We verify representation (4.2) of the basic normal cone to the graph of $\mathcal{F}$ and justify simultaneously the dual norm-stability property of the graph in question.

Let us start with proving the inclusion "$\subset$" in (4.2). Pick an arbitrary limiting normal $(u^*, u) \in N((\bar{x}, \bar{x}^*); \operatorname{gph} \mathcal{F})$ and find by definition (2.4) sequences $(x_k, \widetilde{x}_k^*) \xrightarrow{\operatorname{gph} \mathcal{F}} (\bar{x}, \bar{x}^*)$ and $(u_k^*, u_k) \xrightarrow{w \times w} (u^*, u)$ as $k \to \infty$ satisfying

$$(u_k^*, u_k) \in \widehat{N}\big((x_k, \widetilde{x}_k^*); \operatorname{gph} \mathcal{F}\big) \quad \text{for all} \quad k \in I\!N. \tag{4.3}$$

It follows from (4.3) that $x_k \in \Theta$ and $\widetilde{x}_k^* \in N(x_k; \Theta)$ as $k \in I\!N$. Since Θ has a finite number of faces, assume without loss of generality that there is a common face $M \in \mathcal{M}(\Theta)$ such that $x_k \in \operatorname{ri} M$ for all $k \in I\!N$ and that $\bar{x} \in M$. It gives consequently that

$$I(x_k) = I_M \subset I(\bar{x}) \quad \text{for all} \quad k \in I\!N. \tag{4.4}$$

By representation (3.12) of Proposition 3.2 applied to each normal $\widetilde{x}_k^* \in N(x_k; \Theta)$, we get

$$\widetilde{x}_k^* = A^* y_k^* + \sum_{i \in I_M} \lambda_{ik} x_i^* \quad \text{with some} \quad \lambda_{ik} \geq 0 \quad \text{and} \quad y_k^* \in Y^* \quad \text{as} \quad k \in I\!N. \tag{4.5}$$

With no loss of generality, extracting another subsequence if necessary, select a constant index subset $P \subset I_M \subset I(\bar{x})$ such that

$$P := \mathcal{J}(x_k, \widetilde{x}_k^*) = \big\{i \in I_M \big|\, \lambda_{ik} > 0\big\} \quad \text{whenever} \quad k \in I\!N. \tag{4.6}$$

Observe that the finitely generated sets $A_{I_M,P}$ from (3.20) and $B_{I_M,P}$ from (3.21) are obviously weakly closed in the corresponding spaces. Hence this property holds for the set $B_{I_M,P} \cap \ker A$ and also for the image $A^*(Y^*)$ due to the polarity relationship

$$A^*(Y^*) + A_{I_M,P} = (\ker A)^* + B^*_{I_M,P} = \left[B_{I_M,P} \cap (\ker A)\right]^*.$$

Combining now (4.5) and (4.6), we get that

$$\widetilde{x}^*_k = A^* y^*_k + \sum_{i \in P} \lambda_{ik} x^*_i \in A^*(Y^*) + \mathrm{pos}\{x^*_i \mid i \in P\},$$

which implies in turns by passing to the limit as $k \to \infty$ that $\bar{x}^* \in A^*(Y^*) + \mathrm{pos}\{x^*_i \mid i \in P\}$. This justifies the inclusion $P \in \mathcal{I}$.

Applying the prenormal cone representation (3.22) from Theorem 3.6 to each (u^*_k, u_k) in (4.3) and using the structures of the index sets under consideration, we arrive at

$$u^*_k \in A^*(Y^*) + A_{I_M,P} \quad \text{and} \quad u_k \in B_{I_M,P} \cap (\ker A) \quad \text{for all} \quad k \in I\!N. \tag{4.7}$$

Passing finally to the limit in (4.7) as $k \to \infty$, we conclude that $(u^*, u) \in [A_{I_M,P} + A^*(Y^*)] \times [B_{I_M,P} \cap (\ker A)]$, which proves the inclusion "$\subset$" in (4.2).

To justify now the opposite inclusion "$\supset$" in (4.2), fix an arbitrary element

$$(u^*, u) \in \bigcup_{P \subset I_M \subset I(\bar{x}),\, P \in \mathcal{I},\, M \in \mathcal{M}(\Theta)} \left[A^*(Y^*) + A_{I_M,P}\right] \times \left[B_{I_M,P} \cap (\ker A)\right]$$

and then find a nonempty face M of Θ as well as index subsets $P \subset I_M \subset I(\bar{x})$ such that $P \in \mathcal{I} = \mathcal{I}(\bar{x}, \bar{x}^*)$ with

$$u^* \in A^*(Y^*) + A_{I_M,P} \quad \text{and} \quad u \in B_{I_M,P} \cap (\ker A). \tag{4.8}$$

Take further a point $\widetilde{x} \in \mathrm{ri}\, M$ and construct a sequence $\{x_k\} \subset X$ by

$$x_k := k^{-1}\widetilde{x} + (1 - k^{-1})\bar{x} \to \bar{x} \quad \text{as} \quad k \to \infty. \tag{4.9}$$

Since $\langle x^*_i, \widetilde{x}\rangle = c_i$ for all $i \in I_M$, $\langle x^*_i, \widetilde{x}\rangle < c_i$ for all $i \in I \backslash I_M$, and $A\widetilde{x} = b$, we have $x_k \in \mathrm{ri}\, M$ whenever $k \in I\!N$. This implies that $x_k \in \Theta$ and that the set of active constraint indices $I(x_k)$ at x_k reduces to I_M for each $k \in I\!N$. Then representation (3.12) of Proposition 3.2 gives in this case that

$$N(x_k; \Theta) = A^*(Y^*) + \mathrm{pos}\{x^*_i \mid i \in I_M\} \quad \text{for all} \quad k \in I\!N. \tag{4.10}$$

Furthermore, the inclusion $P \in \mathcal{I} = \mathcal{I}(\bar{x}, \bar{x}^*)$ implies by (4.1) the equality

$$\bar{x}^* = A^* y^* + \sum_{i \in P} \lambda_i x^*_i \quad \text{with some} \quad y^* \in Y^* \quad \text{and} \quad \lambda_i \geq 0.$$

Defining now a sequence $\{\widetilde{x}^*_k\} \subset X^*$ by

$$\widetilde{x}^*_k := A^* y^* + \sum_{i \in P} (\lambda_i + k^{-1}) x^*_i \quad \text{with} \quad \|\widetilde{x}^*_k - \bar{x}^*\| \to 0 \quad \text{as} \quad k \to \infty, \tag{4.11}$$

observe that $\widetilde{x}_k^* \in N(x_k;\Theta)$ for all $k \in I\!N$ due to (4.10) and $P \subset I_M$. Thus Theorem 3.6 applied to each $(x_k,\widetilde{x}_k^*)$ with the index sets I_M and P from (4.8) ensures that $\widehat{N}((x_k,\widetilde{x}_k^*);\operatorname{gph}\mathcal{F}) = [A^*(Y^*)+A_{I_M,P}]\times[B_{I_M,P}\cap(\ker A)]$ and hence

$$(u^*,u)\in\widehat{N}\big((x_k,\widetilde{x}_k^*);\operatorname{gph}\mathcal{F}\big)\quad\text{for all}\quad k\in I\!N. \tag{4.12}$$

Passing to the limit in (4.12) as $k\to\infty$ and using definition (2.4) of the basic normal cone, we get that $(u^*,u)\in N((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F})$, which fully justifies representation (4.2).

It remains to show that the graphical set $\operatorname{gph}\mathcal{F}$ is dually norm-stable at $(\bar{x},\bar{x}^*)$. By the definition of this property in Section 1 we need to check that any basic normal pair $(u^*,u)\in N((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F})$ can be approximated in the norm topology of $X\times X^*$ by prenormals to the graph of $\mathcal{F}$ at points nearby. It fact, it can be observed from the proof of the inclusion "$\supset$" in (4.2) that each such normal (u^*,u) satisfies inclusion (4.12) with the strongly convergent sequences $x_k\to\bar{x}$ by (4.9) and $\widetilde{x}_k^*\to\bar{x}^*$. This completes the proof of the theorem. $\triangle$

Next we establish a simplified representation of the basic normal cone to the graph of $\mathcal{F}$ under the additional verifiable assumptions: the *linear independence* of the generating elements $\{x_i^*\mid i\in I(\bar{x})\}$ and the *qualification condition*

$$(\ker A)^{\perp}\cap\operatorname{span}\{x_i^*|\ i\in I(\bar{x})\}=\{0\}. \tag{4.13}$$

Theorem 4.2 (simplified representation of basic normals to graphs of normal cone mappings). *Let $(\bar{x},\bar{x}^*)\in\operatorname{gph}\mathcal{F}$ in the setting of Theorem 4.1. Assume in addition that the generating elements $\{x_i^*\mid i\in I(\bar{x})\}$ are linearly independent in X^* and that the qualification condition (4.13) is satisfied. Then we have*

$$N\big((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F}\big)=\bigcup_{\mathcal{J}\subset P\subset Q\subset I(\bar{x})}\big[A_{Q,P}+A^*(Y^*)\big]\times\big[B_{Q,P}\cap(\ker A)\big]. \tag{4.14}$$

Proof. First we show that the assumptions made imply that

$$M_Q\in\mathcal{M}(\Theta)\ \text{ with }\ I_{M_Q}=Q\ \text{ for any }\ Q\subset I(\bar{x}), \tag{4.15}$$

where $M_Q:=\{x\in X|\ Ax=b,\ \langle x_i^*,x\rangle\le c_i\ \text{ as }\ i\in I\setminus Q,\ \text{ and }\ \langle x_j^*,x\rangle=c_j\ \text{ as }\ j\in Q\}$.

It is clear that $\bar{x}\in M_Q\in\mathcal{M}(\Theta)$. Furthermore, we can easily check that the linear independence assumption of the theorem and the qualification condition (4.13) ensure that the family $\{x_i^*\mid i\in I(\bar{x})\}$ of linear continuous functions on the linear subspace $\ker A$ are linearly independent as well. Then the linear system

$$\langle x_i^*,x\rangle=0\ \text{ as }\ i\in Q\ \text{ and }\ \langle x_j^*,x\rangle=-1\ \text{ as }\ j\in I(\bar{x})\setminus Q$$

has a solution $\widetilde{x}\in\ker A$. Hence we have

$$A(\bar{x}+t\widetilde{x})=b,\ \langle x_i^*,\bar{x}+t\widetilde{x}\rangle<c_i\ \text{ as }\ i\in I\setminus Q\ \text{ as }\ \langle x_j^*,\bar{x}+t\widetilde{x}\rangle=c_j\ \text{ as }\ j\in Q$$

for all $t>0$ sufficiently small, which implies that $I_{M_Q}=Q$.

To derive next the normal cone representation (4.14) from that of (4.2) in Theorem 4.1, it is sufficient to prove the equivalence

$$P\in\mathcal{I}\iff\mathcal{J}\subset P \tag{4.16}$$

with $\mathcal{I} = \mathcal{I}(\bar{x}, \bar{x}^*)$ defined in (4.1) and with the corresponding index set of positive multipliers as in (3.19). It is easy to see that the implication "$\Longleftarrow$" in (4.16) follows from the definition of $\mathcal{I}$ and the inclusions

$$\bar{x}^* \in A^*(Y^*) + \operatorname{pos}\{x_i^* \big| \, i \in \mathcal{J}\} \subset A^*(Y^*) + \operatorname{pos}\{x_i^* \big| \, i \in P\} \text{ if } \mathcal{J} \subset P.$$

To justify the opposite implication "$\Longrightarrow$" in (4.16), pick any $P \in \mathcal{I}$ and find multipliers $\lambda_i \geq 0$ as $i \in P$ and a dual element $y^* \in Y^*$ such that

$$\bar{x}^* = A^*y^* + \sum_{i \in P} \lambda_i x_i^*. \tag{4.17}$$

Let us show that the index set of positive multipliers $\mathcal{J}(\bar{x}, \bar{x}^*)$ for $(\bar{x}, \bar{x}^*)$ in (4.17) is uniquely determined. Indeed, suppose that $\mathcal{J}_1, \mathcal{J}_2$ are two different such sets sets corresponding to $(\bar{x}, \bar{x}^*)$ in (4.17), i.e., $(\mathcal{J}_2 \setminus \mathcal{J}_1) \cup (\mathcal{J}_1 \setminus \mathcal{J}_2) \neq \emptyset$. Thus

$$\bar{x}^* = A^*{y_1}^* + \sum_{i \in \mathcal{J}_1} \lambda_i x_i^* = A^*{y_2}^* + \sum_{j \in \mathcal{J}_2} \gamma_i x_j^* \text{ for some } {y_1}^*, {y_2}^* \in Y^*.$$

This implies the equality

$$0 = \sum_{l \in \mathcal{J}_1 \cap \mathcal{J}_2} (\lambda_l - \gamma_l) x_l^* + \sum_{i \in \mathcal{J}_1 \setminus \mathcal{J}_2} \lambda_i x_i^* + \sum_{j \in \mathcal{J}_2 \setminus \mathcal{J}_1} \gamma_j x_j^* + A^*({y_1}^* - {y_2}^*),$$

which yields in turn that

$$\sum_{l \in \mathcal{J}_1 \cap \mathcal{J}_2} (\lambda_l - \gamma_l) x_l^* + \sum_{i \in \mathcal{J}_1 \setminus \mathcal{J}_2} \lambda_i x_i^* + \sum_{j \in \mathcal{J}_2 \setminus \mathcal{J}_1} \gamma_j x_i^* = A^*({y_2}^* - {y_1}^*) = 0$$

due to the classical fact that $A^*(Y^*) = (\ker A)^\perp$ and the classification condition (4.13). Applying then the assumed linear independence of the generating elements $\{x_i^* \mid i \in I(\bar{x})\}$ of (3.4), we have that $\lambda_l = \gamma_l$ for all $l \in \mathcal{J}_1 \cap \mathcal{J}_2$, $\lambda_i = 0$ for all $i \in \mathcal{J}_1 \setminus \mathcal{J}_2$, and $\gamma_j = 0$ for all $j \in \mathcal{J}_2 \setminus \mathcal{J}_1$. The latter surely contradicts the multiplier positivity $\lambda_i > 0$ for all $i \in \mathcal{J}_1 \setminus \mathcal{J}_2$ and $\gamma_j > 0$ for all $j \in \mathcal{J}_2 \setminus \mathcal{J}_1$. Hence we get $\mathcal{J} \subset P$ by the definition of the index set of positive multipliers $\mathcal{J}$ in (4.17), and the conclusion of the theorem follows finally from relationships (4.15) and (4.16). $\triangle$

From the normal cone representations of Theorems 4.1 and 4.2 we derive the corresponding representations of the basic coderivative (2.6) of the normal cone mapping (3.1) built upon the generalized polyhedron (3.4).

Corollary 4.3 (coderivative representations for normal cone mappings over generalized polyhedra). *Let $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \mathcal{F}$ for the normal cone mapping* (3.1) *with Θ from* (3.4)*, and let the corresponding index sets be defined in Theorem* 4.1*. Then the normal cone mapping $\mathcal{F}$ is coderivatively normal at $(\bar{x}, \bar{x}^*)$ and the following assertions hold:*

(i) *Under the general assumptions made the basic coderivative* (2.6) *of $\mathcal{F}$ at $(\bar{x}, \bar{x}^*)$ is represented by*

$$\begin{aligned} D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u) = \big\{ u^* \in X^* \big| \, (u^*, -u) \in \big[A_{I_M, P} + A^*(Y^*)\big] \times \big[B_{I_M, P} \cap \ker A\big] \\ \text{for some } P \subset I_M \subset I(\bar{x}) \text{ with } P \in \mathcal{I}(\bar{x}, \bar{x}^*) \text{ and } M \in \mathcal{M}(\Theta) \big\}. \end{aligned} \tag{4.18}$$

(ii) *If in addition the generating elements* $\{x_i^* \mid i \in I(\bar{x})\}$ *of* (3.4) *are linearly independent and the qualification condition* (4.13) *is satisfied, then*

$$D^*\mathcal{F}(\bar{x},\bar{x}^*)(u) = \big\{u^* \in X^* \big| \begin{array}{l}(u^*,-u) \in \big[A_{Q,P} + A^*(Y^*)\big] \times \big[B_{Q,P} \cap (\ker A)\big] \\ \textit{for some } \ \mathcal{J} \subset P \subset Q \subset I(\bar{x})\big\}.\end{array} \tag{4.19}$$

Proof. Representations (4.18) and (4.19) follow directly from the coderivative definition (2.6) and the normal cone representation (4.2) and (4.14), respectively. The coderivative normality (2.9) of the normal cone mapping $\mathcal{F}$ at $(\bar{x},\bar{x}^*)$ is a consequence of the dual norm-stability of the graph of $\mathcal{F}$ at this point justified in Theorem 4.1. $\triangle$

To proceed further, given an active index collection $T \subset I(\bar{x})$ consider the closed set

$$M_T := \big\{x \in X \big| \ Ax = b, \ \langle x_i^*, x\rangle \le c_i \ \text{ as } \ i \in I \setminus T, \ \text{ and } \ \langle x_j^*, x\rangle = c_j \ \text{ as } \ j \in T\big\}, \tag{4.20}$$

which is a face of the generalized polyhedron (3.4), Define the *feature index set* for T by

$$\Upsilon(T) := \big\{i \in I(\bar{x}) \big| \ \langle x_i^*, x\rangle = c_i \ \text{ whenever } \ x \in M_T\big\}. \tag{4.21}$$

It follows from the proof of relationships (4.15) in Theorem 4.2 that $\Upsilon(T) = T$ whenever the generating elements $\{x_i^* \mid i \in I(\bar{x})\}$ of (3.4) are linearly independent and and the qualification condition (4.13) is satisfied.

The following example shows that the feature index set $\Upsilon(T)$ for T is not necessarily equal to T in the general case under consideration.

Example 4.4 (properties of feature index sets). Let

$$x_1^* = (-1,0), \ x_2^* = (0,-1), \ x_3^* = (-1,-1), \ x_4^* = (1,1),$$

let $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, and let $I = \{1,2,3,4\}$. Define a convex polyhedron in $I\!R^2$ by

$$\Theta := \big\{x \in I\!R^2 \big| \ \langle x_i^*, x\rangle \le c_i \ \text{ for } \ i \in I\big\}.$$

Take $\bar{x} = (0,0)$ and select an active index collection $T = \{1,2\} \subset I(\bar{x}) = \{1,2,3\}$. Then we have by (4.20) and (4.21) that, respectively,

$$\begin{array}{l} M_T = \big\{x \in I\!R^2 \big| \ \langle x_3^*, x\rangle = -x_1 - x_2 \le 0, \ \langle x_4^*, x\rangle = x_1 + x_2 \le 1, \\ \langle x_1^*, x\rangle = -x_1 = 0, \ \langle x_2^*, x\rangle = -x_2 = 0\big\} = \big\{(0,0)\big\} \ \text{ and} \end{array}$$

$$\Upsilon(T) = \big\{1,2,3\big\} = I(\bar{x}) \ne T.$$

Similarly it is easy to check that for $T_1 = \{1,3\}$ and $T_2 = \{2,3\}$, we get that

$$M_{T_1} = M_{T_2} = \{(0,0)\} \ \text{ and } \ \Upsilon(T_1) = \Upsilon(T_2) = I(\bar{x}) = \{1,2,3\}.$$

The next result fully characterizes the coderivative domain of the normal cone mapping (3.1) in constructive terms involving the feature index set (4.21)

Theorem 4.5 (description of the coderivative domain for normal cone mappings). *Let* $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \mathcal{F}$ *in the general setting of Theorem* 4.1. *Then* $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ *if and only if we have the relationships*

$$Au = 0,\ \langle x_i^*, u\rangle \geq 0 \ \text{ as } \ i \in \Upsilon(\mathcal{J}) \setminus \mathcal{J}, \ \text{ and } \ \langle x_j^*, u\rangle = 0 \ \text{ as } \ j \in \mathcal{J}, \tag{4.22}$$

where $\mathcal{J} = \mathcal{J}(\bar{x}, \bar{x}^*)$ *is the index set of positive multipliers and* $\Upsilon(\mathcal{J})$ *is defined in* (4.21).

Proof. First we justify the *necessity* of condition (4.22) for $D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u) \neq \emptyset$. Taking $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ and applying the coderivative definition (2.6) and representation (4.2) of the basic normal cone, find $u^* \in X^*$, $M \in \mathcal{M}(\Theta)$, and index sets $P \subset I_M \subset I(\bar{x})$ with $P \in \mathcal{I}$ satisfying the inclusion

$$(u^*, -u) \in \big[A^*(Y^*) + A_{I_M,P}\big] \times \big[B_{I_M,P} \cap (\ker A)\big]. \tag{4.23}$$

Let us now show that $\mathcal{J} \subset I_M$. To proceed, fix $x \in \operatorname{ri} M$ and get by (3.7) that

$$Ax = b, \ \ \langle x_i^*, x\rangle < c_i \ \text{ as } \ i \in I \setminus I_M, \ \text{ and } \ \langle x_j^*, x\rangle = c_j \ \text{ as } \ j \in I_M.$$

Taking into account that $I_M \subset I(\bar{x})$, we have

$$A(x - \bar{x}) = 0, \ \ \langle x_i^*, x - \bar{x}\rangle < 0 \ \text{ as } \ i \in I(\bar{x}) \setminus I_M, \ \text{ and } \ \langle x_j^*, x - \bar{x}\rangle = 0 \ \text{ as } \ j \in I_M. \tag{4.24}$$

Furthermore, the inclusion $P \in \mathcal{I}(\bar{x}, \bar{x}^*)$ allows us to find by (4.1) numbers $\lambda_i \geq 0$ as $i \in P$ and a dual element $y_P^* \in Y^*$ such that

$$\bar{x}^* = A^* y_P^* + \sum_{i \in P} \lambda_i x_i^*.$$

By (4.24) and $P \subset I_M$ the latter implies that

$$\langle \bar{x}^*, x - \bar{x}\rangle = \langle y_P^*, A(x - \bar{x})\rangle + \sum_{i \in P} \lambda_i \langle x_i^*, x - \bar{x}\rangle = 0.$$

On the other hand, we have from the expression of $\bar{x}^*$ in (4.17) in the definition of $\mathcal{J}$ that

$$\begin{aligned} 0 = \langle \bar{x}^*, x - \bar{x}\rangle &= \langle y_{\mathcal{J}}^*, A(x - \bar{x})\rangle + \sum_{i \in \mathcal{J}} \lambda_i \langle x_i^*, x - \bar{x}\rangle \\ &= \sum_{i \in \mathcal{J}} \lambda_i \langle x_i^*, x - \bar{x}\rangle \ \text{ with some } \ y_{\mathcal{J}}^* \in Y^*, \ \text{ and } \ \lambda_i > 0 \ \text{ as } \ i \in \mathcal{J}. \end{aligned}$$

This together with $\mathcal{J} \subset I(\bar{x})$ and (4.24) imply that $\langle x_i^*, x - \bar{x}\rangle = 0$ whenever $i \in \mathcal{J}$, which yields $\mathcal{J} \subset I_M$. Furthermore, from (4.23) and definition (3.21) of the set $B_{I_M,P}$ we get that

$$Au = 0, \ \ \langle x_i^*, u\rangle \geq 0 \ \text{ as } \ i \in I_M \setminus P \ \text{ and } \ \langle x_j^*, u\rangle = 0 \ \text{ as } \ j \in P.$$

It follows from the inclusion $\mathcal{J} \subset I_M$ that $Au = 0$ and $\langle x_i^*, u\rangle \geq 0$ for all $i \in \mathcal{J}$. This allows us to apply to the chosen element u the same arguments as for $x - \bar{x}$ above and conclude that $Au = 0$ and $\langle x_i^*, u\rangle = 0$ whenever $i \in \mathcal{J}$.

To complete the proof of the necessity, it remains to show that $\Upsilon(\mathcal{J}) \subset I_M$. Since $\mathcal{J} \subset I_M$, we have $M \subset M_{\mathcal{J}}$ by definition (4.20), which implies that $\Upsilon(\mathcal{J}) \subset I_M$ by Corollary 3.3. Thus we arrive at (4.22) and justify the necessity in the theorem.

To prove the *sufficiency* part of the theorem, assume that the relationships in (4.22) are satisfied for the given point $u \in X$. Put $M := M_{\mathcal{J}}$ with the active index set $I_M = \Upsilon(\mathcal{J})$ and $P := \mathcal{J} \in \mathcal{I}$. It is clear that $\bar{x} \in M_{\mathcal{J}} \in \mathcal{M}(\Theta)$. Observe that $-u \in B_{I_M,P} \cap (\ker A)$ for the selected pair (I_M, P). By definition (3.20) we have $0 \in A_{I_M,P} + A^*(Y^*)$, even when $P = \emptyset$ and/or $I_M \setminus P = \emptyset$ by the convention made. Thus

$$(0, -u) \in \big[A_{I_M,P} + A^*(Y^*)\big] \times \big[B_{I_M,P} \cap (\ker A)\big],$$

and the sufficiency of condition (4.22) follows from Theorem 4.1. $\triangle$

The next two theorems are the main results of this section providing constructive evaluations of the basic coderivative $D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u)$ of the normal cone mapping (3.1) entirely in terms of the initial data of the generalized polyhedron (3.4). Given $u \in X$, define the *characteristic active index subsets* as follows:

$$I_0(u) := \big\{i \in I(\bar{x}) \big|\ \langle x_i^*, u\rangle = 0\big\} \quad\text{and}\quad I_>(u) := \big\{i \in I(\bar{x}) \big|\ \langle x_i^*, u\rangle > 0\big\}. \tag{4.25}$$

The first main result gives a constructive coderivative upper estimate in the general setting.

Theorem 4.6 (coderivative estimate for normal cone mappings over generalized polyhedra). *Let $(\bar{x}, \bar{x}^*) \in \operatorname{gph} \mathcal{F}$ in the framework of Theorem* 4.5, *and let $I_0(u)$ and $I_>(u)$ be the characteristic active index subsets defined in* (4.25). *Then for all $u \in X$ we have*

$$D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u) \subset A^*(Y^*) + \operatorname{pos}\big\{x_i^* \big|\ i \in I_>(u)\big\} + \operatorname{span}\big\{x_i^* \big|\ i \in I_0(u)\big\}. \tag{4.26}$$

Proof. Estimate (4.26) is trivial when the domain of $D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ is empty. Take further $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ and $u^* \in D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u)$ and find by (2.6) and Theorem 4.1 such a face $M \in \mathcal{M}(\Theta)$ and index subsets $P \subset I_M \subset I(\bar{x})$ that $P \in \mathcal{I}(\bar{x}, \bar{x}^*)$ and

$$u^* \in A^*(Y^*) + A_{I_M,P}, \quad -u \in B_{I_M,P} \cap (\ker A). \tag{4.27}$$

By definition (3.21) of the set $B_{I_M,P}$ the last inclusion in (4.27) is equivalent to

$$Au = 0, \ \langle x_i^*, u\rangle \ge 0 \ \text{ as } \ i \in I_M \setminus P, \ \text{ and } \ \langle x_j^*, u\rangle = 0 \ \text{ as } \ j \in P,$$

which implies the relationships

$$P \subset T := \big\{i \in I_M \big|\ \langle x_i^*, u\rangle = 0\big\} \quad\text{and}\quad \langle x_i^*, u\rangle > 0 \ \text{ for all } \ i \in I_M \setminus T. \tag{4.28}$$

It follows from (4.27), (4.28), and definition (3.20) of the set $A_{I_M,P}$ that

$$\begin{aligned} u^* &\in A^*(Y^*) + \operatorname{pos}\big\{x_i^* \big|\ i \in I_M \setminus P\big\} + \operatorname{span}\big\{x_i^* \big|\ i \in P\big\} \\ &\subset A^*(Y^*) + \operatorname{pos}\big\{x_i^* \big|\ i \in I_M \setminus T\big\} + \operatorname{span}\big\{x_i^* \big|\ i \in T\big\}. \end{aligned} \tag{4.29}$$

Observing finally from the constructions of T in (4.28) and of the characteristic active index subset in (4.25) that $T \subset I_0(u)$ and $I_M \setminus T \subset I_>(u)$, we get (4.26) from (4.29) and thus complete the proof of the theorem. $\triangle$

The second main result of this section contains a *precise formula* for computing the coderivative of the normal cone mapping $\mathcal{F}$ at $(\bar{x}, \bar{x}^*)$ under the linear independence of the generating elements x_i^* in (3.4) and the qualification condition (4.13).

Theorem 4.7 (precise computing coderivatives of normal cone mappings over generalized polyhedra). *Let in the framework of Theorem* 4.6 *the generating elements* $\{x_i^* \mid i \in I(\bar{x})\}$ *of* (3.4) *be linearly independent and the qualification condition* (4.13) *be satisfied. Then for all* $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ *we have*

$$D^*\mathcal{F}(\bar{x}, \bar{x}^*)(u) = A^*(Y^*) + \operatorname{pos}\{x_i^* \,|\, i \in I_>(u)\} + \operatorname{span}\{x_i^* \,|\, i \in I_0(u)\}. \tag{4.30}$$

Proof. By Theorem 4.6 we need to prove the inclusion "$\supset$" opposite to (4.26). It is clear that the imposed linear independence and qualification conditions imply that $\Upsilon(\mathcal{J}) = \mathcal{J}$ for the feature index subset (4.21) of $\mathcal{J} = \mathcal{J}(\bar{x}, \bar{x}^*)$. Take now (u^*, u) satisfying the inclusions $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ and

$$u^* \in A^*(Y^*) + \operatorname{pos}\{x_i^* \,|\, i \in I_>(u)\} + \operatorname{span}\{x_i^* \,|\, i \in I_0(u)\}$$

and then get from (3.20), (3.21), and the latter inclusion that

$$\begin{gathered}(u^*, -u) \in \big[A_{I_M,P} + A^*(Y^*)\big] \times \big[B_{I_M,P} \cap (\ker A)\big] \\ \text{with } \; I_M := I_0(u) \cup I_>(u) \; \text{ and } \; P := I_0(u).\end{gathered} \tag{4.31}$$

Taking finally into account by Theorem 4.5 and the constructions in (4.25) that

$$\mathcal{J} \subset I_0(u) \subset I_0(u) \cup I_>(u) \subset I(\bar{x}),$$

we derive the inclusion "$\supset$" in (4.30) from the relationships in (4.31) and the coderivative representation (4.19) of Corollary 4.3(ii). This completes the proof of the theorem. $\triangle$

5 Robust Stability of Variational Inequalities Over Generalized Polyhedra

The primary goal in this section is to establish constructive characterizations of the Lipschitz-like property of the solution map (1.2) with evaluating the exact Lipschitzian bound in (2.12) entirely in terms of the initial data of the generalized polyhedron (3.4) in reflexive Banach spaces. This will be done by combining the criteria of Theorem 2.1, some calculus results from [19], and the coderivative calculations of Section 4. Observe that the main results and arguments in what follows are significantly more involved in comparison with those in [11] in the case of standard convex polyhedra.

Let us first present a result from [11, Lemma 5.1] showing that the general assumptions imposed in Theorem 2.1 are satisfied for the solution map (1.2) in our setting.

Proposition 5.1 (properties of solutions maps to parametric variational inequalities). *The graph* $\mathrm{gph}\, S \subset Z \times X$ *of the solution map* $S\colon Z \rightrightarrows X$ *is always closed in* $Z \times X$. *Furthermore, the mapping* $S\colon Z \rightrightarrows X$ *is coderivatively normal at every point* $(\bar{p}, \bar{x}) \in \mathrm{gph}\, S$ *where* f *is strictly differentiable and its partial derivative* $\nabla_p f(\bar{p}, \bar{x})\colon Z \to X^*$ *is surjective.*

Next we establish, based on the results in Section 4 and calculus rules of generalized differentiation, constructive representations for the coderivative (2.6) of the solution map (1.2) via the initial data in (1.1) and (3.4).

Proposition 5.2 (coderivatives of solution maps to parametric variational inequalities). *Let* $(\bar{p}, \bar{x}) \in \mathrm{gph}\, S$ *for the solution map* (1.2), *where* f *is strictly differentiable at* $(\bar{p}, \bar{x})$ *with the surjective partial derivative* $\nabla_p f(\bar{p}, \bar{x})$, *and let* $\bar{x}^* := -f(\bar{p}, \bar{x})$. *Then we have the following assertions:*

(i) *The coderivative* $D^*S(\bar{p}, \bar{x})\colon X^* \rightrightarrows Z^*$ *is computed in the general setting by*

$$D^*S(\bar{p}, \bar{x})(x^*) = \left\{ \begin{array}{l} p^* \in Z^* \big|\ \exists u \in X \ \ \textit{with} \ \ P \subset I_M \subset I(\bar{x}) \\ \textit{such that} \ \ P \in \mathcal{I},\ M \in \mathcal{M}(\Theta),\ p^* = \nabla_p f(\bar{p}, \bar{x})^* u, \ \ \textit{and} \\ \big(-x^* - \nabla_x f(\bar{p}, \bar{x})^* u, -u \big) \in \big[A^*(Y^*) + A_{I_M, P} \big] \times \big[B_{I_M, P} \cap (\ker A) \big]. \end{array} \right.$$

(ii) *Assume in addition that the generating element* $\{x_i^* \mid i \in I(\bar{x})\}$ *of* (3.4) *are linearly independent and that the qualification condition* (4.13) *is satisfied. Then the coderivative* $D^*S(\bar{p}, \bar{x})\colon X^* \rightrightarrows Z^*$ *is computed by*

$$D^*S(\bar{p}, \bar{x})(x^*) = \left\{ \begin{array}{l} p^* \in Z^* \big|\ \exists u \in \mathrm{dom}\, D^*\mathcal{F}(\bar{x}, \bar{x}^*) \ \textit{such that} \ \ p^* = \nabla_p f(\bar{p}, \bar{x})^* u, \\ -x^* - \nabla_x f(\bar{p}, \bar{x})^* u \in A^*(Y^*) + \mathrm{pos}\big\{ x_i^* \big|\ i \in I_>(u) \big\} + \mathrm{span}\big\{ x_i^* \big|\ i \in I_0(u) \big\}, \end{array} \right.$$

where the characteristic active index subsets $I_0(u)$ *and* $I_>(u)$ *are defined in* (4.25) *while the coderivative domain* $\mathrm{dom}\, D^*\mathcal{F}(\bar{x}, \bar{x}^*)$ *is computed in Theorem* 4.5.

Proof. Observe first the image rule for basic normals from [19, Theorem 1.17] applied to representation (3.3) ensures the equality

$$N\big((\bar{p}, \bar{x}); \mathrm{gph}\, S\big) = \nabla g(\bar{p}, \bar{x})^* N\big((\bar{x}, -f(\bar{p}, \bar{x})); \mathrm{gph}\, \mathcal{F}\big). \tag{5.1}$$

Combining now (5.1) with the coderivative definition in (2.6) and the basic normal representation (4.2) from Theorem 4.1, we arrive at the equalities

$$\begin{array}{l} D^*S(\bar{p}, \bar{x})(x^*) = \big\{ p^* \in Z^* \big|\ (p^*, -x^*) \\ \in \nabla g(\bar{p}, \bar{x})^* \Big(\displaystyle\bigcup_{P \subset I_M \subset I(\bar{x}),\, P \in \mathcal{I},\, M \in \mathcal{M}(\Theta)} \big[A^*(Y^*) + A_{I_M, P} \big] \times \big[B_{I_M, P} \cap (\ker A) \big] \Big) \big\} \\ = \big\{ p^* \in Z^* \big|\ (p^*, -x^*) \in \begin{pmatrix} 0 & -\nabla_p f(\bar{p}, \bar{x})^* \\ 1 & -\nabla_x f(\bar{p}, \bar{x})^* \end{pmatrix} \begin{pmatrix} A^*(Y^*) + A_{I_M, P} \\ B_{I_M, P} \cap (\ker A) \end{pmatrix} \\ \qquad \text{with} \ \ P \subset I_M \subset I(\bar{x}), \ P \in \mathcal{I}, \ \text{and} \ \ M \in \mathcal{M}(\Theta) \big\} \\ = \big\{ p^* \in Z^* \big|\ \exists u^* \in A^*(Y^*) + A_{I_M, P}, -u \in B_{I_M, P} \cap (\ker A) \\ \qquad \text{with} \ \ P \subset I_M \subset I(\bar{x}), \ P \in \mathcal{I}, \ \text{and} \ \ M \in \mathcal{M}(\Theta) \\ \qquad \text{such that} \ \ p^* = \nabla_p f(\bar{p}, \bar{x})^* u \ \ \text{and} \ \ -x^* = u^* + \nabla_x f(\bar{p}, \bar{x})^* u \big\} \\ = \big\{ p^* \in Z^* \big|\ \exists u \in X, \ P \subset I_M \subset I(\bar{x}) \ \ \text{with} \ \ P \in \mathcal{I}, \ M \in \mathcal{M}(\Theta) \\ \qquad \text{such that} \ \ p^* = \nabla_p f(\bar{p}, \bar{x})^* u \ \ \text{and} \\ \qquad (-x^* - \nabla_x f(\bar{p}, \bar{x})^* u, -u) \in \big[A^*(Y^*) + A_{I_M, P} \big] \times \big[B_{I_M, P} \cap (\ker A) \big] \big\}, \end{array}$$

which imply assertion (i) of this proposition.

To justify next assertion (ii), observe the representation

$$N\big((\bar{x},\bar{x}^*);\operatorname{gph}\mathcal{F}\big)=\big\{(u^*,-u)\big|\, u^*\in D^*\mathcal{F}(\bar{x},\bar{x}^*)(u)\ \text{ for }\ u\in\operatorname{dom}D^*\mathcal{F}(\bar{x},\bar{x}^*)\big\}.$$

Combining the latter with $(\bar{x},-f(\bar{p},\bar{x}))=(\bar{x},\bar{x}^*)$ and the representation of $D^*\mathcal{F}(\bar{x},\bar{x}^*)$ in (4.30) from Theorem 4.7, we have the following equalities:

$$\begin{aligned}
D^*S(\bar{p},\bar{x})(x^*)=\Big\{p^*\in Z^*\Big|\ &\begin{pmatrix} p^*\\ -x^*\end{pmatrix}\in\begin{pmatrix}0 & -\nabla_p f(\bar{p},\bar{x})^*\\ 1 & -\nabla_x f(\bar{p},\bar{x})^*\end{pmatrix}\begin{pmatrix}u^*\\ -u\end{pmatrix}\\
&\text{for some }\ u\in\operatorname{dom}D^*\mathcal{F}(\bar{x},\bar{x}^*)\text{ and }\ u^*\in D^*\mathcal{F}(\bar{x},\bar{x}^*)(u)\Big\}\\
=\Big\{p^*\in Z^*\Big|\ p^*=\nabla_p f(\bar{p},\bar{x})^*u,\ -x^*&=u^*+\nabla_x f(\bar{p},\bar{x})^*u\\
&\text{for some }\ u\in\operatorname{dom}D^*\mathcal{F}(\bar{x},\bar{x}^*)\text{ and }\ u^*\in D^*\mathcal{F}(\bar{x},\bar{x}^*)(u)\Big\}\\
=\Big\{p^*\in Z^*\Big|\ \exists\, u\in\operatorname{dom}D^*\mathcal{F}(\bar{x},\bar{x}^*)\ &\text{ with }\ p^*=\nabla_p f(\bar{p},\bar{x})^*u\ \text{ and}\\
-x^*-\nabla_x f(\bar{p},\bar{x})^*u\in A^*(Y^*)+\operatorname{pos}\big\{x_i^*\big|\, i\in I_>(u)\big\}&+\operatorname{span}\big\{x_i^*\big|\, i\in I_0(u)\big\}\Big\},
\end{aligned}$$

which thus complete the proof of the proposition. △

Now we are ready to obtain verifiable characterizations for robust Lipschitzian stability of solution maps to the parametric variational inequalities (1.1) over generalized polyhedra with evaluating the exact Lipschitzian bound. To proceed in this direction, let us first focus on the case when the parameter space Z is finite-dimensional while the decision variable belongs to an arbitrary reflexive Banach X.

Theorem 5.3 (Lipschitzian stability of variational inequalities over generalized polyhedra with finite-dimensional parameter spaces). *Let $(\bar{p},\bar{x})\in\operatorname{gph}S$ in the framework and notation of Proposition* 5.2, *and let* $\dim Z<\infty$. *Then we have the following:*

(i) *The solution map* (1.2) *is Lipschitz-like around $(\bar{p},\bar{x})$ if and only if*

$$\big[-\nabla_x f(\bar{p},\bar{x})^*u\in A^*(Y^*)+A_{I_M,P},\ -u\in B_{I_M,P}\cap(\ker A)\big]\Longrightarrow u=0 \tag{5.2}$$

for all $P\subset I_M\subset I(\bar{x})$ with $P\in\mathcal{I}(\bar{x},\bar{x}^*)$ and $M\in\mathcal{M}(\Theta)$. Furthermore, we have the precise formula for computing the exact Lipschitzian bound of the solution map S at $(\bar{p},\bar{x})$:

$$\begin{aligned}
&\operatorname{lip}S(\bar{p},\bar{x})=\sup\big\{\|\nabla_p f(\bar{p},\bar{x})^*u\|\,\big|\ u\in -B_{I_M,P}\cap(\ker A),\\
&x^*\in-\nabla_x f(\bar{p},\bar{x})^*u-A^*(Y^*)-A_{I_M,P},\ \|\nabla_x f(\bar{p},\bar{x})^*u+x^*\|\le 1,\\
&\quad \textit{for all}\ \ P\subset I_M\subset I(\bar{x})\ \ \textit{with}\ \ P\in\mathcal{I}(\bar{x},\bar{x}^*)\ \ \textit{and}\ \ M\in\mathcal{M}(\Theta).
\end{aligned}\tag{5.3}$$

(ii) *Assume in addition that the generating element $\{x_i^*\mid i\in I(\bar{x})\}$ of* (3.4) *are linearly independent and that the qualification condition* (4.13) *is satisfied. Then S is Lipschitz-like around $(\bar{p},\bar{x})$ if and only if*

$$\begin{aligned}
\big[-\nabla_x f(\bar{p},\bar{x})^*u\in A^*(Y^*)+\operatorname{pos}\big\{x_i^*\big|\, i\in I_>(u)\big\}&\\
+\operatorname{span}\big\{x_i^*\big|\, i\in I_0(u)\big\}\big]\Longrightarrow u=0&
\end{aligned}\tag{5.4}$$

provided that $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x},\bar{x}^*)$, *where the characteristic active index subsets* $I_0(u)$ *and* $I_>(u)$ *are defined in* (4.25) *while the coderivative domain* $\operatorname{dom} D^*\mathcal{F}(\bar{x},\bar{x}^*)$ *is computed in Theorem* 4.5. *In fact, implication* (5.4) *with* $u \in \operatorname{dom} D^*\mathcal{F}(\bar{x},\bar{x}^*)$ *is equivalent to*

$$\big[-\nabla_x f(\bar{p},\bar{x})^*u \in A^*(Y^*) + A_{I,I},\ -u \in B_{\mathcal{J},\mathcal{J}} \cap (\ker A)\big] \Rightarrow u = 0 \tag{5.5}$$

with $I = I(\bar{x})$ *and* $\mathcal{J} = \mathcal{J}(\bar{x},\bar{x}^*)$. *Furthermore, we have the precise formula for computing the exact Lipschitzian bound of the solution map* S *at* $(\bar{p},\bar{x})$*:*

$$\begin{array}{c}\operatorname{lip} S(\bar{p},\bar{x}) = \sup\big\{\|\nabla_p f(\bar{p},\bar{x})^*u\|\ \big|\ u \in \operatorname{dom} D^*\mathcal{F}(\bar{p},\bar{x}),\\ -x^* - \nabla_x f(\bar{p},\bar{x})^*u \in A^*(Y^*) + \operatorname{span}\{x_i^*|\ i \in I_0(u)\}\\ +\operatorname{pos}\{x_i^*|\ i \in I_>(u)\},\ \|\nabla_x f(\bar{x},\bar{x}^*)^*u + x^*\| \le 1\big\}.\end{array} \tag{5.6}$$

Proof. Let us employ the coderivative characterizations of Theorem 2.1 whose general assumptions are satisfied by Proposition 5.1. Furthermore, the PSNC property of S is automatic due to the finite dimension of the parameter/domain space Z, and the condition $\nabla_p f(\bar{p},\bar{x})^*u = 0$ is equivalent to $u = 0$ by the assumed surjectivity of $\nabla_p f(\bar{p},\bar{x})$. Hence criteria (5.2) and (5.4) for the Lipschitz-like property of S in (i) and (ii), respectively, follow directly from (2.13) and the coderivative formulas for S obtained in Proposition 5.2 as $x^* = 0$. Observe also from the proof of Theorem 4.7 and the monotonicity relationships

$$A_{G,T} \subset A_{G',T'} \quad \text{and} \quad B_{G,T} \supset B_{G',T'} \quad \text{whenever} \quad G \subset G',\ T \subset T' \tag{5.7}$$

for the constructions in (3.20) and (3.21) that the Lipschitzian stability criterion (5.2) can be equivalently written in form (5.5). Indeed, from the proof of Theorem 4.7 we have that $\Upsilon(\mathcal{J}) = \mathcal{J}$ under the assumptions in (ii). Thus

$$\operatorname{dom} D^*\mathcal{F}(\bar{x},\bar{x}^*) = \big\{u \in X\big|\ Au = 0 \ \text{ and } \ \langle x_i^*, u\rangle = 0 \ \text{ for all } \ i \in \mathcal{J}\big\} = B_{\mathcal{J},\mathcal{J}} \cap (\ker A).$$

The exact bound formulas (5.3) and (5.6) follow now from Theorem 2.1 and the coderivative calculations of Proposition 5.2. This completes the proof of the theorem. $\triangle$

Next we study the Lipschitz-like property of the solution map (1.2) over the generalized polyhedron (3.4) when both parameter and decision spaces are infinite-dimensional. The Banach space setting for the parameter space Z makes the situation significantly more difficult in comparison with Theorem 5.3, since it requires to verify the PSNC property of the solution map according to Theorem 2.1. To proceed, we rely on a certain well-posedness of the original variational inequality (1.1) formalized in the following definition, which is an extension of the corresponding property from [11] to the case of generalized polyhedra.

Definition 5.4 (kernel well-posedness of variational inequalities over generalized polyhedra). *We say that the parametric variational inequality* (1.1) *over the generalized polyhedron* (3.4) *exhibits the* KERNEL WELL-POSEDNESS *at the point* $(\bar{p},\bar{x}) \in \operatorname{gph} S$ *of differentiability of the base mapping* f *with respect to the decision variable if*

$$\begin{array}{c}\Big[\sup\big\{|\langle\nabla_x f(\bar{p},\bar{x})u, x_k\rangle|\ \big|\ u \in (\ker A) \cap B(X)\big\} \to 0,\ x_k \xrightarrow{w} 0,\\ x_k \in L := (\ker A) \cap \big(\ker\big\{x_i^*\big|\ i \in \mathcal{J}(\bar{x},\bar{x}^*)\big\}\Big] \Longrightarrow \|x_k\| \to 0 \ \text{ as } \ k \to \infty,\end{array} \tag{5.8}$$

where $\mathcal{J}(\bar{x},\bar{x}^*)$ *is the corresponding index set of positive multipliers with* $\bar{x}^* = -f(\bar{p},\bar{x})$.

It is obvious that the kernel well-posedness holds when the decision space X is finite-dimensional. The next proposition presents some sufficient conditions for the latter property in the case of reflexive Banach spaces X. Observe that the coercivity condition (b) therein significantly improves the one in [11, Proposition 5.6(c)] and its simple proof given below is independent of the Lax-Milgram theorem employed in [11].

Proposition 5.5 (sufficient conditions for kernel well-posedness in generalized polyhedral settings). *Let X be a reflexive Banach space, and let the subspace $L \subset X$ be defined in* (5.8). *Then the variational inequality* (1.1) *over* (3.4) *exhibits the kernel well-posedness under each of the following conditions:*

(a) *The adjoint operator $\nabla_x f(\bar{p},\bar{x})^*\colon X \to X^*$ is injective on L, i.e.,*

$$\big[\nabla_x f(\bar{p},\bar{x})^*(x_1 - x_2) = 0\big] \Longrightarrow \big[x_1 = x_2\big] \quad \text{for any} \quad x_1, x_2 \in L,$$

and the image space $(\nabla_x f(\bar{p},\bar{x})^*(L)$ is closed in X^*; both these properties are automatic when the operator $\nabla_x f(\bar{p},\bar{x})$ is surjective.

(b) *The operator $\nabla_x f(\bar{p},\bar{x})$ is coercive on L, i.e., there is some $\mu > 0$ such that*

$$\mu\|x\|^2 \le \langle \nabla_x f(\bar{p},\bar{x})x, x\rangle \quad \text{for all} \quad x \in L. \tag{5.9}$$

Proof. For case (a) it follows the lines in the proof of [11, Proposition 5.6(b)] with the kernel subspace L defined in (5.8) instead of the one from [11]. To justify the result in case (b), take a sequence $\{x_k\}$ on the left-hand side of (5.8) and observe by (5.9) that

$$\mu\|x_k\| \le \Big|\Big\langle \nabla_x f(\bar{p},\bar{x})\frac{x_k}{\|x_k\|}, x_k\Big\rangle\Big| \le \sup\big\{|\langle \nabla_x f(\bar{p},\bar{x})u, x_k\rangle| \;\big|\; u \in (\ker A) \cap B(X)\big\},$$

which implies that $\|x_k\| \to 0$ as $k \to \infty$ and completes the proof of the proposition. $\triangle$

The following lemma plays a key technical role in the proof of the main stability results of Theorem 5.7, where the parameter space is infinite-dimensional.

Lemma 5.6 (kernel well-posedness implies the PSNC property of solution maps). *In addition to the assumptions of Proposition* 5.2(ii)*, suppose that the kernel well-posedness condition from Definition* 5.4 *is satisfied. Then the solution map* (1.2) *is PSNC at $(\bar{p},\bar{x})$.*

Proof. To verify the PSNC property of the solution map S at $(\bar{p},\bar{x})$ according to its definition (2.11), take sequences $(p_k, x_k) \to (\bar{p},\bar{x})$ such that $(p_k, x_k) \in \operatorname{gph} S$ as $k \in I\!N$ and

$$(p_k^*, v_k^*) \in \widehat{N}\big((p_k, x_k); \operatorname{gph} S\big) \quad \text{with} \quad p_k^* \xrightarrow{w} 0 \quad \text{and} \quad \|v_k^*\| \to 0 \quad \text{as} \quad k \to \infty. \tag{5.10}$$

Since $\nabla_p f(\bar{p},\bar{x})$ is assumed to be surjective, the mapping $g\colon Z \times X \rightrightarrows X \times X^*$ defined in (3.3) has the surjective derivative at $(\bar{p},\bar{x})$. Applying now [19, Lemma 1.16] to the inverse image representation of the graph of S in (3.2), we find sequences $(\widetilde{x}_k, \widetilde{x}_k^*) \xrightarrow{\operatorname{gph}\mathcal{F}} (\bar{x}, -f(\bar{p},\bar{x}))$ such that $(\widetilde{x}_k, \widetilde{x}_k^*) \in \operatorname{gph}\mathcal{F}$ for all $k \in I\!N$ and

$$\begin{cases} (\widetilde{p}_k^*, \widetilde{v}_k^*) \in \nabla g(\bar{p},\bar{x})^*\widehat{N}\big((\widetilde{x}_k, \widetilde{x}_k^*); \operatorname{gph}\mathcal{F}\big) \quad \text{with} \\ \|\widetilde{p}_k^* - p_k^*\| \to 0 \quad \text{and} \quad \|\widetilde{v}_k^* - v_k^*\| \to 0 \quad \text{as} \quad k \to \infty. \end{cases} \tag{5.11}$$

It is easy to observe from (5.11) and the structure of g in (3.3) that there are

$$(u_k^*, u_k) \in \widehat{N}\big((\widetilde{x}_k, \widetilde{x}_k^*); \operatorname{gph} \mathcal{F}\big) \quad \text{for all} \quad k \in I\!N \tag{5.12}$$

satisfying the following relationships with $(\widetilde{p}_k^*, \widetilde{w}_k^*)$ from (5.11):

$$\widetilde{p}_k^* = -\nabla_p f(\bar{p}, \bar{x})^* u_k \quad \text{and} \quad \widetilde{v}_k^* = u_k^* - \nabla_x f(\bar{p}, \bar{x})^* u_k. \tag{5.13}$$

Proceeding now as in the proof of Theorem 4.1 for prenormals (5.12) and defining the active indices subsets $P \subset I_M \subset I(\bar{x})$ as in (4.4) and (4.6), respectively, we get that

$$u_k^* \in A^*(Y^*) + A_{I_M,P} \quad \text{and} \quad u_k \in B_{I_M,P} \cap (\ker A) \tag{5.14}$$

along a subsequence of $k \in I\!N$ with no relabeling. Hence

$$u_k^* = A^* y_k^* + \sum_{i \in I_M \setminus P} \lambda_{ik} x_i^* + \sum_{j \in P} \mu_{jk} x_j^* \tag{5.15}$$

for some $\lambda_{ik} \geq 0$ as $i \in I_M \setminus P$, $\mu_{jk} \in I\!R$ as $j \in P$, and $y_k^* \in Y^*$ via the generating elements $\{x_i^* \mid i \in I_M\}$ of (3.4) and the operator A in the generalized polyhedron description (3.4).

It follows from the convergence $\widetilde{p}_k^* \xrightarrow{w} 0$ due to (5.10) and (5.11) and from the surjectivity of the operator $\nabla_p f(\bar{p}, \bar{x})$ that $u_k \xrightarrow{w} 0$ as $k \to \infty$ by the first equality in (5.13). Employing again the relationships in (5.10) and (5.11), we get that

$$\|\widetilde{v}_k^*\| \leq \|\widetilde{v}_k^* - v_k^*\| + \|v_k^*\| \to 0 \quad \text{as} \quad k \to \infty. \tag{5.16}$$

Combining the latter with the second equality in (5.13) implies the convergence $u_k^* \xrightarrow{w} 0$ as $k \to \infty$. Furthermore, the sequence $\{u_k^*\}$ is bounded by the uniform boundedness principle. It is not hard to conclude from (5.15) by the standard contradiction arguments based on the linear independence assumption on the active generating element $\{x_i^* \mid i \in I(\bar{x})\}$ and the qualification condition (4.13) that the sequences $\{y_k^*\}$, $\{\lambda_{ik}\}$, and $\{\mu_{jk}\}$ are bounded for all $i \in I_M \setminus P$ and all $j \in P$, respectively. This ensures with no loss of generality that $y_k^* \xrightarrow{w} y^* \in Y^*$, $\lambda_{ik} \to \lambda_i \geq 0$, and $\mu_{jk} \to \mu_j \in I\!R$ as $k \to \infty$ whenever $i \in I_M \setminus P$ and $j \in P$. Now passing to the limit in (5.15) as $k \to \infty$, we arrive at

$$\sum_{i \in I_M \setminus P} \lambda_i x_i^* + \sum_{j \in P} \mu_j x_j^* + A^* y^* = 0. \tag{5.17}$$

It follows from (5.17) and the qualification condition (4.13) that

$$\sum_{i \in I_M \setminus P} \lambda_i x_i^* + \sum_{j \in P} \mu_j x_j^* = A^* y^* = 0,$$

which implies that $\lambda_i = 0$ for all $i \in I_M(\bar{x})$ and $\mu_j = 0$ for all $j \in P$ by the assumed linear independence of $\{x_i^* \mid i \in I_M\}$. This gives in turns that

$$\lim_{k \to \infty} \|u_k^* - A^* y_k^*\| = 0.$$

Thus we arrive at the limiting relationship

$$\lim_{k \to \infty} \sup_{u \in (\ker A) \cap B(X)} |\langle u_k^*, u \rangle| = 0,$$

which yields by (5.13) and (5.16) that

$$\lim_{k\to\infty} \sup_{u\in(\ker A)\cap B(X)} |\langle u_k, f(\bar{p},\bar{x})u\rangle| = 0. \tag{5.18}$$

Further, it follows from the construction of $B_{I_M,P}$ in (3.21) and the set monotonicity relationships in (5.7) that the second inclusion in (5.14) can be replaced by

$$u_k \in B_{I_M,P} \cap (\ker A) \subset B_{\mathcal{J},\mathcal{J}} \cap (\ker A) = (\ker A) \cap \ker\big\{x_i^*\big|\ i\in\mathcal{J}(\bar{x},\bar{x}^*)\big\}, \tag{5.19}$$

where the equality in (5.19) is a direct consequence of the definitions. Observe also that property (5.19) together with (5.18) and the kernel well-posedness of (1.1) at $(\bar{p},\bar{x})$ imply that $\|u_k\| \to 0$ and hence $\|\widetilde{p}_k^*\| \to 0$ as $k\to\infty$ by (5.13). Taking finally (5.10) into account allows us to conclude that the relationships in (5.11) imply that $\|p_k^*\|\to 0$ as $k\to\infty$, which thus justifies the PSNC property of S at $(\bar{p},\bar{x})$ and completes the proof of the lemma. $\triangle$

Combining the above pieces together, we now arrive at characterizing the Lipschitz-like property of the solution map (1.2), which is a (not full) counterpart of Theorem 5.3(ii) when the parameter space Z is infinite-dimensional.

Theorem 5.7 (Lipschitzian stability of parametric variational inequalities over generalized polyhedra in infinite dimensions). *Let the parameter space Z be a reflexive Banach space in the framework of Theorem* 5.3(ii). *Assume in addition the fulfillment of the kernel well-posedness condition from Definition* 5.4. *Then all the conclusions of Theorem* 5.3(ii) *holds true in this setting except that equality* (5.6) *is replaced by the following lower estimate of the exact Lipschitzian bound:*

$$\begin{aligned}\operatorname{lip} S(\bar{p},\bar{x}) \ge \sup\big\{\|\nabla_p f(\bar{p},\bar{x})^*u\|\big|\ u\in \operatorname{dom} D^*\mathcal{F}(\bar{p},\bar{x}),\\ -x^*-\nabla_x f(\bar{p},\bar{x})^*u \in A^*(Y^*)+\operatorname{span}\big\{x_i^*\big|\ i\in I_0(u)\big\}\\ +\operatorname{pos}\big\{x_i^*\big|\ i\in I_>(u)\big\},\ \|\nabla_x f(\bar{p},\bar{x})^*u+x^*\|\le 1\big\}.\end{aligned} \tag{5.20}$$

Proof. Observe first that the general assumptions of Theorem 2.1 are satisfied by Proposition 5.1. Furthermore, the PSNC property of S is satisfied under the kernel well-posedness by Lemma 5.6. Based now the coderivative characterization of the Lipschitz-like property from Theorem 2.1 and repeating the proof of Theorem 5.3(ii), we arrive at all the conclusion of this theorem, where the lower estimate for the exact Lipschitzian bound (5.20) follows from the corresponding estimate (2.14) in Theorem 2.1. This completes the proof. $\triangle$

To conclude this section and the whole paper, we derive explicit conditions ensuring the fulfillment of the coderivative criterion (5.5) in Theorems 5.3(ii) and 5.7 together with the kernel well-posedness of the variational inequality under consideration. Observe that the result below improves [11, Corollary 5.8] even in the case of standard convex polyhedra.

Corollary 5.8 (Lipschitzian stability under kernel coercivity over generalized polyhedra). *Let $(\bar{p},\bar{x})\in \operatorname{gph} S$ for the solution map* (1.2) *to the parametric variational inequality* (1.1) *over the generalized polyhedron* (3.4) *in reflexive Banach spaces X and Z, where the base mapping f is strictly differentiable at $(\bar{p},\bar{x})$ with the surjective partial derivative $\nabla_p f(\bar{p},\bar{x})$. Assume that the generating elements $\{x_i^* \mid i\in I(\bar{x})\}$ of* (3.4) *are* *linearly independent, that the qualification condition* (4.13) *is satisfied, that* $I(\bar{x}) = \mathcal{J}(\bar{x}, \bar{x}^*)$ *with* $\bar{x}^* = -f(\bar{p}, \bar{x})$*, and that* $\nabla_x f(\bar{p}, \bar{x})$ *is coercive on the kernel subspace* L *from* (5.8). *Then the solution map* S *is Lipschitz-like around* $(\bar{p}, \bar{x})$.

Proof. Observe first that the kernel well-posedness property from Definition 5.4 is satisfied under the assumed coercivity (5.9) by Proposition 5.5(b). By Theorem 5.7 it remains to check that the coderivative criterion (5.5) holds in this setting. We can easily see that the kernel subspace L from (5.8) admits the representations

$$L = B_{\mathcal{J},\mathcal{J}} \cap (\ker A) = B_{I,I} \cap (\ker A) \tag{5.21}$$

and that the criterion (5.5) can be equivalently written as

$$\left.\begin{array}{r} -\nabla_x f(\bar{p}, \bar{x})^* u \in A^*(Y^*) + \operatorname{span}\{x_i^* \mid i \in I(\bar{x})\} \\ \text{and} \quad -u \in (\ker A) \cap \ker\{x_i^* \mid i \in I(\bar{x})\} \end{array}\right\} \Longrightarrow u = 0. \tag{5.22}$$

By the coercivity (5.9) of $\nabla_x f(\bar{p}, \bar{x})$ on the kernel subspace L from (5.21), we find a constant $\mu > 0$ such that

$$\mu \|u\|^2 \le \langle \nabla_x f(\bar{p}, \bar{x}) u, u \rangle = \langle u, \nabla_x f(\bar{p}, \bar{x})^* u \rangle = 0$$

for any $u \in X$ satisfying the inclusions on the left-hand side of (5.22). The latter justifies the implication in (5.22) and thus completes the proof of the proposition. △

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