12-1-2009

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Department of Mathematics
Research Report

2009 Series
#12

This research was partly supported by the USA National Science Foundation

Well-Posedness of Minimal Time Problem with Constant Dynamics in Banach Spaces *

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Dedicated to Jean-Baptiste Hiriart-Urruty in honor of his 60th birthday

Abstract This paper concerns the study of a general minimal time problem with a convex constant dynamic and a closed target set in Banach spaces. We pay the main attention to deriving efficient conditions for the major well-posedness properties that include the existence and uniqueness of optimal solutions as well as certain regularity of the optimal value function with respect to state variables. Most of the results obtained are new even in finite-dimensional spaces. Our approach is based on advanced tools of variational analysis and generalized differentiation.

Keywords Minimal time function · Minimal time projection · Variational analysis · Generalized differentiation

Mathematical Subject Classifications (2000) 49J53 · 49J52 · 46B20 · 49J50

1 Introduction

A general class of *minimal time* problems with constant dynamics can be described as:

$$\text{minimize } t > 0 \text{ subject to } (x + tF) \cap C \neq \emptyset, \quad x \in X, \tag{1.1}$$

where X is a Banach space of *state variables*, $C \subset X$ is a closed *target* set, and $F \subset X$ is a closed, convex, and bounded set with $0 \in \operatorname{int} F$ reflecting the *constant dynamics* $\dot{x} \in F$ of the differential inclusion. These requirements on the dynamics and target are our *standing assumptions* in this paper. Various properties of optimal solutions to (1.1) were studied in [6, 7, 9, 13] and the references therein in finite and infinite dimensions.

*This work was partially supported by M.I.U.R., project "Viscosity, Metric, and Control Theoretic Methods for Nonlinear Partial Differential Equations" and by CARIPARO "Visiting Professors' Program"

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The major optimal characteristics of problem (1.1) are given by the *optimal value function* (known also as the *minimal value function*) defined by

$$T_C^F(x) = T(x) := \inf\{t > 0 \mid (x + tF) \cap C) \neq \emptyset\} = \inf_{y \in C} \rho_F(y - x), \tag{1.2}$$

where $\rho_F(\cdot)$ is the classical *Minkowski functional/gauge*

$$\rho_F(u) := \inf\{t > 0 \mid t^{-1}u \in F\}, \quad u \in X, \tag{1.3}$$

and by the *generalized/minimal time projection*

$$\Pi_C^F(x) := \{y \in C \mid \rho_F(y - x) = T(x)\}, \quad x \in X, \tag{1.4}$$

which is generally a set-valued mapping $\Pi\colon X \rightrightarrows C$ with possibly empty values.

Observe that the minimal value function (1.2) can be considered as a natural generalization of the classical *distance function*

$$\operatorname{dist}(x; C) := \inf_{y \in C} \|y - x\|, \quad x \in X, \tag{1.5}$$

which corresponds to (1.2) with $F = I\!B$, the closed unit ball of the space in question. In the latter case, the generalized projection (1.4) reduces to the usual *metric projection*

$$\Pi_C(x) := \{y \in C \mid \operatorname{dist}(x; C) = \|y - x\|\}, \quad x \in X, \tag{1.6}$$

of x onto C induced by the norm of the Banach space X under consideration.

The main objective of this paper is to study *well-posedness* of the minimal time problem (1.1) in finite-dimensional and infinite-dimensional spaces. By well-posedness we understand here the *existence* and *uniqueness* of the generalized projection (1.4) in connection with certain *regularity* properties of the minimal time function (1.2). Since the latter function is *intrinsically nonsmooth*, our study strongly involves the usage of appropriate tools of advanced *variational analysis* and *generalized differentiation.*

Concerning most relevant results of the previous investigations, note that the *proximal* and *Fréchet subdifferentials* of the minimal time function in *Hilbert spaces* are computed in [5, 6]. Furthermore, based on these formulas and adapting respective arguments used earlier in [2, 3, 4, 15] to study the metric projection (1.6), the authors of [5, 6] establish some well-posedness properties (existence, uniqueness, and certain regularity *near* the target C) of the minimal time projection (1.4). The assumptions in [5, 6] involve both "external sphere type" conditions on C, called *φ-convexity* or *proximal smoothness*, and a kind of uniform *strict convexity* of the dynamics F. Quite recently [7], these conditions have been sharpened and localized by introducing a certain *regular curvature* of a convex body. Besides that, an alternative hypothesis involving the *duality mapping* (see Section 2) is proposed in [7] requiring neither φ-convexity of the target set C nor rotundity/strict convexity of the dynamics F.

Formulas for evaluating the *proximal* and *Fréchet subdifferentials* of the minimal time function (1.2) in general Banach spaces, extending the corresponding results of [5, 6] from the Hilbert space setting, are obtained in [9]. The most recent results in this direction derived in [13] provide tight upper estimates as well as exact formulas for computing the *ε-subdifferentials* of the Fréchet type and the *limiting/Mordukhovich subdifferential* of the minimal time function at both *in-set* ($\bar{x} \in C$) and *out-of-set* ($\bar{x} \notin C$) points in arbitrary *Banach spaces* X. The results of [13] extend those obtained in [11, 12] for the latter subdifferentials of the distance function (1.5). They are used in what follows to establish some regularity properties of the minimal time function (1.2).

In this paper we develop an approach to the *existence* and *uniqueness* of the minimal time projection (1.4) that is different from the previous investigations discussed above. Namely, we study the existence and uniqueness of (1.4) through *subdifferentiability* of the minimal time function (1.2) at a *fixed* point $\bar{x} \in X \setminus C$ (not necessarily in a neighborhood of the target). A prototype result for the metric projection mapping (1.6) can be found in [1, Lemma 6], which proves the existence and uniqueness of the metric projection of $\bar{x} \in X \setminus C$ onto C provided that the distance function (1.5) is Fréchet subdifferentiable at $\bar{x}$ and that the space X admits a Fréchet smooth renorm. We obtain a similar result in this direction for the general minimal time projection (1.4) by using some *local rotundity* properties of the dynamics F expressed in terms of the *duality mapping* associated with F. We also derive other verifiable conditions, including *necessary and sufficient* ones, for the existence and uniqueness of the minimal time projection that are expressed in terms of the dynamics and target sets in (1.1).

Finally, we derive efficient conditions ensuring the *lower/subdifferential regularity* of (1.2) at both *in-set* and *out-of-set* points of the target. The assumptions imposed and the results obtained are illustrated by examples and counterexamples in both finite-dimensional and infinite-dimensional spaces.

The rest of the paper is organized as follows. Section 2 contains some definitions and *preliminary material* from convex and variational analysis widely used in the paper.

In Section 3 we establish relationships between the minimal time projection mapping (1.4) and its ε-enlargement, from one side, and the duality mapping for the dynamics and the Fréchet subdifferential of the minimal time function at the reference point from the other. This allows us to derive sufficient conditions for the existence and uniqueness of the minimal time projection via some *rotundity* properties of the *dynamics*.

In Section 4, by using a somewhat different approach in comparison with Section 3, we obtain *characterizations* of the existence and uniqueness of the minimal time projection in terms of the *Gâteaux differentiability* of the *support function* associated with an appropriate subset of the *target* C in (1.1). Sufficient conditions of another type are derived under the *Fréchet differentiability* of the support function associated with the ε-enlargement of the minimal time projection onto C.

The final Section 5 provides efficient conditions ensuring the *lower regularity* of the minimal time function (1.2) that imply, in the case of *reflexive* and more general *Asplund* spaces, the Fréchet *subdifferentiability* of (1.2) at the reference points, which is essentially used in the most interesting results of Sections 3 and 4 on the uniqueness of the minimal time projection (1.4). On the other hand, we present examples showing that the lower regularity of (1.2) is *not necessary* for the uniqueness of (1.4), while even

Fréchet *differentiability* alone does not ensure the uniqueness of generalized projections.

2 Basic Definitions and Preliminaries

In this section we present some constructions and facts from convex and variational analysis needed in what follows. The reader can consult the books [14] in the convex case and [10] in the general setting for more details, discussions, and further references.

Unless otherwise stated, the space X under consideration is arbitrary Banach with the norm $\|\cdot\|$, and the canonical pairing $\langle\cdot,\cdot\rangle$ between X and its topological dual X^*. Given a nonempty set $F \subset X$, we recall the construction of its *polar*

$$F^\circ := \{x^* \in X^* \big| \langle x^*, x\rangle \le 1 \text{ for all } x \in F\},$$

which is always a convex (even when F is nonconvex) and weak* closed subset of the dual space X^*. The *support function* $\sigma_F\colon X^* \to \overline{I\!R} := (-\infty, \infty]$ of F is

$$\sigma_F(x^*) = \sup_{x\in F}\langle x^*, x\rangle, \quad x^* \in X^*. \tag{2.1}$$

Note that (2.1) is always convex and lower semicontinuous (l.s.c.) on X^*, and it is *conjugate* to the *indicator function* $\delta_F(\cdot)$ of F equal to 0 for $x \in F$ and ∞ otherwise.

The *duality mapping* $\mathfrak{J}_F\colon F^\circ \rightrightarrows F$ for F is defined by

$$\mathfrak{J}_F(x^*) := \{x \in F \big| \langle x^*, x\rangle = \sigma_F(x^*)\}, \quad x^* \in F^\circ,$$

which reduces to the simplified representation on the boundary $\mathrm{bd}\, F^\circ$ of F°:

$$\mathfrak{J}_F(x^*) = \{x \in F \big| \langle x^*, x\rangle = 1\}, \quad x^* \in \mathrm{bd}\, F^\circ. \tag{2.2}$$

Furthermore, in the case of reflexive spaces X we have the relationship

$$\partial\sigma_F(x^*) = \mathfrak{J}_F(x^*) \text{ for all } x^* \in F^\circ, \tag{2.3}$$

where ∂ stands for the classical *subdifferential* of *convex analysis*. It is worth mentioning that, under our standing assumptions on the dynamics F, both the Minkowski gauge (1.3) and the minimal time function (1.2) are *Lipschitz continuous* on X. Furthermore,

$$\rho_F(x) = \sigma_{F^\circ}(x) \text{ and } \rho_{F^\circ}(x^*) = \sigma_F(x^*) \text{ for all } x \in X,\ x^* \in X^*. \tag{2.4}$$

Using (1.2) and (1.3) allows us to consider, together with the (possibly empty) minimal time projection (1.4), its always nonempty enlargement called ε-*approximate minimal time projection* as $\varepsilon > 0$ and denoted by

$$\Pi_C^F(x,\varepsilon) := \{y \in C \big| \rho_F(y-x) \le T(x) + \varepsilon\} \ne \emptyset \text{ for all } x \in X. \tag{2.5}$$

Note that both the minimal time projection (1.4) and its ε-enlargement (2.5), for all $\varepsilon > 0$, are *closed* subsets of X under our standing assumptions.

Let us further recall two outer/upper limits of the Painlevé-Kuratowski type needed in this paper. Given a set-valued mapping $G\colon Y \rightrightarrows Z$ between two Banach spaces, consider the *weak sequential outer limit* of G at $\bar{y}$ defined and denoted by

$$w\text{-}\mathop{\rm Lim\,sup}_{y\to\bar{y}} G(y) := \Big\{z\in Z\Big|\ \exists \text{ sequences } y_k\to\bar{y} \text{ and } z_k \xrightarrow{w} z, \text{ as } k\to\infty \text{ with } z_k\in G(y_k) \text{ for all } k\in I\!N := \{1,2,\ldots\}\Big\}, \tag{2.6}$$

where $\xrightarrow{w}$ signifies the weak topology of the image space Z.

Another version of the sequential Painlevé-Kuratowski outer limit needed below concerns set-valued mappings $G\colon X \rightrightarrows X^*$ between a Banach space X and its topological dual. We say that the construction

$$w^*\text{-}\mathop{\rm Lim\,sup}_{x\to\bar{x}} G(x) := \Big\{x^*\in X^*\Big|\ \exists \text{ sequences } x_k\to\bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^*\in G(x_k) \text{ for all } k\in I\!N\Big\} \tag{2.7}$$

is the *w^*-sequential outer limit* of G as $x\to\bar{x}$, where w^* signifies the weak* topology of the dual space X^*. Note that we use *sequences* in both constructions (2.6) and (2.7) although neither the weak topology of a Banach space nor the weak* topology of a dual Banach space is generally sequential.

Consider next an extended-real-valued function $\varphi\colon X\to\overline{I\!R}$ on a Banach space X. Given $\varepsilon\ge 0$, the *ε-subdifferential* of φ at $\bar{x}\in\operatorname{dom}\varphi := \{x\in X|\ \varphi(x)<\infty\}$ is defined by

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) := \Big\{x^*\in X^*\Big|\ \liminf_{x\to\bar{x}} \frac{\varphi(x)-\varphi(\bar{x})-\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|}\ge -\varepsilon\Big\} \tag{2.8}$$

with $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0\varphi(\bar{x})$ called the *Fréchet subdifferential* if φ at $\bar{x}$. If φ is Fréchet differentiable at $\bar{x}$, then $\widehat{\partial}\varphi(\bar{x})$ reduces to the classical Fréchet derivative. In general the set $\widehat{\partial}_\varepsilon\varphi(\bar{x})$ is convex for any $\varepsilon\ge 0$ while it may often be empty for nonconvex functions as, e.g., in the case of $\varphi(x)=-|x|$ at $\bar{x}=0\in I\!R$. Observe furthermore that the subdifferential construction (2.8), including $\widehat{\partial}\varphi(\cdot)$, does not satisfy pointwise calculus rules (for sums, compositions, etc.) required in various applications. This is dramatically improved for the sequential limiting construction

$$\partial\varphi(\bar{x}) := w^*\text{-}\mathop{\rm Lim\,sup}_{\substack{x\xrightarrow{\varphi}\bar{x}\\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon\varphi(x), \tag{2.9}$$

known as the *basic/limiting/Mordukhovich subdifferential* of φ at $\bar{x}$, where $\bar{x}\xrightarrow{\varphi}\bar{x}$ stands for $x\to\bar{x}$ with $\varphi(x)\to\varphi(\bar{x})$. We can equivalently put $\varepsilon=0$ in (2.9) if φ is l.s.c. around $\bar{x}$ and the space X is *Asplund*, i.e., each separable subspace of it has a separable dual. The latter subclass of Banach spaces is sufficiently large including, in particular, every reflexive space and every Banach space with a separable dual. Note that the basic subdifferential (2.9) reduces to the classical derivative for smooth functions φ and to the subdifferential of convex analysis when φ is convex, while the set (2.9) may often be nonconvex for simple nonconvex functions; e.g., $\partial\varphi(0)=\{-1,1\}$ for $\varphi(x)=-|x|$. Recall also that $\partial\varphi(\bar{x})\neq\emptyset$ if φ is locally Lipschitzian around $\bar{x}$ and the space X is Asplund.

It is clear from (2.9) that we always have the inclusion

$$\widehat{\partial}\varphi(\bar{x})\subset\partial\varphi(\bar{x}),\quad \bar{x}\in\operatorname{dom}\varphi. \tag{2.10}$$

A function $\varphi\colon X\to\overline{I\!R}$ is called to be *lower regular* at $\bar{x}$ if (2.10) holds as equality. Besides smooth functions, convex functions and the like, the latter property is satisfied for various classes of "nice" functions important in applications. Furthermore, there is a well-developed *calculus* ensuring the preservation of lower regularity under a variety of operations; see [10] for more details. We can easily deduce from the previous discussions that $\widehat{\partial}\varphi(\bar{x})\neq\emptyset$ if φ is locally Lipschitzian around $\bar{x}$ and lower regular at this point provided that either X is Asplund, or X is arbitrarily Banach and φ is convex. It is particularly important in the framework of this paper, where the minimal time function (1.2) is Lipschitz continuous on X while its Fréchet differentiability $\widehat{\partial}T(\bar{x})\neq\emptyset$ is used in the major results established below.

Finally in this section, define the *normal cones*

$$\widehat{N}(\bar{x};\Omega):=\widehat{\partial}\delta_\Omega(\bar{x})\ \text{ and }\ N(\bar{x};\Omega):=\partial\delta_\Omega(\bar{x}) \tag{2.11}$$

to a set $\Omega\subset X$ at $\bar{x}\in\Omega$ generated by the corresponding subdifferentials (2.8) and (2.9) of the indicator function of Ω. Note that for $X=I\!R^n$ the limiting normal cone in (2.11) can be equivalently described via the metric projection (1.6) by

$$N(\bar{x};\Omega)=\operatorname*{Lim\,sup}_{x\to\bar{x}}\big[\operatorname{cone}\big(x-\Pi_\Omega(x)\big)\big],$$

where the symbol "cone" stands for the conic hull of the set. We say that Ω is *normally regular* at $\bar{x}\in\Omega$ if $\widehat{N}(\bar{x};\Omega)=N(\bar{x};\Omega)$. This property holds for convex and "nice" nonconvex sets; it satisfies rich calculus ensuring its preservation under various operations.

3 Minimal Time Projection via Dynamics Rotundity

In this section we first establish general relationships between the minimal time projection (1.4) and its enlargement (2.5), from one side, and the minimal time function (1.2) and the duality mapping (2.2) from the other in arbitrary Banach spaces. Then these relationships are used, in the reflexive space framework, for deriving conditions ensuring the existence and uniqueness of generalized projections via of a certain rotundity property of the dynamics that, as shown by an example, cannot be generally removed.

We begin with the following unconditional inclusion, which is definitely of its own interest, being at the same time crucial for the subsequent applications in this paper.

Theorem 3.1 (approximate projection via minimal time function and duality mapping in Banach spaces). *For each $\bar{x}\notin C$ we have the inclusion*

$$w\text{-}\operatorname*{Lim\,sup}_{\varepsilon\downarrow 0}\Pi_C^F(\bar{x},\varepsilon)\subset\bigcap_{x^*\in\widehat{\partial}T(\bar{x})}\big(\bar{x}+T(\bar{x})\mathfrak{J}_F(-x^*)\big). \tag{3.1}$$

Proof. In the case of $\widehat{\partial}T(\bar{x}) = \emptyset$ inclusion (3.1) is trivial, since the right-hand side of (3.1) is the whole space X by the standard convention. Assuming that $\widehat{\partial}T(\bar{x}) \neq \emptyset$, we have $\widehat{\partial}T(\bar{x}) \subset -\mathrm{bd}\, F^0$ by [9, Theorem 4.2], which is a Banach space extension of the Hilbert space result of [6, Theorem 3.1]. This implies by (2.2) that

$$\mathfrak{J}_F(-x^*) = \{x \in F \mid \langle -x^*, x\rangle = 1\} \text{ for } -x^* \in \widehat{\partial}T(\bar{x}). \tag{3.2}$$

Select now an arbitrary element $y \in w\text{-}\mathop{\mathrm{Lim\,sup}}_{\varepsilon\downarrow 0} \Pi_C^F(\bar{x},\varepsilon)$ and find by (2.6) a sequence $\{y_k\}$ with $y_k \in \Pi_C^F(\bar{x},\varepsilon_k)$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$. Denoting $t_k := \varepsilon_k^{1/2} > 0$, suppose without loss of generality that $T(\bar{x}) \leq \rho_F(y_k - \bar{x}) < T(\bar{x}) + t_k^2$. Taking an arbitrary Fréchet subgradient $x^* \in \widehat{\partial}T(\bar{x})$, we get from definition (2.8) as $\varepsilon = 0$ that

$$\liminf_{x\to\bar{x}} \frac{T(x) - T(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0. \tag{3.3}$$

Define further a sequence $\{x_k\} \subset X$ by

$$x_k := \bar{x} + t_k(y_k - \bar{x}), \quad k \in I\!N,$$

and observe that $x_k \to \bar{x}$ as $k \to \infty$, since $\{y_k\}$ is bounded in X. Furthermore, $x_k \neq \bar{x}$ for all $k \in I\!N$ by construction. It thus follows from (3.3) that there is a sequence $\{\alpha_k\} \subset I\!R$ with $\lim_{k\to\infty} \alpha_k \geq 0$ for which we have the inequality

$$\frac{T(x_k) - T(\bar{x})}{t_k} \geq \langle x^*, y_k - \bar{x}\rangle + \alpha_k, \quad k \in I\!N.$$

The latter implies by the definitions of the minimal time and Minkowski functions and by the constructions of the sequences involved that

$$\langle x^*, y_k - \bar{x}\rangle + \alpha_k \leq \frac{\rho_F(y_k - x_k) - \rho_F(y_k - \bar{x}) + t_k^2}{t_k} = -\rho_F(y_k - \bar{x}) + t_k, \quad k \in I\!N.$$

By passing to the "lim sup" as $k \to \infty$ above, we arrive at

$$\liminf_{k\to\infty} \langle x^*, \bar{x} - y_k\rangle \geq \lim_{k\to\infty} \rho_F(y_k - \bar{x}) = T(\bar{x}). \tag{3.4}$$

On the other hand, by the choice of x^* we have the relationships

$$\langle x^*, \bar{x} - y_k\rangle = \langle -x^*, y_k - \bar{x}\rangle \leq \sigma_{F^0}(y_k - \bar{x}) = \rho_F(y_k - \bar{x}),$$

which imply, by passing to the "lim sup" as $k \to \infty$, that

$$\limsup_{k\to\infty} \langle x^*, \bar{x} - y_k\rangle \leq \lim_{k\to\infty} \rho_F(y_k - \bar{x}) = T(\bar{x}). \tag{3.5}$$

Comparing (3.5) with (3.4) allows us to conclude that the limit $\lim_{k\to\infty} \langle x^*, \bar{x} - y_k\rangle$ exists and equals $T(\bar{x})$. This implies, since the sequence $\{y_k\}$ *weakly converges* in X to the element y fixed above, that

$$\langle x^*, \bar{x} - y_k\rangle \to \langle x^*, \bar{x} - y\rangle \text{ as } k \to \infty,$$

and thus $\langle x^*, \bar{x} - y\rangle = T(\bar{x})$. Furthermore, we have

$$\lim_{k\to\infty} \rho_F(y_k - \bar{x}) = T(\bar{x}) = \langle x^*, \bar{x} - y\rangle \le \sigma_{F^0}(y - \bar{x}) = \rho_F(y - \bar{x}). \tag{3.6}$$

By the *convexity* of $\rho_F(\cdot)$, it is *weakly lower semicontinuous* on X. Hence

$$\liminf_{k\to\infty} \rho_F(y_k - \bar{x}) \ge \rho_F(y - \bar{x}). \tag{3.7}$$

Combining the relationships in (3.6) and (3.7) gives us

$$\lim_{k\to\infty} \rho_F(y_k - \bar{x}) = \rho_F(y - \bar{x}) = T(\bar{x}),$$

which in turn implies the equalities

$$\Big\langle x^*, \frac{\bar{x} - y}{\rho_F(y - \bar{x})}\Big\rangle = 1 \quad\text{and}\quad \frac{\bar{x} - y}{\rho_F(y - \bar{x})} \in -\mathrm{bd}\, F.$$

Since $-x^* \in \mathrm{bd}\, F^0$ by the above, we get the inclusion

$$\frac{y - \bar{x}}{\rho_F(y - \bar{x})} \in \mathfrak{J}_F(-x^*), \quad\text{i.e.,}\quad y \in \bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*),$$

which concludes the proof of the theorem by taking into account that the Fréchet subgradient $-x^* \in \widehat{\partial} T(\bar{x})$ was chosen arbitrarily. △

As a consequence of Theorem 3.1, we obtain a *precise representation* of the (possibly empty) minimal time projection (1.4) in arbitrary Banach spaces under our standing assumptions made on the dynamics and target.

Corollary 3.2 (precise representation of minimal time projection in Banach spaces). *For each $\bar{x} \notin C$ and $x^* \in \widehat{\partial} T(\bar{x})$ the following holds:*

$$\Pi_C^F(\bar{x}) = \big(\bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*)\big) \cap C. \tag{3.8}$$

Proof. It follows from Theorem 3.1 that for any fixed $\bar{x} \notin C$ and $x^* \in \widehat{\partial} T(\bar{x})$ we have

$$\Pi_C^F(\bar{x}) \subset w\text{-}\mathop{\mathrm{Lim\,sup}}_{\varepsilon\downarrow 0} \Pi_C^F(\bar{x}, \varepsilon) \subset \bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*).$$

This implies, by taking into account the obvious one $\Pi_C^F(\bar{x}) \subset C$, that the inclusion "$\subset$" holds in (3.8). To justify the opposite inclusion "$\supset$" in (3.8), observe that for any

$$y \in \big(x + T(\bar{x})\mathfrak{J}_F(-x^*)\big) \cap C \subset (\bar{x} + T(\bar{x})\mathrm{bd}\, F) \cap C$$

we get by the constructions of (1.2) and (2.2) with $x^* \in \widehat{\partial} T(\bar{x}) \subset -\mathrm{bd}\, F^\circ$ that

$$y \in C \quad\text{and}\quad \rho_F(y - \bar{x}) = T(\bar{x}),$$

which gives $y \in \Pi_C^F(\bar{x})$ and thus completes the proof of the corollary. △

Now we are ready to derive the first result of this paper ensuring the *existence* and *uniqueness* of the minimal time projection (1.4) in *reflexive* Banach spaces.

Theorem 3.3 (existence and uniqueness of minimal time projection for rotund dynamics). *Let X be reflexive, and let $\bar{x} \notin C$ with $\widehat{\partial} T(\bar{x}) \neq \emptyset$. Assume in addition that for some $x^* \in \widehat{\partial} T(\bar{x})$ the support function $\sigma_F(\cdot)$ of the dynamics is Fréchet differentiable at $-x^*$. Then the set $\emptyset \neq \Pi_C^F(\bar{x})$ is a singleton.*

Proof. Since the approximate projection (2.5) is nonempty, closed, and bounded subset of X for any $\varepsilon > 0$, we deduce from the *reflexivity* of X that the weak sequential outer limit $w\text{-}\operatorname{Lim}\sup_{\varepsilon \downarrow 0} \Pi_C^F(\bar{x}, \varepsilon)$ is a nonempty subset of X. Pick any element y from the latter set that surely belongs to $\overline{\text{co}}\, C$, the convex closure of C. We have by Theorem 3.1 that $y \in \bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*)$. Furthermore, by the *Fréchet differentiability* of $\sigma_F(\cdot)$ (or, equivalently, of $\rho_{F^o}(\cdot)$) at $-x^*$, it follows from [14, Proposition 5.11] that $\mathfrak{J}_F(-x^*)$ is a *singleton* $\{v\}$, which is a *strongly exposed* point of F; see more discussions in Remark 3.4(ii) below. By the above choice y we conclude that y is a *weak* limit of a minimizing sequence $\{y_k\} \subset C$ with $\rho_F(y_k - \bar{x}) \to T(\bar{x})$ as $k \to \infty$. Then we have the weak convergence

$$\frac{y_k - \bar{x}}{\rho_F(y_k - \bar{x})} \xrightarrow{w} \frac{y - \bar{x}}{T(\bar{x})} = v \quad \text{as} \quad k \to \infty. \tag{3.9}$$

Since v is a strongly exposed point of F, the convergence in (3.9) is indeed *strong* in X, and thus $y_k \to y$ as $k \to \infty$. This gives $y \in C$, which yields that $y \in \Pi_C^F(\bar{x}) \neq \emptyset$ by Corollary 3.2. Observe finally that the above arguments with the usage of Corollary 3.2 ensure in fact that the minimal time projection set $\Pi_C^F(\bar{x})$ is a singleton. $\triangle$

Remark 3.4 (on Fréchet subdifferentiability and differentiability properties).
(i) Since any reflexive space is Asplund and since the minimal time function (1.2) is Lipschitz continuous, $T(\cdot)$ is *Fréchet subdifferentiable* $\widehat{\partial} T(\bar{x}) \neq \emptyset$ at a point $\bar{x}$ if, in particular, it is *lower regular* at this point. Indeed, it follows from the definition of lower regularity in Section 2 and the fact that the basic subdifferential (2.9) is nonempty for any locally Lipschitzian function on an Asplund space; see [10, Corollary 2.25].

(ii) *Fréchet differentiability* of the support function $\sigma_F(\cdot)$ reflects certain (strong) *rotundity* properties of the dynamics F; some characterizations and verifiable sufficient conditions ensuring this property in Hilbert spaces are given in [8]. Note, in particular, that the Fréchet differentiability of $\sigma_F(\cdot)$ at x^* is equivalent to the existence of a unique point $\bar{x} \in \operatorname{bd} F$ such that the *rotundity modulus*

$$\mathfrak{R}(\bar{x}, x^*) := \big\{ \langle x^*, \bar{x} - x \rangle \big| \; x \in F, \; \|x - \bar{x}\| \geq r \big\}$$

is positive for each $r > 0$. In this case we have $\bar{x} = \nabla \sigma_F(x^*)$. In turn, the condition $\mathfrak{R}(\bar{x}, x^*) > 0$ is equivalent to say that $\bar{x}$ is a *strongly exposed point* of the set F, in the sense that the hyperplane

$$H(x^*) := \big\{ x \in X \big| \; \langle x^*, x \rangle = \sigma_F(x^*) \big\} \tag{3.10}$$

touches F at the point $\bar{x}$ only (i.e., $\mathfrak{J}_F(x^*) = \{\bar{x}\}$) and that each sequence $\{x_k\} \subset F$ with $\langle x^*, x_k - \bar{x} \rangle \to 0$ *strongly* converges to $\bar{x}$ as $k \to \infty$; in fact, the latter property is used in the proof of Theorem 3.3. The following condition is sufficient for the equivalent properties above: taking a unique point $\bar{x} \in \operatorname{bd} F$ at which x^* is normal to F, we have

$$\sup_{\|x-\bar{x}\|>r} \rho_F(\bar{x}+x) < 2 \text{ for each } r > 0.$$

Let us show next that the Fréchet differentiability assumption on $\sigma_F(\cdot)$ in Theorem 3.3 *cannot be removed*, for an arbitrary infinite-dimensional Hilbert space X, in order to ensure the *existence* of the minimal time projection (1.4). Furthermore, the generalized projection may exist without this assumption while not being *unique*.

Example 3.5 (non-existence and non-uniqueness of generalized projection for dynamics with nonsmooth support function). *In an arbitrary Hilbert space X there are dynamics sets $F \subset X$ and targets sets $C \subset X$ satisfying our standing assumptions and such that $T(\cdot)$ is Fréchet differentiable at 0, that $\sigma_F(\cdot)$ is not Fréchet differentiable at $\nabla T(0)$ while $\Pi_C^F(0) = \emptyset$. Furthermore, there are F and C with the above properties for which $\Pi_C^F(0) \neq \emptyset$ while the latter projection set in not a singleton.*

Proof. Let X be an arbitrary infinite-dimensional Hilbert space with an orthonormal base $\{e_n\} \subset X$. Define the dynamics and target sets by, respectively,

$$F := \Big\{x \in X \Big| \; \|x\| \le 1, \; \langle x, e_1\rangle \le \frac{1}{2}\Big\}, \tag{3.11}$$

$$C := \Big\{\Big(\frac{1}{2}+\frac{1}{n}\Big)\Big(\frac{e_n}{2}+e_1\Big)\Big| \; n \ge 2\Big\}. \tag{3.12}$$

It is easy to check that the sets F and C satisfy our standing assumptions formulated in Section 1. Observe that $T(x) = 1 - 2\langle x, e_1\rangle$ in a neighborhood of the origin, and so the minimal time function is *Fréchet differentiable* therein. We have $T(0) = 1$ and $\nabla T(0) = -2e_1$ while $\rho_F(y) > 1$ for all $y \in C$. The latter means that the minimum of $\rho_F(\cdot)$ is not attained in C, and thus $\Pi_C^F(0) = \emptyset$. Observe furthermore that

$$\mathfrak{J}_F\big(-\nabla T(0)\big) = \Big\{x \in X \Big| \; \|x\| \le 1, \; \langle x, e_1\rangle = \frac{1}{2}\Big\},$$

which contains, in particular, the points $\frac{1}{2}e_1 + \lambda e_n$ with $\lambda^2 \le 3/4$, $n = 2, 3, \ldots$. Thus the support function $\sigma_F(\cdot)$ is *not Fréchet differentiable* at $-\nabla T(0)$.

By keeping the same dynamics (3.11) while changing the target (3.12) by

$$\widetilde{C} := \Big\{\Big(\frac{1}{2}+\frac{1}{n}\Big)\frac{e_n}{2}\Big| \; n \ge 2\Big\},$$

we easily check that the minimal time projection $\Pi_C^F(0)$ is *nonempty*, but it is *not a singleton*. In this case the minimal time function $T(\cdot)$ remains the same. △

4 Properties of Minimal Time Projection via Target Set

In this section we derive new verifiable conditions for the *existence* and *uniqueness* of the minimal time projection (1.4) that, in contrast to the "dynamics" ones in Section 3, exploit the corresponding properties of the *target* set C. Some of the results obtained are *necessary* and *sufficient* for the generalized projection set $\Pi_C^F(\bar{x})$ to be a *singleton.* As in Section 3, we essentially use here the underlying projection formula established above in Corollary 3.2 applying it to a modified projection operator.

First we use Corollary 3.2 to interpret the minimal time function (1.2) and the minimal time projection (1.4) in a somewhat different way. Assuming that $\widehat{\partial}T(\bar{x}) \neq \emptyset$ for some $\bar{x} \in X$ and picking a Fréchet subgradient $x^* \in \widehat{\partial}T(\bar{x})$, consider the *modified minimal time problem*

$$\text{minimize } t > 0 \text{ subject to } \big(x + t\mathfrak{J}_F(-x^*)\big) \cap C \neq \emptyset, \quad x \in X, \tag{4.1}$$

for which the dynamics F in (1.1) is replaced by the duality mapping (2.2). Denote the *modified minimal time function* in (4.1) by

$$\widetilde{T}_{x^*}(x) = \inf_{y \in C} \rho_{\mathfrak{J}_F(-x^*)}(y - x) \tag{4.2}$$

and the corresponding *modified minimal time projection* operator by

$$\widetilde{\Pi}_C^F(x, x^*) := \Big\{ y \in C \Big| \, \rho_{\mathfrak{J}_F(-x^*)}(y - x) = \widetilde{T}_{x^*}(x) \Big\}. \tag{4.3}$$

The next proposition provides conditions for the *nonemptiness* of the original minimal time projection (1.4) in terms of the modified problem (4.1).

Proposition 4.1 (existence of minimal time projection via modified problem). *For any fixed $\bar{x} \in X$ and $x^* \in \widehat{\partial}T(\bar{x})$ we have that $\Pi_C^F(\bar{x}) \neq \emptyset$ if and only if $T(\bar{x}) = \widetilde{T}_{x^*}(\bar{x})$ and $\widetilde{\Pi}_C^F(\bar{x}, x^*) \neq \emptyset$. In this case $\widetilde{\Pi}_C^F(\bar{x}, x^*) = \Pi_C^F(\bar{x})$.*

Proof. With no loss of generality, suppose that $\bar{x} \notin C$. Assuming that $\Pi_C^F(\bar{x}) \neq \emptyset$ and taking the selected subgradient $x^* \in \widehat{\partial}T(\bar{x})$ give us by Corollary 3.2 that

$$\big(\bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*)\big) \cap C \neq \emptyset, \text{ and hence } \widetilde{T}_{x^*}(\bar{x}) \leq T(\bar{x}).$$

On the other hand, we have the relationships

$$\mathfrak{J}_F(-x^*) \subset F, \text{ and thus } \rho_{\mathfrak{J}_F(-x^*)}(y - \bar{x}) \geq \rho_F(y - \bar{x}) \text{ for all } y \in C. \tag{4.4}$$

Taking the infimum in (4.4) over $y \in C$ gives us $\widetilde{T}_{x^*}(\bar{x}) \geq T(\bar{x})$. Thus we get

$$\widetilde{T}_{x^*}(\bar{x}) = T(\bar{x}) \text{ and } \widetilde{\Pi}_C^F(\bar{x}, x^*) = C \cap \big(\bar{x} + \widetilde{T}_{x^*}(\bar{x})\mathfrak{J}_F(-x^*)\big) = \Pi_C^F(\bar{x}) \neq \emptyset,$$

which proves the "only if" part of the proposition.

To justify the "if" part, suppose that $T(\bar{x}) = \widetilde{T}_{x^*}(\bar{x})$ and $\widetilde{\Pi}_C^F(\bar{x}, x^*) \neq \emptyset$. Then we easily conclude by Corollary 3.2 that

$$\begin{aligned}\Pi_C^F(\bar{x}) &= \big(\bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*)\big) \cap C \\ &= \big(\bar{x} + \widetilde{T}_{x^*}(\bar{x})\mathfrak{J}_F(-x^*)\big) \cap C = \widetilde{\Pi}_C^F(\bar{x}, x^*) \neq \emptyset,\end{aligned}$$

which thus completes the proof of the proposition. △

To proceed with deriving efficient conditions for well-posedness of the minimal time problem (1.1) in terms of the target set C, we need the following representation (Proposition 4.2) of the modified projection (4.3) via the support function of some subset of the target. Given $x^* \in \widehat{\partial} T(\bar{x})$, denote by K_{x^*} the *convex and closed cone* generated by the duality mapping value $\mathfrak{J}_F(-x^*)$. Consider further the set

$$C(\bar{x}, x^*) := C \cap (\bar{x} + K_{x^*}), \tag{4.5}$$

which is nonempty *if and only if* $\widetilde{T}_{x^*}(\bar{x}) < \infty$ for the modified minimal time function (4.2). Thus it follows from Proposition 4.1 that $C(\bar{x}, x^*) \neq \emptyset$ if $\Pi_C^F(\bar{x}) \neq \emptyset$.

Proposition 4.2 (representation of modified minimal time projection in Banach spaces). *Given $\bar{x} \notin C$ and $x^* \in \widehat{\partial} T(\bar{x})$, we have the representation*

$$\widetilde{\Pi}_C^F(\bar{x}, x^*) = \big\{ y \in C(\bar{x}, x^*) \big|\ \sigma_{C(\bar{x}, x^*)}(x^*) = \langle x^*, y \rangle \big\} \tag{4.6}$$

of the modified projection (4.3) *via the support function of* (4.5).

Proof. If y belongs to the right-hand side of (4.6), then

$$\langle x^*, y \rangle \geq \langle x^*, v \rangle \ \text{ for all } \ v \in C \ \text{ with } \ v - \bar{x} \in K_{x^*}. \tag{4.7}$$

Moreover, $y \in C(\bar{x}, x^*)$, i.e., $y \in C$ and there exists $\lambda > 0$ with $y \in \bar{x} + \lambda \mathfrak{J}_F(-x^*)$.

Let us show that $\lambda = \widetilde{T}_{x^*}(\bar{x})$. Indeed, we have $\langle -x^*, y - \bar{x} \rangle = \lambda$, and thus inequality (4.7) can be rewritten as follows:

$$\lambda \leq t \ \text{ for all } \ t > 0 \ \text{ such that } \ C \cap \big(\bar{x} + t\mathfrak{J}_F(-x^*)\big) \neq \emptyset.$$

This implies that $\lambda \leq \widetilde{T}_{x^*}(\bar{x})$, since $\widetilde{T}_{x^*}(\bar{x})$ is the *infimum* of all such $t > 0$ by (4.1) and (4.2). On the other hand, we have

$$y \in \big(\bar{x} + \lambda \mathfrak{J}_F(-x^*)\big) \cap C \neq \emptyset,$$

which yields $\widetilde{T}_{x^*}(\bar{x}) \leq \lambda$. Thus $\lambda = \widetilde{T}_{x^*}(\bar{x}) < \infty$, and furthermore

$$\rho_{\mathfrak{J}_F(-x^*)}(y - \bar{x}) \leq \lambda = \widetilde{T}_{x^*}(\bar{x}), \ \text{ i.e., } y \in \widetilde{\Pi}_C^F(\bar{x}, x^*).$$

This gives $y \in \widetilde{\Pi}_C^F(\bar{x}, x^*)$ and justifies the inclusion "$\supset$" in (4.6).

To prove the inclusion "$\subset$" in (4.6), pick any $y \in \widetilde{\Pi}^F_C(\bar{x}, x^*)$ and get by (4.3) that

$$\rho_{\mathfrak{J}_F(-x^*)}(y - \bar{x}) \le \rho_{\mathfrak{J}_F(-x^*)}(v - \bar{x}) \text{ for all } v \in C.$$

Define further $\lambda := \widetilde{T}_{x^*}(\bar{x}) = \rho_{\mathfrak{J}_F(-x^*)}(y - \bar{x})$ and observe that $y \in \bar{x} + \lambda \mathfrak{J}_F(-x^*)$. On the other hand, take $t > 0$ to be sufficiently small to have $C \cap \big(\bar{x} + t\mathfrak{J}_F(-x^*)\big) \ne \emptyset$. In this case there is $v \in C$ with $t = \langle -x^*, v - \bar{x} \rangle$, and we get $\lambda \le t$. Taking now into account that $\lambda = \langle -x^*, y - \bar{x} \rangle$ by construction, we arrive at

$$\langle x^*, y - \bar{x} \rangle \ge \langle x^*, v - \bar{x} \rangle \text{ for each } v \in C \text{ with } v - \bar{x} \in t\mathfrak{J}_F(-x^*),\ t > 0.$$

It gives $\langle x^*, y \rangle \ge \langle x^*, v \rangle$ whenever $v \in C \cap (\bar{x} + K_{x^*})$, which justifies the inclusion "$\subset$" in (4.6) and completes the proof of the proposition. $\triangle$

Next we use the above propositions to derive *necessary* and *sufficient* conditions for the existence and uniqueness of the minimal time projection (1.4) at the point in question involving the modified value function (4.2) and appropriate subsets of the target set C. Note the results obtained in the following theorem are new for the uniqueness part even in the cases when the existence is already known as, e.g., in finite dimensions as well as for convex target sets in reflexive spaces.

Theorem 4.3 (characterization of well-posedness of minimal time projection via subsets of target). *Let X be a reflexive Banach space, and let $\bar{x} \in X$. The minimal time projection set $\Pi^F_C(\bar{x})$ is a nonempty singleton if and only if there exists a Fréchet subgradient $x^* \in \widehat{\partial} T(\bar{x})$ such that $\widetilde{T}_{x^*}(\bar{x}) = T(\bar{x})$ and the support function $\sigma_{C(\bar{x},x^*)}(\cdot)$ of (4.5) is Gâteaux differentiable at x^*. In this case*

$$\Pi^F_C(\bar{x}) = \big\{\nabla \sigma_{C(\bar{x},x^*)}(x^*)\big\}. \tag{4.8}$$

Proof. By Proposition 4.1 we have that the set $\Pi^F_C(\bar{x})$ is *nonempty* if and only if $\widetilde{\Pi}^F_C(\bar{x}, x^*) \ne \emptyset$ and $\widetilde{T}_{x^*}(\bar{x}) = T(\bar{x})$ for some $x^* \in \widehat{\partial} T(\bar{x})$. It holds furthermore that $\widetilde{\Pi}^F_C(\bar{x}, x^*) = \Pi^F_C(\bar{x})$, and thus the sets $\Pi^F_C(\bar{x})$ and $\widetilde{\Pi}^F_C(\bar{x}, x^*)$ are singletons simultaneously. Employing Proposition 4.2, we get in this case that the right-hand side of (4.6) reduces to $\{\bar{y}\}$ for some $\bar{y} \in C(\bar{x}, x^*)$. On the other hand, the subdifferential of the (always convex) support function $\sigma_{C(\bar{x},x^*)}(\cdot)$ at x^* admits the representation

$$\partial \sigma_{C(\bar{x},x^*)}(x^*) = \partial \sigma_{\overline{\text{co}}\, C(\bar{x},x^*)}(x^*) = \big\{v \in \overline{\text{co}}\, C(\bar{x}, x^*) \big|\ \sigma_{C(\bar{x},x^*)}(x^*) = \langle x^*, v \rangle\big\}. \tag{4.9}$$

It follows from (4.9) that the set $\widetilde{C} := \partial \sigma_{C(\bar{x},x^*)}(x^*)$ is an *exposed face* of the convex closed set $\overline{\text{co}}\, C(\bar{x}, x^*)$. Let us show that $\widetilde{C}$ is bounded.

To proceed, fix $v \in \widetilde{C}$ and note that $\langle x^*, v \rangle \ge \langle x^*, \bar{y} \rangle$. Since $\bar{y} \in \bar{x} + T(\bar{x})\mathfrak{J}_F(-x^*)$, we have by Corollary 3.2 that $\langle -x^*, \bar{y} - \bar{x} \rangle = T(\bar{x})$, which yields $\mu := \langle -x^*, v - \bar{x} \rangle \le T(\bar{x})$. On the other hand, it follows from (4.5) and (4.9) that $v \in \bar{x} + K_{x^*}$, which gives

$$\mu > 0 \text{ and } v \in \bar{x} + [0, T(\bar{x})]\mathfrak{J}_F(-x^*),$$

and so ensures the boundedness of the set $\widetilde{C}$ in X.

The reflexivity of the space X and the convexity and closedness of $\widetilde{C}$ allow us to conclude that the latter set is *weakly compact* in X. By the classical Krein-Milman theorem there is an *extreme point* $\bar{v}$ of $\widetilde{C}$ that is, by transitivity, an extreme point of the set $\overline{\text{co}}\, C(\bar{x}, x^*)$ as well. Thus we have $\bar{v} \in C(\bar{x}, x^*)$, and the equality $\sigma_{C(\bar{x},x^*)}(x^*) = \langle x^*, \bar{v}\rangle$ holds. The latter means that $\bar{v}$ belongs to the set on the right-hand side of (4.6), which is the singleton $\{\bar{y}\}$ in the notation above. This justifies that the *subgradient set* $\widetilde{C} = \partial\sigma_{C(\bar{x},x^*)}(x^*)$ is a *singleton*, and thus—by the classical result of convex analysis—the support function $\sigma_{C(\bar{x},x^*)}(\cdot)$ is *Gâteaux differentiable* at x^* with

$$\Pi_C^F(\bar{x}) = \{\bar{y}\} = \{\bar{v}\} = \big\{\nabla\sigma_{C(\bar{x},x^*)}(x^*)\big\}.$$

This proves the "only if" part of the theorem with the projection representation (4.8).

To justify the "if" part, suppose that the support function $\sigma_{C(\bar{x},x^*)}(\cdot)$ is Gâteaux differentiable at x^*, which gives the relationships

$$\partial\sigma_{C(\bar{x},x^*)}(x^*) = \big\{\nabla\sigma_{C(\bar{x},x^*)}(x^*)\big\} =: \{\bar{y}\}.$$

Arguing as above, we conclude that $\bar{y} \in C(\bar{x}, x^*)$. This yields, by using formula (4.6) from Proposition 4.2, that $\widetilde{\Pi}_C^F(\bar{x}, x^*) = \{\bar{y}\}$. Taking finally into account that $\widetilde{T}_{x^*}(\bar{x}) = T(\bar{x})$ as assumed, we arrive at $\Pi_C^F(\bar{x}) = \{\bar{y}\}$ and complete the proof of the theorem. △

The following remarkable characterization of well-posedness of the minimal time problem (1.1) in the case of *convex targets* follows from Theorem 4.3.

Corollary 4.4 (existence and uniqueness of minimal time projection for convex targets). *Let C be a closed and convex subset of a reflexive space X, and let $\bar{x} \in X$. Then $\widehat{\partial}T(\bar{x}) \neq \emptyset$, and for any subgradient $x^* \in \widehat{\partial}T(\bar{x})$ the minimal time projection $\Pi_C^F(\bar{x})$ is a nonempty singleton if and only if the support function of the set $C(\bar{x}, x^*)$ in (4.5) is Gâteaux differentiable at x^*.*

Proof. Since the minimal time function $T(\bar{x})$ is (Lipschitz) continuous and *convex* on X under the assumptions made, we get that the Fréchet subdifferential $\widehat{\partial}T(\bar{x})$ reduces to $\partial T(\bar{x})$, the subdifferential of convex analysis, and also $T(\bar{x}) = \widetilde{T}_{x^*}(\bar{x})$ for every subgradient $x^* \in \partial T(\bar{x})$. The results of the corollary follow now from Theorem 4.3. △

Remark 4.5 (Gâteaux differentiability versus Fréchet differentiability of convex functions). Observe that, in contrast to the results of Section 3 involving the *Fréchet differentiability* assumption on the support function of the dynamics, we use in Theorem 4.3 and Corollary 4.4 the *Gâteaux differentiability* requirement on the support function of a subset of the target set. As it is well known, a Gâteaux differentiability requirement is essentially weaker than the Fréchet differentiability one for convex continuous functions in infinite dimensions. In particular, there is an equivalent norm in the space ℓ^1, which is nowhere Fréchet differentiable while Gâteaux differentiable at every nonzero point; see, e.g., [14, Example 1.14(c)].

Let us next obtain another *sufficient* condition for well-posedness of the minimal time problem (1.1), different from the previous results of this section, that relates to the *Gâteaux differentiability* of the support function of an *enlargement* of the minimal time projection. We preliminary need the following proposition of its own interest; cf. [6, Theorem 4.2] for the case of Hilbert spaces.

Proposition 4.6 (minimal time function and projection estimates for convex targets in Banach spaces). *Let $C \subset X$ be a closed and convex subset of a Banach space X. Then for any $\bar{x} \in X$ we have the inclusion*

$$\partial T(\bar{x}) \subset N(\bar{y}; C) \quad \text{whenever} \quad \bar{y} \in \Pi_C^F(\bar{x}), \tag{4.10}$$

which can be equivalently written in the dual form

$$\Pi_C^F(\bar{x}) \subset \partial \sigma_C(x^*) \quad \text{for each} \quad x^* \in \partial T(\bar{x}) \neq \emptyset. \tag{4.11}$$

Proof. As mentioned above, the (Lipschitz continuous) minimal time function $T(\cdot)$ is convex, and hence its Fréchet subdifferential $\widehat{\partial} T(\bar{x})$ is nonempty and reduces to the subdifferential $\partial T(\bar{x})$ of convex analysis, i.e.,

$$T(x) - T(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \quad \text{for all} \quad x \in X \quad \text{and} \quad x^* \in \partial T(\bar{x}).$$

By the construction of $T(x)$ in (1.2) the latter implies, in particular, that

$$\langle x^*, x - \bar{x} \rangle \leq -T(\bar{x}) \quad \text{for all} \quad x \in C, \tag{4.12}$$

which implies (4.10) by $T(\bar{x}) \geq 0$. On the other hand, we get from Corollary 3.2 that

$$\left[y \in \Pi_C^F(\bar{x}) \subset \bar{x} + T(\bar{x}) \mathfrak{J}_F(-x^*) \right] \Longrightarrow \left[\langle -x^*, y - \bar{x} \rangle = T(\bar{x}) \right]. \tag{4.13}$$

It follows from (4.12) and (4.13) that $\langle x^*, x - y \rangle \leq 0$ for all $x \in C$, and hence we get $\langle x^*, y \rangle \geq \sigma_C(x^*)$. Consequently, $\langle x^*, y \rangle = \sigma_C(x^*)$ by $y \in C$. Thus $y \in \partial \sigma_C(x^*)$, which justifies (4.11) as well as the equivalence between (4.10) and (4.11). $\triangle$

Now we are ready to derive the aforementioned sufficient condition for well-posedness of the minimal function problem with an arbitrary closed target set (1.1).

Theorem 4.7 (existence and uniqueness of minimal time projection via properties of its enlargement). *Let X be reflexive, let $x \notin C$, and let $\widehat{\partial} T(\bar{x}) \neq \emptyset$. Given any $\varepsilon > 0$, denote by $S(\bar{x}, \varepsilon)$ the closed convex hull of the ε-approximate projection $\Pi_C^F(\bar{x}, \varepsilon)$ in (2.5). Then the minimal time projection $\Pi_C^F(\bar{x})$ is a nonempty singleton if for some $x^* \in \widehat{\partial} T(\bar{x})$ the support function $\sigma_{S(\bar{x},\varepsilon)}(\cdot)$ is Gâteaux differentiable at x^*.*

Proof. Since the space X is reflexive and the set $S(\bar{x}, \varepsilon)$ is convex and closed, the approximate projection set $\Pi_{S(\bar{x},\varepsilon)}^F(\bar{x})$ is nonempty. Moreover, we get from Proposition 4.2 that $\Pi_{S(\bar{x},\varepsilon)}^F(\bar{x}) \subset \partial \sigma_{S(\bar{x},\varepsilon)}(x^*)$, and hence the set $\Pi_{S(\bar{x},\varepsilon)}^F(\bar{x})$ is a singleton

$$\Pi_{S(\bar{x},\varepsilon)}^F(\bar{x}) =: \{\bar{y}\}$$

by the imposed differentiability assumption. Furthermore, it follows from the *Gâteaux differentiability* of $\sigma_{S(\bar{x},\varepsilon)}$ at x^* that $\bar{y}$ is an *exposed point* (not strongly) of the convex and closed set $S(\bar{x},\varepsilon) \subset C$, i.e., the corresponding hyperplane (3.10) touches the set $S(\bar{x},\varepsilon)$ only at $\bar{y}$. This yields that $\bar{y} \in C$.

Let us next justify the inequality in

$$\rho_F(\bar{y}-\bar{x}) = \inf_{y \in S(\bar{x},\varepsilon)} \rho_F(y-\bar{x}) \le T(\bar{x}). \tag{4.14}$$

Indeed, given an arbitrary γ satisfying $0 < \gamma < \varepsilon$ and using definition (1.2), find $y \in C$ such that $T(\bar{x}) \ge \rho_F(y-\bar{x}) - \gamma \ge \rho_F(y-\bar{x}) - \varepsilon$. This gives $y \in S(\bar{x},\varepsilon)$ and thus justifies the inequality in (4.14). Since $\bar{y} \in C$, we get $\rho_F(\bar{y}-\bar{x}) = T(\bar{x})$, i.e., $\bar{y} \in \Pi_C^F(\bar{x}) \neq \emptyset$.

It remains to prove the uniqueness of the minimal time projection. Assume on the contrary that there is $\widetilde{y} \in \Pi_C^F(\bar{x})$ with $\widetilde{y} \neq \bar{y}$. We obviously have $\widetilde{y} \in S(\bar{x},\varepsilon)$ and

$$\langle -x^*, \bar{y}-\bar{x}\rangle = \langle -x^*, \widetilde{y}-\bar{x}\rangle = 1$$

by Corollary 3.2. The latter gives

$$\sigma_{S(\bar{x},\varepsilon)}(x^*) = \langle x^*, \bar{y}\rangle = \langle x^*, \widetilde{y}\rangle,$$

which contradicts the assumed differentiability of the support function $\sigma_{S(\bar{x},\varepsilon)}$ at x^* and this completes the proof of the theorem. $\triangle$

5 Lower Regularity of Minimal Time Function and Uniqueness of Generalized Projections

Observe that the major results of Sections 3 and 4 involve the assumption about the *Fréchet subdifferentiability* of the minimal time function (1.2) at the reference point, i.e., $\widehat{\partial}T(\bar{x}) \neq \emptyset$. As mentioned above, this assumption holds if the minimal time function $T(\cdot)$ is *lower regular* at $\bar{x}$ and the space X is either Asplund (e.g., reflexive), or X is arbitrary Banach and $T(\cdot)$ is convex on X. On the other hand, it is well known (see [10, p. 111] and the references therein) that, in the case of the metric projection (1.6) of $\bar{x} \notin C$ onto a closed subset C of a finite-dimensional Euclidean space, i.e., for $X = I\!R^n$ and $F = I\!B$ in (1.4), the *lower regularity* of $T(\cdot) = \text{dist}(\cdot; C)$ at $\bar{x}$ is *equivalent* to its *differentiability* at this point and provides a *necessary and sufficient* condition for $\Pi_C(\bar{x})$ to be a *singleton*. This characterization is essentially due to the fact that the distance function is *semiconcave* around out-of-set points $\bar{x} \notin C$.

In this section we pursue a twofold goal. First to illustrate by examples that the aforementioned relationships are *no longer true* for the minimal time problem (1.1) with $C \neq I\!B$ in both finite and infinite dimensions. Then we present efficient conditions, which ensure (and some of them are also necessary for) the lower regularity of the minimal time function (1.2) at in-set and out-of set points.

Let us start with an example showing that, already in $X = I\!R^2$, the minimal time projection $\Pi_C^F(\bar{x})$ can be a *singleton* for some $\bar{x} \notin C$ while the minimal time function $T(\cdot)$ is *not differentiable* and even *not lower regular* at this point.

Example 5.1 (single-valuedness of minimal time projection does not imply either differentiability or lower regularity of minimal time function in finite dimensions). *There is a nonconvex target set $C \subset I\!R^2$ and a convex polyhedral dynamics F such that $\Pi^F_C(\bar{x})$ is a singleton at some $\bar{x} \notin C$ while $T(\cdot)$ is not lower regular at $\bar{x}$.*

Proof. Consider an even function $\varphi\colon I\!R \to I\!R$ given on $I\!R_+ := [0,\infty)$ by

$$\varphi(x) := \begin{cases} 0 & \text{for } x = 0 \\ k\Big(x - \dfrac{1}{k}\Big) - \dfrac{1}{k^2} & \text{for } \dfrac{1}{k} \le x \le \dfrac{k^2+1}{k^3},\ k \ge 2 \\ \dfrac{k^3(k-1)}{(k-1)^2(-k^2+k-1)}\Big(x - \dfrac{1}{k-1}\Big) - \dfrac{1}{(k-1)^2} & \text{for } \dfrac{k^2+1}{k^3} \le x \le \dfrac{1}{k-1},\ k \ge 2 \\ \infty & \text{for } x > 1 \end{cases}$$

and define the closed *target* set $C \subset I\!R^2$ in question by the epigraph of this function

$$C := \big\{(x,y) \in I\!R^2 \big|\ y \ge \varphi(x)\big\}$$

depicted at Figure 1. It is easy to see that φ is *differentiable* at $x = 0$ with $\varphi'(0) = 0$, and thus $\widehat{N}((0,0);C) = (0,-1)I\!R_+$. Consider now the *dynamics*

$$F := \text{co}\Big\{\Big(0,\frac{1}{2}\Big),\Big(\frac{1}{2},-\frac{1}{2}\Big),\Big(-\frac{1}{2},-\frac{1}{2}\Big)\Big\}$$

and observe that $\Pi^F_C(0,-1/2) = \{(0,0)\}$ and that $T(0,-1/2) = 1$. Letting further

$$\Omega := \big\{(x,y) \in I\!R^2 \big|\ T(x,y) \le 1\big\}, \tag{5.1}$$

we get the representation of the Fréchet normal cone (2.11) to this set at $(0,-1/2)$:

$$\widehat{N}\big((0,-1/2);\Omega\big) = (0,-1)I\!R_+.$$

In fact, there are no points belonging to Ω below the parabola $y = -1/2 - x^2$ for all x sufficiently close to 0. On the other hand, all the points

$$z_k := \Big(\frac{1}{k}, -\frac{1}{k^2} - \frac{1}{2}\Big), \quad k \ge 2,$$

belong to both the set Ω in (5.1) and the aforementioned parabola for all $k \ge 2$. Thus it follows from [6, Theorem 3.1] that

$$\widehat{\partial} T(0,-1/2) = \{(0,-\lambda)\}, \text{ where } \lambda \in I\!R \text{ is such that } \rho_{F^o}(0,\lambda) = 1.$$

Observe further that for each $k \geq 2$ if the triangle $z_k + F$ is displaced a little in the direction $e = (1,2)$, then it continues to touch the boundary of the target C at the point $(1/k, -1/k^2)$ only; in the other words

$$\Pi_C^F(z_k + te) = \left(\frac{1}{k}, -\frac{1}{k^2}\right) \text{ and } T(z_k + te) = 1$$

for all $t \geq 0$ sufficiently small (see Figure 1). Then for each $t > 0$ we have

$$(2, -1) \in \widehat{N}(z_k + te; \Omega), \quad k \geq 2.$$

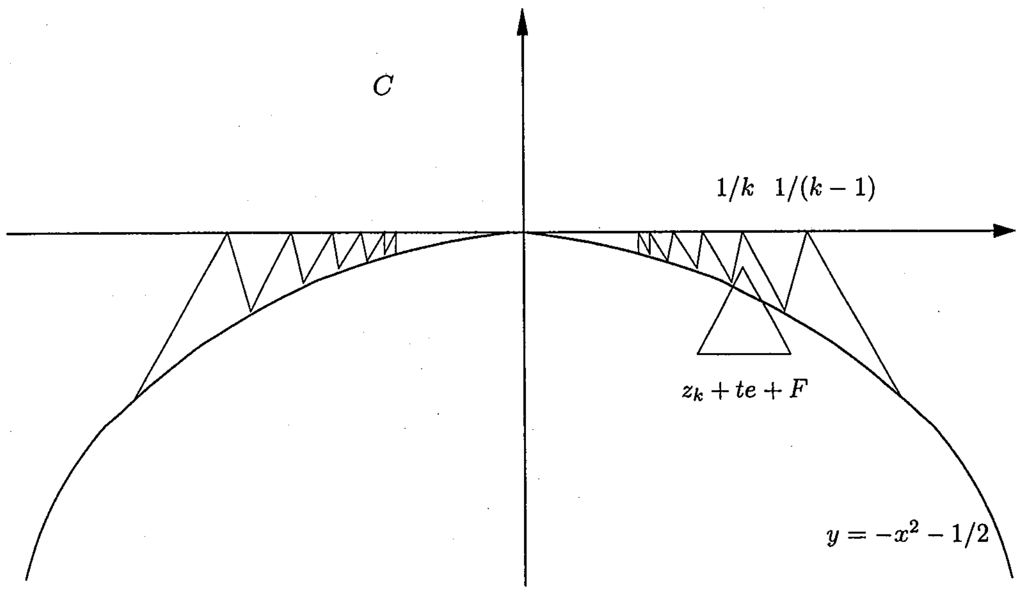

Figure 1

Thus there exists an appropriate constant $\widetilde{\lambda} > 0$ such that

$$\widetilde{\lambda}(2, -1) \in \widehat{\partial} T(z_k + te) \text{ for small } t \geq 0 \text{ and } k \geq 2.$$

Passing to the limit in the latter inclusions as $k \to \infty$ and $t \downarrow 0$, we arrive at

$$\widetilde{\lambda}(2, -1) \in \partial T(0, 1/2)$$

for the limiting subdifferential (2.9). This implies that $\partial T(0, -1/2) \neq \widehat{\partial} T(0, -1/2)$, and hence the minimal time function $T(\cdot)$ is *not lower regular* at $(0, -1/2)$. △

It is worth mentioning further that there are sets $C, F \subset I\!R^2$ and a point $\bar{x} \notin C$ such that the minimal time function is *Fréchet subdifferentiable* at $\bar{x}$ with $\widehat{\partial} T(\bar{x})$ *not being a singleton*, while the minimal time projection $\emptyset \neq \Pi^F(\bar{x}; C)$ is a *singleton*. Indeed, it holds, e.g., in the case of the sets

$$C := \left\{(x_1, x_2) \in I\!R^2 \middle|\, x_1 \leq 0,\ x_2 \leq 0\right\} \text{ and } F := [-1, 1] \times [-1, 1]$$

for any point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ with $\bar{x}_1 = \bar{x}_2 > 0$.

Observe also that, as shown in Example 3.5, the *Fréchet differentiability* of the minimal time function $T(\cdot)$ at $\bar{x} \notin C$ does *not* imply the *nonemptiness* of the minimal time projection $\Pi_C^F(\bar{x})$ in a Hilbert space X. Some modification of that example allows us to examine the situation when $\widehat{\partial}T(\bar{x}) = \emptyset$ (and thus $T(\cdot)$ is definitely *not lower regular* at $\bar{x}$), while $\Pi_C^F(\bar{x})$ is a *nonempty singleton.*

Example 5.2 (existence and uniqueness of minimal time projection do not imply Fréchet subdifferentiability of minimal time function). *There are subsets $C, F \subset X$ of a Hilbert space X satisfying the standing assumptions and a point $\bar{x} \notin C$ such that $\Pi_C^F(\bar{x}) \neq \emptyset$ is a singleton while $\widehat{\partial}T(\bar{x}) = \emptyset$.*

Proof. Let F and C be given as in Example 3.5. Consider now the minimal time problem (1.1) with the same dynamics (3.11) and the modified target

$$C_1 := C \cup \{-e_1\}.$$

It is easy to check that $\Pi_{C_1}^F(0) = \{-e_1\}$, while for all $\lambda \in I\!R$ sufficiently small we compute the corresponding minimal time function by

$$T(\lambda e_1) = \begin{cases} 1 - 2\lambda & \text{if } \lambda \geq 0, \\ 1 + \lambda & \text{if } \lambda < 0. \end{cases} \tag{5.2}$$

Then definition (2.8) with $\varepsilon = 0$ applied to function (5.2) gives us $\widehat{\partial}T(0) = \emptyset$. $\triangle$

In the rest of the section we study the *lower regularity* property of the minimal time function (1.2) in *infinite-dimensional* spaces. This topic is of its own interest as a part of well-posedness of the minimal time problem (1.1) while, as seen above, is important from the viewpoint of existence and uniqueness of the minimal time projection (1.4).

We begin with establishing a *characterization* of the lower regularity of $T(\cdot)$ at *in-set* points of target sets in *arbitrary Banach* spaces.

Proposition 5.3 (characterization of lower regularity of minimal time functions at in-set points of targets in Banach spaces). *Let $\bar{x} \in C$ under the standing assumptions made. Then the minimal time function $T(\cdot)$ is lower regular at $\bar{x}$ if and only if the target set C is normally regular at this point.*

Proof. It follows from [9, Theorem 4.1] that

$$\widehat{\partial}T(\bar{x}) = \widehat{N}(\bar{x}; C) \cap \{x^* \in X^* \mid \rho_{F^\circ}(x^*) \leq 1\}. \tag{5.3}$$

Furthermore, we get from [13, Theorem 3.6] the representation

$$N(\bar{x}; C) = \bigcup_{\lambda \geq 0} \lambda \partial T(\bar{x}). \tag{5.4}$$

Since $0 \in \operatorname{int} F^{\circ}$, equality (5.3) easily implies that

$$\widehat{N}(\bar{x}; C) = \bigcup_{\lambda \geq 0} \lambda \widehat{\partial} T(\bar{x}) \tag{5.5}$$

with the convention that $0 \times \emptyset = 0$. Combining (5.4) and (5.5), we arrive at the equivalence stated in the proposition. $\triangle$

Next we derive sufficient conditions for lower regularity of the minimal time function $T(\cdot)$ at *out-of-set* points $\bar{x} \notin C$ in *arbitrary Banach* spaces. To proceed, let us recall two additional constructions needed in what follows.

Given a target set $C \subset X$ and a point $\bar{x} \notin C$, define the *minimal time enlargement* of C relative to $\bar{x}$ by

$$C_r := \big\{ x \in X \big| \, T(x) \leq r \big\} \quad \text{with} \quad r = T(\bar{x}) > 0. \tag{5.6}$$

Given further a function $\varphi \colon X \to \overline{I\!R}$ on a Banach space with $\varphi(\bar{x}) < \infty$ and following [11], define the *right-sided limiting subdifferential* of φ at $\bar{x}$ by

$$\partial_{\geq} \varphi(\bar{x}) := \Big\{ x^* \in X^* \Big| \begin{array}{l} \exists \text{ sequences } \varepsilon_k \downarrow 0, \; x_k \to \bar{x} \text{ and } x_k^* \stackrel{w^*}{\to} x^* \\ \text{such that } \varphi(x_k) \downarrow \varphi(\bar{x}) \text{ and } x_k^* \in \widehat{\partial}_{\varepsilon_k} \varphi(x_k) \end{array} \Big\}. \tag{5.7}$$

Comparing (5.7) with the basic subdifferential (2.9), we see that the only difference between these two constructions is that $\varphi(x_k) \to \varphi(\bar{x})$ with $\varphi(x_k) \geq \varphi(\bar{x})$ in (5.7) while the latter requirement is omitted in (2.9). It follows from the definitions that

$$\widehat{\partial} \varphi(\bar{x}) \subset \partial_{\geq} \varphi(\bar{x}) \subset \partial \varphi(\bar{x}). \tag{5.8}$$

Note that both inclusions in (5.8) are generally strict even in finite dimensions and that we can equivalently put $\varepsilon_k \equiv 0$ in (5.7) if X is Asplund and φ is lower semicontinuous around $\bar{x}$; see [11, 12] and [10, Subsection 1.3.3] for these and other properties of the right-sided limiting subdifferential (5.7).

Proposition 5.4 (lower regularity of minimal time function at out-of-set points in Banach spaces). *Assume that the minimal time enlargement* (5.6) *is normally regular at $\bar{x} \notin C$. Then the minimal time function $T(\cdot)$ is lower regular at this point.*

Proof. It follows from [9, Theorem 4.2] (see also [13, Theorem 4.2] for another proof and correction) that

$$\widehat{\partial} T(\bar{x}) = \widehat{N}(\bar{x}; C_r) \cap \big\{ x^* \in X^* \big| \, \rho_{F^{\circ}}(x^*) \leq 1 \big\} \tag{5.9}$$

in terms of the enlargement (5.6). On the other hand, we get from [13, Theorem 4.4] by involving the right-sided limiting subdifferential (5.7) that

$$N(\bar{x}; C_r) = \bigcup_{\lambda \geq 0} \lambda \partial_{\geq} T(\bar{x}). \tag{5.10}$$

Comparing (5.9) and (5.10) and taking (5.8) into account allow us to arrive at the conclusion of the proposition. △

Our final result provides efficient conditions of another type ensuring the lower regularity of $T(\cdot)$ at *out-of set* points in terms of their generalized projection onto the target set X under additional assumptions in *Hilbert* spaces.

Recall that the *proximal normal cone* to a set $\Omega \subset X$ at $\bar{x} \in \Omega$ is defined by

$$N_p(\bar{x};\Omega) := \big\{x^* \in X^* \big|\, \exists \gamma > 0 \text{ with } \langle x^*, x - \bar{x}\rangle \le \gamma\|x - \bar{x}\|^2 \text{ for all } x \in \Omega\big\}. \quad (5.11)$$

We always have the inclusions

$$N_p(\bar{x};\Omega) \subset \widehat{N}(\bar{x};\Omega) \subset N(\bar{x};\Omega), \quad \bar{x} \in \Omega, \quad (5.12)$$

both of which can be strict already in finite dimensions. Accordingly, a set Ω is *proximally regular* at a point $\bar{x} \in \Omega$ if

$$N_p(\bar{x};\Omega) = N(\bar{x};\Omega).$$

It follows directly from (5.12) that the proximal regularity of a set implies its normal regularity at the corresponding point.

Now we are ready to establish the aforementioned sufficient conditions for lower regularity of the minimal time function, giving moreover a precise representation of its subdifferential(s) under consideration.

Theorem 5.5 (lower regularity of minimal time function at out-of-set points in Hilbert spaces). *In the framework of problem* (1.1) *let the space X be Hilbert, and let $\bar{x} \notin C$. Assume furthermore that the minimal time projection $\Pi^F_C(\cdot)$ is single-valued around $\bar{x}$ and satisfies the following "one-point" Hölder property at $\bar{x}$: there are constants $K > 0$, $1/2 < \alpha \le 1$, and neighborhood U of $\bar{x}$ such that*

$$\|\Pi^F_C(x) - \Pi^F_C(\bar{x})\| \le K\|x - \bar{x}\|^\alpha \ \textit{for all} \ x \in U. \quad (5.13)$$

If the set C is proximally regular around $\bar{y} := \Pi^F_C(\bar{x})$, then the minimal time function $T(\cdot)$ is lower regular at this point, and we have the representation

$$\partial T(\bar{x}) = N(\bar{y};C) \cap \big(-\partial\rho_F(\bar{y} - \bar{x})\big). \quad (5.14)$$

Proof. Fix an arbitrary point $x \in U$ from the neighborhood U in (5.13). It follows from [6, Theorem 3.3] and the assumed proximal regularity of C around $\bar{y}$ that

$$\widehat{\partial} T(x) \subset N_p(y;C) \cap \big(-\partial\rho_F(y - x)\big) \ \text{ for } \ y = \Pi^F_C(x) \ \text{ and any } \ x \in U. \quad (5.15)$$

Let us now justify the "one-point" version

$$N_p(\bar{y};C) \cap \big(-\partial\rho_F(\bar{y} - \bar{x})\big) \subset \widehat{\partial} T(\bar{x}) \quad (5.16)$$

of the opposite inclusion to (5.15) under all the assumptions made. Picking x^* from the set on the left-hand side of (5.16), we get by the subdifferential construction of convex analysis that the inclusion $-x^* \in \partial\rho_F(\bar{y}-\bar{x})$ reads as

$$\rho_F(u) \geq \rho_F(\bar{y}-\bar{x}) + \langle -x^*, u-\bar{y}+\bar{x}\rangle \text{ for all } u \in X. \tag{5.17}$$

Furthermore, that of $x^* \in N_p(\bar{y};C)$ signifies by (5.11) the existence of $\gamma > 0$ with

$$\langle x^*, v-\bar{y}\rangle \leq \gamma\|v-\bar{y}\|^2 \text{ for all } v \in C. \tag{5.18}$$

Putting then $u = y - x$ for $x \in U$ and $y = \Pi_C^F(x)$, we get from (5.17) the estimate

$$\rho_F(y-x) - \rho_F(\bar{y}-\bar{x}) - \langle x^*, x-\bar{x}\rangle \geq \langle -x^*, y-\bar{y}\rangle. \tag{5.19}$$

Combining (5.18) and (5.19) allows us to conclude that

$$\begin{aligned} &\liminf_{x\to\bar{x}} \frac{T(x)-T(\bar{x})-\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq \liminf_{x\to\bar{x}} \frac{\langle -x^*, y-\bar{y}\rangle}{\|x-\bar{x}\|} \\ &\geq \liminf_{x\to\bar{x}} \frac{-\gamma\|y-\bar{y}\|^2}{\|x-\bar{x}\|} \geq -K\gamma \lim_{x\to\bar{x}} \|x-\bar{x}\|^{2\alpha-1} = 0, \end{aligned} \tag{5.20}$$

where the latter inequality is due to the Hölder condition (5.13). By definition (2.8) with $\varepsilon = 0$, we get from (5.20) that $x^* \in \widehat{\partial}T(\bar{x})$ and thus justify inclusion (5.16).

Using next construction (2.9) of the basic subdifferential in the case of Hilbert spaces and then inclusion (5.15) and the first one in (5.12) gives us the relationships

$$\begin{aligned} \partial T(\bar{x}) = w\text{-}\operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{\partial}T(x) &\subset w\text{-}\operatorname*{Lim\,sup}_{x\to\bar{x}} \big[N_p(y;C) \cap \big(-\partial\rho_F(y-x)\big)\big] \\ &\subset w\text{-}\operatorname*{Lim\,sup}_{x\to\bar{x}} \big[\widehat{N}(y;C) \cap \big(-\partial\rho_F(y-x)\big)\big], \end{aligned} \tag{5.21}$$

where clearly $y = \Pi_C^F(x) \to \bar{y}$ as $x \to \bar{x}$. It follows further from the normal cone definitions (2.11) and the graph closedness of the subdifferential of convex analysis that the right-hand side of the last inclusion in (5.21) reduces to $N(\bar{y};C) \cap (-\partial\rho_F(\bar{y}-\bar{x}))$. This yields by inclusions (5.16), (2.10), and the proximal regularity of C at $\bar{y}$ that

$$N(\bar{y};C) \cap \big(-\partial\rho_F(\bar{y}-\bar{x})\big) = N_p(\bar{y};C) \cap \big(-\partial\rho_F(\bar{y}-\bar{x})\big) \subset \widehat{\partial}T(\bar{x}) \subset \partial T(\bar{x}). \tag{5.22}$$

Combining (5.21) and (5.22), we get the lower regularity of T at $\bar{x}$ and thus complete the proof of theorem. △

Note that efficient conditions ensuring the Hölder property of the minimal time projection (5.13) appear in [8].

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