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OPTIMAL CONTROL OF NONCONVEX DIFFERENTIAL INCLUSIONS[1]

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Abstract. The paper deals with dynamic optimization problems of the Bolza and Mayer types for evolution systems governed by nonconvex Lipschitzian differential inclusions in Banach spaces under endpoint constraints described by finitely many equalities and inequalities with generally nonsmooth functions. We develop a variational analysis of such problems mainly based on their discrete approximations and the usage of advanced tools of generalized differentiation. In this way we establish extended results on stability of discrete approximations and derive necessary optimality conditions for nonconvex discrete-time and continuous-time systems in the refined Euler-Lagrange and Weierstrass-Pontryagin forms accompanied by the appropriate transversality inclusions. In contrast to the case of geometric endpoint constraints in infinite dimensions, the necessary optimality conditions obtained in this paper do not impose any nonempty interiority/finite codimension/normal compactness assumptions.

Key words. variational analysis, dynamic optimization and optimal control, differential inclusions, infinite dimension, discrete approximations, generalized differentiation, necessary optimality conditions

AMS subject classification. 49J53, 49J52, 49J24, 49M25, 90C30

1 Introduction

The paper is devoted to the study of dynamic optimization problems governed by differential inclusions in infinite-dimensional spaces. We pay the main attention to variational analysis of the following *generalized Bolza problem* (P) for differential inclusions in Banach spaces with endpoint constraints described by finitely many equalities and inequalities.

Let X be a Banach *state space* with the *initial state* $x_0 \in X$, and let $T := [a,b] \subset I\!R$ be a fixed *time interval.* Given a set-valued mapping $F\colon X \times T \rightrightarrows X$ and real-valued functions $\varphi_i\colon X \to I\!R$ as $i = 0,\ldots,m+r$ and $\vartheta\colon X \times X \times T \to I\!R$, consider the problem:

$$\text{(1.1)} \qquad \text{minimize} \quad J[x] := \varphi_0\big(x(b)\big) + \int_a^b \vartheta\big(x(t),\dot{x}(t),t\big)\,dt$$

subject to *dynamic constraints* governed by the differential inclusion [1]

$$\text{(1.2)} \qquad \dot{x}(t) \in F\big(x(t),t\big) \quad \text{a.e.} \quad t \in [a,b] \quad \text{with} \quad x(a) = x_0$$

with *functional endpoint constraints* of the inequality and equality types given by

$$\text{(1.3)} \qquad \varphi_i\big(x(b)\big) \le 0, \quad i = 1,\ldots,m,$$

$$\text{(1.4)} \qquad \varphi_i\big(x(b)\big) = 0, \quad i = m+1,\ldots,m+r.$$

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Dynamic optimization problems for differential inclusions with the *finite-dimensional* state space $X = I\!R^n$ have been intensively studied over the years, especially during the last decade, mainly from the viewpoint of deriving necessary optimality conditions; see [3, 8, 11, 13, 15, 17] for various results, methods, and more references. Dynamic optimization problems governed by infinite-dimensional *evolution equations* have also been much investigated, motivating mainly by applications to optimal control of partial differential equations; see, e.g., the books [7, 9] and the references therein. To the best of our knowledge, deriving necessary optimality conditions in dynamic optimization problems for evolution systems governed by differential inclusions in *infinite-dimensional spaces* has not drawn attention in the literature till the very recent time.

In [13], the author developed the method of *discrete approximations* to study optimal control problems of minimizing the Bolza functional (1.1) over appropriate solutions to evolution systems governed by infinite-dimensional differential inclusions of type (1.2) with endpoint constrains given in the *geometric form*

$$x(b) \in \Omega \subset X \tag{1.5}$$

via closed subsets of Banach spaces satisfying certain requirements. The major assumption on Ω made in [13] is the so-called *sequential normal compactness* (SNC) property of Ω at the optimal endpoint $\bar{x}(b) \in \Omega$; see [12] for a comprehensive theory for this and related properties, which play a major role in infinite-dimensional variational analysis. Loosely speaking, the SNC property means that Ω should be "sufficiently fat" around the reference point; e.g., it never holds for singletons unless X is finite-dimensional, where the SNC property is satisfied for every nonempty set. For *convex* sets in infinite-dimensional spaces, the SNC property always holds when $\operatorname{int}\Omega \neq \emptyset$. Furthermore, it happens to be closely related [13] to the so-called "finite-codimension" property of convex sets, which is known to be essential for the fulfillment of an appropriate counterpart of the Pontryagin maximum principle for infinite-dimensional systems of optimal control; see, e.g., [7, 9].

In this paper we show that the dynamic optimization problem (P) formulated above, with the *functional* endpoint constraints (1.3) and (1.4), admits necessary optimality conditions in the extended Euler-Lagrange form accompanied by the corresponding Weierstrass-Pontryagin/maximum and transversality relations with *no SNC* and similar assumptions imposed on the underlying endpoint constraint set in infinite dimensions. Moreover, the case of endpoint constraints (1.3) and (1.4) allows us to partly avoid some other rather restrictive assumptions (like "strong coderivative normality," which may not hold in infinite-dimensional spaces; see Sections 6, 7 for more details) imposed in [13] in the general case of geometric constraints (1.5). Our approach is based, in addition to [13], on refined properties of appropriate *subdifferentials* of locally Lipschitzian functions on infinite-dimensional spaces, as well as on *dual/coderivative characterizations* of Lipschitzian and metric regularity properties of set-valued mappings.

The rest of the paper is organized as follows. In Section 2 we formulate the standing assumptions on the initial data of (P), and discuss the relaxation procedure used for some results and proofs in the paper. The main attention in this paper is paid to the so-called *intermediate local minimizers*, which occupy an intermediate position between the classical weak and strong minima.

In Section 3 we construct a sequence of the *well-posed discrete approximations* (P_N) to the original Bolza problem (P) involving *consistent perturbations* of the endpoint constraints in the discrete approximation procedure. Then we present a major result on the *strong* *stability* of discrete approximations that justifies the $W^{1,2}$-norm convergence of optimal solutions for (P_N) to the fixed local minimizer for the original problem (P).

Section 4 contains an overview of the basic tools of *generalized differentiation* needed to perform the subsequent variational analysis of the discrete-time and continuous-time evolution systems under consideration in infinite-dimensional spaces. Most of the material in this section is taken from the author's book [12], where the reader can find more results and commentaries in this direction and related topics.

Section 5 is devoted to deriving necessary optimality conditions for the constrained *discrete-time* problems arising from the discrete approximation procedure. These problems are reduced to constrained problems of mathematical programming in infinite dimensions, which happen to be *intrinsically nonsmooth* and involve finitely many functional and geometric constraints generated by those in (1.2)–(1.4) via the discrete approximation procedure. Variational analysis of such problems requires applications of the full power of the generalized differential calculus in infinite-dimensional spaces developed in [12].

In Section 6 we derive necessary optimality conditions of the extended *Euler-Lagrange* type for *relaxed* intermediate minimizers to the original Bolza problem (P) by passing to the limit from those obtained for discrete-time problems. It worth emphasizing that the realization of the limiting procedure requires not only the strong convergence of optimal trajectories to discrete approximation problems but also justifying an appropriate convergence of *adjoint trajectories* in necessary optimality conditions for discrete-time systems. The latter becomes passible due to specific properties of the basic generalized differential constructions reviewed in Section 4, which include complete *dual characterizations* of Lipschitzian and metric regularity properties of set-valued mappings.

The concluding Section 7 concerns necessary optimality conditions for arbitrary (*non-relaxed*) intermediate minimizers to problem (P) that are established in terms of the *extended Euler-Lagrange* inclusion accompanied by the *Weierstrass-Pontryagin/maximum* and transversality relations without imposing any SNC assumptions on the target/endpoint constraint set. The approach is based on an additional approximation procedure that allows us to reduce (P) to an unconstrained Bolza problem of the type treated in Section 6 for which *any* intermediate local minimizer happens to be a relaxed one.

Our notation is basically standard; cf. [12, 13]. Unless otherwise stated, all the spaces considered are Banach with the norm $\|\cdot\|$ and the canonical dual pairing $\langle\cdot,\cdot\rangle$ between the space in question, say X, and its topological dual X^* whose weak* topology is denoted by w^*. We use the symbols $I\!B$ and $I\!B^*$ to signify the closed unit balls of the space under consideration and its dual, respectively. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, its *sequential Painlevé-Kuratowski upper/outer limit* at $\bar{x}$ is defined by

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^*\in X^*\Big|\ \exists \text{ sequences } x_k\to\bar{x},\ x_k^*\stackrel{w^*}{\to}x^* \text{ with } x_k^*\in F(x_k) \text{ as } k\in I\!N:=\{1,2,\ldots\}\Big\}. \tag{1.6}$$

2 Bolza Problem for Differential Inclusions

Just for brevity and simplicity, we consider in this paper the Bolza problem (P) with *autonomous* (time-independent) data. By a *solution* to inclusion (1.2) we understand (as in [1, 6] a mapping $x\colon T\to X$, which is Fréchet differentiable for a.e. $t\in T$ satisfying (1.2) and the *Newton-Leibniz formula*

$$x(t) = x_0 + \int_a^t \dot{x}(s)\,ds \quad \text{for all} \quad t \in T,$$

where the integral in taken in the BOCHNER SENSE.

Recall that a Banach space X is *Asplund* if any of its separable subspaces has a separable dual. This is a major subclass of Banach spaces that particularly includes every space with a *Fréchet differentiable renorm* off the origin (i.e., every *reflexive* space), every space with a separable dual, etc.; see [4] for more details, characterizations, and references. There is a deep relationship between spaces having the *Radon-Nikodým property* (RNP) and Asplund spaces, which is used in what follows: *given a Banach space X, the dual space X^* has the RNP if and only if X is Asplund.*

It has been well recognized that differential inclusions (1.2), which are certainly of their own interest, provide a useful generalization of *control systems* governed by differential/evolution *equations* with control parameters:

$$\dot{x} = f(x,u), \quad u \in U, \tag{2.1}$$

where the control sets $U(\cdot)$ may also depend on time and *state* variables via $F(x,t) = f(x, U(x,t), t)$. In some cases, especially when the sets $F(\cdot)$ are convex, the differential inclusions (1.2) admit parametric representations of type (2.1), but in general they cannot be reduced to parametric control systems and should be studied for their own sake; see [1]. Note also that the *ODE form* (2.1) in Banach spaces is strongly related to various control problems for evolution *partial differential equations* of parabolic and hyperbolic types, where solutions may be understood in some other appropriate senses [7, 9].

In what follows, we pay the main attention to the study of *intermediate local minimizers* for problem (P) introduced in [11]. Recall that a *feasible arc* to (P) is a solution to the differential inclusion (1.2) for which $J[x] < \infty$ in (1.1) and the endpoint constraints (1.3) and (1.4) are satisfied.

Definition 2.1 (intermediate local minimizers). *A feasible arc $\bar{x}(\cdot)$ is an* INTERMEDIATE LOCAL MINIMIZER *(i.l.m.) of rank $p \in [1,\infty)$ for (P) if there are numbers $\epsilon > 0$ and $\alpha \ge 0$ such that $J[\bar{x}] \le J[x]$ for any feasible arcs to (P) satisfying the relationships*

$$\|x(t) - \bar{x}(t)\| < \epsilon \quad \text{for all} \quad t \in [a,b] \quad \text{and} \tag{2.2}$$

$$\alpha \int_a^b \|\dot{x}(t) - \dot{\bar{x}}(t)\|^p\,dt < \epsilon. \tag{2.3}$$

In fact, relationships (2.2) and (2.3) mean that we consider a neighborhood of $\bar{x}(\cdot)$ in the Sobolev space $W^{1,p}([a,b];X)$ with the norm

$$\|x(\cdot)\|_{W^{1,p}} := \max_{t\in[a,b]} \|x(t)\| + \Big(\int_a^b \|\dot{x}(t)\|^p\,dt\Big)^{1/p},$$

where the norm on the right-hand side is taken in the space X. If there is only the requirement (2.2) in Definition 2.1, i.e., $\alpha = 0$ in (2.3), then we get the classical *strong* local minimum corresponding to a neighborhood of $\bar{x}(\cdot)$ in the norm topology of $C([a,b];X)$. If instead of (2.3) one puts the more restrictive requirement

$$\|\dot{x}(t)-\dot{\bar{x}}(t)\|<\epsilon \ \text{ a.e. } \ t\in[a,b],$$

then we have the classical *weak* local minimum in the framework of Definition 2.1. Thus the introduced notion of i.l.m. takes, for any $p\in[1,\infty)$, an *intermediate* position between the classical concepts of strong ($\alpha=0$) and weak ($p=\infty$) local minima, being indeed different from both classical notions; see various examples in [18, 13]. Clearly all the necessary conditions for i.l.m. automatically hold for strong (and hence for global) minimizers.

Considering the autonomous Bolza problem (P) in this paper, we impose the following *standing assumptions* on its initial data along a given intermediate local minimizer $\bar{x}(\cdot)$:

(H1) There are a open set $U\subset X$ and a number $\ell_F>0$ such that $\bar{x}(t)\in U$ for all $t\in[a,b]$, the sets $F(x)$ are nonempty and compact for all $x\in U$ and satisfy the inclusion

$$F(x)\subset F(u)+\ell_F\|x-u\|I\!B \ \text{ whenever } \ x,u\in U, \tag{2.4}$$

which implies the uniform boundedness of the sets $F(x)$ on U, i.e., the existence of some constant $\gamma>0$ such that

$$F(x)\subset\gamma I\!B \ \text{ for all } \ x\in U.$$

(H2) The integrand ϑ is Lipschitzian continuous on $U\times(\gamma I\!B)$.

(H3) The endpoint functions φ_i, $i=0,\ldots,m+r$, are locally Lipschitzian around $\bar{x}(b)$ with the common Lipschitz constant $\ell>0$.

Observe that (2.4) is equivalent to say that the set-valued mapping F is *locally Lipschitzian* around $\bar{x}(\cdot)$ with respect to the classical Hausdorff metric on the space of nonempty and compact subsets of X.

In what follows, along with the original problem (P), we consider its "relaxed" counterpart significantly used in some results and proofs of the paper. Roughly speaking, the relaxed problem is obtained from (P) by a *convexification* procedure with respect to the *velocity* variable. It follows the route of Bogolyubov and Young in the classical calculus of variations and of Gamkrelidze and Warga in optimal control; see the books [1, 13] and the references therein for more details and commentaries.

To construct an appropriate relaxation of the Bolza problem (P) under consideration, we first consider the extended-real-valued function

$$\vartheta_F(x,v):=\vartheta(x,v)+\delta(v;F(x)),$$

where $\delta(\cdot;\Omega)$ is the *indicator function* of the set Ω. Denote

$$\widehat{\vartheta}_F(x,v):=(\vartheta_F)^{**}_v(x,v),\quad (x,v)\in X\times X,$$

the *biconjugate/bypolar* function to $\vartheta_F(x,\cdot)$, i.e., the greatest proper, convex, and lower semicontinuous (l.s.c.) function with respect to v that is majorized by ϑ_F. Then the *relaxed problem* (R) to (P), or the *relaxation* of (P), is defined as follows:

$$\text{minimize } \ \widehat{J}[x]:=\varphi(x(b))+\int_a^b\widehat{\vartheta}_F(x(t),\dot{x}(t))\,dt \tag{2.5}$$

over a.e. differentiable arcs $x\colon [a,b] \to X$ that are Bochner integrable on $[a,b]$ together with $\vartheta_F\big(x(t),\dot x(t)\big)$, satisfy the Newton-Leibniz formula and the endpoint constraints (1.3), (1.4).

Note that the feasibility requirement $\widehat{J}[x] < \infty$ in (2.5) is fulfilled only if $x(\cdot)$ is a solution to the *convexified differential inclusion*

$$\dot x(t) \in \operatorname{clco} F\big(x(t),\dot x(t)\big) \quad \text{a.e. } t \in [a,b] \text{ with } x(a) = x_0, \tag{2.6}$$

where "clco" stands for the convex closure of a set in X. Thus the relaxed problem (R) can be considered under explicit dynamic constraints given by the convexified differential inclusion (2.6). Any trajectory for (2.6) is called a *relaxed trajectory* for (1.2), in contrast to the *ordinary* (or *original*) trajectories for the latter inclusion.

There are deep relationships between relaxed and ordinary trajectories for differential inclusions, which reflect the fundamental *hidden convexity* inherent in continuous-time (nonatomic measure) dynamic systems defined by differential and integral operators. In particular, *any relaxed trajectory* of (1.2) under assumption (H1) can be *uniformly approximated* (in the $C\big([a,b];X\big)$-norm) by a sequence of ordinary trajectories; see, e.g., [1, 6, 16]. We need the following version [5] of this approximation/density property involving not only differential inclusions but also minimizing functionals.

Lemma 2.2 (approximation property for the relaxed Bolza problem). *Let $x(\cdot)$ be a relaxed trajectory for the differential inclusion* (1.2) *with a separable state space X, where F and ϑ satisfy assumptions* (H1) *and* (H2), *respectively. Then there is sequence of the ordinary trajectories $x_k(\cdot)$ for* (1.2) *such that $x_k(\cdot) \to x(\cdot)$ in $C\big([a,b];X\big)$ as $k \to \infty$ and*

$$\liminf_{k\to\infty} \int_a^b \vartheta\big(x_k(t),\dot x_k(t)\big)\,dt \le \int_a^b \widehat{\vartheta}_F\big(x(t),\dot x(t)\big)\,dt.$$

Note that Theorem 2.2 does *not* assert that the approximating trajectories $x_k(\cdot)$ satisfy the endpoint constraints (1.3) and (1.4). Indeed, there are examples showing that the latter may not be possible and, moreover, the property of *relaxation stability*

$$\inf(P) = \inf(R) \tag{2.7}$$

is violated; in (2.7) the infima of the cost functionals (1.1) and (2.5) are taken over all the feasible arcs in (P) and (R), respectively.

An obvious sufficient condition for the relaxation stability is the *convexity* of the sets $F(x,t)$ and of the integrand ϑ in v. However, the relaxation stability goes *far beyond* the standard convexity due to the *hidden convexity* property of continuous-time differential systems. In particular, Theorem 2.2 ensures the relaxation stability of nonconvex problems (P) with no constraints on the endpoint $x(b)$. There are various efficient conditions for the relaxation stability of nonconvex problems with endpoint and other constraint; see [13, Subsection 6.1.2] with the commentaries therein for more details, discussions, and references.

A *local* version of the relaxation stability property (2.7) regarding intermediate minimizers for the Bolza problem (P) is postulated as follows.

Definition 2.3 (relaxed intermediate local minimizers). *A feasible arc $\bar x(\cdot)$ to the Bolza problem (P) is a* RELAXED INTERMEDIATE LOCAL MINIMIZER *(r.i.l.m.) of rank $p \in [1,\infty)$ for (P) if it is an intermediate local minimizer of this rank for the relaxed problem (R) providing the same value of the cost functionals: $J[\bar x] = \widehat{J}[\bar x]$.*

It is not hard to observe that, under the standing assumptions formulated above, the notions of intermediate local minima and relaxed intermediate local minima do not actually depend on rank p. In what follows we always take (unless otherwise stated in Section 7) $p = 2$ and $\alpha = 1$ in (2.3) for simplicity.

The principal method of our study in this paper involves *discrete approximations* of the original Bolza problem (P) for constrained continuous-time evolution inclusions by a family of dynamic optimization problems of the Bolza type governed by discrete-time inclusions with endpoint constraints. We show that this method generally leads to necessary optimality conditions for *relaxed* intermediate local minimizers of (P). Then an additional approximation procedure allows us to establish necessary optimality conditions for *arbitrary* (non-relaxed) intermediate local minimizers by reducing them to problems, which are *automatically* stable with respect to relaxation.

3 Discrete Approximations

In this section we present basic constructions of the method of discrete approximations in the theory of necessary optimality conditions for differential inclusions following the scheme of [11, 13] developed for the case of geometric constraints, with certain modifications required for the functional endpoint constraints (1.3) and (1.4).

For simplicity we use the replacement of the derivative by the *uniform Euler scheme*:

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h}, \quad h \to 0.$$

To formalize this process, we take any natural number $N \in I\!N$ and consider the *discrete grid/mesh* on T defined by

$$T_N := \{a, a + h_N, \ldots, b - h_N, b\}, \quad h_N := (b - a)/N,$$

with the *stepsize of discretization* h_N and the *mesh points* $t_j := a + jh_N$ as $j = 0, \ldots, N$, where $t_0 = a$ and $t_N = b$. Then the differential inclusion (1.2) is replaced by a sequence of its *finite-difference/discrete approximations*

$$(3.1) \qquad x_N(t_{j+1}) \in x_N(t_j) + h_N F(x_N(t_j)), \quad j = 0, \ldots, N-1, \quad x(t_0) = x_0.$$

Given a discrete trajectory $x_N(t_j)$ satisfying (3.1), we consider its *piecewise linear extension* $x_N(t)$ to the continuous-time interval $T = [a, b]$, i.e., the *Euler broken lines*. We also define the *piecewise constant extension* to T of the corresponding *discrete velocity* by

$$v_N(t) := \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}, \quad t \in [t_j, t_{j+1}), \quad j = 0, \ldots, N-1.$$

It follows from the very definition of the Bochner integral that

$$x_N(t) = x_0 + \int_a^t v_N(s)\, ds \quad \text{for} \quad t \in T.$$

The next result establishes the *strong* $W^{1,2}$*-norm approximation* of *any* trajectory for the differential inclusion (1.2) by extended trajectories of the sequence of discrete inclusions (3.1). Note that the norm convergence in $W^{1,2}([a, b]; X)$ implies the *uniform* convergence of the trajectories on $[a, b]$ and the *pointwise*, for a.e. $t \in [a, b]$, convergence of (some subsequence of) their *derivatives*. The latter is crucial for the study of *nonconvex*-valued differential inclusions. The proof of this result in given in [13, Theorem 6.4], which is an infinite-dimensional counterpart of the one in [11, Theorem 3.1].

Lemma 3.1 (strong $W^{1,2}$-approximation by discrete trajectories). *Let $\bar{x}(\cdot)$ be an arbitrary solution to the differential inclusion* (1.2) *under the assumptions in* (H1), *where X is a general Banach space. Then there is a sequence of solutions $\widehat{x}_N(t_j)$ to the discrete inclusions* (3.1) *such that their extensions $\widehat{x}_N(t)$, $a \le t \le b$, converge to $\bar{x}(t)$ strongly in the space $W^{1,2}([a,b];X)$ as $N \to \infty$.*

Now fix an *intermediate local minimizer* $\bar{x}(\cdot)$ for the Bolza problem (P) and construct a sequence of discrete approximation problems (P_N), $N \in I\!N$, admitting optimal solutions $\bar{x}_N(\cdot)$ whose extensions converge to $\bar{x}(\cdot)$ in the norm topology of $W^{1,2}([a,b];X)$ as $N \to \infty$.

To proceed, we take a sequence of the discrete trajectories $\widehat{x}_N(\cdot)$ approximating by Lemma 3.1 the given local minimizer $\bar{x}(\cdot)$ to (P) and denote

$$\eta_N := \max_{j\in\{1,\ldots,N\}} \|\widehat{x}_N(t_j) - \bar{x}(t_j)\| \to 0 \ \text{ as } \ N \to \infty. \tag{3.2}$$

Having $\epsilon > 0$ from relations (2.2) and (2.3) for the intermediate minimizer $\bar{x}(\cdot)$ with $p = 2$ and $\alpha = 1$, we always suppose that

$$\bar{x}(t) + \epsilon/2 \in U \ \text{ for all } \ t \in [a,b],$$

where U is a neighborhood of $\bar{x}(\cdot)$ from (H1). Let $\ell > 0$ be the common Lipschitz constant of φ_i, $i = 1,\ldots,m+r$, from (H3). Construct problems (P_N), $N \in I\!N$, as follows: minimize

$$\begin{aligned} J_N[x_N]: \ &= \varphi_0(x_N(t_N)) + h_N \sum_{j=0}^{N-1} \vartheta\Big(x_N(t_j), \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}\Big) \\ &+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \Big\| \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} - \dot{\bar{x}}(t) \Big\|^2 dt \end{aligned} \tag{3.3}$$

over discrete trajectories $x_N = x_N(\cdot) = (x_0, x_N(t_1), \ldots, x_N(t_N))$ for the difference inclusions (3.1) subject to the constraints

$$\varphi_i(x_N(t_N)) \le \ell\eta_N \ \text{ for } \ i = 1,\ldots,m, \tag{3.4}$$

$$-\ell\eta_N \le \varphi_i(x_N(t_N)) \le \ell\eta_N \ \text{ for } \ i = m+1,\ldots,m+r, \tag{3.5}$$

$$\|x_N(t_j) - \bar{x}(t_j)\| \le \frac{\epsilon}{2} \ \text{ for } \ j = 1,\ldots,N, \quad \text{and} \tag{3.6}$$

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \Big\| \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} - \dot{\bar{x}}(t) \Big\|^2 dt \le \frac{\epsilon}{2}. \tag{3.7}$$

Considering in the sequel (without mentioning any more) piecewise linear extension of $x_N(\cdot)$ to the whole interval $[a,b]$, we observe the relationships:

$$\begin{cases} x_N(t) = x_0 + \displaystyle\int_a^t \dot{x}_N(s)\,ds \quad \text{for all } t \in [a,b] \quad \text{and} \\ \dot{x}_N(t) = \dot{x}_N(t_j) \in F(x_N(t_j)), \ \ t \in [t_j, t_{j+1}), \ \ j = 0,\ldots,N-1. \end{cases} \tag{3.8}$$

In the next theorem, we establish that the given *relaxed* intermediate local minimizer (r.i.l.m.) $\bar{x}(\cdot)$ to (P) can be *strongly* in $W^{1,2}$ approximated by *optimal solutions* to (P_N); the latter implies the a.e. *pointwise* convergence of the derivatives significant for the main results of the paper. To justify such an approximation, we need to impose the *Asplund* structure on *both* X and its dual X^*, which is particularly the case when X is *reflexive.*

Theorem 3.2 (strong convergence of discrete optimal solutions). *Let $\bar{x}(\cdot)$ be an r.i.l.m. for the Bolza problem (P) under the standing assumptions* (H1)–(H3) *in the Banach state space X, and let (P_N), $N \in I\!N$, be a sequence of discrete approximation problems built above. The following hold:*

(i) *Each (P_N) admits an optimal solution.*

(ii) *If in addition both X and X^* are Asplund, then any sequence $\{\bar{x}_N(\cdot)\}$ of optimal solutions to (P_N) converges to $\bar{x}(\cdot)$ strongly in $W^{1,2}([a,b];X)$.*

The proof of this theorem follows the arguments in [12, Theorem 6.13] and the estimates

$$\left|\varphi_i(\widehat{x}_N(t_N)) - \varphi_i(\bar{x}(b))\right| \le \ell\,\|\widehat{x}(t_N) - \bar{x}(t_N)\| \le \ell\eta_N \quad \text{for all } i = 1,\ldots,m+r$$

due to (3.2), which are needed for (3.4) and (3.5).

The strong convergence result of Theorem 3.2 *makes a bridge* between the original continuous-time dynamic optimization problem (P) and its discrete-time counterparts (P_N), which allows us to derive necessary optimality conditions for (P) by passing to the limit from those for (P_N). The latter ones are *intrinsically nonsmooth* and require appropriate tools of generalized differentiation for their variational analysis.

4 Generalized Differentiation

In this section, we define the main constructions of generalized differentiation used in what follows. Since our major framework in this paper is the class *Asplund spaces*, we present simplified definitions and some properties held in this setting. All the material reviewed and employed below is taken from the author's book [12], where the reader can find more details and references.

We start with generalized normals to closed sets $\Omega \subset X$. Given $\bar{x} \in \Omega$, the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x \to \bar{x}} \widehat{N}(x;\Omega), \tag{4.1}$$

where "Lim sup" stands for the *sequential* upper/outer limit (1.6) of the *Fréchet normal cone* (or the *prenormal cone*) to Ω at $x \in \Omega$ given by

$$\widehat{N}(x;\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0\Big\}, \tag{4.2}$$

where $x \xrightarrow{\Omega} \bar{x}$ signifies that $x \to \bar{x}$ with $x \in \Omega$, and where $\widehat{N}(x;\Omega) := \emptyset$ for $x \notin \Omega$.

Given a set-valued mapping $F\colon X \rightrightarrows Y$ of closed graph

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\},$$

define its *normal coderivative* and *Fréchet coderivative* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ by, respectively,

$$(4.3)\qquad D^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* |\ (x^*,-y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} F)\},$$

$$(4.4)\qquad \widehat{D}^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* |\ (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y}); \operatorname{gph} F)\}.$$

If $F = f\colon X \to Y$ is *strictly differentiable* at $\bar{x}$ (in particular, if $f \in C^1$), then

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad y^* \in Y^*,$$

i.e., both coderivatives (4.3) and (4.4) are positively homogeneous extensions of the classical *adjoint* derivative operator to nonsmooth and set-valued mappings.

Finally, consider a function $\varphi\colon X \to I\!R$ *locally Lipschitzian* around $\bar{x}$; in this paper we do not use more general functions. Then the (basic, limiting) *subdifferential* of φ at $\bar{x}$ is

$$(4.5)\qquad \partial\varphi(\bar{x}) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{\partial}\varphi(x),$$

where the sequential outer limit (1.6) of the *Fréchet subdifferential* mapping $\widehat{\partial}\varphi(\cdot)$ is

$$(4.6)\qquad \widehat{\partial}\varphi(x) := \left\{x^* \in X^* \middle|\ \frac{\varphi(u)-\varphi(x)-\langle x^*, u-x\rangle}{\|u-x\|} \geq 0\right\}.$$

We are not going to review in this section appropriate properties of the generalized differential constructions (4.1)–(4.6) used in Sections 5–7: these properties will be invoked with the exact references to [12] in the corresponding places of the proofs in the subsequent sections. Just note here that our basic/limiting constructions (4.1), (4.3), and (4.5) enjoy *full calculus* in the framework of Asplund spaces, while the Fréchet-like ones (4.2), (4.4), and (4.6) satisfy certain rules of "fuzzy calculus."

5 Necessary Conditions for Discrete Inclusions

In this section we derive necessary optimality conditions for the sequence of discrete approximation problems (P_N) defined in (3.1) and (3.3)–(3.7). We only present results in the "fuzzy" form, which are more convenient to derive necessary conditions for the original problem (P) by the limiting procedure in Section 6.

Observe first that each discrete optimization problem (P_N) can be equivalently written in a special form of *constrained mathematical programming (MP)*:

$$\begin{cases} \text{minimize } \psi_0(z) \text{ subject to} \\ \psi_j(z) \leq 0, \quad j = 1,\ldots,s, \\ f(z) = 0, \\ z \in \Theta_j \subset Z, \quad j = 1,\ldots,l, \end{cases}$$

where ψ_j are real-valued functions on some Banach space Z, where $f\colon Z \to E$ is a mapping between Banach spaces, and where $\Theta_j \subset Z$. To see this, we let

$$z := (x_1,\ldots,x_N,v_0,\ldots,v_{N-1}) \in Z := X^{2N},\ E := X^N,\ s := N+2+m+2r,\ l := N-1$$

and rewrite (P_N) as an (MP) problem (5.1) with the following data:

$$\psi_0(z) := \varphi_0(x_N) + h_N \sum_{j=0}^{N-1} \vartheta(x_j, v_j) + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \|v_j - \dot{\bar{x}}(t)\|^2 \, dt,$$

$$\psi_j(z) := \begin{cases} \|x_{j-1} - \bar{x}(t_{j-1})\| - \epsilon/2, \; j = 1, \ldots, N+1, \\ \displaystyle\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|v_i - \dot{\bar{x}}(t)\|^2 \, dt - \epsilon/2, \; j = N+2, \\ \varphi_i(x_N) - \ell\eta_N, \; j = N+2+i, \quad i = 1, \ldots, m+r, \\ -\varphi_i(x_N) - \ell\eta_N, \; j = N+2+m+r+i, \; i = m+1, \ldots, m+r; \end{cases}$$

$$\begin{cases} f(z) = \big(f_0(z), \ldots, f_{N-1}(z)\big) \text{ with} \\ f_j(z) := x_{j+1} - x_j - h_N v_j, \quad j = 0, \ldots, N-1, \end{cases}$$

$$\Theta_j := \Big\{ z \in X^{2N} \Big| \, v_j \in F(x_j) \Big\} \text{ for } j = 0, \ldots, N-1.$$

The next theorem establishes necessary optimality conditions for each problem (P_N) in the *approximate/fuzzy* form of refined *Euler-Lagrange* and *transversality inclusions* expressed in terms of Fréchet-like normals and subgradients. The proof is based on applying the corresponding *fuzzy calculus* rules and *neighborhood criteria* for *metric regularity and Lipschitzian behavior* of set-valued mappings; cf. [13, Theorem 6.19].

Theorem 5.1 (fuzzy Euler-Lagrange conditions for discrete approximations). *Let $\bar{x}_N(\cdot) = \{\bar{x}_N(t_j) | \, j = 0, \ldots, N\}$ be local optimal solutions to problems (P_N) as $N \to \infty$ under the assumptions* (H1)–(H3) *with the Asplund state space X. Consider the quantities*

$$\theta_{Nj} := 2 \int_{t_j}^{t_{j+1}} \Big\| \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}(t) \Big\| \, dt, \quad j = 0, \ldots, N-1.$$

Then there exists a sequence $\varepsilon_N \downarrow 0$ along some $N \to \infty$, and there are sequences of Lagrange multipliers λ_{iN}, $i = 0, \ldots, m+r$, and adjoint trajectories $p_N(\cdot) = \{p_N(t_j) \in X^* | \, j = 0, \ldots, N\}$ satisfying the following relationships:

—the sign and nontriviality conditions

$$\lambda_{iN} \ge 0 \text{ for all } i = 0, \ldots, m+r, \quad \sum_{i=0}^{m+r} \lambda_{iN} = 1;$$

—the complementary slackness conditions

$$\lambda_{iN} \Big[\varphi_i\big(\bar{x}_N(t_N)\big) - \ell\eta_N \Big] = 0 \text{ for } i = 1, \ldots, m;$$

—the extended Euler-Lagrange inclusion in the approximate form

$$\Big(\frac{p_N(t_{j+1}) - p_N(t_j)}{h_N}, p_N(t_{j+1}) - \lambda_{0N} \frac{\theta_{Nj}}{h_N} b^*_{Nj} \Big) \in \lambda_{0N} \widehat{\partial} \vartheta \Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} \Big)$$
$$+ \widehat{N} \Big(\Big(\bar{x}_N(t_j), \frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} \Big); \text{gph}\, F \Big) + \varepsilon I\!B^* \text{ with } b^*_{Nj} \in I\!B^*, \quad j = 0, \ldots, N-1;$$

—the approximate transversality inclusion

$$-p_N(t_N) \in \sum_{i=0}^{m} \lambda_{iN} \widehat{\partial} \varphi_i(\bar{x}_N(t_N)) + \sum_{i=m+1}^{m+r} \lambda_{iN} \Big[\widehat{\partial} \varphi_i(\bar{x}_N(t_N)) \bigcup \widehat{\partial}(-\varphi_i)(\bar{x}_N(t_N)) \Big] + \varepsilon I\!B^*.$$

6 Euler-Lagrange Conditions for Relaxed Minimizers

This section contains necessary optimality conditions in the refined forms of the extended Euler-Lagrange and transversality inclusions for *relaxed* intermediate local minimizers of the original problem (P). The proof is based on the passing to the limit from the necessary optimality conditions for discrete approximation problems obtained in Section 5 and on the usage of the *strong stability* of discrete approximations established in Section 3. A crucial part of the proof involves the justification of an appropriate convergence of *adjoint arcs*; the latter becomes possible due to the *coderivative characterization* of Lipschitzian set-valued mappings; cf. [13, Theorem 6.21]

Theorem 6.1 (extended Euler-Lagrange and transversality inclusions for relaxed intermediate minimizers). *Let $\bar{x}(\cdot)$ be a relaxed intermediate local minimizer for the Bolza problem (P) given in* (1.1)–(1.4) *under the standing assumptions of Section* 2, *where the spaces X and X^* are Asplund. Then there are nontrivial Lagrange multipliers $0 \neq (\lambda_0, \ldots, \lambda_{m+r}) \in I\!R^{m+r+1}$ and an absolutely continuous mapping p: $[a,b] \to X^*$ such that the following necessary conditions hold:*

—the sign conditions

(6.1) $$\lambda_i \geq 0 \ \textit{for all} \ i = 0, \ldots, m+r,$$

—the complementary slackness conditions

(6.2) $$\lambda_i \varphi_i(\bar{x}(b)) = 0 \ \textit{for} \ i = 1, \ldots, m,$$

—the extended Euler-Lagrange inclusion, for a.e., $t \in [a,b]$,

(6.3) $$\dot{p}(t) \in \operatorname{clco}\Big\{u \in X^* \Big| \ (u, p(t)) \in \lambda_0 \partial \vartheta(\bar{x}(t), \dot{\bar{x}}(t)) + N\big((\bar{x}(t), \dot{\bar{x}}(t)); \operatorname{gph} F\big)\Big\},$$

—and the transversality inclusion

(6.4) $$-p(b)) \in \sum_{i=0}^{m} \lambda_i \partial \varphi_i(\bar{x}(b)) + \sum_{i=m+1}^{m+r} \lambda_i \Big[\partial \varphi_i(\bar{x}(b)) \bigcup \partial(-\varphi_i)(\bar{x}(b))\Big].$$

Note that the results obtained in Theorem 6.1 are different from those derived in [13, Subsection 6.1.5] not only by the *absence* of any SNC-like assumptions on the target/constraint set but also by *not* imposing the "coderivative normality" property on F needed in [13] in similar settings. Observe also that the arguments developed above allow us to provide the correspondent improvements in the case of Lipschitzian endpoint constraints of the Euler-Lagrange type necessary optimality conditions derived in [14] for evolution models governed by *semilinear inclusions*

(6.5) $$\dot{x}(t) \in Ax(t) + F(x(t), t),$$

where A is an *unbounded* infinitesimal generator of a *compact C_0-semigroup* on X, and where continuous solutions to (6.5) are understood in the *mild* sense.

7 Necessary Conditions without Relaxation

In this section we establish necessary optimality conditions for intermediate local minimizers $\bar{x}(\cdot)$ of evolution inclusions *without any relaxation.* It can be done under certain more restrictive assumptions on the initial data in comparison with those in Theorem 6.1. For simplicity, consider here the *Mayer version* (P_M) of problem (P) with $\vartheta = 0$ in (1.1). In this case, the *Euler-Lagrange inclusion* (6.3) admits the *coderivative form*

$$\dot{p}(t) \in \operatorname{clco} D^*F\big(\bar{x}(t), \dot{\bar{x}}(t)\big)\big(-p(t)\big) \quad \text{a.e. } t \in [a,b], \tag{7.1}$$

which easily implies, due to the extremal property for coderivatives of convex-valued mappings given in [12, Theorem 1.34], the *Weierstrass-Pontryagin maximum condition*

$$\big\langle p(t), \dot{\bar{x}}(t)\big\rangle = \max_{v \in F(\bar{x}(t))} \big\langle p(t), v\big\rangle \quad \text{a.e. } t \in [a,b] \tag{7.2}$$

provided that the sets $F(x)$ are *convex* near $\bar{x}(t)$ for a.e. $t \in [a,b]$. Our goal is to justify the above Euler-Lagrange and Weierstrass-Pontryagin conditions, together with the other necessary optimality conditions of Theorem 6.1, for intermediate minimizers of the Mayer problem (P_M) subject to the Lipschitzian endpoint constraints (1.3) and (1.4), *without any convexity or relaxation* assumptions and with *no SNC-like* requirements imposed on the endpoint constraint set. To accomplish this goal, we employ a certain approximation technique involving *Ekeland's variational principle* combined with other advanced results of variational analysis and generalized differentiation, which allow us to reduce the constrained problem under consideration to an unconstrained (and thus *stable with respect to relaxation*) Bolza problem studied in Section 6. However, this requires additional assumptions on the initial data of (P_M) imposed in what follows.

Recall that a set-valued mapping $F\colon X \rightrightarrows Y$ is *strongly coderivatively normal* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its normal coderivative (4.3) admits the representation

$$\begin{aligned} D^*F(\bar{x},\bar{y})(y^*) = \Big\{ x^* \in X^* \Big|\, &\exists \text{ sequences } (x_k, y_k) \to (\bar{x},\bar{y}),\ x_k^* \xrightarrow{w^*} x^*, \text{ and } y_k^* \to y^* \\ &\text{with } y_k \in F(x_k) \text{ and } x_k^* \in \widehat{D}^*F(x_k,y_k)(y_k^*) \text{ as } k \to \infty \Big\} =: D_M^*F(\bar{x},\bar{y})(y^*), \end{aligned}$$

where $D_M^*F(\bar{x},\bar{y})$ is called the *mixed coderivative* of F at $(\bar{x},\bar{y})$. Observe that the only difference between the normal and mixed coderivatives of F at $(\bar{x},\bar{y})$ is that the *mixed* weak* convergence of $x_k^* \xrightarrow{w^*} x^*$ and the norm convergence of $y_k^* \to y^*$ is used for $D_M^*F(\bar{x},\bar{y})$ in (7.3), in contrast to the weak* convergence of *both* components $(x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$ for $D^*F(\bar{x},\bar{y})$ in (4.3) via (4.1). Besides the obvious case of $\dim Y < \infty$, the strong coderivative normality holds in many important infinite-dimensional settings, and the property is preserved under various compositions; see [12, Proposition 4.9] describing major classes of mappings satisfying this property.

A mapping $F\colon X \rightrightarrows Y$ is called *sequentially normally compact* (SNC) at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if for any sequences $(x_k, y_k) \xrightarrow{\operatorname{gph} F} (\bar{x},\bar{y})$ and $(x_k^*, y^*) \in \widehat{N}\big((x_k,y_k); \operatorname{gph} F\big)$ one has

$$(x_k^*, y_k^*) \xrightarrow{w^*} 0 \Longrightarrow \|(x_k^*, y_k^*)\| \to 0 \quad \text{as } k \to \infty.$$

As discussed in Section 1, this property is a far-going extension of the "finite-codimension" and other related properties of sets and mappings. It always holds in finite dimensions, while in reflexive spaces agrees with the"compactly epi-Lipschitzian" property by Borwein and Strójwas; see [12] for more details, discussions, and calculus.

Finally, recall that the given norm on a Banach space X is *Kadec* if the strong and weak convergences agree on the boundary of the unit sphere of X. It is well known that every reflexive space admits an equivalent Kadec norm.

Theorem 7.1 (Euler-Lagrange and Weierstrass-Pontryagin conditions for intermediate local minimizers with no relaxation). *Let $\bar{x}(\cdot)$ be an intermediate local minimizer for the Mayer problem (P_M) in* (1.1)–(1.4) *under the standing hypotheses* (H1) *and* (H3) *on F and φ_i. Assume in addition that:*

(a) *the state space X is separable and reflexive with the Kadec norm on it;*

(b) *the velocity mapping F is SNC at $(\bar{x}(t), \dot{\bar{x}}(t))$ and strongly coderivatively normal with weakly closed graph around this point for a.e. $t \in [a,b]$.*

Then there are nontrivial Lagrange multipliers $0 \neq (\lambda_0, \ldots, \lambda_{m+r}) \in I\!R^{m+r+1}$ and an absolutely continuous mapping $p\colon [a,b] \to X^*$ satisfying the following relationships:

—the sign and complementarity slackness conditions in (6.1) *and* (6.2)*;*

—the Euler-Lagrange inclusion (7.1)*, where the closure operation is redundant;*

—the Weierstrass-Pontryagin maximum condition (7.2)*; and*

—the transversality inclusion (6.4).

Proof. Denote

$$\text{(7.3)} \quad \varphi_0^+(x,\nu) := \max\{\varphi_0(x) - \nu, 0\}, \quad \varphi_i^+(x) := \max\{\varphi_i(x), 0\}, \ i = 1, \ldots, m,$$

and, by the *method of metric approximations* [10], consider the parametric cost functional

$$\text{(7.4)} \quad \theta_\nu[x] := \Big[(\varphi_0^+)^2(x(b),\nu) + \sum_{i=1}^{m} (\varphi_i^+)^2(x(b)) + \sum_{i=m+1}^{m+r} \varphi_i^2(x(b))\Big]^{1/2}, \ \nu \in I\!R,$$

over trajectories for (1.1) with *no endpoint constraints.* Since $\bar{x}(\cdot)$ is an *intermediate local minimizer* for (P_M) and due to the constructions in (7.3) and (7.4), we have

$$\theta_\nu[x] > 0 \ \text{ for any } \ \nu < \bar{\nu} := \varphi_0(\bar{x}(b))$$

provided that $x(\cdot)$ is a trajectory for (1.2) belonging to the prescribed $W^{1,1}$-neighborhood of the given intermediate local minimizer and such that $x(t) \in U$ for all $t \in [a,b]$, where the open set $U \subset X$ is taken from the requirements in (H1) imposed on $\bar{x}(\cdot)$.

Then following the proof of [13, Theorem 6.27], we find an absolute continuous arc $x_\varepsilon(\cdot)$ satisfying the estimate

$$\int_a^b \|\dot{\bar{x}}(t) - \dot{x}_\varepsilon(t)\| \, dt \le \varepsilon$$

and such that x_ε provides an intermediate minimum to the *unconstrained* Bolza problem with *Lipschitzian* data:

$$\text{(7.5)} \qquad \text{minimize } \ \varphi_\varepsilon(x(b)) + \int_a^b \vartheta_\varepsilon(x(t), \dot{x}(t), t) \, dt$$

over absolutely continuous arcs $x(\cdot)$ satisfying $x(a)=x_0$ and lying in a $W^{1,1}$-neighborhood of $\bar{x}(\cdot)$, where the functions $\varphi_\varepsilon\colon X\to I\!R$ and $\theta_\varepsilon\colon X\times X\times[a,b]\to I\!R$ are given by

$$\varphi_\varepsilon(x):=\Big[(\varphi_0^+)^2(x,\nu_\varepsilon)+\sum_{i=1}^{m}\big(\varphi_i^+\big)^2(x)+\sum_{i=m+1}^{m+r}\varphi_i^2(x)\Big]^{1/2}, \tag{7.6}$$

$$\vartheta_\varepsilon(x,v,t):=\eta\sqrt{1+\ell_F^2}\,\mathrm{dist}\big((x,v);\mathrm{gph}\,F\big)+\sqrt{\varepsilon}\|v-\dot{x}_\varepsilon(t)\|. \tag{7.7}$$

Applying the optimality conditions of Theorem 6.1 to problem (7.5) with the initial data (7.6) and (7.7), for all small $\varepsilon>0$ we find an absolutely continuous adjoint arc $p_\varepsilon\colon[a,b]\to X^*$ satisfying

$$\text{(7.8)}\quad \dot{p}_\varepsilon(t)\in\mathrm{co}\Big\{u\in X^*\Big|\ \big(u,p_\varepsilon(t)\big)\in\mu\,\partial\mathrm{dist}\big((x_\varepsilon(t),\dot{x}_\varepsilon(t));\mathrm{gph}\,F\big)+\sqrt{\varepsilon}\big(0,I\!B^*\big)\Big\}$$

for a.e. $t\in[a,b]$ with $\mu:=\eta\sqrt{1+\ell_F^2}$ and

$$-p_\varepsilon(b)\in\partial\Big[(\varphi_0^+)^2(\cdot,\nu_\varepsilon)+\sum_{i=1}^{m}\big(\varphi_i^+\big)^2(\cdot)+\sum_{i=m+1}^{m+r}\varphi_i^2(\cdot)\Big]^{1/2}\big(x_\varepsilon(b)\big). \tag{7.9}$$

Passing to the limit in (7.8), (7.9) and using the calculus rules of generalized differentiation as in the proof of [13, Theorem 6.27], we arrive at the Euler-Lagrange and transversality conditions of the theorem *without any relaxation*.

Observe that in the general nonconvex setting the Euler-Lagrange inclusion (7.1) does not automatically imply the maximum condition (7.2). To establish the latter condition supplementing the other necessary conditions of the theorem, we follow the proof of [17, Theorem 7.4.1] given for a Mayer problem of the type (P_M) involving nonconvex differential inclusions in finite-dimensional spaces; it holds with minor changes in infinite-dimensions under the assumptions imposed. The proof of the latter theorem is based on reducing the constrained Mayer problem for nonconvex differential inclusions to an unconstrained Bolza (finite Lagrangian) problem, which in turn is reduced to a problem of optimal control with *smooth dynamics* and *nonsmooth endpoint constraints* first treated in [10] via the nonconvex normal cone (4.1) and the corresponding subdifferential (4.5) introduced therein to describe the appropriate transversality conditions in the maximum principle. $\triangle$

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