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PROGRAMS**

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# ROBUST STABILITY AND OPTIMALITY CONDITIONS FOR PARAMETRIC INFINITE AND SEMI-INFINITE PROGRAMS<sup>1</sup>

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**Abstract.** This paper primarily concerns the study of parametric problems of infinite and semi-infinite programming, where functional constraints are given by systems of infinitely many linear inequalities indexed by an arbitrary set  $T$ , where decision variables run over Banach (infinite programming) or finite-dimensional (semi-infinite case) spaces, and where objectives are generally described by nonsmooth and nonconvex cost functions. The parameter space of admissible perturbations in such problems is formed by all bounded functions on  $T$  equipped with the standard supremum norm. Unless the index set  $T$  is finite, this space is intrinsically infinite-dimensional (nonreflexive and nonseparable) of the  $l_\infty$ -type. By using advanced tools of variational analysis and generalized differentiation and largely exploiting underlying specific features of linear infinite constraints, we establish complete characterizations of robust Lipschitzian stability (with computing the exact bound of Lipschitzian moduli) for parametric maps of feasible solutions governed by linear infinite inequality systems and then derive verifiable necessary optimality conditions for the infinite and semi-infinite programs under consideration expressed in terms of their initial data. A crucial part of our analysis addresses the precise computation of coderivatives and their norms for infinite systems of parametric linear inequalities in general Banach spaces of decision variables. The results obtained are new in both frameworks of infinite and semi-infinite programming.

**Key words.** semi-infinite and infinite programming, parametric optimization, variational analysis, linear infinite inequality systems, robust stability, necessary optimality conditions, generalized differentiation, coderivatives

**AMS subject classification.** 90C34, 90C05, 49J52, 49J53, 65F22

## 1 Introduction

This paper deals with problems of *parametric optimization* formalized as

$$\text{minimize } \varphi(p, x) \text{ subject to } x \in \mathcal{F}(p), \quad (1.1)$$

where  $x \in X$  is a *decision* variable belonging to an arbitrary Banach space  $X$  (which may be finite-dimensional), where  $p = (p_t)_{t \in T} \in P$  is a functional *parameter* taking values in the prescribed Banach space  $P$  of perturbations specified below, where  $\varphi: P \times X \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$  is an extended-real-valued *cost* function finite at reference points, and where the

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feasible set mapping  $\mathcal{F}: P \rightrightarrows X$  associated with the *parameterized/indexed linear inequality system* is defined by

$$\mathcal{F}(p) := \{x \in X \mid \langle a_t^*, x \rangle \leq b_t + p_t, t \in T\}, \quad p = (p_t)_{t \in T}, \quad (1.2)$$

involving an *arbitrary* (possibly infinite) *index set*  $T$ . For the sake of simplicity, from now on the linear inequality system in (1.2) will be also referred to as  $\mathcal{F}(p)$ .

The data of (1.2) are given as follows:

- $a_t^* \in X^*$  for all  $t \in T$ , where the space  $X^*$  is topologically dual to  $X$  with the canonical pairing  $\langle \cdot, \cdot \rangle$  between  $X$  and  $X^*$ . If no confusion arises, we use the *same notation*  $\|\cdot\|$  for the given norm in  $X$  and the corresponding dual norm in  $X^*$  defined by

$$\|x^*\| := \sup \{ \langle x^*, x \rangle \mid \|x\| \leq 1 \}, \quad x^* \in X^*.$$

We always assume that  $a_t^*$  are *fixed* and *arbitrary* in  $X^*$  for all  $t \in T$ .

- $b_t \in \mathbb{R}$  for all  $t \in T$ . We identify the collection  $\{b_t \mid t \in T\}$  with the real-valued function  $b: T \rightarrow \mathbb{R}$  and assume that it is *fixed* and *arbitrary*.

- $p_t = p(t) \in \mathbb{R}$  for all  $t \in T$ . These functional parameters  $p: T \rightarrow \mathbb{R}$  are our varying *perturbations*, which are taken from the Banach *parameter space*  $P := l_\infty(T)$  of all bounded functions on  $T$  with the supremum norm

$$\|p\|_\infty := \sup_{t \in T} |p_t| = \sup \{ |p(t)| \mid t \in T \} \quad (1.3)$$

(see, e.g., [14]), where the subscript “ $\infty$ ” is omitted if no confusion arises. When the index set  $T$  is compact (which is not assumed in this paper) and the perturbations  $p(\cdot)$  are restricted to be continuous on  $T$ , the maximum is realized in (1.3), and thus the parameter space  $l_\infty(T)$  reduces to the classical space  $\mathcal{C}(T)$  of continuous functions over a compact set.

Observe that the parameter space  $l_\infty(T)$  is finite-dimensional *if and only if* the index set  $T$  is finite. In the latter case our problem (1.1) is a usual problem of parametric mathematical programming (in finite or infinite dimensions, depending on the dimensionality of  $X$ ) with *finitely many* linear inequality constraints and possibly nondifferentiable objectives. However, the main emphasis of this paper is addressed to the case when the index set  $T$  is *infinite*, and thus problem (1.1) is *intrinsically* infinite-dimensional with respect to the parameter variable. According to the conventional terminology (see, e.g., [1, 15]), we say that the mathematical program (1.1) is a problem of *semi-infinite programming* (SIP) if the decision space  $X$  is finite-dimensional, and that it belongs to the *infinite programming* framework if  $X$  is an infinite-dimensional space.

Note also that problem (1.1) is written in the format of the so-called *abstract MPECs* (mathematical programs with equilibrium constraints) [22, 29, 30], but the main emphasis there is the *generalized equation/variational condition* (in Robinson’s sense [32]) structure of the set-valued mapping  $\mathcal{F}$  in (1.1) given by

$$\mathcal{F}(p) := \{x \in X \mid 0 \in f(x, p) + Q(x)\}$$

with a single-valued mapping  $f: X \times P \rightarrow Y$  and a set-valued mapping  $Q: X \rightrightarrows Y$ , which particularly encompasses solution maps to the classical variational inequalities and complementarity problems when  $Q(x) = N(x; \Omega)$  is the normal cone mapping to a convex set  $\Omega \subset X$ . The underlying *infinite inequality* structure (1.2) of the mapping  $\mathcal{F}$  in our framework is completely different from the MPEC case. Its comprehensive study and applications from the viewpoint of variational analysis and generalized differentiation are among the main goals and achievements of this paper.

To the best of our knowledge, this generalized differentiation approach is new in the literature despite many publications related to various properties and applications of linear infinite inequality systems, most of which concern the case of finite-dimensional spaces  $X$  of decisions variables (i.e., in the semi-infinite programming framework); see [1] and [15] for a comprehensive overview on this field and [4] when confined to the parameter space of continuous perturbations  $P = \mathcal{C}(T)$ , provided that the index set  $T$  is a compact Hausdorff space. We refer the reader to [9] for the study of *qualitative stability* (formalized through certain semicontinuity properties of feasible solution and optimal solution mappings) in the framework of  $X = \mathbb{R}^n$ , an arbitrary index set  $T$ , and arbitrary perturbations. In the same semi-infinite context, for a *quantitative perspective* (through Lipschitzian properties), the reader is addressed to [6], and to [5] for the case of continuous perturbations. Let us mention [11] addressing the case of infinite linear programming from the viewpoint of qualitative stability. We also refer the reader to, e.g., [7, 8, 10, 12] for the study of convex semi-infinite/infinite inequality systems, to [19, 20, 35] for their smooth nonlinear counterparts, and to the recent paper [37] for necessary optimality conditions of the Lagrangian type in some classes of nonsmooth semi-infinite optimization problems.

In this paper we develop *coderivative analysis* for the linear infinite inequality systems  $\mathcal{F}$  in (1.2), which turns out to be a major part of variational analysis for the infinite/semi-infinite programming problems under consideration, eventually leading us to complete characterizations of *robust stability* of the parametric sets of *feasible solutions* and to deriving verifiable *necessary conditions* for *optimal solutions* to (1.1) in terms of the initial data. By robust stability we understand the fulfillment of the so-called *Lipschitz-like* (known also as Aubin) property of the solution map *around* the reference point, which is stable with respect to small perturbations of parameters; see Section 2 for more detail.

Coderivatives of set-valued mappings introduced in [24] have been well recognized as a powerful tool of variational analysis and its numerous applications, particularly to problems of optimization and control; see, e.g., the books [3, 21, 28, 29, 33, 34, 37] and the references therein. However, we are not familiar with any implementation of coderivatives in problems of infinite or semi-infinite programming as well as with their application to analyze stability of linear infinite inequality systems of type (1.2) in finite or infinite dimensions.

The power of coderivatives in variational analysis and its applications comes, first of all, from the possibility to obtain in their terms verifiable *pointwise characterizations* of *robust Lipschitzian* properties of set-valued mappings (as well as of the equivalent properties of *metric regularity* and *linear openness* for the inverse mappings) and deriving *necessary optimality conditions* in rather general settings. These developments are strongly supported by comprehensive *coderivative calculus* based on *variational/extremal principles* of advanced

variational analysis; see [28, 29] and the references therein. However, a number of the results in this vein are limited in infinite dimensions. In particular, the available coderivative characterizations of the Lipschitz-like property of closed-graph mappings  $F: Z \rightrightarrows Y$  obtained in [28, Theorem 4.10] require that both spaces  $Z$  and  $Y$  are *Asplund* (i.e., every separable subspace of them has a separable dual), while the precise coderivative formula for computing the *exact Lipschitzian bound* is established therein via the coderivative norm under the *finite-dimensionality* assumption on  $Z$ . But this is *never the case* for our infinite inequality system  $\mathcal{F}: P \rightrightarrows X$  from (1.2), where the parameter space  $(Z =)P = l_\infty(T)$  is always *infinite-dimensional* and *not Asplund* unless the index set  $T$  is finite!

This paper contains new and fairly comprehensive results in the aforementioned directions for the infinite/semi-infinite problems under consideration, which essentially take into account underlying specific features of the infinite inequality constraints (1.2) largely related to the possibility of employing an appropriate extended version of the fundamental *Farkas Lemma* for infinite systems of linear inequalities in *general Banach spaces*.

The rest of paper is organized as follows. Section 2 presents some preliminary material from convex and variational analysis widely used in formulations and proofs of the subsequent main results. In Section 3 we provide *precise calculations* of the basic *coderivative*  $D^*\mathcal{F}$  and its *norm* at the reference/nominal point for the feasible solution map  $\mathcal{F}: l_\infty(T) \rightrightarrows X$  in (1.1) defined by the infinite inequality system (1.2) via the initial data of  $\mathcal{F}$  in the general case of an arbitrary index set  $T$  and an arbitrary Banach space  $X$  of decision variables.

Section 4 is devoted to deriving *coderivative characterizations* of *robust stability* for the feasible solution system (1.2) of infinite inequalities with an arbitrary index set  $T$ , which are explicitly expressed in terms of the initial data  $\{a_t^*, b_t, t \in T\}$ . We establish verifiable *criteria* (i.e., necessary and sufficient conditions) for the fulfillment of the Lipschitz-like (and hence the classical local Lipschitzian) property of  $\mathcal{F}$  around the reference points and derive furthermore the *precise formulas* for computing the *exact bounds* of Lipschitzian moduli in the case of general Banach spaces  $X$ . It is worth mentioning that the criteria and exact bound formulas obtained in this section are represented in the conventional *coderivative form* of variational analysis as in [28, Theorem 4.10] for the case of abstract set-valued mappings but with *no Asplund* space and *finite-dimensionality* requirements imposed therein, which are *never* satisfied for the infinite inequality systems (1.2) under consideration in either infinite programming or semi-infinite programming framework.

In the concluding Section 5 we derive new *necessary optimality conditions* for infinite and semi-infinite programming problems (1.1) with generally *nonsmooth* cost functions. Without imposing additional assumptions on the sets of feasible solutions in (1.2), we present *two types* of necessary conditions for optimal solutions to (1.1) given in a verifiable *qualified asymptotic* form, which are independent of each other in the case of nonsmooth cost functions. The first type of *lower subdifferential* conditions corresponds to traditional usage of (lower) subgradients of convex and nonconvex functions. The other *upper subdifferential* type provides trivial information for the case of convex cost functions while leads to significantly stronger optimality conditions, in comparison with lower subdifferential ones, for classes of “upper regular” functions, e.g., for minimization problems involving concave objectives; see more details below. We refer the reader to [29, Sections 5.1 and 5.2] for

conditions of these types in various optimization problems, mainly in Asplund spaces, that include the case of (1.2) with finitely many inequality constraints. The results of both lower and upper subdifferential types obtained in Section 5 take advantages of precise computing the coderivative of the solution feasible map  $\mathcal{F}$  and thus express necessary optimality conditions in terms of the initial data of (1.1) and (1.2).

Our notation is basically standard and conventional in the areas of variational analysis and infinite/semi-infinite programming; see, e.g., [15, 28, 33]. Unless otherwise stated, all the spaces under consideration are *Banach* with the corresponding norm  $\|\cdot\|$ . Recall that  $w^*$  indicates the weak\* topology of a dual space, and we use the symbol  $w^*$ -lim for the weak\* *topological* limit, which generally means the weak\* convergence of *nets* denoted usually by  $\{x_\nu^*\}_{\nu \in \mathcal{N}}$ . In the case of sequences we use the standard notation  $\mathbb{N} := \{1, 2, \dots\}$  for the collections of all natural numbers.

Given a subset  $\Omega \subset Z$  of a Banach space, the symbols  $\text{int } \Omega$ ,  $\text{cl } \Omega$ ,  $\text{co } \Omega$ , and  $\text{cone } \Omega$  stand, respectively, for the interior, closure, convex hull, and conic convex hull of  $\Omega$ ; the notation  $\text{cl}^* \Theta$  signifies the weak\* closure of a subset  $\Theta \subset Z^*$  in the dual space. Given a set-valued mapping  $F: Z \rightrightarrows Y$ , we denote its *domain*, *graph*, and *inverse* by, respectively,

$$\text{dom } F = \{z \in Z \mid F(z) \neq \emptyset\}, \quad \text{gph } F := \{(z, y) \in Z \times Y \mid y \in F(z)\},$$

and  $F^{-1}(y) := \{z \in Z \mid (z, y) \in \text{gph } F\}$ . Considering finally an arbitrary index set  $T$ , let  $\mathbb{R}^T$  be the *product space* of  $\lambda = (\lambda_t \mid t \in T)$  with  $\lambda_t \in \mathbb{R}$  for all  $t \in T$ , let  $\mathbb{R}^{(T)}$  be the collection of  $\lambda \in \mathbb{R}^T$  such that  $\lambda_t \neq 0$  for *finitely many*  $t \in T$ , and let  $\mathbb{R}_+^{(T)}$  be the *positive cone* in  $\mathbb{R}^{(T)}$  defined by

$$\mathbb{R}_+^{(T)} := \{\lambda \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\}.$$

## 2 Basic Definitions and Preliminaries

In this section we discuss the basic notions and tools needed for our subsequent analysis and results. As mentioned in Section 1, a major attention of this paper is paid to *robust stability* of the feasible solution map for (1.1) given by the linear infinite inequality system (1.2). By robust stability we understand Lipschitzian behavior *around* (i.e., in a neighborhood) of the reference point. The most natural formalization of this behavior widely recognized in variational (as well as in general nonlinear) analysis is known as the Lipschitz-like or Aubin property, which can be viewed as a graphical localization (in the set-valued case) of the classical local Lipschitzian property of single-valued and set-valued mappings.

Given a set-valued mapping  $F: Z \rightrightarrows Y$  between Banach spaces, we say the  $F$  is *Lipschitz-like* around  $(\bar{z}, \bar{y}) \in \text{gph } F$  with modulus  $\ell \geq 0$  if there are neighborhoods  $U$  of  $\bar{z}$  and  $V$  of  $\bar{y}$  such that

$$F(z) \cap V \subset F(u) + \ell \|z - u\| \mathbb{B} \text{ for any } z, u \in U, \quad (2.1)$$

where  $\mathbb{B}$  stands for the closed unit ball in the space in question. The infimum of moduli  $\{\ell\}$  over all the combinations of  $\{\ell, U, V\}$  satisfying (2.1) is called the *exact Lipschitzian bound* of  $F$  around  $(\bar{z}, \bar{y})$  and is labeled as  $\text{lip } F(\bar{z}, \bar{y})$ . If  $V = Y$  in (2.1), this relationship

signifies the classical (Hausdorff) *local Lipschitzian* property of  $F$  around  $\bar{z}$  with the exact Lipschitzian bound denoted by  $\text{lip } F(\bar{z})$  in this case.

We can easily observe from the definition (2.1) that the exact Lipschitzian bound of  $F$  around  $(\bar{z}, \bar{y})$  admits the following limiting representation via the *distance function* to a set:

$$\text{lip } F(\bar{z}, \bar{y}) = \limsup_{(z, y) \rightarrow (\bar{z}, \bar{y})} \frac{\text{dist}(y; F(z))}{\text{dist}(z; F^{-1}(y))}, \quad (2.2)$$

where  $\inf \emptyset = \infty$  (and hence  $\text{dist}(x; \emptyset) = \infty$ ) as usual and where  $0/0 := 0$ . We have accordingly that  $\text{lip } F(\bar{z}, \bar{y}) = \infty$  if  $F$  is not Lipschitz-like around  $(\bar{z}, \bar{y})$ .

It is worth mentioning that the Lipschitz-like property of an arbitrary mapping  $F: Z \rightrightarrows Y$  between Banach spaces is equivalent to other two fundamental properties in nonlinear analysis but for the *inverse* mapping  $F^{-1}: Y \rightrightarrows Z$ ; namely, to the *metric regularity* of  $F^{-1}$  and to the *linear openness* of  $F^{-1}$  around  $(\bar{y}, \bar{z})$ , with the corresponding relationships between their exact bounds (see, e.g., [17, 28, 33]).

A remarkable fact consists of the possibility to *characterize* pointwisely the (derivative-free) Lipschitz-like property of  $F$  around  $(\bar{z}, \bar{y})$ —and hence its local Lipschitzian, metric regularity, and linear openness counterparts—in terms of a *dual-space* construction of generalized differentiation called the *coderivative* of  $F$  at  $(\bar{z}, \bar{y}) \in \text{gph } F$ . The latter is a positively homogeneous multifunction  $D^*F(\bar{z}, \bar{y}): Y^* \rightrightarrows Z^*$  defined by

$$D^*F(\bar{z}, \bar{y})(y^*) := \{z^* \in Z^* \mid (z^*, -y^*) \in N((\bar{z}, \bar{y}); \text{gph } F)\}, \quad y^* \in Y^*, \quad (2.3)$$

where  $N(\cdot; \Omega)$  stands for the collection of generalized normals to a set at a given point known as the basic, or limiting, or Mordukhovich normal cone; see, e.g., [23, 28, 33, 34] and the references therein. When both  $Z$  and  $Y$  are *finite-dimensional*, it is proved in [25] (cf. also [33, Theorem 9.40]) that a closed-graph mapping  $F: Z \rightrightarrows Y$  is Lipschitz-like around  $(\bar{z}, \bar{y}) \in \text{gph } F$  *if and only if*

$$D^*F(\bar{z}, \bar{y})(0) = \{0\} \quad (2.4)$$

and the exact Lipschitzian bound of moduli  $\{\ell\}$  in (2.1) is computed by

$$\text{lip } F(\bar{z}, \bar{y}) = \|D^*F(\bar{z}, \bar{y})\| := \sup \{\|z^*\| \mid z^* \in D^*F(\bar{z}, \bar{y})(y^*), \|y^*\| \leq 1\}. \quad (2.5)$$

The situation is significantly more involved in infinite dimension. It is proved in [26] (see also [28, Theorem 4.10]) that a closed-graph mapping  $F: Z \rightrightarrows Y$  is Lipschitz-like around  $(\bar{z}, \bar{y}) \in \text{gph } F$  *if and only if* the coderivative condition (2.4) holds in terms of the so-called “mixed coderivative” (which reduces to (2.3) in finite dimensions and the setting considered in this paper) together with a certain “partial sequential normal compactness” condition (which is automatic in finite dimensions and in the setting of this paper) provided that both spaces  $Z$  and  $Y$  are *Asplund*. The latter property is defined in Section 1; we also refer the reader to [14, 28, 31] for more details and various characterizations of this remarkable and well investigated subclass of Banach spaces that includes, in particular, all reflexive ones while does not include, e.g., the classical spaces  $\mathcal{C}$ ,  $l_1$ ,  $L_1$ ,  $l_\infty$ , and  $L_\infty$ .

The situation is even more complicated with infinite-dimensional extensions of the exact bound formula in (2.5). The aforementioned results of [26, 28] give merely upper and lower



estimates for  $\text{lip } F(\bar{z}, \bar{y})$ , which ensure the precise equality in (2.5) in the setting of this paper provided that  $Y$  is *Asplund* while  $Z$  is *finite-dimensional*.

The set-valued mapping  $F: Z \rightrightarrows Y$  considered in this paper is  $\mathcal{F}: l_\infty(T) \rightrightarrows X$  defined by the infinite system of linear inequalities (1.2); in what follows we always assume that the index set  $T$  is *infinite*, which is a characteristic feature of infinite and semi-infinite programs. In this setting the domain/parameter space  $Z = l_\infty(T)$  must be *infinite-dimensional* Banach that is *never Asplund*. Also, we do not suppose in this paper that our decision space  $X$  is anything but *arbitrary Banach*.

In this general setting for (1.2) we show that the coderivative condition (2.4) is *necessary and sufficient* for the Lipschitz-like property of  $F = \mathcal{F}$  around the reference/nominal solution  $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$  and (2.5) is a *precise formula* for computing the exact Lipschitzian bound  $\text{lip } \mathcal{F}(\bar{p}, \bar{x})$ . This is *exactly what we have in finite dimensions*, while it is far removed from being a part of the infinite-dimensional variational theory in [28]. Moreover, we express the relationships in (2.4) and (2.5) *explicitly* in terms of the *initial data* of (1.2).

To proceed further, observe that the graph

$$\text{gph } \mathcal{F} = \{(p, x) \in l_\infty(T) \times X \mid \langle a_t^*, x \rangle \leq b_t + p_t \text{ for all } t \in T\} \quad (2.6)$$

of the mapping  $\mathcal{F}: l_\infty(T) \rightrightarrows X$  in (1.2) is *convex*. Hence the basic normal cone to  $\text{gph } \mathcal{F}$  at  $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$  reduces to

$$N((\bar{p}, \bar{x}); \text{gph } \mathcal{F}) = \{(p^*, x^*) \in l_\infty(T)^* \times X^* \mid \langle (p^*, x^*), (p, x) - (\bar{p}, \bar{x}) \rangle \leq 0 \text{ for } (p, x) \in \text{gph } \mathcal{F}\}$$

and the coderivative (2.3) of  $\mathcal{F}$  admits the representation

$$D^* \mathcal{F}(\bar{p}, \bar{x})(x^*) = \{p^* \in l_\infty(T)^* \mid \langle p^*, \bar{p} \rangle - \langle x^*, \bar{x} \rangle = \max_{(p, x) \in \text{gph } \mathcal{F}} [\langle p^*, p \rangle - \langle x^*, x \rangle]\}. \quad (2.7)$$

Let us now present two preliminary results that play an important role in our subsequent analysis. The first one taken from [10, Lemma 2.4] can be viewed as an *extended Farkas lemma* for infinite inequality systems in Banach spaces.

**Lemma 2.1 (extended Farkas lemma).** *Let  $p \in \text{dom } \mathcal{F}$  for the infinite system (1.2) with a Banach decision space  $X$ , and let  $(x^*, \alpha) \in X^* \times \mathbb{R}$ . The following are equivalent:*

(i) *We have  $\langle x^*, x \rangle \leq \alpha$  whenever  $x \in \mathcal{F}(p)$ , i.e.,*

$$[\langle a_t^*, x \rangle \leq b_t + p_t \text{ for all } t \in T] \implies [\langle x^*, x \rangle \leq \alpha].$$

(ii) *The pair  $(x^*, \alpha)$  satisfies the inclusion*

$$(x^*, \alpha) \in \text{cl}^* \text{cone}[\{ \langle a_t^*, b_t + p_t \rangle \mid t \in T \} \cup \{(0, 1)\}] \text{ with } 0 \in X^*.$$

Throughout the paper we largely use the parametric *characteristic sets*

$$C(p) := \text{co}\{ \langle a_t^*, b_t + p_t \rangle \mid t \in T \}, \quad p \in l_\infty(T), \quad (2.8)$$

and suppose with no loss of generality that our *nominal parameter* is the zero function  $\bar{p} = 0$  in the parameter space  $l_\infty(T)$ .

Let us recall a well-recognized *qualification condition* for linear infinite inequalities, which is often used in problems of infinite and semi-infinite programming.

**Definition 2.2 (strong Slater condition).** We say that the infinite system (1.2) satisfies the STRONG SLATER CONDITION (SSC) at  $p = \{p_t\}_{t \in T}$  if there is  $\hat{x} \in X$  such that

$$\sup_{t \in T} [\langle a_t^*, \hat{x} \rangle - b_t - p_t] < 0. \quad (2.9)$$

Furthermore, every point  $\hat{x} \in X$  satisfying condition (2.9) is a STRONG SLATER POINT for system (1.2) at  $p = \{p_t\}_{t \in T}$ .

The next result contains several equivalent descriptions and interpretations of the strong Slater condition used in what follows; the most important is the equivalence (i)  $\iff$  (ii). Note that a similar equivalence can be found in [11] for more general convex systems in locally convex spaces with different spaces of associated parameters.

**Lemma 2.3 (equivalent descriptions of the strong Slater condition).** Let  $X$  be a Banach space, and let  $p \in \text{dom } \mathcal{F}$  for the linear infinite inequality system (1.2). Then the following properties are equivalent:

- (i)  $\mathcal{F}$  satisfies the strong Slater condition at  $p$ .
- (ii)  $(0, 0) \notin \text{cl}^* C(p)$  via the characteristic set from (2.8).
- (iii)  $p \in \text{int}(\text{dom } \mathcal{F})$ .
- (iv)  $\mathcal{F}$  is Lipschitz-like around  $(p, x)$  for all  $x \in \mathcal{F}(p)$ .

**Proof.** We begin with the proof of (i)  $\implies$  (ii). Arguing by contradiction, assume that  $(0, 0) \in \text{cl}^* C(p)$ . Then there is a net  $\{\lambda_\nu\}_{\nu \in N} \in \mathbb{R}_+^{(T)}$  satisfying  $\sum_{t \in T} \lambda_{t\nu} = 1$  for all  $\nu \in N$  and the limiting condition

$$(0, 0) = w^* \text{-} \lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t + p_t). \quad (2.10)$$

If  $\hat{x}$  is a strong Slater point for system (1.2) at  $p$ , we find  $\vartheta > 0$  such that

$$\langle a_t^*, \hat{x} \rangle - b_t - p_t \leq -\vartheta \text{ for all } t \in T.$$

Then (2.10) leads to the following contradiction:

$$0 = \langle 0, \hat{x} \rangle + 0 \cdot (-1) = \lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (\langle a_t^*, \hat{x} \rangle + (b_t + p_t) \cdot (-1)) \leq -\vartheta.$$

Let us next justify the converse implication (ii)  $\implies$  (i). By [10, Theorem 3.1] we have

$$p \in \text{dom } \mathcal{F} \iff (0, -1) \notin \text{cl}^* \text{cone}\{(a_t^*, b_t + p_t) \mid t \in T\}.$$

Then the strong separation theorem ensures the existence of  $(0, 0) \neq (v, \alpha) \in X \times \mathbb{R}$  with

$$\langle a_t^*, v \rangle + \alpha(b_t + p_t) \leq 0 \text{ for all } t \in T, \quad (2.11)$$

$$\langle 0, v \rangle + (-1)\alpha = -\alpha > 0.$$

At the same time by (ii) we have  $(0, 0) \neq (z, \beta) \in X \times \mathbb{R}$  and  $\gamma \in \mathbb{R}$  for which

$$\langle a_t^*, z \rangle + \beta(b_t + p_t) \leq \gamma < 0 \text{ whenever } t \in T. \quad (2.12)$$

Consider further the combination

$$(u, \eta) := (z, \beta) + \lambda(v, \alpha)$$

and select  $\lambda > 0$  to be sufficiently large to ensure that  $\eta < 0$ . Defining now  $\hat{x} := -\eta^{-1}u$ , we observe from (2.11) and (2.12) that

$$\langle a_t^*, \hat{x} \rangle - b_t - p_t = -\eta^{-1}(\langle a_t^*, u \rangle + \eta(b_t + p_t)) \leq -\eta^{-1}\gamma < 0.$$

This allows us to conclude that  $\hat{x}$  is a strong Slater point for system (1.2) at  $p$ .

To prove implication (i)  $\implies$  (iii), assume that  $\hat{x}$  is a strong Slater point for system (1.2) at  $p$  and find  $\vartheta > 0$  such that

$$\langle a_t^*, \hat{x} \rangle - b_t - p_t \leq -\vartheta \text{ for all } t \in T.$$

Then it is obvious that for any  $q \in l_\infty(T)$  with  $\|q\| < \vartheta$  we have  $\hat{x} \in \mathcal{F}(p + q)$ . Therefore  $p + q \in \text{dom } \mathcal{F}$ , and thus (iii) holds.

Let us further proceed with justifying implication (iii)  $\implies$  (i). If  $p \in \text{int}(\text{dom } \mathcal{F})$ , then  $p + q \in \text{dom } \mathcal{F}$  provided that  $q_t = -\vartheta$  as  $t \in T$  and that  $\vartheta > 0$  is sufficiently small. Thus every  $\hat{x} \in \mathcal{F}(p + q)$  is a strong Slater point for the infinite system (1.2) at  $p$ .

The remaining equivalence between (iii) and (iv) is a consequence of the classical Robinson-Ursescu closed graph/metric regularity theorem; see, e.g., [17] with more discussions and references therein.  $\triangle$

The major space for our consideration in this paper is the parameter space  $l_\infty(T)$  of bounded functions  $p: T \rightarrow \mathbb{R}$  on  $T$  with the supremum norm (1.3). It is obviously a Banach space that is *never finite-dimensional* when the index set  $T$  is infinite, which is our standing assumption. Let us show that is *never Asplund*.

**Proposition 2.4 (parameter space is never Asplund).** *The parameter space  $l_\infty(T)$  is Asplund if and only if the index set  $T$  is finite.*

**Proof.** If  $T$  is countable (i.e.,  $T = \mathbb{N}$  and the parameter space is the classical space of sequences  $l_\infty$ ), the proof can be found in [31, Example 1.21]; in fact, this space is not even weak Asplund. The same arguments can be adapted for any infinite index set  $T$ .  $\triangle$

Finally in this section, recall a convenient description of the topological dual space  $l_\infty(T)^*$  to the parameter space  $l_\infty(T)$ . According to [13], there is an isometric isomorphism between  $l_\infty(T)^*$  and the space of *bounded and additive measures*

$$ba(T) = \{ \mu: 2^T \rightarrow \mathbb{R} \mid \mu \text{ is bounded and additive} \}$$

satisfying the relationship

$$\langle \mu, p \rangle = \int_T p_t \mu(dt) \text{ with } p = (p_t)_{t \in T}.$$

The *dual norm* on  $ba(T)$  corresponding to (1.3) is the *total variation* of  $\mu \in ba(T)$  on the index set  $T$  defined by

$$\|\mu\| := \sup_{ACT} \mu(A) - \inf_{BCT} \mu(B).$$

In what follows we always identify the measure space  $ba(T)$  with the dual parameter space  $l_\infty(T)^*$  and use, for the notational unification,  $p^* \in l_\infty(T)^*$  instead of  $\mu \in ba(T)$ .

### 3 Computing Coderivatives and Coderivative Norms for Linear Infinite Constraint Systems

In this section we establish a constructive representation of the coderivative  $D^*\mathcal{F}(0, \bar{x})$  for the feasible solution map  $\mathcal{F}$  at the nominal point  $(0, \bar{x})$  and compute its norm  $\|D^*\mathcal{F}(0, \bar{x})\|$  in terms of the initial data of the linear infinite system of inequalities (1.2). Let us first describe the *normal cone* to the convex graph (2.6) employing the *extended Farkas lemma* presented above. In what follows  $\delta_t$  denote the classical *Dirac measure* at  $t \in T$  satisfying

$$\langle \delta_t, p \rangle = p_t \text{ as } t \in T \text{ for } p = (p_t)_{t \in T} \in l_\infty(T).$$

**Proposition 3.1 (computing normals to the graphical set of feasible solutions).**

Let  $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$  for the graphical set (2.6) with a Banach decision space  $X$ , and let  $(p^*, x^*) \in l_\infty(T)^* \times X^*$ . Then we have  $(p^*, x^*) \in N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})$  if and only if

$$(p^*, x^*, \langle p^*, \bar{p} \rangle + \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone}[\{(-\delta_t, a_t^*, b_t) \mid t \in T\} \cup \{(0, 0, 1)\}], \quad (3.1)$$

where  $0 \in l_\infty(T)^*$  and  $0 \in X^*$  stand for the first and second entry of the last triple in (3.1).

**Proof.** Observe from (2.6) and the definition of the Dirac measure that the graph of  $\mathcal{F}$  admits the representation

$$\text{gph } \mathcal{F} = \{(p, x) \in l_\infty(T) \times X \mid \langle a_t^*, x \rangle - \langle \delta_t, p \rangle \leq b_t \text{ for all } t \in T\}. \quad (3.2)$$

Therefore we have  $(p^*, x^*) \in N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})$  if and only if

$$\langle p^*, p \rangle + \langle x^*, x \rangle \leq \langle p^*, \bar{p} \rangle + \langle x^*, \bar{x} \rangle$$

for every  $(p, x)$  satisfying (3.2). Employing now the equivalence between (i) and (ii) in Lemma 2.1, we conclude that  $(p^*, x^*) \in N((\bar{p}, \bar{x}); \text{gph } \mathcal{F})$  if and only if inclusion (3.1) holds. This completes the proof of the proposition.  $\triangle$

Based on the above proposition and the general coderivative definition, we now obtain a constructive representation of the coderivative  $D^*\mathcal{F}(0, \bar{x})$  in question.

**Theorem 3.2 (coderivative of the feasible solution map).** Let  $\bar{x} \in \mathcal{F}(0)$  for the feasible solution map  $\mathcal{F}: l_\infty(T) \rightrightarrows X$  from (1.2) with a Banach decision space  $X$ . Then  $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$  if and only if

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone}\{(-\delta_t, a_t^*, b_t) \mid t \in T\}. \quad (3.3)$$

**Proof.** By the coderivative construction (2.3) applied to  $\mathcal{F}$  and by the normal cone formula from Proposition 3.1 as  $\bar{p} = 0$  we get that  $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$  if and only if

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone}[\{(-\delta_t, a_t^*, b_t) \mid t \in T\} \cup \{(0, 0, 1)\}]. \quad (3.4)$$

To justify the coderivative representation claimed in the theorem, we need to show that inclusion (3.4) implies in fact the “smaller” one in (3.3). Assuming indeed that (3.4) holds,

we find by the structure of the right-hand side on (3.4) some nets  $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^{(T)}$  and  $\{\gamma_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+$  satisfying the limiting relationship

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu} \left( \sum_{t \in T} \lambda_{t\nu} (-\delta_t, a_t^*, b_t) + \gamma_\nu (0, 0, 1) \right), \quad (3.5)$$

where  $\lambda_{t\nu}$  stands for the  $t$ -entry of  $\lambda_\nu = (\lambda_{t\nu})_{t \in T}$  as  $\nu \in \mathcal{N}$ . It follows directly from the component structure in (3.5) that

$$0 = \langle p^*, 0 \rangle + \langle -x^*, \bar{x} \rangle + (-\langle x^*, \bar{x} \rangle)(-1) = \lim_{\nu} \left( \sum_{t \in T} \lambda_{t\nu} (\langle a_t^*, \bar{x} \rangle - b_t) - \gamma_\nu \right). \quad (3.6)$$

Taking finally into account the definition of the positive cone  $\mathbb{R}_+^{(T)}$  and the fact that the pair  $(0, \bar{x})$  satisfies the infinite inequality system in (1.2), we conclude from (3.6) that  $\lim_{\nu} \gamma_\nu = 0$ . This justifies (3.3) and thus completes the proof of the theorem.  $\triangle$

Our further intention is to provide the *exact calculation* of the *coderivative norm*

$$\|D^*\mathcal{F}(0, \bar{x})\| := \sup \{ \|p^*\| \mid p^* \in D^*\mathcal{F}(0, \bar{x})(x^*), \|x^*\| \leq 1 \} \quad (3.7)$$

in terms of the initial data of the linear infinite inequality system (1.2). A part of our analysis in this direction is the following lemma on properties of the characteristic set (2.8) at  $p = 0$ , which is also used in Section 4 to compute the exact Lipschitzian bound  $\text{lip } \mathcal{F}(0, \bar{x})$ .

**Lemma 3.3 (properties of the characteristic set).** *Let  $X$  be an arbitrary Banach space. Suppose also that  $\bar{x} \in \mathcal{F}(0)$  is not a strong Slater point for the infinite system (1.2) at  $p = 0$  and that the collection  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ . Then the set*

$$S := \{x^* \in X^* \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^*C(0)\} \quad (3.8)$$

*built upon the characteristic set  $C(0)$  in (2.8) is nonempty and  $w^*$ -compact at  $X^*$ .*

**Proof.** Since  $\bar{x}$  is not a strong Slater point for the infinite system (1.2) at  $p = 0$ , there is a sequence  $\{t_k\}_{k \in \mathbb{N}} \subset T$  such that  $\lim_k (\langle a_{t_k}^*, \bar{x} \rangle - b_{t_k}) = 0$ . The boundedness of  $\{a_t^* \mid t \in T\}$  implies by the classical Alaoglu-Bourbaki theorem that this set is relatively  $w^*$ -compact in  $X^*$ , i.e., there is a *subnet*  $\{a_{t_\nu}^*\}_{\nu \in \mathcal{N}}$  of the latter sequence that  $w^*$ -converges to some element  $u^* \in \text{cl}^*\{a_t^* \mid t \in T\}$ . This gives  $\lim_{\nu} b_{t_\nu} = \langle u^*, \bar{x} \rangle$  and therefore

$$(u^*, \langle u^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu} (a_{t_\nu}^*, b_{t_\nu}) \in \text{cl}^*C(0),$$

which justifies the *nonemptiness* of the set  $S$  in (3.8). Next we prove that  $S$  is  $w^*$ -compact.

Indeed, by our assumption the set  $A := \{a_t^* \mid t \in T\}$  is bounded in  $X^*$ , and so is  $\text{cl}^*\text{co}A$ ; the latter is actually  $w^*$ -compact due to its automatic  $w^*$ -closedness. Observe further that the set  $S$  in (3.8) is a *preimage* of  $\text{cl}^*C(0)$  under the  $w^*$ -continuous mapping  $x^* \mapsto (x^*, \langle x^*, \bar{x} \rangle)$ , and thus it is  $w^*$ -closed in  $X^*$ . Since  $S$  is obviously a subset of  $\text{cl}^*\text{co}A$ , it is also bounded and hence  $w^*$ -compact in  $X^*$ . This completes the proof of the lemma.  $\triangle$

Now we are ready to compute the coderivative norm  $\|D^*\mathcal{F}(0, \bar{x})\|$  at the reference point.

**Theorem 3.4 (computing the coderivative norm).** *Let  $\bar{x} \in \text{dom } \mathcal{F}$  for the infinite system (1.2) with an arbitrary Banach space  $X$  of decision variables. Assume that  $\mathcal{F}$  satisfies the strong Slater condition at  $p = 0$  and that the coefficient set  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ . The following assertions hold:*

(i) *If  $\bar{x}$  is a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then  $\|D^*\mathcal{F}(0, \bar{x})\| = 0$ .*

(ii) *If  $\bar{x}$  is not a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then the coderivative norm (3.7) is positive and is computed by*

$$\|D^*\mathcal{F}(0, \bar{x})\| = \max \left\{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^*C(0) \right\} \quad (3.9)$$

via the  $w^*$ -closure of the characteristic set (2.8) at  $p = 0$ .

**Proof.** We begin with considering case (i). Let us show that the strong Slater condition (2.9) with  $\hat{x} = \bar{x}$  and the boundedness of  $\{a_t^* \mid t \in T\}$  imply that the point  $(0, \bar{x})$  belongs to the interior of the set  $\text{gph } \mathcal{F}$ . Since  $\bar{x}$  is a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , there is a number  $\varepsilon > 0$  for which we have

$$\langle a_t^*, \bar{x} \rangle \leq b_t - \varepsilon \text{ for all } t \in T. \quad (3.10)$$

We show now that (3.10) implies the existence of  $\eta > 0$  such that

$$[\|p\| \leq \eta, \|x - \bar{x}\| \leq \eta] \implies [\langle a_t^*, x \rangle \leq b_t + p_t, t \in T], \quad (3.11)$$

and hence  $x \in \mathcal{F}(p)$ . To justify (3.11), select  $\eta > 0$  by

$$\eta := \frac{\varepsilon}{1 + \sup_{t \in T} \|a_t^*\|} \in (0, \infty),$$

which is well defined due to the boundedness of  $\{a_t^* \mid t \in T\}$ . Taking  $p \in l_\infty(T)$  with  $\|p\| \leq \eta$  and  $x \in X$  with  $\|x - \bar{x}\| \leq \eta$ , we then get from (3.10) that

$$\langle a_t^*, x \rangle - b_t - p_t \leq -\varepsilon + \langle a_t^*, x - \bar{x} \rangle + |p_t| \leq -\varepsilon + \left(1 + \sup_{t \in T} \|a_t^*\|\right) \eta \leq 0.$$

Once we have  $(0, \bar{x}) \in \text{int}(\text{gph } \mathcal{F})$ , this gives  $N((0, \bar{x}); \text{gph } \mathcal{F}) = \{(0, 0)\}$ , and thus

$$p^* \in D^*\mathcal{F}(0, \bar{x})(x^*) \implies (p^*, x^*) = (0, 0).$$

The latter ensures that  $\|D^*\mathcal{F}(0, \bar{x})\| = 0$  by definition (3.7).

Next we consider case (ii), where  $\bar{x}$  is not a strong Slater point for  $\mathcal{F}$  at  $p = 0$ . Let us show that in this case the coderivative norm (3.7) can be equivalently expressed by

$$\|D^*\mathcal{F}(0, \bar{x})\| = \sup \{ \|p^*\| \mid p^* \in D^*\mathcal{F}(0, \bar{x})(x^*), \|x^*\| = 1 \}. \quad (3.12)$$

To proceed, we observe first that

$$D^*\mathcal{F}(0, \bar{x})(x^*) \neq \emptyset \text{ for some } x^* \in X^* \text{ with } \|x^*\| = 1 \quad (3.13)$$

which is equivalent to the coderivative nonemptiness in (3.13) with some nonzero  $x^* \in X^*$ , since the coderivative is positively homogeneous. Indeed, the fact that  $\bar{x}$  is not a strong

Slater point for  $\mathcal{F}$  at  $p = 0$  easily implies that  $(0, \bar{x})$  belongs to the boundary of the graph of  $\mathcal{F}$ . Furthermore, as in case (i), the strong Slater condition for  $\mathcal{F}$  at  $p = 0$  and the boundedness of  $\{a_t^* \mid t \in T\}$  imply that  $\text{int}(\text{gph } \mathcal{F}) \neq \emptyset$ . By the classical *separation theorem* we find  $(p^*, x^*) \neq (0, 0)$  such that

$$\langle (p^*, x^*), (0, \bar{x}) \rangle \leq \langle (p^*, x^*), (p, x) \rangle \text{ for all } (p, x) \in \text{gph } \mathcal{F}, \quad (3.14)$$

which means by (2.7) that  $-p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$ . If  $x^* = 0$  for this  $x^* \in X^*$ , then we must have  $p^* \neq 0$  in (3.14), and hence

$$D^*\mathcal{F}(0, \bar{x})(0) \neq \{0\}. \quad (3.15)$$

The latter yields by [28, Theorem 1.44] that  $\mathcal{F}$  is *not Lipschitz-like* around  $(0, \bar{x})$ , and therefore it cannot satisfy the strong Slater condition by implication (i) $\implies$ (iv) in Lemma 2.3, which contradicts our assumption.

Having thus (3.13), we can justify (3.12). Indeed, the opposite means that

$$\sup \{\|p^*\| \mid p^* \in D^*\mathcal{F}(0, \bar{x})(0)\} > \sup \{\|p^*\| \mid p^* \in D^*\mathcal{F}(0, \bar{x})(x^*), \|x^*\| = 1\} \geq 0$$

thus ensuring the existence of  $0 \neq p^* \in l_\infty(T)^*$  with  $p^* \in D^*\mathcal{F}(0, \bar{x})(0)$ , which again gives (3.15) and contradicts the assumed strong Slater condition.

Operating subsequently with the coderivative norm formula (3.12), pick any  $x^* \in X^*$  with  $\|x^*\| = 1$  as in (3.13) and fix some  $p^* \in D^*\mathcal{F}(0, \bar{x})(x^*)$ . By the coderivative representation of Theorem 3.2 we find a net  $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^{(T)}$  such that

$$(p^*, -x^*, -\langle x^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (-\delta_t, a_t^*, b_t). \quad (3.16)$$

Let us show that the assumed uniform boundedness of the coefficients  $\{a_t^*\}$  and the strong Slater condition imposed on  $\mathcal{F}$  imply the relationships

$$0 < \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} < \infty, \quad (3.17)$$

which contain the assertion that the limit exists in (3.17). Indeed, the *existence* and *finiteness* of the limit in (3.17) follows from considering just the first component of the limiting equality in (3.16), which gives the expression

$$\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} = \langle p^*, -e \rangle < \infty, \text{ where } e = (e_t)_{t \in T} \text{ with } e_t = 1 \text{ for all } t \in T.$$

Using further the lower semicontinuity of the norm on  $X^*$  in the weak\* topology, we have from the second component in (3.16) that

$$1 = \|x^*\| \leq \liminf_{\nu \in \mathcal{N}} \left\| \sum_{t \in T} \lambda_{t\nu} a_t^* \right\| \leq \left( \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} \right) \sup_{t \in T} \|a_t^*\|,$$

which implies the *positivity* of the limit in (3.17) due to the uniform boundedness of  $\{a_t^*\}$ .

It follows from (3.16) and (3.17) with  $\lambda_\nu \in \mathbb{R}_+^{(T)}$  that

$$\|p^*\| = \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu},$$

and therefore the norm expression (3.12) can be rewritten as

$$\begin{aligned} & \|D^* \mathcal{F}(0, \bar{x})\| \\ = & \sup \left\{ \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} \left| \begin{array}{l} \exists x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ (-x^*, -\langle x^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t) \\ \text{and } w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (-\delta_t) \text{ exists} \end{array} \right. \right\}. \end{aligned} \quad (3.18)$$

We can see in fact that the following coderivative norm representation holds:

$$\begin{aligned} & \|D^* \mathcal{F}(0, \bar{x})\| \\ = & \sup \left\{ \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} \left| \begin{array}{l} \exists x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ (-x^*, -\langle x^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t) \end{array} \right. \right\}. \end{aligned}$$

Indeed, take  $x^* \in X^*$  with  $\|x^*\| = 1$  such that for some net  $\mathcal{N}$  we have

$$(-x^*, -\langle x^*, \bar{x} \rangle) = w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t). \quad (3.19)$$

It will be enough to see that for some subnet (with no relabeling) the limit

$$w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (-\delta_t)$$

exists, and from this we can employ to the Alaoglu-Bourbaki theorem. Note that the whole net  $\{\sum_{t \in T} \lambda_{t\nu} (-\delta_t)\}_{\nu \in \mathcal{N}}$  might be unbounded, but we show the existence of  $\nu_0 \in \mathcal{N}$  such that  $\{\sum_{t \in T} \lambda_{t\nu} (-\delta_t)\}_{\nu \succ \nu_0}$  is bounded. It is clear, appealing again to the vector  $-e$  defined above, that  $\|\sum_{t \in T} \sum_{t \in T} \lambda_{t\nu} t\nu (-\delta_t)\| = \sum_{t \in T} \lambda_{t\nu}$ , since  $\lambda_\nu \in \mathbb{R}_+^{(T)}$ . If such an index  $\nu_0 \in \mathcal{N}$  does not exist, we would have  $\lim_{\nu \in \mathcal{M}} \sum_{t \in T} \lambda_{t\nu} = \infty$  for some cofinal subset  $\mathcal{M} \subset \mathcal{N}$  and, dividing both sides of (3.19) by  $\sum_{t \in T} \lambda_{t\nu}$  and then taking the limit as  $\nu \in \mathcal{M}$ , we would obtain the relationships

$$(0, 0) = \lim_{\nu \in \mathcal{M}} \sum_{t \in T} \frac{\lambda_{t\nu}}{\sum_{t \in T} \lambda_{t\nu}} (a_t^*, b_t) \in \text{cl}^* C(0),$$

which contradict the strong Slater condition at  $p = 0$  by Lemma 2.3(ii).

Once we establish the boundedness of the net  $\{\sum_{t \in T} \lambda_{t\nu} (-\delta_t)\}_{\nu \succ \nu_0}$  for some  $\nu_0 \in \mathcal{N}$ , it contains by the Alaoglu-Bourbaki theorem a  $w^*$ -convergent subnet, with no relabeling. This implies that the limit  $\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu}$  does not actually exceed the supremum in (3.18),



and thus we have the coderivative norm representation

$$\begin{aligned}
& \|D^*\mathcal{F}(0, \bar{x})\| \\
&= \sup \left\{ \gamma > 0 \left| \begin{array}{l} \exists x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ (-x^*, -\langle x^*, \bar{x} \rangle) = \gamma [w^*\text{-}\lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t)] \\ \text{for some } \{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^{(T)} \text{ with } \sum_{t \in T} \lambda_{t\nu} = 1 \end{array} \right. \right\} \\
&= \sup \left\{ \gamma > 0 \left| \begin{array}{l} \exists x^* \in X^* \text{ with } \|x^*\| = 1 \text{ such that} \\ \gamma^{-1}(-x^*, -\langle x^*, \bar{x} \rangle) \in \text{cl}^*C(0) \end{array} \right. \right\} \\
&= \sup \left\{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^*C(0) \right\} = \sup \left\{ \|x^*\|^{-1} \mid x^* \in S \right\},
\end{aligned}$$

where the set  $S$  is defined in (3.8).

To justify finally (3.9), it remains to show that the supremum is realized in the last expression above under the assumptions in (ii). Indeed, it is proved in Lemma 3.3 that the set  $S$  is  $w^*$ -compact in  $X^*$  under the assumptions made. Observe further that  $0 \notin S$  due to the strong Slater condition by Lemma 2.3(ii) and that the function  $x^* \mapsto \|x^*\|^{-1}$  is  $w^*$ -upper semicontinuous on  $S$  due to the  $w^*$ -lower semicontinuity of the norm. Employing the classical Weierstrass theorem in the  $w^*$ -topology of  $X^*$ , we conclude that the maximum is attained in (3.9) and thus  $\|D^*\mathcal{F}(0, \bar{x})\| > 0$ . This ends the proof of the theorem.  $\triangle$

## 4 Characterizations of Robust Lipschitzian Stability for Feasible Solution Maps

In this section we employ the above coderivative analysis combined with appropriate techniques developed in linear semi-infinite/infinite programming to establish a *coderivative characterization* of robust Lipschitzian stability, in the sense discussed in Section 2, for the infinite inequality system  $\mathcal{F}$  in (1.2) at the reference point  $(0, \bar{x})$  with computing the *exact Lipschitzian bound*  $\text{lip } \mathcal{F}(0, \bar{x})$ .

The first result of this section establishes the coderivative *necessary and sufficient* condition in form (2.4) for the Lipschitz-like property of  $\mathcal{F}$  around  $(0, \bar{x}) \in \text{gph } \mathcal{F}$  in the general setting under consideration.

**Theorem 4.1 (coderivative criterion for robust Lipschitzian stability of linear infinite inequalities).** *Let  $\bar{x} \in \mathcal{F}(0)$  for the infinite inequality system (1.2) with a Banach space  $X$  of decision variables. Then  $\mathcal{F}$  is Lipschitz-like around  $(0, \bar{x})$  if and only if*

$$D^*\mathcal{F}(0, \bar{x})(0) = \{0\}. \quad (4.1)$$

**Proof.** The “only if” part follows from [28, Theorem 1.44] specified for the mapping  $\mathcal{F}: l_\infty(T) \rightrightarrows X$  under consideration. Let us now prove the “if” part of the theorem.

Arguing by contradiction, suppose that  $D^*\mathcal{F}(0, \bar{x})(0) = \{0\}$  while the mapping  $\mathcal{F}$  is *not* Lipschitz-like around  $(0, \bar{x})$ . Then, by the equivalence between properties (ii) and (iv) of Lemma 2.3, we get the inclusion

$$(0, 0) \in \text{cl}^*\text{co}\{(a_t^*, b_t) \in X^* \times \mathbb{R} \mid t \in T\},$$

which means that there is a net  $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{(T)}$  such that  $\sum_{t \in T} \lambda_{t\nu} = 1$  for all  $\nu \in \mathcal{N}$  and

$$w^* \text{-} \lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t) = (0, 0). \quad (4.2)$$

Since the net  $\{\sum_{t \in T} \lambda_{t\nu} (-\delta_t)\}_{\nu \in \mathcal{N}}$  is obviously bounded in  $l_\infty(T)^*$ , the Alaoglu-Bourbaki theorem ensures the existence of its subnet (with no relabeling) that  $w^*$ -converges to some element  $p^* \in l_\infty(T)^*$ , i.e.,

$$p^* = w^* \text{-} \lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (-\delta_t). \quad (4.3)$$

It follows from (4.3) by the Dirac function definition that

$$\langle p^*, -e \rangle = \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} = 1, \text{ where } e = (e_t)_{t \in T} \text{ with } e_t = 1 \text{ for all } t \in T,$$

which implies that  $p^* \neq 0$ . It follows from (4.2) and (4.3) that

$$(p^*, 0, 0) = w^* \text{-} \lim_{\nu} \sum_{t \in T} \lambda_{t\nu} (-\delta_t, a_t^*, b_t) \text{ with } p^* \neq 0,$$

and therefore, by the explicit coderivative description of Theorem 3.2, we get the inclusion  $p^* \in D^* \mathcal{F}(0, \bar{x})(0) \setminus \{0\}$ , which contradicts the assumed condition (4.1). This justifies the sufficiency of the coderivative condition (4.1) for the Lipschitz-like property of  $\mathcal{F}$  around  $(0, \bar{x})$  and thus completes the proof of the theorem.  $\triangle$

Our further goal is to compute the *exact Lipschitzian bound*  $\text{lip } \mathcal{F}(0, \bar{x})$  of  $\mathcal{F}$  around  $(0, \bar{x})$ . We are going to do it on the base of the *distance representation* (2.2) for the exact Lipschitzian bound. To proceed, denote

$$H(x^*, \alpha) := \{x \in X \mid \langle x^*, x \rangle \leq \alpha\} \text{ for } (x^*, \alpha) \in X^* \times \mathbb{R} \quad (4.4)$$

and observe the following representation (known as the *Ascoli formula*; see, e.g., [2]):

$$\text{dist}(x; H(x^*, \alpha)) = \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}, \quad (4.5)$$

where  $[\gamma]_+ := \max\{\gamma, 0\}$  for  $\gamma \in \mathbb{R}$ . The next result, which is certainly of its own interest, provides a significant extension of the Ascoli formula (4.4) to the case of *infinite* systems of linear inequalities instead of the single one as in (4.4). This result is essentially employed in what follows for computing the exact Lipschitzian bound  $\text{lip } \mathcal{F}(0, \bar{x})$ . We refer the reader to [5, Lemma 2.3] and [6, Lemma 1] for related results in the framework of semi-infinite programming and observe that in infinite dimensions we use the  $w^*$ -closure of the characteristic sets  $C(p)$  from (2.8); see also Remark 4.3 and Example 4.4 below for more discussions.

**Lemma 4.2 (distance to infinite inequalities in Banach spaces).** *Consider the infinite system  $\mathcal{F}$  of linear inequalities (1.2) with an arbitrary Banach space  $X$  of decision variables. Assume that  $\mathcal{F}$  satisfies the strong Slater condition at  $p = 0$  and that the set  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ . Then there is  $\eta > 0$  such that*

$$\text{dist}(x; \mathcal{F}(p)) = \sup_{(x^*, \alpha) \in \text{cl}^* C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|} \quad (4.6)$$

for any  $x \in X$  and any  $p \in l_\infty(T)$  with  $\|p\| \leq \eta$ .

**Proof.** Observe first that, under the assumed boundedness of  $\{a_t^* \mid t \in T\}$ , the strong Slater condition is *robust* with respect to small parameter perturbations, i.e., the validity of the SSC at  $p = 0$  implies the existence of  $\eta > 0$  such that this condition is fulfilled at any  $p \in l_\infty(T)$  with  $\|p\| \leq \eta$ ; specifically,  $\eta$  can be chosen as one half of that used in the proof of Theorem 3.4. In the rest of the proof we fix such a parameter  $p$  and establish formula (4.6) for this  $p$  and every  $x \in X$ .

It follows directly from Lemma 2.1 and the structures of the sets  $\mathcal{F}(p)$  in (1.2),  $H(x^*, \alpha)$  in (4.4), and  $C(p)$  in (2.8) that

$$\mathcal{F}(p) \subset H(x^*, \alpha) \text{ whenever } (x^*, \alpha) \in \text{cl}^* C(p),$$

which immediately implies the inequality “ $\geq$ ” in (4.6) by the Ascoli formula (4.5).

To justify the converse inequality “ $\leq$ ” in (4.6), it is sufficient to consider the nontrivial case of  $x \notin \mathcal{F}(p)$  when we have

$$\text{dist}(x; \mathcal{F}(p)) = \inf_{u \in \mathcal{F}(p)} \langle x^*, x - u \rangle \text{ for some } x^* \in X^* \text{ with } \|x^*\| = 1 \quad (4.7)$$

by [18, Corollary 23.14]. Fixing  $x^*$  from (4.7) in what follows, we get therefore that

$$\langle x^*, u \rangle \leq \langle x^*, x \rangle - \text{dist}(x; \mathcal{F}(p)) \text{ for all } u \in \mathcal{F}(p).$$

This implies, by using Lemma 2.1 again and representing the  $w^*$ -closure of the characteristic set  $C(p)$  in (2.8), that there are nets  $\{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^{(T)}$  and  $\{\gamma_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+$  such that

$$(x^*, \langle x^*, x \rangle - \text{dist}(x; \mathcal{F}(p))) = w^* \text{-} \lim_{\nu} \left\{ \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t + p_t) + \gamma_\nu (0, 1) \right\} \quad (4.8)$$

with  $0 \in X^*$ . Let us show that  $\lim_{\nu} \gamma_\nu = 0$  in (4.8), i.e., we can equivalently remove the term “ $\gamma_\nu(0, 1)$ ” from this representation.

To proceed, fix an arbitrary number  $\varepsilon > 0$  and pick any  $x_\varepsilon \in \mathcal{F}(p)$  satisfying the estimate

$$\langle x^*, x - x_\varepsilon \rangle - \text{dist}(x; \mathcal{F}(p)) < \varepsilon. \quad (4.9)$$

It follows from (4.8), by applying both sides there as linear functionals at  $(-x_\varepsilon, 1)$ , that

$$(x^*, x - x_\varepsilon) - \text{dist}(x; \mathcal{F}(p)) = \lim_{\nu} \left\{ \sum_{t \in T} \lambda_{t\nu} [b_t + p_t - \langle a_t^*, x_\varepsilon \rangle] + \gamma_\nu \right\} \geq \limsup_{\nu} \gamma_\nu,$$

since the term  $[\cdot]$  in the latter sum is nonnegative due the feasibility of  $x_\varepsilon$  to the inequality system (1.2). By (4.9) we now conclude that  $\limsup_{\nu} \gamma_\nu < \varepsilon$ , where  $\varepsilon > 0$  was chosen arbitrarily. Thus  $\lim_{\nu} \gamma_\nu = 0$ , which justifies the representation

$$(x^*, \langle x^*, x \rangle - \text{dist}(x; \mathcal{F}(p))) = w^* \text{-} \lim_{\nu} \left\{ \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t + p_t) \right\}. \quad (4.10)$$

It follows from the proof of Theorem 3.4 in a similar situation that

$$0 < \theta := \limsup_{\gamma} \sum_{t \in T} \lambda_{t\gamma} < \infty \quad (4.11)$$

under the imposed boundedness of  $\{a_t^* | t \in T\}$  and the strong Slater condition for  $\mathcal{F}(p)$ . Then we have from (4.10), for a suitable subnet with no relabeling, that

$$\frac{(x^*, \langle x^*, x \rangle - \text{dist}(x; \mathcal{F}(p)))}{\theta} = w^* \text{-} \lim_{\nu} \left\{ \sum_{t \in T} \frac{\lambda_{t\nu}}{\sum_{t \in T} \lambda_{t\nu}} (a_t^*, b_t + p_t) \right\}. \quad (4.12)$$

By passing finally to the limit in (4.12) along this subnet and taking into account relationships (4.5) and (4.11), we find a pair  $(\tilde{x}^*, \tilde{\alpha}) \in X^* \times \mathbb{R}$  such that

$$(\tilde{x}^*, \tilde{\alpha}) := \frac{1}{\theta} (x^*, \langle x^*, x \rangle - \text{dist}(x; \mathcal{F}(p))) \in \text{cl}^* C(p), \quad \text{dist}(x; \mathcal{F}(p)) = \frac{[\langle \tilde{x}^*, x \rangle - \tilde{\alpha}]_+}{\|\tilde{x}^*\|},$$

which justifies the inequality “ $\leq$ ” in (4.6) and completes the proof of the lemma.  $\triangle$

**Remark 4.3 (simplified distance formula to infinite inequalities in reflexive spaces).** As mentioned above, some analogs of the distance formula (4.6) can be found in [5, Lemma 2.3] and [6, Lemma 1] in the case of finite-dimensional decision spaces  $X$ . It is actually proved therein that in  $X = \mathbb{R}^n$  we have

$$\text{dist}(x; \mathcal{F}(p)) = \sup_{(x^*, \alpha) \in C(p)} \frac{[\langle x^*, x \rangle - \alpha]_+}{\|x^*\|}, \quad (4.13)$$

for all  $x \in X$  and the corresponding parameter perturbations  $p$  sufficiently small.

Let us show that this *simplified distance formula* holds true in the infinite-dimensional framework of Lemma 4.2 provided in addition that the decision space  $X$  is *reflexive*. This means that in reflexive spaces  $X$  we have a counterpart of Lemma 4.2 with replacing (4.6) by (4.13), where the  $w^*$ -closure of the characteristic set  $C(p)$  is *not involved*. This result is certainly of *independent interest* while it is not used in the rest of the paper.

Due to the above results and discussions, it is sufficient to justify the inequality “ $\leq$ ” in (4.13) if  $X$  is reflexive. Considering the nontrivial case of  $x \notin \mathcal{F}(p)$ , we find by [36, Theorems 3.8.1(i) and 3.8.4(ii)] elements  $v \in \mathcal{F}(p)$  and  $x^* \in X^*$  with  $\|x^*\| = 1$  such that

$$\text{dist}(x; \mathcal{F}(p)) = \|x - v\| = \langle x^*, x - v \rangle \quad \text{and} \quad \langle x^*, u - v \rangle \leq 0 \quad \text{for all } u \in \mathcal{F}(p).$$

Proceeding as in the proof of Lemma 4.2 while using the classical Mazur theorem ensuring that, in duals to reflexive spaces, the  $w^*$ -closure of a convex set agrees with its *norm closure*, we find *sequences*  $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+^{(T)}$  and  $\{\gamma_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$  satisfying the limiting equality

$$(x^*, \langle x^*, v \rangle) = \lim_{k \rightarrow \infty} \left\{ \sum_{t \in T} \lambda_{tk} (a_t^*, b_t + p_t) + \gamma_k (0, 1) \right\} \quad \text{with } 0 \in X^*, \quad (4.14)$$

where the convergence is in the *norm* topology of  $X^*$ . This implies the relationships

$$1 = \|x^*\| = \lim_{k \rightarrow \infty} \left\| \sum_{t \in T} \lambda_{tk} a_t^* \right\| \quad \text{and} \quad (4.15)$$

$$\langle x^*, v \rangle = \lim_{k \rightarrow \infty} \left\{ \sum_{t \in T} \lambda_{tk} (b_t + p_t) + \gamma_k \right\}. \quad (4.16)$$

Similarly to the proof of Lemma 4.2 we show that  $\lim_{k \rightarrow \infty} \gamma_k = 0$ , i.e., we can equivalently remove  $\gamma_k(0, 1)$  in (4.14) and  $\gamma_k$  in (4.16). Denoting further

$$\theta_k := \sum_{t \in T} \lambda_{tk} \text{ for all } k \in \mathbb{N},$$

observe that  $\theta_k > 0$  for all  $k$  sufficiently large due to (4.15). Finally, we define

$$(x_k^*, \alpha_k) := \frac{1}{\theta_k} \sum_{t \in T} \lambda_{tk} (a_t^*, b_t + p_t) \in C(p) \quad (4.17)$$

and conclude from (4.14) with  $\gamma_k = 0$  and from (4.17) that

$$\text{dist}(x; \mathcal{F}(p)) = \langle x^*, x - v \rangle = \lim_{k \rightarrow \infty} \frac{[\langle x_k^*, x \rangle - \alpha_k]_+}{\|x_k^*\|},$$

which justifies the inequality “ $\leq$ ” in (4.13) and thus shows that the  $w^*$ -closure of  $C(p)$  can be dropped in the setting of Lemma 4.2 under the additional assumption on the reflexivity of the decision spaces  $X$ .  $\triangle$

The following example shows that the *reflexivity* of the decision space  $X$  is an *essential* requirement for the validity of the simplified distance formula (4.13), even in the framework of *Asplund* spaces.

**Example 4.4 (failure of the simplified distance formula in nonreflexive Asplund spaces).** Consider the classical space  $c_0$  of sequences of real numbers converging to zero endowed with the supremum norm. This space is well known to be *Asplund* while *not reflexive*; see, e.g., [14]. Let us show that the simplified distance formula (4.13) *fails* in  $X = c_0$  for a rather plain linear system of *countable* inequalities. Of course, we need to demonstrate that the inequality “ $\leq$ ” is generally violated in (4.13), since the opposite inequality holds in any Banach space. Form the infinite (countable) linear inequality system

$$\mathcal{F}(0) := \{x \in c_0 \mid \langle e_1^* + e_t^*, x \rangle \leq -1, t \in \mathbb{N}\}, \quad (4.18)$$

where  $e_t^* \in l_1$  has 1 as its  $t$ -th component while all the remaining components are 0. System (4.18) can be rewritten as

$$x \in \mathcal{F}(0) \iff x(1) + x(t) \leq -1 \text{ for all } t \in \mathbb{N}.$$

Observe that for the origin  $z = 0$  we have  $\text{dist}(0; \mathcal{F}(0)) = 1$ , and the distance is realized at, e.g.,  $u = (-1, 0, 0, \dots)$ . Indeed, passing to the limit in the inequality

$$x(1) + x(t) \leq -1 \text{ as } t \rightarrow \infty$$

and taking into account that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , by the structure of the space of  $c_0$ , we get  $x(1) \leq -1$ . Furthermore, it can be checked that

$$(e_1^*, -1) \in \text{cl}^* C(0), \quad \langle e_1^*, x - u \rangle \leq 0 \text{ for all } x \in \mathcal{F}(0),$$

$$\text{dist}(z; \mathcal{F}(0)) = \|z - u\| = \langle e_1^*, z - u \rangle = \frac{\langle e_1^*, z \rangle - (-1)}{\|e_1^*\|}.$$

On the other hand, for the pair  $(x^*, \alpha) \in X^* \times \mathbb{R}$  given by

$$(x^*, \alpha) := \left( e_1^* + \sum_{t \in \mathbb{N}} \lambda_t e_t^*, -1 \right) \in C(0) \text{ with } \lambda \in \mathbb{R}_+^{(\mathbb{N})} \text{ and } \sum_{t \in \mathbb{N}} \lambda_t = 1,$$

we can directly check that  $\|x^*\| = 2$  and hence

$$\frac{[\langle x^*, z \rangle - \alpha]_+}{\|x^*\|} = \frac{1}{2},$$

which shows that the equality in (4.13) is violated for the countable system (4.18) in the nonreflexive Asplund space  $X = c_0$  of decision variables.  $\triangle$

Now we are ready to establish a verifiable precise formula for computing the exact Lipschitzian bound  $\text{lip } \mathcal{F}(0, \bar{x})$  for the infinite system (1.2) in the general Banach space  $X$ .

**Theorem 4.5 (computing the exact Lipschitzian bound).** *Let  $\bar{x} \in \mathcal{F}(0)$  for the linear infinite inequality system (1.2) with a Banach decision space  $X$ . Assume that  $\mathcal{F}$  satisfies the strong Slater condition at  $p = 0$  and that the coefficient set  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ . The following assertions hold:*

- (i) *If  $\bar{x}$  is a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then  $\text{lip } \mathcal{F}(0, \bar{x}) = 0$ .*
- (ii) *If  $\bar{x}$  is not a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then the exact Lipschitzian bound is computed by*

$$\text{lip } \mathcal{F}(0, \bar{x}) = \max \{ \|x^*\|^{-1} \mid (x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* C(0) \} > 0 \quad (4.19)$$

via the  $w^*$ -closure of the characteristic set (2.8) at  $p = 0$ .

**Proof.** Let us first justify (i). It is shown in the proof of Theorem 3.4(i) that our current assumptions imply that  $(0, \bar{x}) \in \text{int}(\text{gph } \mathcal{F})$ , which in turn yields by the definition of the exact Lipschitzian bound that  $\text{lip } \mathcal{F}(0, \bar{x})$  in this case.

Next we prove the more difficult assertion (ii) of the theorem assuming that  $\bar{x}$  is *not* a strong Slater point for  $\mathcal{F}$  at  $p = 0$ . Observe that by Lemma 3.3 the set (3.8) under the maximum operation on the right-hand side in (4.19) is nonempty and  $w^*$ -compact in  $X^*$  and the maximum over this set is realized and hence it is finite. The inequality “ $\geq$ ” in (4.19) follows from the estimate

$$\text{lip } \mathcal{F}(0, \bar{x}) \geq \|D^* \mathcal{F}(0, \bar{x})\|$$

established for general mappings between Banach space in [28, Theorem 1.44] and formula (3.9) for computing the coderivative norm of the infinite inequality system  $\mathcal{F}$  in (1.2) derived above in Theorem 3.4. It remains to prove the opposite inequality “ $\leq$ ” in (4.19).

To proceed, we use the *distance representation* (2.2) of the exact Lipschitzian bound. It is easy to see directly from the structure of  $\mathcal{F}$  in (1.2) that

$$\text{dist}(p; \mathcal{F}^{-1}(x)) = \sup_{t \in T} [\langle a_t^*, x \rangle - b_t - p_t]_+. \quad (4.20)$$

Furthermore, we can represent the right-hand side expression in (4.20) via the  $w^*$ -closure of the characteristic set (2.8) as follows:

$$\sup_{t \in T} [\langle a_t^*, x \rangle - b_t - p_t]_+ = \sup_{(x^*, \alpha) \in \text{cl}^* C(p)} [\langle x^*, x \rangle - \alpha]_+. \quad (4.21)$$

Indeed, picking  $(x^*, \alpha) \in \text{cl}^* C(p)$ , representing it the net limiting form

$$(x^*, \alpha) = w^* \text{-} \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \lambda_{t\nu} (a_t^*, b_t) \quad \text{with} \quad \{\lambda_\nu\}_{\nu \in \mathcal{N}} \subset \mathbb{R}_+^{(T)}, \quad \sum_{t \in T} \lambda_{t\nu} = 1,$$

and applying both sides of the latter equality at  $(x, -1)$ , we get

$$[\langle x^*, x \rangle - \alpha]_+ = \lim_{\nu \in \mathcal{N}} \left[ \sum_{t \in T} \lambda_{t\nu} (\langle a_t^*, x \rangle - b_t - p_t) \right]_+ \leq \sup_{t \in T} [\langle a_t^*, x \rangle - b_t - p_t]_+,$$

which justifies representation (4.21).

Now substituting the distance formula (4.6) from Lemma 4.2 as well as those in (4.20) and (4.21) into the limiting distance representation (2.2) of the exact Lipschitzian bound for  $F = \mathcal{F}$  at  $(0, \bar{x})$ , we get

$$\begin{aligned} \text{lip } \mathcal{F}(0, \bar{x}) &= \limsup_{(p, x) \rightarrow (0, \bar{x})} \frac{\sup_{(x^*, \alpha) \in \text{cl}^* C(p)} [\langle x^*, x \rangle - \alpha]_+ \|x^*\|^{-1}}{\sup_{(x^*, \alpha) \in \text{cl}^* C(p)} [\langle x^*, x \rangle - \alpha]_+} \\ &= \lim_{k \rightarrow \infty} \frac{\sup_{(x^*, \alpha) \in \text{cl}^* C(p_k)} [\langle x^*, x_k \rangle - \alpha]_+ \|x^*\|^{-1}}{\sup_{(x^*, \alpha) \in \text{cl}^* C(p_k)} [\langle x^*, x_k \rangle - \alpha]_+} \end{aligned}$$

along some sequence  $\{(p_k, x_k)\}$  converging to  $(0, \bar{x})$  as  $k \rightarrow \infty$ . In what follows we write  $p_k = (p_{tk})_{t \in T}$  for each  $k \in \mathbb{N}$ . Since  $\bar{x}$  is not strong Slater point for  $p = 0$ , we have  $\sup_{t \in T} [\langle a_t^*, \bar{x} \rangle - b_t] = 0$  and hence

$$\left| \sup_{t \in T} [\langle a_t^*, x_k \rangle - b_t - p_{tk}] \right| \leq \|\bar{x} - x_k\| \sup_{t \in T} \|a_t^*\| + \|p_k\|.$$

Thus we can choose  $\{t_k\}_{k \in \mathbb{N}}$  in  $T$  such that  $\lim_{k \rightarrow \infty} [\langle a_{t_k}^*, x_k \rangle - b_{t_k} - p_{t_k k}] = 0$ . Given any  $m \in \mathbb{N}$ , choose now  $k_m \in \mathbb{N}$  such that

$$\langle a_{t_k}^*, x_k \rangle - b_{t_k} - p_{t_k k} \geq -\frac{1}{m} \quad \text{whenever} \quad k \geq k_m, \quad m \in \mathbb{N}.$$

Consequently we have the relationship

$$\left\{ (x^*, \alpha) \in C(p_k) \mid \langle x^*, x_k \rangle - \alpha \geq -\frac{1}{m} \right\} \neq \emptyset \quad \text{for all } k \geq k_m, \quad m \in \mathbb{N}.$$

Furthermore, we can always choose  $k_m$  to be so large that  $\langle a_t^*, x_k \rangle - b_t - p_{tk} \leq (1/m)$  for all  $t \in T$  whenever  $k \geq k_m$ , and thus

$$\left| \langle a_{t_k}^*, x_k \rangle - b_{t_k} - p_{t_k k} \right| \leq \frac{1}{m} \quad \text{for all } k \geq k_m. \quad (4.22)$$

This allows us, with no loss of generality, to select a sequence of  $k_1 < k_2 < \dots$  along which

$$\begin{aligned}
\text{lip } \mathcal{F}(0, \bar{x}) &= \lim_{m \rightarrow \infty} \frac{\sup_{(x^*, \alpha) \in \text{cl}^* C(p_{k_m}), \langle x^*, x_{k_m} \rangle - \alpha \geq -\frac{1}{m}} [\langle x^*, x_{k_m} \rangle - \alpha]_+ \|x^*\|^{-1}}{\sup_{(x^*, \alpha) \in \text{cl}^* C(p_{k_m}), \langle x^*, x_{k_m} \rangle - \alpha \geq -\frac{1}{m}} [\langle x^*, x_{k_m} \rangle - \alpha]_+} \\
&\leq \limsup_{m \rightarrow \infty} \left\{ \sup_{(x^*, \alpha) \in \text{cl}^* C(p_{k_m}), \langle x^*, x_{k_m} \rangle - \alpha \geq -\frac{1}{m}} \|x^*\|^{-1} \right\} \\
&= \lim_{m \rightarrow \infty} \|x_m^*\|^{-1} \tag{4.23}
\end{aligned}$$

for some sequence  $\{x_m^*\} \subset X^*$  and the corresponding one  $\{\alpha_m\} \subset \mathbb{R}$  satisfying

$$(x_m^*, \alpha_m) \in \text{cl}^* C(p_{k_m}) \text{ and } \langle x_m^*, x_{k_m} \rangle - \alpha_m \geq -\frac{1}{m} \text{ for all } m \in \mathbb{N}. \tag{4.24}$$

Note that, as shown in Example 4.4, the  $w^*$ -closure  $\text{cl}^* C(p_{k_m})$  of the characteristic set *cannot* be dropped in (4.24) unless the decision space  $X$  is *reflexive*. Observe also from the inequality preceding (4.22) and from (4.24) that we have in fact the equality

$$\lim_{m \rightarrow \infty} [\langle x_m^*, x_{k_m} \rangle - \alpha_m] = 0 \tag{4.25}$$

for the sequence of triples  $\{(x_m^*, x_{k_m}, \alpha_m)\} \subset X^* \times X \times \mathbb{R}$  selected above.

Since  $\{x_m^*\}_{m \in \mathbb{N}}$  belongs to the  $w^*$ -compact set  $\text{cl}^* \text{co}\{a_t^* \mid t \in T\}$ , we can select a *subnet*  $\{x_\nu^*\}_{\nu \in \mathcal{N}}$  of this sequence that  $w^*$ -converges to some  $\bar{x}^* \in \text{cl}^* \text{co}\{a_t^* \mid t \in T\}$ . Using the  $w^*$ -upper semicontinuity of the reciprocal norm  $x^* \mapsto \|x^*\|^{-1}$ , we get from (4.23) that

$$\text{lip } \mathcal{F}(0, \bar{x}) \leq \|\bar{x}^*\|^{-1}. \tag{4.26}$$

It follows from (4.25) that  $\lim_\nu \alpha_\nu = \langle \bar{x}^*, \bar{x} \rangle$  along the chosen subnet; note that the corresponding subnet  $\{x_\nu\}$  *strongly* converges to  $\bar{x}$ . Let us now show that

$$(\bar{x}^*, \langle \bar{x}^*, \bar{x} \rangle) \in \text{cl}^* C(0), \tag{4.27}$$

which implies, by the the equivalent version of the strong Slater condition in Lemma 2.3(ii), that  $\bar{x}^* \neq 0$  and thus justifies by (4.26) the desired inequality “ $\leq$ ” in (4.19).

Associated with the above subnet index  $\nu \in \mathcal{N}$ , we denote by  $\{p_\nu\}_{\nu \in \mathcal{N}}$  the corresponding subnet of the sequence of perturbations  $\{p_{k_m}\}_{m \in \mathbb{N}}$  and write  $p_\nu = (p_{t\nu})_{t \in T}$ . To prove now (4.27), fix an index  $\nu \in \mathcal{N}$  of the subnet under consideration and select for this  $\nu$  another net, denoted by  $\kappa \in \mathcal{K}$ , such that

$$(x_\nu^*, \alpha_\nu) = w^* \text{-} \lim_{\kappa \in \mathcal{K}} \sum_{t \in T} \lambda_{t\nu\kappa} (a_t^*, b_t + p_{t\nu}) \text{ for each } \nu \in \mathcal{N}, \tag{4.28}$$

where  $\{\lambda_{t\nu\kappa}\}_{\kappa \in \mathcal{K}} \subset \mathbb{R}_+^{(T)}$  with  $\sum_{t \in T} \lambda_{t\nu\kappa} = 1$  for all  $\kappa \in \mathcal{K}$ . Our goal now is to show that for each  $\nu \in \mathcal{N}$  there is  $\kappa_\nu \in \mathcal{K}$  realizing the limiting procedure to get (4.27) by

$$(\bar{x}^*, \langle \bar{x}^*, \bar{x} \rangle) = w^* \text{-} \lim_{\nu \in \mathcal{N}} \sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t) \in \text{cl}^* C(0) \text{ with } \tilde{\lambda}_{t\nu} := \lambda_{t\nu\kappa_\nu}. \tag{4.29}$$



To proceed, fix an arbitrary  $w^*$ -neighborhood  $V_{A,\varepsilon} \subset X^* \times \mathbb{R}$  of  $(\bar{x}^*, \langle \bar{x}^*, \bar{x} \rangle)$  that, by definition, is determined by a number  $\varepsilon > 0$  and a finite subset of the primal product space  $A := \{(x_i, \beta_i) \in X \times \mathbb{R} \mid i = 1, \dots, n \text{ with some } n \in \mathbb{N}\}$  as follows:

$$V_{A,\varepsilon} = \{(x^*, \alpha) \in X^* \times \mathbb{R} \mid \rho_A((x^*, \alpha), (\bar{x}^*, \langle \bar{x}^*, \bar{x} \rangle)) < \varepsilon\},$$

where  $\rho_A$  is a *seminorm* on  $X^* \times \mathbb{R}$  defined by

$$\rho_A((x_1^*, \alpha_1), (x_2^*, \alpha_2)) := \max_{1 \leq i \leq n} |\langle x_1^* - x_2^*, x_i \rangle + (\alpha_1 - \alpha_2)\beta_i|.$$

Since  $(x_\nu^*, \alpha_\nu) \rightarrow (\bar{x}^*, \langle \bar{x}^*, \bar{x} \rangle)$  for the subnet constructed above, we choose  $\nu \in \mathcal{N}$  with  $(x_\nu^*, \alpha_\nu) \in V_{A,\varepsilon/3}$ . Moreover, for each  $\nu$  we can select  $\kappa_\nu \in \mathcal{K}$  such that

$$\sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t + p_{t\nu}) \in V_{A,2\varepsilon/3} \quad (4.30)$$

with the notation from (4.28) and (4.29). Indeed, inclusion (4.30) is satisfied by letting

$$\rho_A\left((x_\nu^*, \alpha_\nu), \sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t + p_{t\nu})\right) < \frac{\varepsilon}{3}.$$

Observe further, by seminorm properties and the above choice of nets and notation, that

$$\begin{aligned} & \rho_A\left(\sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t + p_{t\nu}), \sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t)\right) \\ &= \max_{1 \leq i \leq n} \left| \beta_i \sum_{t \in T} \tilde{\lambda}_{t\nu} p_{t\nu} \right| \leq \left( \max_{1 \leq i \leq n} \beta_i \right) \|p_\nu\|. \end{aligned}$$

Thus letting  $\beta_i \|p_\nu\| < \varepsilon/3$  for all  $i = 1, \dots, n$  and taking into account that

$$\left\| \sum_{t \in T} \tilde{\lambda}_{t\nu} p_{t\nu} \right\| \leq \|p_\nu\| \rightarrow 0$$

along the selected net, we conclude that

$$\sum_{t \in T} \tilde{\lambda}_{t\nu} (a_t^*, b_t) \in V_{A,\varepsilon},$$

which implies (4.29) and completes the proof of the theorem.  $\triangle$

Comparing finally the results on computing the coderivative norm in Theorem 3.4 and the exact Lipschitzian bound in Theorem 4.5, we arrive at the *unconditional* relationship between the coderivative norm and exact Lipschitzian bound of the infinite system  $\mathcal{F}$  in infinite dimensions known before only for mappings between finite-dimensional spaces; cf. formula (2.5) and the corresponding discussions in Section 2.

**Corollary 4.6 (relationship between the exact Lipschitzian bound and coderivative norm).** *Let  $\bar{x} \in \mathcal{F}(0)$  for the infinite system (1.2) satisfying the strong Slater condition at  $p = 0$ . Assume that the decision space  $X$  is arbitrary Banach and that the coefficient set  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ . Then*

$$\text{lip } \mathcal{F}(0, \bar{x}) = \|D^* \mathcal{F}(0, \bar{x})\|. \quad (4.31)$$

**Proof.** If  $\bar{x}$  is a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then we have equality (4.31) directly by comparing assertions (i) in Theorem 3.4 and Theorem 4.5 that ensure that

$$\text{lip } \mathcal{F}(0, \bar{x}) = \|D^* \mathcal{F}(0, \bar{x})\| = 0.$$

Note that the boundedness assumption on  $\{a_t^* \mid t \in T\}$  is not needed in Theorem 3.4(i) while it is needed in Theorem 4.5(i). On the other hand, if  $\bar{x}$  is not a strong Slater point for  $\mathcal{F}$  at  $p = 0$ , then equality (4.31) follows from comparing assertions (ii) in Theorem 3.4 and Theorem 4.5 that justify the *same formula* for computing  $\|D^* \mathcal{F}(0, \bar{x})\|$  and  $\text{lip } \mathcal{F}(0, \bar{x})$  in (3.9) and (4.19), respectively.  $\triangle$

## 5 Necessary Optimality Conditions for Nonsmooth Infinite and Semi-Infinite Programs

This section is devoted to deriving necessary optimality conditions for the parametric infinite programming problem (1.1) with the constraint inequality system (1.2) of feasible solutions belonging to an arbitrary Banach space  $X$  of decision variables. We keep the same standing assumptions on the initial data of the feasible solution map  $\mathcal{F}$  as in the preceding sections while imposing fairly general requirements on the cost function  $\varphi$ , which may be *nonsmooth* and heavily *nonconvex*. To the best of our knowledge, not much has been done in the theory of necessary optimality conditions for problems of infinite and semi-infinite programs with nonsmooth and nonconvex objectives. The results obtained in this section seem to be new in both semi-infinite and infinite frameworks even for programs with smooth objectives; see Remarks 5.3 and 5.6 for more discussions and references.

As mentioned in Section 1, we derive necessary optimality conditions of two independent types: *lower subdifferential* and *upper subdifferential* ones depending on the type and “direction” of subdifferentials used for the cost function; cf. [29, Chapter 5] for the classification and developments in other optimization frameworks. For the infinite and semi-infinite programs under consideration, computing the *coderivative* of the feasible solution map  $\mathcal{F}$  in Section 3 plays a crucial role in formulating and proving the most suitable qualification requirements and constructive necessary optimality conditions of both lower and upper subdifferential types expressed entirely via the initial data.

To proceed, let us briefly overview some notions of generalized differentiation employed in our subsequent results. We begin with *lower* subdifferentials that extend the classical subdifferential of convex analysis to nonconvex settings and are conventionally used in the study of *minimization* problems with “*less or equal* ( $\leq$ )” inequality constraints as in (1.2). Since our infinite/semi-infinite setup is intrinsically in general Banach spaces by the unavoidable  $l_\infty$  nature of the parameter space  $l_\infty(T)$  (see Proposition 2.4) independently of the dimension and nature of the decision space  $X$ , and since we have to work in the product space  $l_\infty(T) \times X$  due to the parametric constrained structure of (1.1), we cannot employ the well-developed Asplund space theory from [28, 29]. The most appropriate subdifferential construction in our framework is the so-called approximate subdifferential by Ioffe [16, 17], which is a general (while more complicated, topological) Banach space extension of the (sequential) basic/limiting subdifferential by Mordukhovich [23, 28] that may be larger than

the latter even for locally Lipschitzian functions on nonseparable Asplund spaces while it is always smaller than the Clarke subdifferential; see [28, Subsection 3.2.3] for more details.

The approximate subdifferential constructions on arbitrary Banach spaces are defined by the following multistep procedure. Given a function  $\varphi: Z \rightarrow \overline{\mathbb{R}}$  finite at  $\bar{z}$ , we first consider its *lower Dini* (or Dini-Hadamard) *directional derivative*

$$d^- \varphi(\bar{z}; v) := \liminf_{\substack{u \rightarrow v \\ t \downarrow 0}} \frac{\varphi(\bar{z} + tu) - \varphi(\bar{z})}{t}, \quad v \in Z,$$

and then define the *Dini  $\varepsilon$ -subdifferential* of  $\varphi$  at  $\bar{z}$  by

$$\partial_\varepsilon^- \varphi(\bar{z}) := \{z^* \in Z^* \mid \langle z^*, v \rangle \leq d^- \varphi(\bar{z}; v) + \varepsilon \|v\| \text{ for all } v \in Z\}, \quad \varepsilon \geq 0.$$

As usual, put  $\partial_\varepsilon^- \varphi(\bar{z}) := \emptyset$  if  $\varphi(\bar{z}) = \infty$ . The *A-subdifferential* of  $\varphi$  at  $\bar{z}$  is defined via topological limits involving finite-dimensional reductions of  $\varepsilon$ -subgradients by

$$\partial_A \varphi(\bar{z}) := \bigcap_{\substack{L \in \mathcal{L} \\ \varepsilon > 0}} \overline{\text{Lim sup}}_{z \xrightarrow{\varphi} \bar{z}} \partial_\varepsilon^-(\varphi + \delta(\cdot; L))(z),$$

where  $\mathcal{L}$  is the collection of all finite-dimensional subspaces of  $Z$ , where  $\delta(\cdot; L)$  is the indicator function of a set equal 0 on the set and  $\infty$  otherwise, where  $z \xrightarrow{\varphi} \bar{z}$  means that  $z \rightarrow \bar{z}$  with  $\varphi(z) \rightarrow \varphi(\bar{z})$ , and where  $\overline{\text{Lim sup}}$  stands for the *topological Painlevé-Kuratowski* upper/outer limit of a mapping  $F: Z \rightrightarrows Z^*$  as  $z \rightarrow \bar{z}$  defined by

$$\overline{\text{Lim sup}}_{z \rightarrow \bar{z}} F(z) := \left\{ z^* \in Z^* \mid \exists \text{ a net } (z_\nu, z_\nu^*)_{\nu \in \mathcal{N}} \subset Z \times Z^* \text{ with } z_\nu^* \in F(z_\nu) \text{ and } (z_\nu, z_\nu^*) \rightarrow (\bar{z}, z^*) \text{ in the } \|\cdot\| \times w^* \text{ topology of } Z \times Z^* \right\}.$$

Then the *approximate G-subdifferential* of  $\varphi$  at  $\bar{z}$  (the main construction called the “nucleus of the  $G$ -subdifferential” in [16]) is defined by

$$\partial_G \varphi(\bar{z}) := \left\{ z^* \in X^* \mid (z^*, -1) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{z}, \varphi(z)); \text{epi } \varphi) \right\}, \quad (5.1)$$

where  $\text{epi } \varphi := \{(z, \mu) \in Z \times \mathbb{R} \mid \mu \geq \varphi(z)\}$ . This construction, in any Banach space  $Z$ , reduces to the classical derivative in the case of smooth functions and to the classical subdifferential of convex analysis if  $\varphi$  is convex.

In what follows we also need the *singular G-subdifferential* of  $\varphi$  at  $\bar{x}$  defined by

$$\partial_G^\infty \varphi(\bar{x}) := \left\{ z^* \in X^* \mid (z^*, 0) \in \bigcup_{\lambda > 0} \lambda \partial_A \text{dist}((\bar{x}, \varphi(z)); \text{epi } \varphi) \right\}. \quad (5.2)$$

Note that  $\partial_G^\infty \varphi(\bar{x}) = \{0\}$  if  $\varphi$  is locally Lipschitzian around  $\bar{x}$ .

Now we are ready to establish the first result of this section providing lower subdifferential necessary optimality conditions for the original infinite programming problem in (1.1), (1.2) with a general nonsmooth cost function  $\varphi$  in Banach spaces. These conditions and the subsequent results of this section address an arbitrary local minimizer  $(\bar{p}, \bar{x}) \in \text{gph } \mathcal{F}$  to the problem under consideration. Following our convention in the previous sections, we suppose without loss of generality that  $\bar{p} = 0$ .

**Theorem 5.1 (lower subdifferential optimality conditions for nonsmooth infinite programming in Banach spaces).** *Let  $(0, \bar{x}) \in \text{gph } \mathcal{F}$  be a local minimizer for problem (1.1) with the constraint system (1.2) given by the infinite linear inequalities. Assume that the decision space  $X$  is Banach and that the cost function  $\varphi: l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$  is lower semicontinuous around  $(0, \bar{x})$  with  $\varphi(0, \bar{x}) < \infty$ . Suppose furthermore that:*

- (a) *either  $\varphi$  is locally Lipschitzian around  $(0, \bar{x})$ ,*
- (b) *or  $\mathcal{F}$  satisfies the strong Slater condition at  $p = 0$ , the coefficient set  $\{a_t^* \mid t \in T\}$  is bounded in  $X^*$ , and the system*

$$(p^*, x^*) \in \partial_G^\infty \varphi(0, \bar{x}), \quad -(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone}\{(-\delta_t, a_t^*, b_t) \mid t \in T\} \quad (5.3)$$

*has only the trivial solution  $(p^*, x^*) = (0, 0)$ .*

*Then there is a  $G$ -subgradient pair  $(p^*, x^*) \in \partial_G \varphi(0, \bar{x})$  such that*

$$-(p^*, x^*, \langle x^*, \bar{x} \rangle) \in \text{cl}^* \text{cone}\{(-\delta_t, a_t^*, b_t) \mid t \in T\}. \quad (5.4)$$

**Proof.** The original problem (1.1) can be rewritten as a mathematical program with geometric constraints:

$$\text{minimize } \varphi(p, x) \text{ subject to } (p, x) \in \text{gph } \mathcal{F}, \quad (5.5)$$

which is equivalently described by unconstrained minimization with “infinite penalties”:

$$\text{minimize } \varphi(p, x) + \delta((p, x); \text{gph } \mathcal{F})$$

via the indicator function of the graph of the feasible set  $\mathcal{F}$ . Applying the  $G$ -generalized Fermat stationary rule to the latter problem at its local minimizer  $(0, \bar{x})$ , we have

$$(0, 0) \in \partial_G [\varphi + \delta(\cdot; \text{gph } \mathcal{F})](0, \bar{x}). \quad (5.6)$$

Employing the  $G$ -subdifferential sum rule to (5.6), formulated in [16, Theorem 7.4] for the “nuclei”, we obtained from (5.6) that

$$(0, 0) \in \partial_G \varphi(0, \bar{x}) + N((0, \bar{x}); \text{gph } \mathcal{F}) \quad (5.7)$$

provided that either  $\varphi$  is locally Lipschitzian around  $(0, \bar{x})$ , or the interior of  $\text{gph } \mathcal{F}$  is nonempty and the qualification condition

$$\partial_G^\infty \varphi(0, \bar{x}) \cap [-N((0, \bar{x}); \text{gph } \mathcal{F})] = \{0, 0\} \quad (5.8)$$

is satisfied. Note that the strong Slater condition (2.9) and the boundedness of  $\{a_t^* \mid t \in T\}$  surely imply that the interior of  $\text{gph } \mathcal{F}$  is nonempty. Observe furthermore that, by the coderivative definition (2.3), the optimality condition (5.7) can be equivalently written as

$$\text{there is } (p^*, x^*) \in \partial_G \varphi(0, \bar{x}) \text{ with } -p^* \in D^* \mathcal{F}(0, \bar{x})(x^*). \quad (5.9)$$

Employing now in (5.9) the precise coderivative calculation from Theorem 3.2, we arrive at (5.4). It follows by similar arguments that the qualification condition (5.8) can be written in the explicit form (5.3). This completes the proof of the theorem.  $\triangle$

The result of Theorem 5.1 is represented in a much simpler form for smooth programs (1.1), which also seems to be new in the framework of infinite programming under consideration. Recall that a function  $\varphi: Z \rightarrow \mathbb{R}$  is *strictly differentiable* at  $\bar{z}$ , with its gradient at this point denoted by  $\nabla\varphi(\bar{z}) \in Z^*$ , if we have

$$\lim_{z, u \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(u) - \langle \nabla\varphi(\bar{z}), z - u \rangle}{\|z - u\|} = 0,$$

which surely holds if  $\varphi$  is continuously differentiable around  $\bar{z}$ .

**Corollary 5.2 (necessary optimality conditions for smooth infinite programs).** *Given a local minimizer  $(0, \bar{x}) \in \text{gph } \mathcal{F}$  for problem (1.1) with the feasible solution map (1.2) in a Banach space  $X$ , assume that the cost function  $\varphi: l_\infty(T) \times X \rightarrow \overline{\mathbb{R}}$  is strictly differentiable at  $(0, \bar{x})$ . Then we have the inclusion*

$$-\left(\nabla_p\varphi(0, \bar{x}), \nabla_x\varphi(0, \bar{x}), \langle \nabla_x\varphi(0, \bar{x}), \bar{x} \rangle\right) \in \text{cl}^*\text{cone}\left\{(-\delta_t, a_t^*, b_t) \mid t \in T\right\}. \quad (5.10)$$

**Proof.** It is easy to check that

$$\partial_G\varphi(0, \bar{x}) = \left\{(\nabla_p\varphi(0, \bar{x}), \nabla_x\varphi(0, \bar{x}))\right\}$$

if  $\varphi$  is strictly differentiable at  $(0, \bar{x})$ . Furthermore, a function strictly differentiable at some point is well known to be locally Lipschitzian around this point. Thus assumption (a) of Theorem 5.1 is satisfied (observe that the strong Slater condition on  $\mathcal{F}$  and the boundedness of  $\{a_t^* \mid t \in T\}$  are not needed), and we arrive at (5.10) from (5.4).  $\triangle$

**Remark 5.3 (qualified asymptotic optimality conditions).** Observe that the necessary optimality conditions obtained in Theorem 5.1 and its Corollary 5.2 for general classes of infinite programs are given in the *qualified/KKT form* (nonzero multiplier for the cost function) with *no constraint qualification* imposed in the case of Lipschitzian and hence of smooth objectives. This differs the conditions obtained from those known in the literature, mainly for smooth semi-infinite programs and for convex infinite and semi-infinite ones; see Section 1. We particularly refer the reader to the recent paper [37] (probably the first one on nonsmooth and nonconvex semi-infinite optimization), which contains necessary optimality conditions of the Lagrangian type, under certain constraint qualifications, for nonsmooth and nonconvex semi-infinite programs with a compact index set that are expressed in terms of Clarke's generalized gradient. On the other hand, our necessary optimality conditions are given in the new *asymptotic form* that involves the  $w^*$ -closure of the set on the right-hand sides in (5.4) and (5.10), which does not seem to be used in the previous publications on optimality conditions in infinite or semi-infinite programming.

Next we derive qualified asymptotic necessary optimality condition of a new *upper sub-differential* type, initiated in [27] for other classes of optimization problems with finitely

many constraints. The upper subdifferential optimality conditions presented below are generally independent of Theorem 5.1 while reduce to Corollary 5.2 for smooth ( $C^1$ ) problems and give significantly stronger results for nonsmooth objectives of important special structure; see Remark 5.6 for more details and discussions. The main characteristic feature of upper subdifferential conditions is that they apply to *minimization* problems but not to the expected framework of maximization.

To proceed, we recall the notion of the *Fréchet upper subdifferential* (known also as the Fréchet or viscosity superdifferential) of  $\varphi: X \rightarrow \mathbb{R}$  at  $\bar{z}$  defined by

$$\widehat{\partial}^+ \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \limsup_{z \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leq 0 \right\}, \quad (5.11)$$

which reduces to the classical gradient  $\nabla \varphi(\bar{z})$  if  $\varphi$  is Fréchet differentiable at  $\bar{z}$  and to the (upper) subdifferential of *concave* functions in the framework of convex analysis. Note that we always have the relationship  $\widehat{\partial}^+ \varphi(\bar{z}) = -\widehat{\partial}(-\varphi)(\bar{z})$  between the upper subdifferential (5.11) and its *lower* (usual) Fréchet counterpart

$$\widehat{\partial} \varphi(\bar{z}) := \left\{ z^* \in Z^* \mid \liminf_{z \rightarrow \bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \geq 0 \right\}. \quad (5.12)$$

We have the following upper subdifferential optimality conditions for the general infinite and semi-infinite programs (1.1), (1.2) under consideration.

**Theorem 5.4 (upper subdifferential optimality conditions for nonsmooth infinite programming in Banach spaces).** *Let  $(0, \bar{x}) \in \text{gph } \mathcal{F}$  be a local minimizer for problem (1.1) with the infinite constraint system (1.2) in Banach spaces. Then EVERY upper subgradient  $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$  satisfies inclusion (5.4) in Theorem 5.1.*

**Proof.** Pick any  $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$  and, employing [28, Theorem 1.88(i)] held in arbitrary Banach spaces, construct a function  $s: l_\infty(T) \times X \rightarrow \mathbb{R}$  such that

$$s(0, \bar{x}) = \varphi(0, \bar{x}), \quad \varphi(p, x) \leq s(p, x) \text{ for all } (p, x) \in l_\infty(T) \times X, \quad (5.13)$$

and  $s(\cdot)$  is Fréchet differentiable at  $(0, \bar{x})$  with the gradient  $\nabla s(0, \bar{x}) = (p^*, x^*)$ . Taking into account that  $(0, \bar{x})$  is a local minimizer for (1.1) and that

$$s(0, \bar{x}) = \varphi(0, \bar{x}) \leq \varphi(p, x) \leq s(p, x) \text{ for all } (p, x) \in \text{gph } \mathcal{F} \text{ near } (0, \bar{x})$$

by (5.13), we conclude that  $(0, \bar{x})$  is a *local minimizer* for the auxiliary problem

$$\text{minimize } s(p, x) \text{ subject to } (p, x) \in \text{gph } \mathcal{F} \quad (5.14)$$

with the objective  $s(\cdot)$  that is Fréchet differentiable at  $(0, \bar{x})$ . Rewriting (5.14) in the infinite-penalty unconstrained form

$$\text{minimize } s(p, x) + \delta((p, x); \text{gph } \mathcal{F})$$

via the indicator function of  $\text{gph } \mathcal{F}$ , observe directly from definition (5.12) of the Fréchet subdifferential at a local minimizer that

$$(0, 0) \in \widehat{\partial}[s + \delta(\cdot; \text{gph } \mathcal{F})](0, \bar{x}). \quad (5.15)$$

Since  $s(\cdot)$  is Fréchet differentiable at  $(0, \bar{x})$ , we easily get from (5.15) that

$$(0, 0) \in \nabla s(0, \bar{x}) + N((0, \bar{x}); \text{gph } \mathcal{F}),$$

which implies by  $\nabla s(0, \bar{x}) = (p^*, x^*)$  and the coderivative definition (2.3) that

$$-p^* \in D^* \mathcal{F}(0, \bar{x})(x^*).$$

Employing finally the coderivative description for the infinite system  $\mathcal{F}$  at  $(0, \bar{x})$  from Theorem 3.2, we arrive at the optimality condition (5.4) held for *every*  $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$  and thus complete the proof of the theorem.  $\triangle$

As a simple consequence of Theorem 5.4, we get an improvement of Corollary 5.2, where the cost function  $\varphi$  is assumed to be merely *Fréchet differentiable* at the optimal point  $(0, \bar{x})$  instead of the more restrictive assumption on its strict differentiability at this point.

**Corollary 5.5 (necessary optimality conditions for infinite programs with Fréchet differentiable objectives).** *Let  $(0, \bar{x}) \in \text{gph } \mathcal{F}$  be a local minimizer for problem (1.1), (1.2) with a Banach decision space  $X$ , where the cost function  $\varphi$  is Fréchet differentiable at  $(0, \bar{x})$  with the gradient  $\nabla \varphi(0, \bar{x})$ . Then the necessary optimality condition (5.10) is satisfied.*

**Proof.** It follows directly from Theorem 5.4 due to the fact that  $\widehat{\partial}^+ \varphi(0, \bar{x}) = \{\nabla \varphi(0, \bar{x})\}$  when  $\varphi$  is Fréchet differentiable at  $(0, \bar{x})$ .  $\triangle$

Finally, let us discuss the major relationships between the lower and upper subdifferential optimality conditions obtained in this paper.

**Remark 5.6 (comparison between lower and upper subdifferential optimality conditions for infinite and semi-infinite programs).** We can see that the necessary optimality conditions in Theorem 5.1 and Theorem 5.4 are formulated in the similar formats with two visible distinctions:

(i) There are *no assumptions* imposed on  $\varphi$  and  $\mathcal{F}$  in Theorem 5.4 in contrast to those in Theorem 5.1.

(ii) The resulting inclusion (5.4) is proved to hold for *every* Fréchet upper subgradient  $(p^*, x^*) \in \widehat{\partial}^+ \varphi(0, \bar{x})$  in Theorem 5.4 instead of *some*  $G$ -subgradient  $(p^*, x^*) \in \partial_G \varphi(0, \bar{x})$  in the lower subdifferential result of Theorem 5.1.

The underlying issue to draw the reader's attention is that the Fréchet upper subdifferential  $\widehat{\partial}^+ \varphi(0, \bar{x})$  may be *empty* in many important situations (e.g., for convex cost functions) while the  $G$ -subdifferential is *nonempty* at least for every function on a Banach space that is locally Lipschitzian around  $(0, \bar{x})$ . Note that the optimality condition of Theorem 5.4 holds trivially if  $\widehat{\partial}^+ \varphi(0, \bar{x}) = \emptyset$ , while even in this case it provides an easily checkable information on optimality without taking constraints into account. Of course, a real strength of upper subdifferential optimality conditions as in Theorem 5.4 should be exhibited for *nonsmooth* cost functions admitting Fréchet upper subgradients at the point in question.

There are remarkable classes of nonsmooth functions enjoying the latter property. First we mention *concave* continuous functions on arbitrary Banach spaces and also *DC* (*difference of convex*) functions whose minimization can be reduced to minimizing concave functions

subject to convex constraints. Another important class of functions admitting a nonempty set of Fréchet upper subgradients consists of the so-called *semiconcave* functions, known also under various other names (e.g., upper subsmooth, paraconcave, approximately concave, etc.) and being particularly important for applications to optimization, viscosity solutions of the Hamilton-Jacobi partial differential equations, optimal control, and differential games; see more discussions and references in [29, Commentary 5.5.4, pp. 135–136].

Since the  $G$ -subdifferential  $\partial_G\varphi(0, \bar{x})$  is smaller than the Clarke generalized gradient  $\partial_C\varphi(0, \bar{x})$  for lower semicontinuous functions in any Banach space, Theorem 5.1 immediately implies its counterparts with a  $C$ -subgradient  $(p^*, x^*) \in \partial_C\varphi(0, \bar{x})$  therein. It is worth emphasizing that the latter lower subdifferential optimality condition is *significantly weaker* than the upper subdifferential one in Theorem 5.4 for concave and other “upper regular” functions (see [28]) including those mentioned above. Considering for simplicity the case of concave continuous functions, we have

$$\widehat{\partial}^+\varphi(\bar{z}) = -\widehat{\partial}(-\varphi)(\bar{z}) = -\partial_C(-\varphi)(\bar{z}) = \partial_C\varphi(\bar{z}) \neq \emptyset$$

due to the plus-minus symmetry of the generalized gradient for locally Lipschitzian functions. Thus Theorem 5.4 dramatically strengthens the  $C$ -counterpart of Theorem 5.1 in such cases justifying the necessary optimality condition held for *every*  $(p^*, x^*) \in \partial_C\varphi(0, \bar{x})$  instead of just *one* element from this set.

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