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Abstract. The paper concerns minimax control problems for linear multidimensional parabolic systems with distributed uncertain perturbations and control functions acting in the Dirichlet boundary conditions. The underlying parabolic control system is functioning under hard/pointwise constraints on control and state variables. The main goal is to design a feedback control regulator that ensures the required state performance and robust stability under any feasible perturbations and minimize an energy-type functional under the worst perturbations from the given area. We develop an efficient approach to the minimax control design of constrained parabolic systems that is based on certain characteristic features of the parabolic dynamics including the transient monotonicity with respect to both controls and perturbations and the turnpike asymptotic behavior on the infinite horizon. In this way, solving a number of associated open-loop control and approximation problems, we justify an easily implemented suboptimal structure of the feedback boundary regulator and compute its optimal parameters ensuring the required state performance and robust stability of the closed-loop, highly nonlinear parabolic control system on the infinite horizon.

Key words. parabolic systems, Dirichlet boundary controls, state constraints, uncertainty perturbations, feedback control, suboptimality, minimax synthesis, robust stability

AMS subject classifications. 49K20, 49K35, 49N35, 93B50, 93D09

Abbreviated title. Feedback control of parabolic systems

1 Introduction and Problem Formulation

This paper is devoted to developing an efficient procedure of design a *suboptimal feedback control* regulator acting in the *Dirichlet boundary conditions* of a multidimensional linear *parabolic system* with *hard/pointwise constraints* on the state and control variables under distributed *uncertain perturbations.* Problems of this type are among the most challenging and difficult in control theory while being among the most important for various applications. The original motivation for our development came from practical design problems of automatic control of the soil groundwater regime in irrigation engineering networks functioning under uncertain weather and environmental conditions; see [11] for technological descriptions and modeling.

The system dynamics in the problem under consideration is given by the multidimensional *linear parabolic equation*

$$(1.1)\qquad \begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \quad \text{a.e. in } \; Q := [0,T]\times\Omega, \\ y(0,x) = 0, \quad x\in\Omega, \\ y(t,x) = u(t), \quad (t,x)\in\Sigma := [0,T]\times\partial\Omega \end{cases}$$

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with *controls* $u(\cdot)$ acting in the Dirichlet boundary conditions and distributed *perturbations* $w(\cdot)$ on the right-hand side of the parabolic equation. In (1.1), A is a *self-adjoint* and *uniformly strongly elliptic operator* on $L^2(\Omega)$ defined by

$$(1.2) \qquad A := -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} - c,$$

(see Section 2 for the precise assumptions), where $\Omega \subset I\!R^n$ is an open bounded domain with the closure $\mathrm{cl}\,\Omega$ and the boundary $\partial\Omega$ that is supposed to be a sufficiently smooth $(n-1)$-dimensional manifold, and where $T > 0$ is a fixed time bound.

The sets of *admissible controls* U and *admissible perturbations* W are given, respectively, by the relationships

$$(1.3) \qquad U := \Big\{ u \in L^\infty[0,T] \Big| \leq -\alpha \leq u(t) \leq \alpha \ \text{ a.e. } \ t \in [0,T] \Big\},$$

$$(1.4) \qquad W := \Big\{ w \in L^\infty[0,T] \Big| -\beta \leq w(t) \leq \beta \ \text{ a.e. } t \in [0,T] \Big\}$$

with some fixed bounds $\alpha, \beta > 0$. Note that control and perturbation functions look similarly via the *pointwise* constraints in (1.3) and (1.4)—except they are situated in the different parts of the parabolic system (1.1)—while their roles in the feedback control problem formulated below are completely opposite.

It has been well recognized that the *Dirichlet boundary conditions* as in (1.1) offer the *least regularity* properties of the parabolic dynamics and occur to be the *most challenging* in control theory; see, e.g., [2, 8, 14, 17, 20] with various results, discussions, and references therein. In particular, a lower regularity of feasible controls in (1.3) is *not sufficient* for the existence of classical solutions to the initial-boundary value problem in (1.1), while for any feasible pair $(u, w) \in U \times W$ there is a *unique generalized solution* $y \in L^2(Q)$ to the parabolic system (1.1); see, e.g., [9]. Having this in mind, fix a point $x_0 \in \Omega$ from the space domain and suppose that we are able to *collect information* about the system motion/performance $y(t, x_0)$ at this point. A crucial requirement on the system performance (originally motivated by the groundwater control problem in [11]) is to keep the motion $y(t, x_0)$ within the given distance $\eta > 0$ from the initial equilibrium state $y(x, 0) \equiv 0$ for the whole dynamic process. This means imposing the *pointwise state constraints* on the motion under observation

$$(1.5) \qquad -\eta \leq y(t, x_0) \leq \eta \ \text{ a.e. } \ t \in [0,T].$$

As mentioned, perturbations $w(\cdot)$ in (1.1) are *uncertain*, i.e., they are not known a priori; the only information available on perturbations is the *bound* β of their admissible variations. The main goal of boundary controls $u(\cdot)$ in (1.1) is to keep the motion $y(t, x_0)$ within the state constraints (1.5) for *all admissible perturbations* $w(\cdot)$ from (1.4). Clearly, it *cannot* be done in any (prescribed) *open loop* $u = u(t)$, and so control actions in the boundary conditions of (1.1) should be formed depending on the *current position* $y(t, x_0)$ under observation. This means that we have to design a *feedback control regulator* in the boundary conditions as a function of the state position $\xi \in I\!R^n$, where ξ is generated by the dynamic system (1.1) via the moving point of observation $y(t, x_0)$ for each $t \in [0,T]$.

To formalize this procedure, we consider a function $f\colon I\!R \to I\!R$ satisfying the composite *summability condition*

$$(1.6) \qquad \big|f(\gamma(t))\big| \in L^1[0,T] \ \text{ whenever } \ \gamma(t) \in L^2[0,T]$$

and construct boundary controls in (1.1) via the *feedback law*

$$u(t) := f\big(y(t,x_0)\big), \quad t \in [0,T]. \tag{1.7}$$

Thus boundary controls $u(t)$ in (1.1) are fully determined via (1.7) by the choice of a *feedback function/regulator* $f = f(\xi)$. We say that such a function f defines a *feasible regulator* if it satisfies the summability condition (1.6), generates controls $u(t)$ by (1.7) belonging to the admissible set U from (1.3), and keeps the corresponding motions $y(t,x_0)$ of the parabolic system (1.1) within the prescribed constraint area (1.5) for *every* admissible perturbation $w \in W$ from (1.4). The set of all feasible regulators is labeled as $\mathcal{F}$.

To estimate the *quality* of feasible regulators $f = f(\xi)$, we consider the *cost functional*

$$J(f) := \max_{w \in W} \Big\{ \int_0^T \big| f\big(y(t,x_0)\big)\big| \, dt \Big\}, \tag{1.8}$$

which is an *energy-type* functional with respect to controls (1.7) in the boundary conditions of (1.1) subject to the symmetric constraints (1.3). The *maximum* operation in (1.8) reflects the required *control energy* needed to neutralize the adverse effect of the *worst perturbations* from (1.4) and to keep the state performance within the prescribed area (1.5).

The *minimax feedback control problem* (P) studied in this paper is as follows:

$$\text{minimize } J(f) \text{ over } f \in \mathcal{F}, \tag{1.9}$$

i.e., to find an *optimal feedback control* $\overline{f} = \overline{f}(\xi)$ that minimizes the energy-type cost functional (1.8) over the set $\mathcal{F}$ of all feasible regulators, provided of course that $\mathcal{F} \neq \emptyset$.

It has been well recognized in control theory and applications that *feedback* control problems are the most challenging and important for any type of dynamical systems, while PDE systems provide additional difficulties and much less investigated in comparison, e.g., with the ODE dynamics; see more discussions and references in [14]. Furthermore, significant complications come from *pointwise/hard constraints* on control and (much more) state functions; the latter are of high nontriviality even for open-look control problems, especially in the case of Dirichlet boundary control (see, in particular, the afore-mentioned publications [2, 17, 14, 20]). We are not familiar with any device applicable to the problem (P) under consideration among a variety of approaches and results available in the theories of differential games, H_∞-control, Riccati's feedback synthesis, etc.; see, e.g., [1, 5, 6, 8] and the references therein.

In this paper we develop and significantly extend the approach to solving the feedback control problem (P), which was initiated in [12] for the case of the one-dimensional heat equation in (1.1); see also [13, 14, 16] for partial results reported for Dirichlet boundary controls of multidimensional parabolic systems and [18] for the cases of controls in the Neumann and mixed (Robin) boundary conditions.

Our approach is essentially based on certain underlying features of the parabolic dynamics, particularly on the *monotonicity property* of transients, which is eventually related to the fundamental *Maximum Principle* for parabolic equations; see Section 2. Due to this property and the specific structures of the cost functional (1.8) and boundary controls in (1.1), we are able to select the *worst perturbations* in the area (1.4) for the class of *non-increasing* and *odd feedbacks* (1.7) and then to study the corresponding *open-loop* optimal control problem with *pointwise state constraints* as a reaction of the parabolic system to the worst perturbations. Using the *spectral* Fourier type representation of solutions to the parabolic system (1.1) and assuming the *positivity* of the *first eigenvalue* of the elliptic operator A in (1.2)—which is often the case— we observe the *dominance* of the *first term* in the exponential series representation of solutions to (1.1) as $t \to \infty$. This allows us to justify an *efficient approximation* of the open-loop optimal control problem for the parabolic system under consideration by that for the corresponding *ODE system* with state constraints on a sufficiently *large* time interval. Moreover, the approximating ODE optimal control problem is solved *exactly*—under some requirements on the initial data of (P)— by constructing *yet another approximation* of state constraints, employing the *Pontryagin maximum principle* that provides *necessary and sufficient* optimality conditions for the unconstrained approximating problems with both *bang-bang* and *singular modes* of optimal controls, and then by passing to the limit while meeting the state constraints. It happens in this way (due to specific features of the ODE problems under consideration approximating the parabolic dynamics) that the *state constraints* surprisingly occur to be a *regularization factor*, which simplifies the structure of optimal controls, especially when the time interval becomes bigger and bigger ($T \to \infty$)—this reveals the fundamental *turnpike property* of such dynamic systems expanding to the *infinite horizon*.

Thus using the ODE approximation described above, we justify an easily implemented *suboptimal/near-optimal structures* of optimal controls in both *open-loop* and *closed-loop* modes and then *optimize their parameters* along the *parabolic dynamics*. This allows us arrive at a *there-positional feedback regulator* $f = f(\xi)$ in (1.7) acting via the Dirichlet boundary conditions of (1.1) that ensures the required state performance (1.5) under the fulfillments of all the constraints in (P) for *every feasible perturbation* from (1.4) providing a *near-optimal response* of the closed-loop control system in the case of *worst perturbations.*

The feedback control design constructed in this way leads us to the *highly nonlinear* closed-loop system (1.1) and (1.7), where $f(\xi)$ is a *discontinuous* three-positional regulator. The system may loose *robust stability* (in the large) and maintain the state performance (1.5) in a unacceptable *self-vibrating regime*. Developing a *variational approach* to robust stability that reduces the stability issue to a certain open-loop optimal control problem on the *infinite horizon*, we establish efficient conditions for robust stability of the closed-loop system whenever $t \geq 0$ in terms of the initial data of problem (P) and parameters of the three-positional feedback regulator.

The rest of the paper is organized as follows. In Section 2 we formulate the *standing assumptions* on the parabolic system (1.1) and then present efficient conditions ensuring the fulfillment of these assumptions and also certain important properties of the parabolic dynamics following from them.

Section 3 includes the underlying *monotonicity property* of solutions to (1.1) with respect to both controls and perturbations, which is a consequence of the fundamental *Maximum Principle* for the parabolic dynamics. Based on this result and on the specific features of the minimax problem (P) from (1.9), we justify that the *worst perturbations* (P) occur to be the *extreme* ones from (1.4) for every *nonincreasing* and *odd* feedback $f(\xi)$ in (1.7).

This allows us to consider next an *open-loop* parabolic control problem arising from (P) under the worst/extreme perturbations and then to approximate it by appropriate *ODE systems* subject to pointwise state and control constraints. Observing that the *first-order approximation* is sufficiently adequate to the parabolic dynamic on large time intervals (due to the afore-mentioned *first-term dominance* in the spectral representation of trajectories), we concentrate in Section 4 on the corresponding state-constrained ODE optimal control problem and *solve it completely* by using a *penalty-type approximation* of the state constraints and the *Pontryagin maximum principle* providing in this case *necessary and* *sufficient* conditions for the open-loop control optimality.

In Section 5 we take the optimal control structure computed for the ODE constrained problem in Section 4 and impose it as a *suboptimal structure* of open-loop boundary controls for the *parabolic system* (1.1) acting under the worst perturbations. Furthermore, we *optimize* the parameters of this structure along the constrained parabolic dynamics.

Section 6 is devoted to computing parameters of the *minimax control design* for the parabolic system (1.1) with feedback controls of type (1.7) defining by *nonincreasing* and *odd* functions $f(\xi)$. We justify the structure of a *there-positional* feedback regular and compute its parameters in such a way that it gives the open-loop *(sub)optimal* control realization of Section 4 for the case of *worst* perturbations while keeping the dynamics within the prescribed constraints (1.5) for *any* feasible perturbation $w \in W$.

In Section 7 we compute *optimal parameters* of the *closed-loop* nonlinear control system from Section 6, which ensure *robust stability* of the *stabilizing equilibrium state* of the system for all $t > 0$. This is done by reducing the stability issue to an open-loop constrained *optimal control* problem on the *infinite horizon*. Finally, we establish verifiable *reliability conditions* for the feedback control design that simultaneously ensures *controllability, stability,* and *minimax optimality* of the closed-loop constrained parabolic system.

2 Standing Assumptions and Preliminary Results

Consider the parabolic system (1.1) with the operator A defined in (1.2), where $c \in I\!R$ and where the functions $a_{ij}\colon \mathrm{cl}\,\Omega \to I\!R$ satisfy the properties:

$$\begin{aligned} &a_{ij} \in C^\infty(\mathrm{cl}\,\Omega), \;\; a_{ij}(x) = a_{ji}(x) \;\text{ for all }\; x \in \Omega, \quad i,j = 1,\ldots,n,\\ &\sum_{i,j=1}^{n} a_{ij}(x)v_i v_j \ge \nu \sum_{i=1}^{n} v_i^2 \;\text{ with some } \nu > 0 \end{aligned} \tag{2.1}$$

whenever $x \in \Omega$ and $v = (v_1,\ldots,\, v_n) \in I\!R^n$. Then the differential operator (1.2) is *self-adjoint* and *uniformly strongly elliptic* on $L^2(\Omega)$.

Observe that the input data $(u,w) \in L^\infty[0,T] \times L^\infty[0,T]$ are *irregular* to ensure the existence of the classical smooth solutions y to (1.1). Nevertheless, for all admissible pairs $(u,w) \in U \times W$ system (1.1) admits a *unique generalized solution* $y = y(t,x) \in L^2(Q)$; this is proved, e.g., in [9]. In what follows we present a convenient series representation of generalized solutions to the parabolic equation (1.1) generated by admissible pairs (u,w), while first we discuss some properties of uniformly strongly elliptic operators crucial for establishing the main results of the paper.

Given the operator A in (1.2), consider the *homogeneous boundary value problem*

$$\begin{cases} -A\phi + \lambda\phi = 0,\\ \phi|_{\partial\Omega} = 0 \end{cases} \tag{2.2}$$

and recall that the number component λ in the nontrivial pair (λ,ϕ) satisfying (2.2) is an *eigenvalue*, while ϕ is the corresponding *eigenfunction* for the operator A under the Dirichlet boundary condition. According to [3, Theorems 8.37, 8.38], the assumptions imposed on the operator A in (2.1) ensure the following properties:

(a) The eigenvalues λ_i, $i = 1,2,\ldots$, are real and form a nondecreasing sequence, which accumulates only at ∞;

(b) The corresponding orthonormal system $\{\phi_i(x)\} \subset C^\infty(\Omega)$ of eigenfunctions is complete in $L^2(\Omega)$.

(c) The first eigenvalue λ_1 is simple and has the positive eigenfunction $\phi_1(x)$.

In addition to these underlying properties, the next proposition offers one more important consequence of the strong ellipticity.

Proposition 2.1 (consequence of strong ellipticity). *Let λ be an eigenvalue of the operator A in* (1.2) *satisfying the assumptions in* (2.1). *Then for any constant $c \in I\!R$ in* (2.2) *the sum $c + \lambda$ is positive.*

Proof. It follows from [10, Theorem 3.20] that $c + \lambda \geq 0$. Thus it remains to show that $c + \lambda \neq 0$. Assuming the contrary, i.e., that $c + \lambda = 0$ and substituting the latter into (2.2), we get that the eigenfunction ϕ corresponding to λ is a solution to the homogeneous elliptic boundary value problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \phi}{\partial x_j}\right) = 0, \\ \phi|_{\partial\Omega} = 0. \end{cases}$$

Due to well-known *uniqueness* of solutions to the latter problem, we have that $\phi \equiv 0$, which contradicts the above properties (a) and (c) and completes the proof. $\triangle$

In addition to (2.1), the basis hypothesis needed to develop our approach is as follows:

(H) The first/principal eigenvalue λ_1 of the operator A in (1.2) is *positive.*

Along with properties (a) and (c), the latter implies that the eigenvalues of A satisfy the series of inequalities

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \ldots. \tag{2.3}$$

Let us provide an efficient condition on parameters of the operator A in connection with the size of the domain Ω that ensures the fulfillment of (2.3).

Proposition 2.2 (positivity of the first eigenvalue). *Let*

$$d := \sup_{x_1, x_2 \in \Omega} \|x_1 - x_2\|$$

be the diameter of the domain $\Omega \subset I\!R^n$ in (1.1), *and let the relationship*

$$c < \frac{2n\nu}{d^2} \tag{2.4}$$

hold for the parameters of the operator A from (1.2), *where $\nu > 0$ is the constant of strong ellipticity in* (2.1). *Then $\lambda_1 > 0$ for the first eigenvalue of A.*

Proof. Let the pair (λ, ϕ) satisfy (2.2). Multiplying the equation in (2.2) by ϕ and then *integrating by parts*, we come up to the equality

$$(c + \lambda)\int_\Omega \phi^2(x)\,dx = \int_\Omega \sum_{i=1}^{n} a_{ij}(x)\frac{\partial \phi}{\partial x_i}(x)\frac{\partial \phi}{\partial x_j}(x)\,dx.$$

The latter implies, due to the strong ellipticity in (2.1), that

$$(c+\lambda)\int_0^T \|\phi(x)\|^2\,dx \geq \nu \sum_{i=1}^{n} \int_0^T \left\|\frac{\partial \phi}{\partial x_i}(x)\right\|^2 dx. \tag{2.5}$$

Taking Proposition 2.1 into account and employing the *Poincaré inequality* (see, e.g., [10, Lemma 3.3]) in (2.5), we get

$$\lambda \geq \frac{2n\nu}{d^2} - c > 0$$

by (2.4), which ensures that $\lambda_1 > 0$ and completes the proof. △

As mentioned above, for every feasible input pair $(u,w) \in L^\infty[0,T] \times L^\infty[0,T]$ the parabolic system (1.1) admits a unique generalized solution $y \in L^2(Q)$. The next proposition gives a convenient *spectral representation* of this solution via a Fourier-like series involving the eigenvalues and eigenfunctions (λ_i, ϕ_i) of the operator A. This representation, which is widely used in what follows, is essentially related to the given setting of system (1.1), where both Dirichlet boundary controls $u = u(t)$ and distributed perturbations $w = w(t)$ are *spatially constant*, i.e., independent of state variables.

Proposition 2.3 (spectral representation of transients). *Let* $(u,w) \in L^\infty[0,T] \times L^\infty[0,T]$ *in* (1.1) *under assumptions* (2.1) *on the strongly elliptic operator* A, *and let* (λ_i, ϕ_i) *be the corresponding eigenvalues and eigenfunctions of* A *with the weights*

$$\mu_i := \int_\Omega \phi_i(x)\,dx, \quad i = 1, 2, \ldots.$$

Then the unique solution $y \in L^2(Q)$ *to* (1.1) *admits the spectral representation*

$$y(t,x) = \sum_{i=1}^{\infty} \mu_i \Big(\int_0^t w(\theta) e^{\lambda_i \theta} d\theta + (c+\lambda_i) \int_0^t u(\theta) e^{\lambda_i \theta} d\theta \Big) e^{-\lambda_i t} \phi_i(x), \tag{2.6}$$

where the series in (2.6) *strongly converges in the space* $L^2(Q)$.

Proof. The pointwise representation (2.6) is well known for the classical solutions to (1.1) corresponding to *smooth* inputs $(u,w) \in C^\infty[0,T] \times C^\infty[0,T]$; see, e.g., [7]. Its justification in the general (*irregular*) case under consideration is directly based on the limiting procedure, which takes into account the *continuity* of the linear *solution operator* $(u,w) \mapsto y$ from $L^\infty[0,T] \times L^\infty[0,T]$ into $L^2(Q)$ established in [9]. △

3 Monotonicity of Transients and the Worst Perturbations

In this section we begin our study of the minimax feedback control problem (P) given in (1.9) assuming that it has at least one *feasible solution*, i.e., $\mathcal{F} \neq \emptyset$. Observe that it is not always the case; see, in particular, examples in [15], where this issue is considered from the viewpoint of asymmetric games. Later in Section 6 we present efficient conditions ensuring the existence of feasible feedback controls and actually find a *suboptimal* one.

The main result of this section justifies that for *any feasible feedback law* (1.7), defined by a *nonincreasing* and *odd* function $f(\xi)$, the *worst perturbations* occur to be the *extreme* ones

$w \equiv \beta$ and, symmetrically, $w \equiv -\beta$. It is a significant observation allowing us to *decompose* the minimax problem and to develop an efficient approach to the minimax control design starting with the *open-loop* system reaction to the revealed worst perturbations.

This fact is rather *surprising* from the viewpoint of general minimax/game theory, which usually identifies minimax optimal solutions with *saddle points* consisting, in the setting of problem (P), of *interrelated* pairs of worst perturbations and optimal controls. The underlying fact for the problem (P) under consideration is partly due to the *specific structures* of the cost functional and boundary controls in (P), related to each other, while largely due to the following *monotonicity property* of solutions (transients) to the parabolic system (1.1) with respect to both perturbations and controls; cf. [13]. The latter property is based on the fundamental *Maximum Principle* for the parabolic dynamics and plays a crucial role in this and other aspects of the feedback control design developed in this paper.

Theorem 3.1 (monotonicity property of the parabolic dynamics). *Let (u_1, w_1) and (u_2, w_2) be admissible control-perturbation pairs from $U \times W$ such that*

$$u_1(t) \geq u_2(t) \;\; \text{and} \;\; w_1(t) \geq w_2(t) \;\; \text{a.e.} \;\; t \in [0,T],$$

and let $y_1, y_2 \in L^2(Q)$ be the corresponding generalized solutions to the parabolic system (1.1). Then we have

$$y_1(t,x) \geq y_2(t,x) \;\; \text{a.e.} \;\; (t,x) \in Q. \tag{3.1}$$

Proof. We derive this from the Maximum Principle for the *classical solutions* to the parabolic equations [7] by using an additional *smooth approximation procedure*. Denoting

$$u(t) := u_1(t) - u_2(t), \quad w(t) := w_1(t) - w_2(t), \quad y(t,x) := y_1(t,x) - y_2(t,x),$$

we conclude that $y \in L^2(Q)$ is the generalized solution to the parabolic system (1.1) corresponding to the *nonnegative* L^∞-inputs

$$u(t) \geq 0 \;\; \text{and} \;\; w(t) \geq 0 \;\; \text{a.e.} \;\; t \in [0,T]. \tag{3.2}$$

Take an arbitrary C^∞-function $\rho\colon I\!R \to I\!R$ with the properties

(a) $\rho(r) = 0$ if $|r| \geq 1$,
(b) $\rho(r) \geq 0$ if $|r| \leq 1$,
(c) $\displaystyle\int_{I\!R} \rho(r)dr = 1.$

Then for any $v \in L^2[0,T]$ and $\varepsilon > 0$ define

$$v_\varepsilon(t) := \frac{1}{\varepsilon}\int_{I\!R} \rho\Big(\frac{t-r}{\varepsilon}\Big) v(r)\, dr, \tag{3.3}$$

which is a C^∞-function such that $v_\varepsilon \to v$ strongly in $L^2[0,T]$ as $\varepsilon \downarrow 0$. Furthermore, $v_\varepsilon(t) \geq 0$ for all $t \in [0,T]$ whenever $v(t) \geq 0$ for a.e. $t \in [0,T]$.

Applying now the smooth approximation procedure (3.3) to the functions $u(t)$ and $w(t)$ from (3.2), we construct *nonnegative* functions $u_\varepsilon(t)$ and $w_\varepsilon(t)$ that generate, for each $\varepsilon > 0$, the *classical solution* $y_\varepsilon \in C^{1,2}([0,T] \times \mathrm{cl}\,\Omega)$ to (1.1) satisfying

$$y_\varepsilon(t,x) \geq 0 \;\; \text{for all} \;\; (t,x) \in \mathrm{cl}\, Q \tag{3.4}$$

by the parabolic Maximum Principle. Passing to the limit in (3.4) as $\varepsilon \downarrow 0$ and taking into account that $u_\varepsilon \to u$ and $w_\varepsilon \to w$ strongly in $L^2[0,T]$ which apply by (2.6) the strong convergence of $y_\varepsilon \to y$ in $L^2(Q)$, we arrive at (3.1). $\triangle$

Next we clarify the structure of the *worst perturbations* for a large class of feasible feedbacks $f \in \mathcal{F}$ in the minimax problem (P) given by (1.9). We confine our consideration by a class of feedbacks in (1.7) defined by *nonincreasing* and *odd* functions $f = f(\xi)$. This choice allows us to justify that for *any* feedback control the worst perturbations occur to be the *extreme* ones $w \equiv \beta$ and $w \equiv -\beta$.

Theorem 3.2 (worst perturbations). *Under the standing assumptions on problem (P), suppose that a feasible feedback $f \in \mathcal{F}$ is a nonincreasing and odd function on $I\!R$. Then the worst perturbations $w(t)$ providing the maximum value to the integral functional in* (1.8) *over all $w \in W$ from* (1.4) *are the extreme ones from the admissible area:*

$$w \equiv \beta \quad \textit{and} \quad w \equiv -\beta. \tag{3.5}$$

Proof. Observe first that both admissible control and perturbation areas in (1.3) and (1.4) are *fully symmetric* with respect to the origin, and they both enter *linearly* to the linear parabolic system (1.1) with the homogeneous initial condition. The state constraint area in (1.5) is symmetric as well. To keep this symmetry in the feedback system (1.1) and (1.7) with the cost functional (1.8), we consider feedback laws in (1.7) given by *odd* functions $f = f(\xi)$, i.e., by those having the symmetry $f(-\xi) = -f(\xi)$ whenever $\xi \in I\!R$. Observe furthermore that the transients $\xi(t) := y(t, x_0)$ of the system (1.1) generated by any admissible controls $u(t)$ and perturbations $w(t)$ admit the *convolution representation*

$$\xi(t) = \varphi(t) * w(t) + \psi(t) * u(t), \quad 0 \le t \le T, \tag{3.6}$$

where both functions ψ and φ are *positive* on $[0,T]$ (due to the Maximum Principle) and are the same for all $(u, w) \in U \times W$; see [15] for more details. It follows from (3.6) and the *symmetry* of (P) discussed above that without loss of generality we can consider only the *one-sided* case of controls and perturbations in (P) when

$$0 \le w(t) \le \beta \ \text{ and } \ -\alpha \le u(t) \le 0 \ \text{ a.e. } \ t \in [0,T]; \tag{3.7}$$

the other case is completely symmetric. We will use this observation in what follows.

From (3.7) and the *monotonicity property* of Theorem 3.1 we conclude that *the bigger magnitude of a perturbation is, the more control of the opposite sign should be applied* to neutralize the perturbation ensuring the required *state performance* (1.5). Furthermore, the *nonincreasing* assumption on the feedback law $f(\xi)$ gives

$$f(\xi_1) \le f(\xi_2) \ \text{ whenever } \ \xi_1 \ge \xi_2.$$

Combining all the above and taking into account the boundary control and cost functional structures in (1.7) and (1.8), we conclude that

$$\int_0^T \left| f\big(y_1(t, x_0)\big) \right| dt \ge \int_0^T \left| f\big(y_2(t, x_0)\big) \right| dt$$

if either $y_1(t, x_0) \ge y_2(t, x_0) \ge 0$ or $y_1(t, x_0) \le y_2(t, x_0) \le 0$ for a.e. $t \in [0,T]$. This means that the compensation of bigger (by magnitude) perturbations requires more control energy with respect to the cost functional in (1.8) and justifies that the extreme perturbations (3.5) are indeed the worst to problem (P) for any feasible feedback $f(\xi)$ with the properties assumed in the theorem. △

Our next step is to consider problem (P) under the *worst perturbations* (3.5). By the above discussions it is sufficient to study only the one-sided case (3.7), since the other case is completely symmetric. In case (3.7) the worst perturbation is $w \equiv \beta$, and thus problem (P) reduces under this perturbation to the following *open-loop* optimal control problem $(\overline{P})$:

$$\text{minimize } J(u) := -\int_0^T u(t)\,dt \tag{3.8}$$

along the parabolic system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = \beta \quad \text{a.e. } (t,x) \in Q, \\ y(0,x) = 0, \quad x \in \Omega, \\ y(t,x) = u(t) \quad \text{a.e. } (t,x) \in \Sigma \end{cases} \tag{3.9}$$

with the *fixed* perturbation $w \equiv \beta$ and subject to the *pointwise control* and *state constraints*

$$u(\cdot) \in L^\infty[0,T] \text{ with } -\alpha \le u(t) \le 0 \text{ a.e. } t \in [0,T], \tag{3.10}$$

$$y(\cdot, x_0) \in L^2[0,T] \text{ with } y(t,x_0) \le \eta \text{ a.e. } t \in [0,T]. \tag{3.11}$$

Problem $(\overline{P})$ is a *state-constrained Dirichlet boundary control* problem, which was considered in [14, 17] in more generality; cf. also [2, 20]. In [14, 17] we obtained *necessary optimality conditions* for $(\overline{P})$ that involve the adjoint operator to the so-called *Dirichlet map* and *Borel measures*. These conditions are rather complicated and *do not* allow us to compute or even efficiently estimate an optimal control.

In this paper we develop, following [12, 13], another approach to solve $(\overline{P})$ based first on *ODE approximations* of the parabolic system (3.9) and then on subsequent *penalty-type approximations of state constraints*. To proceed, we use the *spectral representation*

$$y(t,x_0) = \sum_{i=1}^{\infty} \mu_i \Big(\frac{\beta}{\lambda_i}(e^{\lambda_i t} - 1) + (c + \lambda_i) \int_0^t u(\theta) e^{\lambda_i \theta} d\theta \Big) e^{-\lambda_i t} \phi_i(x_0) \tag{3.12}$$

of solutions to the parabolic system (3.9) at $x = x_0$ following from Proposition 2.3 (by straightforward integration), where the series in (3.12) converges strongly in $L^2[0,T]$. Now taking any natural $N = 1, 2, \ldots$, we replace series (3.12) by the *finite N-sum*

$$y^N(t,x_0) = \sum_{i=1}^{N} \mu_i \Big(\frac{\beta}{\lambda_i}(e^{\lambda_i t} - 1) + (c + \lambda_i) \int_0^t u(\theta) e^{\lambda_i \theta} d\theta \Big) e^{-\lambda_i t} \phi_i(x_0) \tag{3.13}$$

for which $y^N(t,x_0) \to y(t,x_0)$ strongly in $L^2[0,T]$. Furthermore, it is easy to observe that $y^N(t,x_0)$ in (3.13) is represented as the sum

$$y^N(t,x_0) = \sum_{i=1}^{N} y_i(t), \quad 0 \le t \le T, \tag{3.14}$$

where each $y_i(t)$, $i = 1, \ldots, N$, satisfies the corresponding *ordinary differential equation*

$$\dot{y}_i = -\lambda_i y_i + \mu_i \phi_i(x_0)\big(\beta + (c + \lambda_i)u(t)\big) \ \text{ a.e. } \ t \in [0,T], \ \ y_i(0) = 0. \tag{3.15}$$

Due to assumption (H) in Section 2 and the inequalities in (2.3), the *first terms* in (3.12)–(3.14) *dominate* the exponential series and finite sums as $t \to \infty$, which is the case of a sufficiently large time interval $[0,T]$ of the dynamic process. Besides optimal control theory for systems on the *infinite horizon*, the latter case seems to be of a particular interest and importance, e.g., from the viewpoint of *robust stability* of the closed-loop control system studied in Section 7 via a variational approach. We refer the reader to [16] for more theoretical and numerical justifications of the first term dominance in (3.12)–(3.14), which allow us pay a special attention to the case of $N = 1$ in (3.13)–(3.15) for determining an appropriate *suboptimal control* structure in problem $(\overline{P})$ for the parabolic equation (3.9) and then for its implementation into the feedback control system. In the next section we obtain the *exact optimal solution* to the open-loop control problem corresponding to the described ODE approximation of the original parabolic PDE system (3.9) with $N = 1$ subject to all the imposed pointwise control and state constraints.

4 Exact Solutions to Approximating ODE Control Problems

According to the preceding discussions, we concentrate in this section on the study of the following *open-loop optimal control problem* $(\overline{P}_1)$: minimize the cost functional (3.8) over admissible controls $u(t)$ satisfying the constraints in (3.10) and generating absolutely continuous trajectories $y\colon [0,T] \to I\!R$ of the ODE system

$$\dot{y} = -\lambda_1 y + \mu_1 \phi_1(x_0)\big(\beta + (c + \lambda_1)u(t)\big) \ \text{ a.e. } \ t \in [0,T], \ \ y(0) = 0, \tag{4.1}$$

subject to the pointwise *state constraints*

$$y(t) \le \eta \ \text{ for all } \ t \in [0,T]. \tag{4.2}$$

Observe that the presence of the state constraints (4.2) places problem $(\overline{P}_1)$ among the most challenging problems for ODE control. Available optimality conditions for such problems involve *Borel measures* that make them very difficult for implementations and applications; see, e.g., [4] and the references therein.

We develop a different approach to solve $(\overline{P}_1)$, which employs a *penalty-type* procedure to *approximate state constraints*, then deals with solving approximating problems in the absence of state constraints, and finally derives optimal solutions to the state-constrained problem $(\overline{P}_1)$ by passing to the limit from optimal solutions to the approximating problems.

This approach occurs to be *highly efficient* for the class of problems under consideration. It allows us to find *exact optimal solutions* to the approximating problems based on the *Pontryagin maximum principle* [19], which provides necessary and sufficient optimality conditions for these problems, and then to compute by passing to the limit the *exact optimal control* to the constrained problem $(\overline{P}_1)$. It surprisingly happens that the optimal control for the state-constraint problem enjoys a *simpler structure* in comparison with the unconstrained approximating problems, and that overall the state constraint (4.2) turns out to be a *regularization factor* in this setting.

Given $\varepsilon > 0$, consider the *approximating optimal control problem* $(\overline{P}_{1\varepsilon})$ with *no state constraints* that is defined as follows:

$$\text{minimize } \ J_\varepsilon(u) := \int_0^T \Big(-u(t) + \frac{1}{\varepsilon}\big(\max\big\{0, y(t) - \eta\big\}\big)^2 \Big)\, dt \tag{4.3}$$

over controls $u(\cdot)$ satisfying (3.10) and the corresponding trajectories $y(\cdot)$ of the differential equation (4.1). It is easy to see that any control from (3.10) is *feasible* to $(\overline{P}_{1\varepsilon})$. The following theorem fully describes *optimal controls* to problems $(\overline{P}_{1\varepsilon})$ with $\varepsilon > 0$ sufficiently small in the major cases needed for further applications.

Theorem 4.1 (optimal solutions to ODE approximation problems with no state constraints). *Optimal controls $u_\varepsilon(t)$ to problems $(\overline{P}_{1\varepsilon})$ always exist and for every small $\varepsilon > 0$ are determined as follows:*

(i) *Let either $\mu_1\phi_1(x_0)\beta \le \lambda_1\eta$, or the conditions*

$$\mu_1\phi_1(x_0)\beta > \lambda_1\eta \quad \text{and} \quad \frac{\mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)\beta - \lambda_1\eta} \ge e^{\lambda_1 T} \tag{4.4}$$

be satisfied. Then $u_\varepsilon(t) \equiv 0$ for a.e. $t \in [0,T]$.

(ii) *Let $\mu_1\phi_1(x_0)\beta > \lambda_1\eta$ and both conditions*

$$\frac{\mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)\beta - \lambda_1\eta} < e^{\lambda_1 T}, \quad \mu_1\phi_1(x_0)\big(\beta - \alpha(c+\lambda_1)\big) \le \lambda_1\eta \tag{4.5}$$

be fulfilled. Then optimal controls $u_\varepsilon(t)$ to $(\overline{P}_{1\varepsilon})$ are computed by

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in [0,\tau_{1\varepsilon}) \cup (\tau_{2\varepsilon}, T], \\ \Big[\dfrac{\lambda_1\eta}{\mu_1\phi_1(x_0)(c+\lambda_1)} - \dfrac{\beta}{c+\lambda_1}\Big] + \dfrac{\lambda_1^2\varepsilon}{2\mu_1^2\phi_1^2(x_0)(c+\lambda_1)^2} & \text{if } t \in [\tau_{1\varepsilon}, \tau_{2\varepsilon}], \end{cases}$$

where the switching times $0 < \tau_{1\varepsilon} < \tau_{2\varepsilon} < T$ are given by

$$\tau_{1\varepsilon} = \frac{1}{\lambda_1}\ln\Big[1 - \frac{\lambda_1}{\mu_1\phi_1(x_0)\beta}\Big(\frac{\lambda_1\varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)} + \eta\Big)\Big]^{-1}, \tag{4.6}$$

$$\tau_{2\varepsilon} = T + \frac{1}{\lambda_1}\ln\Big[1 - \lambda_1\sqrt{\frac{\varepsilon}{\mu_1\phi_1(x_0)(c+\lambda_1)\big(\mu_1\phi_1(x_0)\beta - \lambda_1\eta\big)}}\Big]. \tag{4.7}$$

Proof. The *existence of optimal controls* is ensured by the linearity of (4.1) and (3.8) with respect to the control variable and the convexity of the control region in (3.10); see [4].

To justify the *precise formulas* to compute optimal controls asserted in the theorem, consider first the case of

$$\mu_1\phi_1(x_0)\beta \le \lambda_1\eta \tag{4.8}$$

in assertion (i). In this case the solution to the differential equation (4.1) corresponding to $u(t) \equiv 0$ on $[0,T]$ is given by

$$y(t) = \frac{\mu_1\phi_1(x_0)\beta}{\lambda_1}\Big(1 - e^{-\lambda_1 t}\Big) \quad \text{for all } t \in [0,T]. \tag{4.9}$$

It is easy to see that (4.8) and (4.9) with $\lambda_1 > 0$ imply that the state constraint (4.2) holds for this $y(t)$, i.e. $u(t) \equiv 0$ gives the *absolute minimum* to the functional (4.3) over all $u(t) \le 0$.

In case (4.4) of assertion (i) we again use the solution formula (4.9) for $y(t)$ corresponding to $u(t) \equiv 0$ on $[0,T]$ and confirm that

$$y(t) \le y(T) \le \eta \quad \text{for all } t \in [0,T],$$

which ensures the global optimality of $u(t) \equiv 0$ for al $t \in [0,T]$ to $(\overline{P}_{1\varepsilon})$ and thus completes the proof of assertion (i).

To prove assertion (ii) of the theorem, we use the *Pontryagin maximum principle* [19], which provides *necessary and sufficient conditions* for optimal controls to the problems under consideration. According to this result, optimal controls $u_\varepsilon(t)$ to each problem $(\overline{P}_{1\varepsilon})$ are *fully determined* by the *maximum condition*

$$(4.10)\quad \big(\mu_1\phi_1(x_0)(c+\lambda_1)p_\varepsilon(t)+1\big)u_\varepsilon(t) = \max_{-\alpha\le u\le 0}\Big[\big(\mu_1\phi_1(x_0)(c+\lambda_1)p_\varepsilon(t)+1\big)u\Big]$$

for a.e. $t \in [0,T]$, where $p_\varepsilon(t)$ is an absolute continuous trajectory to the *adjoint system*

$$(4.11)\qquad \dot{p}_\varepsilon(t) = \lambda_1 p_\varepsilon(t) + \frac{2}{\varepsilon}\max\Big\{0, y_\varepsilon(t)-\eta\Big\}\ \text{ a.e. } t\in[0,T],\quad p_\varepsilon(T)=0,$$

with $y_\varepsilon(t)$ generated by $u_\varepsilon(t)$ via (4.1). Thus each problem $(\overline{P}_{1\varepsilon})$ equivalently reduces to that of finding feasible controls $u_\varepsilon(t)$ satisfying the maximum condition (4.10) along with the corresponding trajectories $y_\varepsilon(t)$ and $p_\varepsilon(t)$ of the primal (4.1) and adjoint (4.11) systems.

First observe that the maximum condition (4.10) uniquely determines the control $u_\varepsilon(t)$ at any point $t \in [0,T]$ where the expression

$$\mu_1\phi_1(x_0)(c+\lambda_1)p_\varepsilon(t)+1$$

is either positive (then $u_\varepsilon(t) = 0$) or negative (then $u_\varepsilon(t) = -\alpha$). These are the so-called *bang-bang points*, where the optimal control takes one of the *extreme* values from the admissible control region $[-\alpha, 0]$. On the other hand, the maximum condition does not provide any information for points $t \in [0,T]$ at which the latter expression vanishes; these are the so-called *singular points*.

In what follows we are going to verify that for each $\varepsilon > 0$ small enough the control $u_\varepsilon(t)$ asserted in (ii) with the switching times $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$ from (4.6) and (4.7), respectively, is *feasible* to problem $(\overline{P}_{1\varepsilon})$ and satisfies the relationships of the Pontryagin maximum principle. This would justify the control optimality as discussed above.

It is easy to check that $u_\varepsilon(t) \le 0$ for all $t \in [0,T]$ due to the assumed condition $\mu_1\phi_1(x_0)\beta > \lambda_1\eta$ and that $u_\varepsilon(t) \ge -\alpha$ for all $t \in [0,T]$ due to the second condition in (4.5). This confirms the *feasibility* of $u_\varepsilon(t)$ to $(\overline{P}_{1\varepsilon})$.

To prove the *optimality* of the given control $u_\varepsilon(t)$ with the switching times $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$, we can directly substitute this control into (4.1) and (4.11), solve these equations for $y_\varepsilon(t)$ and $p_\varepsilon(t)$, and then check that the maximum condition (4.10) is fulfilled along the triple $(u_\varepsilon, y_\varepsilon, p_\varepsilon)$. Let us provide some calculations showing eventually how we *come up* to the precise formulas for computing the above optimal control $u_\varepsilon(t)$ in (ii).

One can easily verify that the switching times $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$ calculated in (4.6) and (4.7), respectively, satisfy the inequalities

$$(4.12)\qquad 0 < \tau_{1\varepsilon} < \tau_{2\varepsilon} < T.$$

Indeed, these inequalities directly follow from first two conditions assumed in (ii). Furthermore, let us demonstrate that the *singularity condition*

$$(4.13)\qquad \mu_1\phi_1(x_0)(c+\lambda_1)p_\varepsilon(t)+1 \equiv 0 \ \text{ on } \ [\tau_{1\varepsilon}, \tau_{2\varepsilon}]$$

is fulfilled along the adjoint trajectory $p_\varepsilon(t)$ generated by the control $u_\varepsilon(t)$ under consideration with the switching times $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$ computed and (4.6) and (4.7). This would signify that the control $u_\varepsilon(t)$ is *singular* on the interval $[\tau_{1\varepsilon}, \tau_{2\varepsilon}]$. We show in fact that the singularity condition (4.13) allows us to *find* $u_\varepsilon(t)$ with the switching times $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$ asserted in (ii).

To proceed, observe that the singularity condition (4.13) held on *some* interval $[\tau_{1\varepsilon}, \tau_{2\varepsilon}]$, not necessarily generated by (4.6) and (4.7), yields by differentiating identity (4.13) via the adjoint system (4.11) that

$$0 < \frac{\lambda_1 \varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)} = \max\{0, y_\varepsilon(t) - \eta\} \quad \text{for all } t \in [\tau_{1\varepsilon}, \tau_{2\varepsilon}],$$

and thus the corresponding singular arc $y_\varepsilon(\cdot)$ must be *constant* on $[\tau_{1\varepsilon}, \tau_{2\varepsilon}]$ and equal to

$$y_\varepsilon(t) = \eta + \frac{\lambda_1 \varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)}, \quad t \in [\tau_{1\varepsilon}, \tau_{2\varepsilon}]. \tag{4.14}$$

Combining (4.14) with (4.1), we inevitably arrive at the *intermediate* value of $u_\varepsilon(t)$—within the admissible region $[-\alpha, 0]$—on the singularity interval $[\tau_{1\varepsilon}, \tau_{2\varepsilon}]$ from (4.13). To justify the exact formulas for $\tau_{1\varepsilon}$ and $\tau_{2\varepsilon}$ asserted in(4.6) and (4.7), let us compute the switching times from (4.14) providing that the control $u_\varepsilon(t)$ is given by the expression in (ii) with some number $\{\tau_{1\varepsilon}, \tau_{2\varepsilon}\}$ from (4.13). Indeed, we easily get from (4.14) and formula (4.9) for $y_\varepsilon(t)$ on $[0, \tau_{1\varepsilon}]$ generated by $u_\varepsilon(t) \equiv 0$ that $\tau_{1\varepsilon}$ must satisfy the equation

$$\frac{\lambda_1 \varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)} + \eta = \frac{\mu_1\phi_1(x_0)\beta}{\lambda_1}\left(1 - e^{-\lambda_1\tau_{1\varepsilon}}\right),$$

which elementary leads to formula (4.6) for computing the asserted time $\tau_{1\varepsilon}$.

Next we take the control $u_\varepsilon(t)$ from (ii) with $\tau_{1\varepsilon}$ computed by (4.6) and with some (not precisely known so far) switching time $\tau_{2\varepsilon}$ satisfying the singularity condition (4.13) and find $\tau_{2\varepsilon}$ *explicitly* from the latter condition. Using $y_\varepsilon(t)$ on $[\tau_{1\varepsilon}, \tau_{2\varepsilon}]$ from (4.14), we compute $y_\varepsilon(t)$ on $[\tau_{2\varepsilon}, T]$ with $\tau_{1\varepsilon}$ from (4.14) and an unknown time $\tau_{2\varepsilon}$ by integrating the following system:

$$\dot{y}_\varepsilon = -\lambda_1 y_\varepsilon + \mu_1\phi_1(x_0)\beta \text{ on } [\tau_{2\varepsilon}, T], \quad y_\varepsilon(\tau_{2\varepsilon}) = \frac{2\eta\mu_1\phi_1(x_0)(c+\lambda_1) + \lambda_1\varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)}.$$

The solution of the latter initial value problem is

$$y_\varepsilon(t) = \frac{\mu_1\phi_1(x_0)\beta}{\lambda_1} + e^{\lambda_1(\tau_{2\varepsilon}-t)}\left(\frac{2\eta\mu_1\phi_1(x_0)(c+\lambda_1) + \lambda_1\varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)} - \frac{\mu_1\phi_1(x_0)\beta}{\lambda_1}\right) \tag{4.15}$$

on the time interval $[\tau_{2\varepsilon}, T]$. Substituting (4.15) into the adjoint system (4.11), we find the corresponding adjoint trajectory $p_\varepsilon(t)$ on $[\tau_{2\varepsilon}, T]$. Taking into account that

$$p_\varepsilon(\tau_{2\varepsilon}) = \frac{1}{\mu_1\phi_1(x_0)(c+\lambda_1)},$$

we arrive at the following transcendental equation for $\tau_{2\varepsilon}$:

$$\begin{aligned}\frac{\lambda_1\varepsilon}{2\mu_1\phi_1(x_0)(c+\lambda_1)} &= \frac{\mu_1\phi_1(x_0)\beta}{2\lambda_1} - \frac{\mu_1\phi_1(x_0)\beta}{2\lambda_1}e^{\lambda_1(\tau_{2\varepsilon}-T)} \\ &-\left(\frac{2\eta\mu_1\phi_1(x_0)(c+\lambda_1) + \lambda_1\varepsilon}{4\mu_1\phi_1(x_0)(c+\lambda_1)} - \frac{\mu_1\phi_1(x_0)\beta}{\lambda_1}\right)e^{2\lambda_1(\tau_{2\varepsilon}-T)} \\ &+\eta e^{\lambda_1(\tau_{2\varepsilon}-T)} + \frac{\lambda_1\varepsilon}{4\mu_1\phi_1(x_0)(c+\lambda_1)} - \frac{\eta}{2},\end{aligned}$$

which is actually the quadratic equation with respect to the variable $z := e^{\lambda_1 \tau_{2\varepsilon}}$. The latter equation has *the only* solution $\tau_{2\varepsilon}$ belonging to the required interval $(\tau_{1\varepsilon}, T)$—it is given by the explicit formula (4.7); the second solutions gives $\tau_{2\varepsilon} > T$ for all $\varepsilon > 0$, which is not acceptable. This completes the proof of the theorem. $\triangle$

Theorem 4.1 establishes that, in the most interesting case of assertion (ii), optimal controls to the approximating problems $(\overline{P}_{1\varepsilon})$ are *piecewise constant* functions consisting of *there parts*: they start for all $\varepsilon > 0$ from the *upper extreme* value $u = 0$ (minimal resource), then switch to the *intermediate* positions (depending on ε) from the admissible control region, and finally come back to the same extreme value $u = 0$. Note that the other extreme value $u = -\alpha$ (maximum resource) is never used in the optimal control designed in (ii), and that the intermediate control values in (ii) are manifestations of *singular modes.*

Observe also that we did not consider in Theorem 4.1 one more—remaining—possibility of parameter combinations for $(\overline{P}_{1\varepsilon})$, namely:

$$
(4.16) \qquad \begin{cases} \mu_1\phi_1(x_0)\beta > \lambda_1\eta, \quad \dfrac{\mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)\beta - \lambda_1\eta} < e^{\lambda_1 T}, \\ \mu_1\phi_1(x_0)\big(\beta - \alpha(c+\lambda_1)\big) > \lambda_1\eta. \end{cases}
$$

It follows from the proof of Theorem 4.1 that this case *does not* allow any *singular mode* of optimal control, which cannot also take the lower recourse $u \equiv 0$ on the whole interval $[0,T]$. According to the Pontryagin maximum principle, optimal controls $u_\varepsilon(t)$ to $(\overline{P}_{1\varepsilon})$ in this case must be *bang-bang* on the whole $[0,T]$ changing their positions from 0 to $-\alpha$. A detailed study of this case is not of our interest, since the parameter combinations in (4.16) exclude in fact the fulfillment of the *state constraint* (4.2), i.e., there are *no feasible controls* to our main problem $(\overline{P}_1)$ studied in this section; see the next theorem for the full description of feasibility/controllability and optimality in the state-constrained problem $(\overline{P}_1)$.

Theorem 4.2 (full description of controllability and optimality in the state-constrained ODE control problem). *The state constrained problem $(\overline{P}_1)$ is controllable, i.e., there is a feasible control to $(\overline{P}_1)$, if and only if one of the following cases holds:*

(a) $\mu_1\phi_1(x_0)\beta \le \lambda_1\eta$;

(b) *both conditions*

$$
\mu_1\phi_1(x_0)\beta > \lambda_1\eta, \quad \frac{\mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)\beta - \lambda_1\eta} \ge e^{\lambda_1 T}
$$

are satisfied simultaneously;

(c) *all the three conditions*

$$
\mu_1\phi_1(x_0)\beta > \lambda_1\eta, \quad \frac{\mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)\beta - \lambda_1\eta} < e^{\lambda_1 T}, \quad \mu_1\phi_1(x_0)\big(\beta - \alpha(c+\lambda_1)\big) \le \lambda_1\eta
$$

are fulfilled simultaneously.

Furthermore, the constant function

$$
(4.17) \qquad \overline{u}(t) \equiv 0 \quad on \quad [0,T]
$$

is an optimal control to problem $(\overline{P}_1)$ in both cases (a) *and* (b). *In case* (c) *an optimal control to $(\overline{P}_1)$ is given by the two-positional piecewise constant function*

$$
(4.18) \qquad \overline{u}(t) = \begin{cases} 0 & \text{if } t \in [0,\tau), \\ v := \dfrac{\lambda_1\eta - \mu_1\phi_1(x_0)\beta}{\mu_1\phi_1(x_0)(c+\lambda_1)} & \text{if } t \in [\tau, T], \end{cases}
$$

where the switching time τ is computed by

$$0 < \tau := \frac{1}{\lambda_1} \ln \frac{\mu_1 \phi_1(x_0)\beta}{\mu_1 \phi_1(x_0)\beta - \lambda_1 \eta} < T. \tag{4.19}$$

Proof. The feasibility of the trivial control (4.17) to problem $(\overline{P}_1)$ in both cases (a) and (b) is proved in Theorem 4.1, where it is shown that the corresponding trajectory $y(t)$ to (4.1) satisfies the state constraint (4.2) under the assumptions imposed in these cases. The optimality of (4.17) to $(\overline{P}_1)$ obviously follows from the structure of the cost functional (3.8) in $(\overline{P}_1)$. To verify the feasibility of control (4.18) to $(\overline{P}_1)$ in case (c), we observe that $0 < \tau < T$ in (4.19) under the first two conditions imposed in (c), that $-\alpha < \overline{u}(t) \le 0$ on $[0, T]$ due to the first and third conditions therein, and that the state constraint (4.2) holds for the corresponding trajectory $y(t)$ to (4.1) as one can easily verify by substituting (4.18) into (4.1) and elementary integration.

To justify that the union of the conditions in (a)–(c) is not only *sufficient* but also *necessary* for the existence of *feasible controls* to $(\overline{P}_1)$, i.e., it fully describes the *controllability* of $(\overline{P}_1)$, we now show that in the remaining case (4.16) there is no control from the admissible region (3.10) that generates a trajectory to (4.1) satisfying the state constraint (4.2). By the *monotonicity property* of Theorem 3.1, it is sufficient to check that the trajectory $y(t)$ generated by the control $u(t) \equiv -\alpha$ with the *maximum resource* violates the state constraint (4.2). To proceed, we substitute $u(t) \equiv -\alpha$ into (4.1) and get by integration of (4.1) that

$$y(t) = \frac{\mu_1 \phi_1(x_0)\big(\beta - (c + \lambda_1)\alpha\big)}{\lambda_1}\Big(1 - e^{-\lambda_1 t}\Big) \quad \text{on} \quad t \in [0, T].$$

This immediately implies by the lower condition in (4.16) that $y(T) > \eta$, which shows that problem $(\overline{P}_1)$ is *not controllable* in case (4.16).

To complete the proof of the theorem, it remains to demonstrate that the feasible control $\overline{u}(t)$ from (4.18) with the switching time τ computed in (4.19) is *optimal* to problem $(\overline{P}_1)$ in case (c). Indeed, it easy to confirm from the formulas (4.6) for $\tau_{1\varepsilon}$, (4.7) for $\tau_{2\varepsilon}$, and (4.19) for τ that we have

$$\tau_{1\varepsilon} \downarrow \tau \quad \text{and} \quad \tau_{2\varepsilon} \uparrow T \quad \text{as} \quad \varepsilon \downarrow 0.$$

Furthermore, it obvious that the intermediate positions of the controls $u_\varepsilon(t)$ in Theorem 4.1(ii) converge to $v \in [-\alpha, 0)$ in (4.18) as $\varepsilon \downarrow 0$. Thus

$$\int_0^T u_\varepsilon(t)\, dt \to \int_0^T \overline{u}(t)\, dt \quad \text{as} \quad \varepsilon \downarrow 0.$$

The latter immediately implies the optimality of $\overline{u}(t)$ to $(\overline{P}_1)$ by the optimality of $u_\varepsilon(t)$ to $(\overline{P}_{1\varepsilon})$ established in Theorem 4.1. $\triangle$

We can see, by comparison the optimal control $\overline{u}(t)$ from (4.8) derived for the state-constrained problem $(\overline{P}_1)$ in Theorem 4.2 with the ones $u_\varepsilon(t)$ derived in Theorem 4.1(ii) for the approximating problems $(\overline{P}_{1\varepsilon})$ with no state constraints, that the two-part piecewise constant optimal control to the state-constrained problem does not depend on any Borel measure in the adjoint system and turns out to be even *simpler* than those to $(\overline{P}_{1\varepsilon})$ consisting of three parts. It is a *surprising conclusion* that fully relies on the specific features of approximating ODE systems to the *parabolic dynamics* and signifies a *regularization role of state constraints* for the optimal control problems under consideration.

5 Open-Loop Optimal Control of the Parabolic System under Worst Perturbations

The results on controllability and on computing the optimal control to problem $(\overline{P}_1)$ derived in Theorem 4.2 can be treated as the *first-order approximation* to the general case of problem $(\overline{P})$ for the parabolic system under consideration. This approximation is *fairly adequate* to the general setting on a *long time interval* due to the basic assumption (H) on the positivity of the first eigenvalue, which ensures the *dominance* of the first term in the solution representation for the parabolic system; see Section 2 for more details and discussions.

In this section we address the open-loop optimal control problem $(\overline{P})$ involving the *parabolic dynamic* and pointwise state constraints formulated in Section 3 while confine our study to *optimizing* the *two-positional control structure* well justified in Section 4. It means that we now consider the following *dynamic optimization* problem $(\widehat{P})$ depending in fact on *two control parameters*:

$$\text{(5.1)} \qquad \text{minimize } J(v,\tau) := -\int_0^T u(t)\,dt$$

over admissible Dirichlet boundary controls of the form

$$\text{(5.2)} \qquad u(t) = \begin{cases} 0 & \text{if } t \in [0,\tau], \\ -v & \text{if } t \in (\tau, T] \end{cases}$$

subject to the constraints on control recourses v and switching times τ given by

$$\text{(5.3)} \qquad 0 \le v \le \alpha, \quad 0 \le \tau \le T$$

and the pointwise state constraint (3.11) along the corresponding trajectories of the parabolic system (3.9). As seen in Section 4, the *intermediate* position v in (5.2) is a characteristic feature of the *singular* control mode that leads us to the *simple* (while rigorously justified) *suboptimal control structure* in (5.2), which is significantly more convenient for further applications and implementations in comparison with those arising from the complicated and not efficiently verifiable necessary optimality conditions established in [2, 14, 17, 20] that involve, in particular, Borel measures.

In what follows we find an *exact optimal solution* to problem $(\widehat{P})$, which therefore provides a *suboptimal solution* to the general open-loop control problem $(\overline{P})$ formulated in Section 3, at least for all T *sufficiently large*. Furthermore, we derive—in the process of optimization—constructive and simple conditions on the given parameters of the original parabolic system and imposed constraints that ensure the *controllability* in $(\widehat{P})$, i.e., the existence of *feasible controls* to this problem and hence to problem $(\overline{P})$. The *sufficient conditions* obtained in this way turn out to be also *necessary for controllability* of $(\widehat{P})$ on *any* time interval $[0,T]$, i.e., when the problem is considered on the *infinite horizon* $[0,\infty)$.

To proceed, we define an *aggregate spectral parameter* of the strongly uniformly elliptic operator A from (1.2) by

$$\text{(5.4)} \qquad \gamma := \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} > 0$$

whose positivity follows from properties (a)–(c) and the standing assumption (H) formulated in Section 2. The next theorem contains *controllability* conditions for $(\widehat{P})$ and provides computing *optimal control* parameters for this problem.

Theorem 5.1 (controllability and optimal parameters of open-loop suboptimal control structure for the constrained parabolic system). *Under the standing assumptions made, the following hold:*

(i) *The parabolic system* (3.9) *is controllable on* $[0,T]$ *by piecewise constant Dirichlet boundary control functions* (5.2) *with parameters* (v,τ) *from* (5.3) *subject to the state constraint* (3.11) *if one has the conditions*

$$0 < \gamma\beta - \eta \le \min\Big\{\alpha(1+c\gamma),\ \frac{\beta(1+c\gamma)}{c+\lambda_1}\Big\}. \tag{5.5}$$

Moreover, conditions (5.5) *are necessary and sufficient for the controllability of* (3.9) *by* (5.2) *subject to* (5.3) *and* (3.11) *with any* $T>0$*, i.e., on the infinite horizon* $[0,\infty)$.

(ii) *Imposing conditions* (5.5)*, consider the transcendental equation*

$$\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)}{\lambda_i}e^{-\lambda_i T}\Big[(c+\lambda_i)(\gamma\beta-\eta)e^{\lambda_i\tau}-\beta(1+c\gamma)\Big]=0, \tag{5.6}$$

which has the unique solution $\tau=\tau(T)\in(0,T)$ *for all* T *sufficiently large. Then the boundary control* $u(t)$ *from* (5.2) *with the recourse*

$$v := \frac{\gamma\beta-\eta}{1+c\gamma} \tag{5.7}$$

is feasible to $(\widehat{P})$ *for all positive switching times* $\tau\le\tau(T)$ *being optimal to this problem when* $\tau=\tau(T)$*. Furthermore, there is the limit* $\tau(T)\downarrow\overline{\tau}$ *as* $T\to\infty$*, where the asymptotically optimal switching time* $\overline{\tau}$ *is computed by*

$$\overline{\tau} := \frac{1}{\lambda_1}\ln\frac{\beta(1+c\gamma)}{(c+\lambda_1)(\gamma\beta-\eta)} \tag{5.8}$$

and turns out to be maximal among all the switching times $\tau\ge 0$ *in* (5.2) *ensuring the fulfillment of the state constraint* (3.11) *whenever* $t\ge 0$*—i.e., on the infinite horizon* $[0,\infty)$*—along the corresponding solutions* $y(t,x_0)$ *to the parabolic system* (3.9) *generated by the Dirichlet boundary controls* (3.2) *with the recourse* $v\in(0,\alpha]$ *from* (5.7).

Proof. Let $y(t,x)$ be the trajectory of the parabolic system (3.9) generated by some piecewise continuous Dirichlet boundary control $u(t)$ from (5.2) with parameters (v,τ) satisfying (5.3). For convenience we denote

$$y(t) := y(t,x_0)\ \text{ on }\ [0,\tau]\ \text{ and }\ y(t;\tau) := y(t,x_0)\ \text{ on }\ [\tau,T].$$

By the spectral representation of Proposition 2.3 we have

$$y(t) = \beta\Big(\gamma-\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)}{\lambda_i}e^{-\lambda_i t}\Big)\ \text{ for }\ t\in[0,\tau], \tag{5.9}$$

$$y(t;\tau) = \gamma\beta-(1+c\gamma)v+\sum_{i=1}^{\infty}\frac{\mu_i\phi_i(x_0)}{\lambda_i}e^{-\lambda_i t}\Big[(c+\lambda_i)ve^{\lambda_i\tau}-\beta\Big] \tag{5.10}$$

for $t\in[\tau,T]$ with the same value $y(\tau)=y(\tau;\tau)$ at the common point of (5.9) and (5.10).

Let $t = T_0 > 0$ be a solution to the equation $y(t) = \eta$, which exists by $\gamma\beta - \eta > 0$ and is unique due to (5.9) under the standing assumptions of Section 2. When $T \leq T_0$, the control $u(t) \equiv 0$ on $[0, T]$ is obviously *feasible* and hence *optimal* to both problems $(\widehat{P})$ and $(\overline{P})$. In what follows we consider the case when the final time $T > T_0$ is *sufficiently large*. Since

$$(5.11) \qquad y(t;\tau) \longrightarrow \gamma\beta - (1 + c\gamma)v \quad \text{as} \quad t \to \infty,$$

every control from (5.2) with any $\tau \geq 0$ *stabilizes* the corresponding transient (5.10) at the upper boundary $y = \eta$ of the state constraint region *exponentially* approaching the stabilized level (5.11) as $t \to \infty$. Selecting the control recourse v by (5.7) and using the controllability conditions

$$(5.12) \qquad 0 < \gamma\beta - \eta \leq \alpha(1 + c\gamma)$$

from (5.5), we conclude that every control (5.2) is admissible by (5.3). It immediately follows from (5.11) that *otherwise* there is *no control* of type (5.2) that is *feasible* by keeping the pointwise state constraint (3.11) along the parabolic system (3.9) for T sufficiently large, i.e., there is definitely *no controllability* in the problem $(\widehat{P})$ under consideration. Furthermore, the controllability may be *violated* even under conditions (5.12)—in the sense that the pointwise state constraint (3.11) is not preserved *whenever* $t \in [0, T]$—if the switching time τ is not properly selected in (5.2). Let us now demonstrate that the choice of τ as the (unique) solution to the transcendental equation (5.6) ensures—under *all* the conditions in (5.5)—first the *controllability* in problem $(\widehat{P})$ and, moreover, the *optimality* of the feasible control (5.2) with its parameters computed in (5.6) and (5.7) to this problem.

Indeed, consider all τ from (5.3) with v computed in (5.7) such that $y(t;\tau) \leq \eta$ whenever $t \in [0, T]$ for the transient (5.10) generated by $u(t)$ from (5.2) corresponding to this pair (v, τ). Due to the *monotonicity property* of $y(t;\tau)$ with respect to τ—this follows from Theorem 3.1, the *maximal* among all such τ is the one satisfying the equation

$$y(T, \tau(T)) = \eta,$$

which is exactly that of (5.6). It is easy to observe from the explicit expression (5.10) for $y(t;\tau)$ that the transcendental equation (5.6) has the *unique solution* $\tau(T) < T$ for all $T > T_0$. Moreover, we can check that $\tau(T) > 0$ if

$$\gamma\beta - \eta \leq \frac{\beta(1 + c\gamma)}{c + \lambda_1},$$

which finally ensures the *controllability* in $(\widehat{P})$ by controls (5.2) on the given interval $[0, T]$ under the validity of *all* the conditions in (5.5). Further, it follows from structure (5.1) of the cost functional in $(\widehat{P})$ that the control $u(t)$ from (5.2) and (5.7) with the *maximal* $\tau = \tau(T)$ keeping the state constraint (3.11) is indeed *optimal* to $(\widehat{P})$.

It remains to consider the behavior of problem $(\widehat{P})$ when $T \to \infty$. It follows from the *Maximum Principle* for parabolic equations (similarly to the proof of Theorem 3.1) that the *optimal switching time* function $\tau(T)$ is *strictly decreasing* in T being obviously bounded from below. Thus $\tau(T)$ converges as $T \to \infty$, and its limit $\overline{\tau}$ reduces to that computed in (5.8) due to the eigenvalue properties (2.3), which reflect the *first eigenvalue dominance.*

Finally, we observe directly from (5.10) that the control $u(t)$ from (5.2) with the resource v computed in (5.7) and the switching time $\tau = \overline{\tau}$ computed in (5.8) preserves—with the *strict* inequality— the state constraint (3.11) for the corresponding transient in (5.9) and

(5.10) *whenever* $t \geq 0$, i.e., it is *feasible* to problem $(\widehat{P})$ on the *infinite horizon* $[0, \infty)$. Furthermore, $\overline{\tau}$ is the *maximal* τ in (5.2) satisfying this property. The latter can be shown by applying the *Fermat stationary rule* to (5.10) on the open interval (τ, ∞) via differentiation of $y(t;\tau)$ in t and checking that the maximum of $y(t;\tau)$ over (τ, ∞) is bigger than η whenever $\tau > \overline{\tau}$. Since $\overline{\tau} = 0$ in (5.8) when

$$\gamma\beta - \eta = \frac{\beta(1 + c\gamma)}{c + \lambda_1}, \tag{5.13}$$

we thus confirm that the conditions in (5.5) are *necessary and sufficient* for the *controllability* in $(\widehat{P})$ on the infinite horizon $[0, \infty)$ and that the control $u(t)$ from (5.2) with v from (5.7) and $\tau = \overline{\tau}$ is *optimal* to $(\widehat{P})$ when $T = \infty$. This completes the proof of the theorem. $\triangle$

Observe that the *asymptotically optimal* switching time $\overline{\tau}$ in (5.8) can be computed directly from the condition of *vanishing the first term* in the series of (5.10), i.e., from

$$(c + \lambda_1)ve^{\lambda_1 \tau} - \beta = 0$$

with v given in (5.7). This justifies the simple and convenient *first term rule* to deal with the parabolic dynamics under the basic assumption (H) as $t \to \infty$; see also Section 7 below.

The results derived in Theorem 5.1 particularly demonstrate that the *passage to the infinite horizon* allows us to significantly simplify optimal solutions to the open-loop control problems under consideration and to arrive at the convenient analytic formulas for computing their optimal parameters. The discovered phenomenon reveals a certain *turnpike property*, which is a characteristic feature of such *state-constrained* control problems governed by the *parabolic dynamics*.

6 Feedback Suboptimal Control of the Parabolic System

In the previous section we computed *optimal parameters* of the *suboptimal* two-positional control structure for the open-loop control problem $(\overline{P})$, which describes the best possible reaction of the control system to keep the required state constraints under the realization of the *upper* case of the *worst/maximal perturbations* $w(t) \equiv \beta$ on $[0, T]$. Due to the *full symmetry* of the initial problem (P) discussed above, the *lower case* $w(t) \equiv -\beta$ of the worst perturbations on $[0, T]$ can be considered similarly by using open-loop Dirichlet boundary controls

$$u(t) = \begin{cases} 0 & \text{if } t \in [0, \tau], \\ v \in [0, \alpha] & \text{if } t \in (\tau, T] \end{cases} \tag{6.1}$$

for the linear parabolic system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = -\beta \quad \text{a.e. } (t, x) \in Q, \\ y(0, x) = 0, \quad x \in \Omega, \\ y(t, x) = u(t) \quad \text{a.e. } (t, x) \in \Sigma \end{cases} \tag{6.2}$$

subject to the pointwise state constraint

$$y(\cdot, x_0) \in L^2[0, T] \text{ with } y(t, x_0) \geq -\eta \text{ a.e. } t \in [0, T]. \tag{6.3}$$

Then, taking into account the sign changes in (6.1)–(6.3), the *controllability* conditions and formulas for computing the *optimal control parameters* in problem of

$$\text{(6.4)} \qquad \text{minimize} \int_0^T u(t)\,dt \quad \text{over constraints (6.1)–(6.3)}$$

are *exactly* the same as in Theorem 5.1 for problem $(\widehat{P})$ formulated in Section 5.

Now we are going to employ these results to construct a *suboptimal feedback regulator* for the original *minimax feedback control problem* (P) described in Section 1. Recall that the purpose of feedback controls in (P) is to keep transients within the given state constraint region (1.5) for *all* uncertain perturbations $w \in W$ from (1.4) subject to the imposed constrains on controls in such a way that the cost functional (1.8) is *minimized* under the realization of the *worst perturbations.* The results obtained above for computing (sub)optimal *open-loop* controls in the case of the worst perturbations allow us to justify the following *suboptimal structure* $f = f(\xi)$ of *feedback controls* (1.7) in the Dirichlet boundary conditions of the parabolic system (1.1):

$$\text{(6.5)} \qquad f(\xi) = \begin{cases} -v & \text{if } \xi \geq \sigma, \\ 0 & \text{if } -\sigma < \xi < \sigma, \\ v & \text{if } \xi \leq -\sigma \end{cases}$$

describing a *three-positional feedback regulator* with the "dead region" $(-\sigma, \sigma)$. Observe that the three-positional feedback law $f(\xi)$ in (6.5) is given by a *nonincreasing* and *odd* function satisfying the requirements of Theorem 3.2.

By the structure of the boundary control dynamics in (1.1) with the feedback law of (1.7) and (6.5), the *closed-loop control system* under consideration is functioning as follows:

Whatever perturbation $w = w(t)$ is realized in the dynamical system, the control reacts only to the *current state position* $\xi = y(t, x_0)$ under observation, applying its *zero recourse* $u = 0$ if $y(t, x_0)$ is within the *dead region* $-\sigma < \xi < \sigma$. When the state position ξ reaches the *upper bound* σ of the dead region, the control applies its *lower recourse* $u = -v$ in (6.5) and keeps it all the time while the state position ξ *exceeds* the upper admissible level $\xi = \sigma$; then it applies again its zero recourse $u = 0$ whenever the state position comes back to the dead region. The control system behavior is *fully symmetric* when the state position ξ reaches (and then exceeds from below) the *lower bound* $\xi = -\sigma$ of the dead region.

The *feedback control synthesis* design—in the *minimax sense* of problem (P)—reduces now to determining appropriate parameters (v, σ) in (6.5) such that the resulting closed-loop control system keeps the state position $\xi = y(t, x_0)$—starting with the initial *equilibrium state* $y(0, x) \equiv 0$ on Ω—within the admissible state constraint area (1.5) *whatever uncertain perturbation* $w \in W$ is realized and then ensures the *minimum value* of the cost functional (1.8) under the realization of the *worst perturbations.*

According to the results obtained above for the *open-loop* control problems $(\widehat{P})$ and (6.4), we employ in what follows the control recourse v in (6.5) computed by formula (5.7) for all T sufficiently large, which is in fact *necessary* for *stabilizing* transients at the boundary of the admissible state constraint region as $T \to \infty$ by using feedback regulators of the *suboptimal* three-positional structure (6.5). Note that the value v in (5.7) is *not the maximal* available control recourse from the admissible region $[0, \alpha]$—besides the extreme case in the controllability conditions in (5.5)—while, being a characteristics of the *singular* control mode, it ensures the *optimal* control response, with respect to minimizing the cost functional (1.8), to the worst perturbations. Our intention now is to find verifiable conditions on the remaining parameter $\sigma > 0$ in (6.5) such that the resulting closed-loop control system meets the *controllability/feasibility* and *minimax optimality* requirements formulated above.

The next theorem answers both controllability and optimality questions providing in fact the *exact calculation* of the optimal value $\sigma(T)$ on the given time interval $[0,T]$ and fully describes its *limiting/asymptotic* behavior as $T \to \infty$, which corresponds to problem (P) on the *infinite horizon.*

Theorem 6.1 (feasible and optimal parameters of the three-positional regulator in the minimax feedback control problem for the parabolic system). *Consider the minimax feedback control problem (P) formulated in Section 1 under the standing assumptions on its initial data imposed in Section 2. Let the feedback boundary control regulator $f(\xi)$ in (1.7) and (1.1) have the suboptimal three-positional structure (6.5) justified above. Then the following assertions hold:*

(i) *The controllability conditions (5.5) are necessary and sufficient for the existence of a feasible feedback control of type (6.5) to problem (P) on any time interval $[0,T]$. More precisely, let $\gamma\beta - \eta > 0$ and let $T \le T_0$, where $T_0 > 0$ is the unique solution to the equation*

$$\beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i T_0}\Big) = \eta. \tag{6.6}$$

Then the trivial feedback control $f(\xi) \equiv 0$ on $I\!R$ (i.e., $v = 0$ in (6.5), where σ is thus irrelevant) is feasible by the state and control constraints in (P) on the time interval $[0,T]$ and hence optimal to this problem.

(ii) *Let further $T > T_0$ while the right-hand side inequality in (5.5) is fulfilled, let the recourse $v \in [0,\alpha]$ in (6.5) be computed by (5.7), and let*

$$\sigma(T) := \beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i \tau(T)}\Big), \tag{6.7}$$

where $\tau(T)$ is the unique solution to the transcendental equation (5.6). Then the feedback control (6.5) is feasible to (P) on the time interval $[0,T]$ whenever $0 < \sigma \le \sigma(T)$ being in fact optimal to (P) on $[0,T]$ if $\sigma = \sigma(T)$.

(iii) *The control recourse $v \in (0,\alpha]$ in (6.5) computed by formula (5.7) is necessary for stabilizing the observed trajectories $y(t,x_0)$ as $T \to \infty$ at one of the boundaries $\xi = \eta$ and $\xi = -\eta$ under the realization of the corresponding worst perturbations in the closed-loop control system (1.1), (1.7), and (6.5). Furthermore, we have $\sigma(T) \downarrow \overline{\sigma}$ as $T \to \infty$, where the number $\overline{\sigma} \ge 0$ is computed by*

$$\overline{\sigma} := \beta\Big(\gamma - \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} \Big[\frac{(c+\lambda_1)(\gamma\beta - \eta)}{\beta(1+c\gamma)}\Big]^{\frac{\lambda_i}{\lambda_1}}\Big) \tag{6.8}$$

and satisfies the following properties:

(a) $\overline{\sigma} = 0$ *if the extreme case (5.13) is realized in the controllability conditions (5.5). In this case the trivial feedback regulator $f(\xi) \equiv 0$ on $I\!R$ is feasible and thus optimal to (P) on the infinite horizon $[0,\infty)$.*

(b) $\overline{\sigma} > 0$ *if the strict inequality*

$$\gamma\beta - \eta < \frac{\beta(1+c\gamma)}{c+\lambda_1} \tag{6.9}$$

holds in the controllability conditions (5.5). *In this case the three-positional regulator* (6.5) *with the resource* $v \in (0, \alpha]$ *computed in* (5.7) *is feasible to* (P) *on* $[0, \infty)$ *whenever* $0 < \sigma \le \overline{\sigma}$ *being in fact optimal to* (P) *on the infinite horizon* $[0, \infty)$ *when* $\sigma = \overline{\sigma}$.

Proof. A large part of the proof has been already given above and/or follows directly from the previous considerations. Indeed, the controllability conditions (5.5) ensure—by Theorem 5.1 for $(\widehat{P})$ and its symmetric counterparts for problem (6.4)— the existence of feasible open-loop controls, which satisfy the control constraints (1.3) and keep the observed transients $y(t, x_0)$ within the state constraint region (1.5) under the realization of the *worst* system perturbations $w(t) \equiv \beta$ and $w(t) \equiv -\beta$ on $[0, T]$. Due to Theorem 3.1 on the *monotonicity* of transients with respect to *controls* (and thus with respect to switching times τ) and also due to the *time-monotonicity* of $y(t)$ in (5.9), the *feasible* and *optimal* values of σ asserted in the theorem directly relate, concerning the worst perturbations, to the corresponding values of $y(t)$ at T_0 and $\tau(T)$ determined in Theorem 5.1 and its proof. On the other hand, these values of σ found for the case of the worst perturbations happen to be appropriate for *any* perturbations from the admissible area (1.4) due to the *extremality* of the worst perturbations by Theorem 3.2 and due to the *monotonicity* of transients with respect to *perturbations* by Theorem 3.1. In this way we arrive at all the conclusions of assertions (i) and (ii).

The conclusion on the control recourse v in (iii) and also the statements in (a) are established in fact in the proof of Theorem 5.1. The value of $\overline{\sigma}$ in (6.8) corresponds to $\overline{\sigma} = y(\overline{\tau})$ with $y(t)$ from (5.9) and the asymptotically optimal switching time $\overline{\tau}$ computed by (5.8) due to the above arguments based on the monotonicity results of Theorem 3.1. The limiting conclusion $\sigma(T) \downarrow \overline{\sigma}$ as $T \to \infty$ can be checked directly, while all the statements in (b) follow from the above discussions due to the crucial *transient monotonicity*. $\triangle$

For further simplifications of the results obtained and also for the corresponding developments in the next Section 7, we impose the following assumption:

$$
\begin{aligned}
&\sum_{i=2}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} e^{-\lambda_i t} < 0 \ \text{ whenever } \ t = \frac{1}{\lambda_1} \ln \theta \ \text{ with } \ \theta > 1, \ \text{ i.e.} \\
&\sum_{i=2}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} \theta^{-\frac{\lambda_i}{\lambda_1}} < 0 \ \text{ whenever } \ \theta > 1,
\end{aligned}
\tag{6.10}
$$

which definitely holds for various standard parabolic equations in the presence of *symmetry*, e.g., for the multidimensional heat equation defined on rectangulars, balls, etc.; see, in particular, [3, 7, 10] and the references therein.

Consider now the *first term* in the series (6.8), which is

$$
\overline{\sigma}_1 := \beta\Big(\gamma - \frac{\mu_1 \phi_1(x_0)(c + \lambda_1)(\gamma\beta - \eta)}{\lambda_1 \beta (1 + c\gamma)}\Big) > 0 \tag{6.11}
$$

in the nontrivial case (6.9) of the controllability conditions (5.5). If in addition assumption (6.10) holds, then $\overline{\sigma}_1 < \overline{\sigma}$, and hence the interval $[-\overline{\sigma}_1, \overline{\sigma}_1]$ lies *entirely* within the optimal dead region $(-\overline{\sigma}, \overline{\sigma})$ of the three-positional regulator (6.5) by Theorem 6.1(iii). Thus we arrive at the following consequence of the theorem.

Corollary 6.2 (first-order feasible approximation of optimal feedback control). *In addition to the standing assumptions of Section* 2, *suppose that conditions* (6.9) *and* (6.10) *are satisfied. Then the three-positional feedback regulator* (6.5) *with* $v \in (0, \alpha]$ *computed in* (5.7) *and with* $\sigma = \overline{\sigma}_1$ *computed in* (6.11) *is feasible to problem* (P) *on the infinite horizon.*

Proof. Follows from case (b) of Theorem 6.1(iii) due to $0 < \overline{\sigma}_1 < \overline{\sigma}$. △

7 Robust Stability and Reliability of the Closed-Loop Constrained Parabolic System

The concluding section of the paper is devoted to the study of *robust stability* (or *stability in the large*) of the *closed-loop* control system

$$
\left\{
\begin{array}{l}
\dfrac{\partial y}{\partial t} + Ay = w(t), \quad x \in \Omega, \quad t \geq 0, \\
y(0,x) = 0, \quad x \in \Omega, \\
y(t,x) = f\big(y(t,x_0)\big), \quad x \in \partial\Omega, \quad t \geq 0,
\end{array}
\right.
\tag{7.1}
$$

where $f = f(\xi)$ is a three-positional feedback regulator with parameters (v, σ) given in (6.5). Our goal is to derive efficient conditions ensuring the robust stability of system (7.1), (6.5) in the sense precisely defined below and then to combine these conditions with the relationships on (v, σ) established in Section 6 from the viewpoint of controllability and minimax (sub)optimality in the feedback control problem (P) for the parabolic system (7.1) subject to the control and state constraints. In this way we arrive at the *reliable* feedback control design ensuring the required suboptimal performance of the closed-loop control system in a stable regime acceptable for applications.

Note that the minimax design results of Section 6 establish relationships between parameters of the parabolic dynamics, feedback boundary controls, perturbations, and imposed constraints under which the closed-loop control system allows us to keep all the transients at the point of observation within the prescribed state constraint region for any uncertain perturbations from the admissible area, with the optimal effect in the worst perturbation case. However, the above minimax control design does not address *stability issues* for the resulting closed-loop control system, which are of crucial importance for practical applications and are studied in detail in what follows.

We indicate the following *two major sources* that may cause possible *instability* of the closed-loop control system (7.1), (6.5):

(a) System (7.1) with $f(\xi)$ from (6.5) is *highly nonlinear*, despite the linearity of its parabolic dynamics. Of course, the source of nonlinearity is the *discontinuous* three-positional regulator (6.5) in the Dirichlet boundary conditions of (7.1).

(b) System (7.1) is of *distributed parameters*, which is the common name for control systems governed, in particular, by partial differential equations. In the framework of (7.1), the most significant and visible manifestation of the distributed parameter nature is that the control acts in the *boundary conditions* while the feedback is formed by observing the current state position $\xi = y(t, x_0)$ at the *intermediate point* $x_0 \in \Omega$ of the space domain. The latter generates the *inertia* of the closed-loop control system and essentially effect stability.

One can easily see that if the current state position $\xi = y(t, x_0)$ lies *inside* the dead region $(-\sigma, \sigma)$ after terminating all the perturbations, then the closed-loop system (7.1) with the three-positional regulator (6.5) maintains the *stationary equilibrium regime* $y \equiv 0$ as $t \to \infty$. This signifies *stability in the small* of the initial equilibrium state $y \equiv 0$ in this system for any dead region $(-\sigma, \sigma)$ as $\sigma > 0$. However, the latter property is *not sufficient* for the acceptable functioning of the nonlinear control system (7.1), (6.5) with distributed parameters. We need *robust stability*, or *stability in the large*, of the equilibrium state $y \equiv 0$ for the closed-loop system under consideration, which in our case means that $y(t, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$ even if the current state ξ of (7.1) is *outside* the dead region of (6.5) *after terminating* all the perturbations. The presence of perturbations $w(t)$ on some interval $[0, T]$ is clearly *irrelevant* to this stability issue, which is an *internal property* of the parabolic dynamics generated by the elliptic operator A from (1.2) on the *infinite horizon* and the three-positional feedback regulator (6.5) in the Dirichlet boundary conditions of (7.1).

It has been well recognized in the literature that stability *in the large* (or robust stability) issues are among the most challenging in stability theory for nonlinear dynamics, even in the case of finite-dimensional control systems governed by ordinary differential equations. We are not familiar with *any results* in this direction for the parabolic systems studied in this paper. To derive efficient conditions for stability in the large of the equilibrium state $y \equiv 0$ in the closed-loop control system (7.1) with the three-positional feedback regulator (6.5), we develop a *variational approach* to such robust stability, which is largely based on the *monotonicity* properties of the parabolic dynamics and reduces the stability issue to solving an open-loop *optimal control* problem for the initial system (1.1) on the *infinite horizon.*

To proceed, observe from the structure of the closed-loop control system under consideration that the required robust stability of its stationary equilibrium state $y \equiv 0$ can be *lost* if the dead region in (6.5) is *not sufficiently wide.* Indeed, in such cases the transients $\xi = y(t, x_0)$ would move back and forth between the dead region boundaries under switching control positions in (6.5) with *no external perturbations*, just by *inertia* of the control system. This means that the closed-loop control system (7.1), (6.5) may start functioning in a non-acceptable *self-vibrating regime* as $t \longrightarrow \infty$ thus signifying *instability in the large* of the initial equilibrium stare $y = 0$. We intend to find conditions that exclude such instability.

It follows from the above discussions that the unstable self-vibrating regime will *not occur* if the transient $y(t, x_0)$ starting at one boundary of the dead region *does not reach* the other boundary whenever $t > 0$ under the control switching in (6.5) with no external perturbations. Moreover, the *limiting stability resource* of the system relates to the *minimal width* of the dead region ensuring the afore-mentioned property. This allows us to derive efficient stability conditions by solving an open-loop optimal control problem for (1.1) on the *infinite horizon* as is done in the proof of the next theorem.

Theorem 7.1 (robust stability of the closed-loop parabolic control system). *Let* (7.1) *be a closed-loop parabolic system under the standing assumptions of Section* 2, *and let* (6.5) *be a three-positional feedback regulator in the boundary conditions of* (7.1) *with arbitrary parameters* $v > 0$ *and* $\sigma > 0$. *Then the control system* (7.1), (6.5) *exhibits robust stability in the above sense if its parameters satisfy the relationship*

$$\sigma \geq -\frac{v(1+c\gamma)}{2} + \frac{v+\sigma}{2} \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)(c+\lambda_i)}{\lambda_i} \left(\frac{v}{v+\sigma}\right)^{\frac{\lambda_i}{\lambda_1}}, \tag{7.2}$$

where the right-hand side is always positive. If furthermore assumption (6.10) *is fulfilled, then the stability condition can be simplified as*

$$\sigma \geq \frac{v}{2\lambda_1}\Big[\mu_1 \phi_1(x_0)(c+\lambda_1) - \lambda_1(1+c\gamma)\Big], \tag{7.3}$$

where the right-hand side in (7.3) *is always greater than the one in* (7.2) *whenever* $v, \sigma > 0$.

Proof. Developing a *variational approach* to robust stability, we consider the following *open-loop* control system on the *infinite horizon*:

$$\text{(7.4)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + Ay = 0, & x \in \Omega, \quad t > 0, \\ y(0,x) = 0, & x \in \Omega, \\ y(t,x) = u(t), & x \in \partial\Omega, \quad t > 0, \end{cases}$$

with piecewise constant Dirichlet boundary controls given by

$$\text{(7.5)} \qquad u(t) = \begin{cases} h + \Delta h & \text{if } 0 \le t \le \tau, \\ h & \text{if } t > \tau, \end{cases}$$

where h and Δh are some *positive* numbers (to be specified later) while τ is the control switching time to be determined. For formulating the other data (cost functional and state constraint) of the optimal control problem to study in what follows with the application to robust stability, we first employ Proposition 2.3 that gives the spectral representation of the transients $y(t,x)$ to (7.4) generated by controls (7.5). Since

$$\sum_{i=1}^{\infty} \mu_i \phi_i(x) = 1 \text{ in } L^2(0,T) \text{ and } \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)}{\lambda_i} = \gamma > 0$$

by (5.4), we represent via (2.6) the corresponding solution to the boundary value problem in (7.4) and (7.5) at the point of observation $x = x_0$ as

$$\text{(7.6)} \qquad \begin{aligned} y_\tau(t,x_0) &= \sum_{i=1}^{\infty} \mu_i \phi_i(x_0)(c + \lambda_i) e^{-\lambda_i t} \int_0^t u(\theta) e^{\lambda_i \theta}\, d\theta \\ &= \sum_{i=1}^{\infty} \mu_i \phi_i(x_0)(c + \lambda_i) e^{-\lambda_i t} \Big(\int_0^\tau (h + \Delta h) e^{\lambda_i \theta}\, d\theta + \int_\tau^t h e^{\lambda_i \theta}\, d\theta \Big) \\ &= (1 + \gamma c) h + \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)(c + \lambda_i)}{\lambda_i} \Big[\Delta h e^{\lambda_i \tau} - (h + \Delta h) \Big] e^{-\lambda_i t}. \end{aligned}$$

It is easy to see from (7.6) that

$$\text{(7.7)} \qquad y_\tau(t,x_0) \to (1 + c\gamma) h \text{ as } t \to \infty \text{ whenever } \tau > 0.$$

However, the transient $y(t,x_0)$ may intersect the *stabilization level* (7.7) if the switching time τ is not properly chosen. We intend to find efficient conditions under which the latter situation does *not occur*. These conditions, being of certain interest for their own sake, ensure the required *robust stability* of the closed-loop system (7.1), (6.5) when the control levels h and Δh in (7.5) are specified appropriately.

To proceed, consider the following auxiliary *dynamic optimization problem* for the parabolic system (7.4) on the *infinite horizon*:

$$\text{(7.8)} \qquad \begin{cases} \text{minimize } J(\tau) := (1 + c\gamma) h - y_\tau(\tau, x_0) \\ \text{subject to } (7.4), (7.5), \text{ and the state constraint} \\ y_\tau(t,x_0) < (1 + c\gamma) h \text{ for all } t > 0. \end{cases}$$

The meaning of this problem is to find an *optimal switching time* $\tau = \underline{\tau} > 0$ in (7.5) such that the corresponding trajectory $y_{\underline{\tau}}(t,x_0)$ to (7.4) lies *strictly below* the stabilization level

(7.7) for all $t > 0$ (i.e., does not reach this level whenever $t > 0$) and that the *distance* between the stabilization level (7.7) and the *switching level*

$$y(\underline{\tau}, x_0) := y_{\underline{\tau}}(\underline{\tau}, x_0)$$

is *minimal* in comparison with any other switching time τ satisfying all the constraints in (7.8). According to the discussions on robust stability presented right before the formulation of Theorem 7.1, solving this optimal control problem directly leads us to required stability conditions; see below for more details.

It follows from the *monotonicity property* of Theorem 3.1 with respect to controls that

$$y_{\tau_1}(t, x_0) \leq y_{\tau_2}(t, x_0) \quad \text{whenever} \quad t > 0 \quad \text{and} \quad \tau_1 \leq \tau_2$$

for the transients $y_\tau(t, x_0)$ generated in (7.6) by the switching controls (7.5). Thus the *optimal* switching time $\underline{\tau}$ to (7.8) is the *largest* one under which the corresponding transient $y_\tau(t, x_0)$ does not intersect the stabilization level $(1 + c\gamma)h$ for all $t > 0$.

The *exact* solution to the open-loop control problem (7.8) on the *infinite horizon* is given in Theorem 5.1(ii)—more precisely, in its proof. It is provided by the *first term rule*, i.e., by vanishing the first term in the last series of (7.6). By this result we have the simple (while rigorously justified) formula for the *optimal switching time* to (7.8):

$$\underline{\tau} = \frac{1}{\lambda_1} \ln\Big(\frac{h + \Delta h}{\Delta h}\Big) > 0 \quad \text{whenever} \quad v, \sigma > 0,$$

and hence the *exact optimal value* of the cost functional in this problem is computed by:

$$\begin{aligned} \underline{\vartheta} : &= J(\underline{\tau}) = (1 + c\gamma)h - y(\underline{\tau}, x_0) \\ &= -\Delta h(1 + c\gamma) + (h + \Delta h) \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)(c + \lambda_i)}{\lambda_i} \Big(\frac{\Delta h}{h + \Delta h}\Big)^{\frac{\lambda_i}{\lambda_1}} > 0. \end{aligned} \tag{7.9}$$

Imposing assumption (6.10), in addition to the standing hypotheses of Section 2, we get the feasible *first-order approximation*

$$\underline{\vartheta}_1 := \Delta h \Big[\frac{\mu_1 \phi_1(x_0)(c + \lambda_1)}{\lambda_1} - (1 + c\gamma)\Big] > \underline{\vartheta} > 0 \tag{7.10}$$

to (7.9), which happens to be *independent* of the control level h in (7.5).

According to the description of the instability (in the large) phenomenon given before the formulation of Theorem 7.1, robust stability of the closed-loop control system (7.1), (6.5) is ensured if the width of the dead region 2σ is *not smaller* than the value $\underline{\vartheta}$ in (7.9) with $h = \sigma$ and $\Delta h = v$. Substituting these data into (7.9), we arrive at the stability condition (7.2) of the theorem. The explicit first-order approximation condition (7.3) corresponds to substituting the above values of h and Δh into formula (7.10) for $\underline{\vartheta}_1$ via the sufficient stability requirement $2\sigma \geq \underline{\vartheta}_1$. This completes the proof of the theorem. $\triangle$

Finally, we combine the feedback control results derived in Section 6 from the viewpoint of controllability and minimax optimality with the robust stability conditions obtained in this section; thus we establish *reliability* relationships between all the parameters of the feedback control, parabolic dynamics, imposed constraints, and perturbations that ensure feasible and then (sub)optimal behavior of the closed-loop control system under consideration in a *stable regime.* Since the control resource v in (6.5) is uniquely determined by (5.7), the remaining issue is to justify the *existence* of the reliable dead region $(-\sigma, \sigma)$ in (6.5) and to describe further the *reliability range* of the acceptable variety for the characteristic parameter σ of the feedback regulator.

Theorem 7.2 (reliability of the feedback control design). *Consider the closed-loop control parabolic system* (7.1) *with uncertain perturbations* $w \in W$ *from* (1.4) *and with the there-positional feedback regulator* (6.5) *in the Dirichlet boundary conditions. Assume that the standing hypotheses of Section* 2 *and the controllability conditions* (5.5) *are satisfied. The following assertions hold:*

(i) *Let the extreme case* (5.13) *be realized in the controllability conditions* (5.5). *Then the trivial feedback regulator* $f(\xi) \equiv 0$ *in* (6.5) *ensures both robust stability and optimality to the minimax problem (P).*

(ii) *Let the controllability conditions* (5.5) *hold excluding the extreme case* (5.13), *and let the control resource* $v \in (0, \alpha]$ *in* (6.5) *be computed by* (5.7). *Denote*

$$\begin{aligned}\underline{\sigma}: &= \frac{\gamma\beta - \eta + \eta(1+c\gamma)}{2(1+c\gamma)} \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x_0)(c+\lambda_i)}{\lambda_i} \Big(\frac{\gamma\beta - \eta}{\gamma\beta - \eta + \eta(1+c\gamma)} \Big)^{\frac{\lambda_i}{\lambda_1}} \\ &+ \frac{\eta - \gamma\beta}{2} > 0\end{aligned} \tag{7.11}$$

and suppose that $\underline{\sigma} \le \overline{\sigma}$, *where* $\overline{\sigma} > 0$ *is computed by* (6.8). *Then the feedback control system* (7.1), (6.5) *with the dead region parameter* $\sigma > 0$ *belonging to the nonempty interval*

$$\underline{\sigma} \le \sigma \le \overline{\sigma} \tag{7.12}$$

is reliable on the infinite horizon in the sense that it is feasible by all the constraints in (P) on $[0, \infty)$ *for any perturbations* $w \in W$ *enjoying simultaneously robust stability. Furthermore, the upper bound* $\sigma = \overline{\sigma}$ *of the reliable range* (7.12) *optimizes the suboptimal feedback structure* (6.5) *to the minimax problem (P) on* $[0, \infty)$ *under the worst perturbations.*

(iii) *Suppose in addition to the hypotheses in* (ii) *that the first-order approximation assumption* (6.10) *is fulfilled. Denote*

$$\underline{\sigma}_1 := \frac{\gamma\beta - \eta}{2\lambda_1(1+c\gamma)} \Big[\mu_1 \phi_1(x_0)(c+\lambda_1) - \lambda_1(1+c\gamma) \Big] > \underline{\sigma} \tag{7.13}$$

and impose further the first-order reliability condition

$$3\mu_1 \phi_1(x_0)(c+\lambda_1)(\gamma\beta - \eta) \le \lambda_1(3\gamma\beta - \eta)(1+c\gamma). \tag{7.14}$$

Then $\underline{\sigma}_1 \le \overline{\sigma}_1$, *where* $\overline{\sigma}_1 \in (0, \overline{\sigma})$ *is given in* (6.11), *and the feedback control system* (7.1), (6.5) *is reliable on the infinite horizon in the sense described in* (ii) *with the control resource* $v \in (0, \alpha]$ *from* (5.7) *and the dead region parameter* $\sigma > 0$ *satisfying*

$$\underline{\sigma}_1 \le \sigma \le \overline{\sigma}_1. \tag{7.15}$$

Furthermore, the reliability condition (7.14) *can be equivalently described directly via the suboptimal first-order value* $\overline{\sigma}_1$ *from* (6.11) *as* $\overline{\sigma}_1 \ge \eta/3$.

Proof. This theorem unifies and summarizes, to a large extent, the feedback control design results derived above. To begin with, observe that assertion (i) follows directly from case (a) of Theorem 6.1(iii), since the closed-loop system (7.1) with the trivial regulator $f(\xi) \equiv 0$ in (6.5) obviously exhibits robust stability (no control switching).

To justify assertion (ii), we apply Theorem 7.1 in order to check the robust stability of system (7.1) with the three-positional regulator (6.5), where v is computed by (5.7)—due to case (b) of Theorem 6.1(iii)—and where $\sigma \ge \underline{\sigma}$ with $\underline{\sigma}$ computed by (7.11). This follows from the observation that the value of $\underline{\sigma}$ in (7.11) is in fact obtained by substituting v from (5.7) into the right-hand side of (7.2) and by replacing σ with η therein. We easily conclude that $\underline{\sigma}$ satisfies inequality (7.2) whenever $0 < \sigma \le \eta$ in the right-hand side of it, which is the case under consideration. The other statements in (ii) are proved in Theorem 6.1(iii).

To justify assertion (iii) of the theorem, we first observe that the value of $\underline{\sigma}_1$ in (7.13) is obtained by substituting v from (5.7) into the right-hand of (7.3). Furthermore, the first-order reliability condition (7.14) is directly derived from the condition $\underline{\sigma}_1 \le \overline{\sigma}_1$ by substituting there $\overline{\sigma}_1$ from (6.11) and $\underline{\sigma}_1$ from (7.13). Thus the feasibility of the three-positional regulator (6.5) with v from (5.7) and σ from (7.15) follow from Corollary 6.2, while the corresponding robust stability of system (7.1), (6.5) with σ in (7.15) follows from the last part of Theorem 7.1. Finally, we can directly check that the reliability condition (7.14)—ensuring robust stability— can be surprisingly rewritten in the very simple form $\overline{\sigma}_1 \ge \eta/3$ via just the first-order suboptimality value $\overline{\sigma}_1$ computed by (6.11). Note that the equality therein can be used as an additional equation for *shape optimization* to determine, e.g., the best parameters of the domain Ω ensuring a *reliable feedback control design* under the other given data of the minimax problem (P) and the feedback regulator (6.5). $\triangle$

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