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CAN WE HAVE SUPERCONVERGENT GRADIENT RECOVERY UNDER ADAPTIVE MESHES?*

HAIJUN WU ^{††} AND ZHIMIN ZHANG[‡]

Abstract. We study adaptive finite element methods for elliptic problems with domain corner singularities. Our model problem is the two dimensional Poisson equation. Results of this paper are two folds. First, we prove that there exists an adaptive mesh (gauged by a discrete mesh density function) under which the recovered gradient by the Polynomial Preserving Recovery (PPR) is superconvergent. Secondly, we demonstrate by numerical examples that an adaptive procedure with a *a posteriori* error estimator based on PPR does produce adaptive meshes satisfy our mesh density assumption, and the recovered gradient by PPR is indeed superconvergent in the adaptive process.

Key words. finite element method, adaptive, superconvergence, gradient recovery

AMS subject classifications. 65N30, 65N15, 45K20

1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygon with boundary $\partial\Omega$. Consider the Dirichlet boundary problem: find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$A(u, v) = \int_{\Omega} \nabla u \cdot \nabla v = f(v), \quad \forall v \in H_0^1(\Omega), \quad (1.1)$$

where $f \in H^{-1}(\Omega)$.

Assume the solution u has a singularity at the origin O and can be decomposed as a sum of a singular part and a smooth part [8]:

$$u = v + w, \quad (1.2)$$

where

$$\left| \frac{\partial^m v}{\partial x^i \partial y^{m-i}} \right| \lesssim r^{\delta-m} \text{ and } \left| \frac{\partial^m w}{\partial x^i \partial y^{m-i}} \right| \lesssim 1, \quad m = 1, \dots, k+2, \quad i = 0, \dots, m, \quad (1.3)$$

where $r = \sqrt{x^2 + y^2}$ and $0 < \delta < k+1$ is a constant. Here $k = 1$ for linear finite element methods and $k = 2$ for quadratic finite element methods.

Next, we briefly explain the rational of the above regularity assumption. When Ω is a polygonal domain, the solution of the Poisson equation with the Dirichlet boundary condition:

$$-\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = g$$

with sufficiently smooth data f and g , has the following decomposition, see, e.g., [8] and [3], at a corner with angle ω :

$$u(r, \theta) = \sum_{j=1}^J c_j r^{\alpha_j} \ln^{s_j} r \sin \alpha_j \theta + w, \quad \alpha_j = \frac{j\pi}{\omega},$$

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where w is smoother than the terms in the sum, and

$$s_j = \begin{cases} 1 & \alpha_j \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

Specially, for the L -shaped domain, $\omega = 3\pi/2$ at the re-entrance corner and the expansion there is

$$u = c_1 r^{2/3} \sin \frac{2}{3} \theta + c_2 r^{4/3} \sin \frac{4}{3} \theta + c_3 r^2 \ln r \sin 2\theta + c_4 r^{8/3} \sin \frac{8}{3} \theta + w;$$

and for a cracked domain, $\omega = 2\pi$ at the crack tip and the expansion there is

$$\begin{aligned} u = & c_1 r^{1/2} \sin \frac{1}{2} \theta + c_2 2r \ln r \sin \theta + c_3 r^{3/2} \sin \frac{3}{2} \theta \\ & + c_4 r^2 \ln r \sin 2\theta + c_5 r^{5/2} \sin \frac{5}{2} \theta + c_6 r^3 \ln r \sin 3\theta + w. \end{aligned}$$

These are the two cases we shall test numerically in the last section.

Let \mathcal{M}_h be a regular triangulation of the domain Ω , \mathcal{E}_h be the set of all interior edges, and \mathcal{N}_h be the set of all nodal points. Assume that the origin $O \in \mathcal{N}_h$. Remember that any triangle $\tau \in \mathcal{M}_h$ is considered as closed. Let $V_h^k = \{v_h : v_h \in H^1(\Omega), v_h|_\tau \in P_k(\tau)\}$, $k = 1, 2$, be the conforming finite element space associated with \mathcal{M}_h , and $\mathring{V}_h^k = V_h^k \cap H_0^1(\Omega)$. Here P_k denotes the set of polynomials with degree $\lesssim k$. Denote by $I_h^k : C(\bar{\Omega}) \rightarrow V_h^k$ the standard finite element interpolation operator. The finite element solution $u_h \in V_h^k$ satisfies $u_h = I_h^k u$ on $\partial\Omega$ and

$$A(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h = f(v_h), \quad \forall v_h \in \mathring{V}_h^k. \quad (1.4)$$

In adaptive finite element methods, the convergence rate is measured by the total degrees of freedom N , since the mesh is not quasi-uniform. For a two-dimensional second-order elliptic equation, the optimal convergence rates are

$$O(N^{-1/2}) \quad k = 1; \quad O(N^{-1}) \quad k = 2,$$

where $k = 1$ is for linear element and $k = 2$, quadratic.

Starting from a fundamental work of [6], in the last decade, convergence proof of residual based adaptive finite element method has been well established [1, 2, 10, 11]. On the contrary, there is no convergence proof for using recovery based error estimators. By shifting the error estimator from residual based to recovery based, we obtain the same numerical convergence rate following the same mark-up, refinement procedure. Furthermore, the recovered gradient $G_h u_h$ is superconvergence in the sense

$$\|\nabla u - G_h u_h\| \lesssim \frac{1}{N^{1/2+\rho}} \quad k = 1; \quad \|\nabla u - G_h u_h\| \lesssim \frac{1}{N^{1+\rho}} \quad k = 2.$$

where $\rho > 0$ is a constant which depends on the quality of the adaptive mesh.

Throughout the paper, we use the notation $A_1 \lesssim B_1$ to represent the inequality $A_1 \leq \text{constant} \times B_1$, where the *constant* may only depends on the minimum angle of the triangles in the mesh \mathcal{M}_h , the constant δ , and the domain Ω . The notation $A_1 \approx B_1$ is equivalent to the statement $A_1 \lesssim B_1$ and $B_1 \lesssim A_1$.

2. Preliminary. Following the argument in [5], we consider in Figure 2.1, an edge e , two elements τ and τ' sharing e , and $\Omega_e = \tau \cup \tau'$ the patch of e . For an element $\tau \subset \Omega_e$, θ_e denotes the angle opposite to the edge e . h_e , h_{e+1} , and h_{e-1} denote the length of e and the other two edges of τ . The subscript $e+1$ or $e-1$ is for orientation. All triangles in the triangulation are orientated counterclockwise. t_e is the unit tangent vector of e with counterclockwise orientation and n_e is the unit outward normal vector. An index $'$ is add for the corresponding quantity in τ' . Notice that $t_e = -t'_e$ and $n_e = -n'_e$ because of the orientation. For any $\tau \in \mathcal{M}_h$, we denote by h_τ its diameter and by r_τ the distance from the origin to the barycenter of τ , and by $|\tau|$ the area of the triangle τ . For any $e \in \mathcal{E}_h$, let r_e be the distance from the origin O to the midpoint of e .

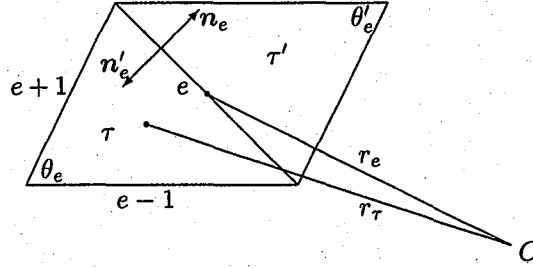


FIG. 2.1. Notation in the patch Ω_e .

Let $e \in \mathcal{E}_h$ be an interior edge. Recall that Ω_e , the patch of e , consists two adjacent triangles sharing e . We say that Ω_e is an $O(\varepsilon)$ approximate parallelogram if the lengths of any two opposite edges differ only by $O(\varepsilon)$.

Definition: The triangulation \mathcal{M}_h is said to satisfy *Condition* (α, σ, μ) if there exist constants $\alpha > 0$, $\sigma \geq 0$, and $\mu > 0$ such that the interior edges can be separated into two parts $\mathcal{E}_h = \mathcal{E}_{1,h} \oplus \mathcal{E}_{2,h}$: Ω_e forms an $O(h_e^{1+\alpha}/r_e^{\alpha+\mu(1-\alpha)})$ parallelogram for $e \in \mathcal{E}_{1,h}$ and the number of edges in $\mathcal{E}_{2,h}$ satisfies $\#\mathcal{E}_{2,h} \lesssim N^\sigma$.

Remark 2.1. The meaning of *Condition* (α, σ, μ) is the following. The edges can be grouped into “good” ($\mathcal{E}_{1,h}$) and “bad” ($\mathcal{E}_{2,h}$), where the number of bad edges are much smaller than good edges. The ratio is

$$\frac{\#\mathcal{E}_{2,h}}{\#\mathcal{E}_{1,h}} \lesssim \frac{N^\sigma}{N} = \frac{1}{N^{1-\sigma}}.$$

When $r_e = O(1)$, i.e., an edge e is far away from the singular point O , more restrictions are put on the adjacent triangles with the common edge e . This condition requires that they form an $O(h_e^{1+\alpha})$ parallelogram, which is the same as in previous works [13]. When e is in a neighborhood of O , where $r_e^{1+\mu(1-\alpha)/\alpha} \lesssim h_e$, the condition $O(h_e)$ implies $O(h_e^{1+\alpha}/r_e^{\alpha+\mu(1-\alpha)})$. In other words, two adjacent triangles that share e are allow to distort $O(h_e)$ from a parallelogram, which implies no restriction on them. Roughly speaking, number of edges in $\mathcal{E}_{1,h}$ that have no restriction imposed are $O(N^{1-\alpha})$ if $h_\tau \approx r_\tau^{1-\mu} \underline{h}^\mu$ for any $\tau \in \mathcal{M}_h$. Here \underline{h} and μ are positive constants. An explanation is given below after Lemma 2.1.

We see from the above discussion, the closer to the singular point, the less restriction is imposed on the mesh. Indeed, for an adaptively refined mesh, the closer to the singular point, the worse of the mesh quality in terms of forming parallelogram triangular pairs. \square

LEMMA 2.1. Assume that $h_\tau \approx r_\tau^{1-\mu} \underline{h}^\mu$ for any $\tau \in \mathcal{M}_h$, where \underline{h} and μ are positive

constants. Then the degree of freedoms N of the finite element equation (1.4) satisfies

$$N \approx \frac{1}{\underline{h}^{2\mu}}. \quad (2.1)$$

Proof.

$$\begin{aligned} N &\approx \sum_{\tau \in \mathcal{M}_h} \frac{h_\tau^2}{h_\tau^2} \approx \frac{1}{\underline{h}^{2\mu}} \sum_{\tau \in \mathcal{M}_h} \frac{1}{r_\tau^{2-2\mu}} \cdot |\tau| \\ &\approx \frac{1}{\underline{h}^{2\mu}} \int_\Omega \frac{1}{r^{2-2\mu}} \approx \frac{1}{\underline{h}^{2\mu}} \int_0^1 \frac{1}{r^{2-2\mu}} \cdot r \, dr \approx \frac{1}{\underline{h}^{2\mu}}. \end{aligned}$$

This completes the proof of the lemma. \square

Remark 2.2. For linear element, $\mu = \delta/2$, $N \approx 1/\underline{h}^\delta$, and for quadratic element $\mu = \delta/3$, $N \approx 1/\underline{h}^{2\delta/3}$. The condition $h_\tau \approx r_\tau^{1-\mu}\underline{h}^\mu$ can be thought of as a discrete mesh density function. The positive number $\underline{h} \approx \min_{\tau \in \mathcal{M}_h} h_\tau$, the size of the minimum element because for an element τ neighboring O , $r_\tau \approx h_\tau$ and the condition $h_\tau \approx r_\tau^{1-\mu}\underline{h}^\mu$ implies that $h_\tau \approx \underline{h}$. It is clear that the condition $h_\tau \approx r_\tau^{1-\mu}\underline{h}^\mu$ for any $\tau \in \mathcal{M}_h$ is equivalent to the condition $h_e \approx r_e^{1-\mu}\underline{h}^\mu$ for any $e \in \mathcal{E}_h$. We recall that *Condition* (α, σ, μ) means no restriction on Ω_e if $r_e^{1+\mu(1-\alpha)/\alpha} \lesssim h_e$. Furthermore, if $h_\tau \approx r_\tau^{1-\mu}\underline{h}^\mu$, i.e., $h_e \approx r_e^{1-\mu}\underline{h}^\mu$, then $r_e \lesssim \underline{h}^\alpha$. Therefore if the mesh \mathcal{M}_h satisfies *Condition* (α, σ, μ) and $h_\tau \approx r_\tau^{1-\mu}\underline{h}^\mu$, then no restriction is imposed on edges within the ball of radius $R \lesssim \underline{h}^\alpha$. The number of edges in the ball is $O(N^{1-\alpha})$ by a similar argument as the proof of Lemma 2.1. \square

3. Superconvergence between the finite element solution and linear interpolation. We now define a quadratic interpolation of ϕ based on moment conditions on edges. Let $\phi_Q = \Pi_Q \phi$ be a quadratic element defined by

$$(\Pi_Q \phi)(z) = \phi(z), \quad \text{and} \quad \int_e \Pi_Q \phi = \int_e \phi, \quad \forall z \in \mathcal{N}_h, e \in \mathcal{E}_h. \quad (3.1)$$

The following fundamental identity is proved in [5] for $v_h \in P_1(\tau)$:

$$\int_\tau \nabla(\phi - \phi_I) \cdot \nabla v_h = \sum_{e \subset \partial\tau} \left(\beta_e \int_e \frac{\partial^2 \phi_Q}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 \phi_Q}{\partial t_e \partial n_e} \frac{\partial v_h}{\partial t_e} \right), \quad (3.2)$$

where

$$\beta_e = \frac{1}{12} \cot \theta_e (h_{e+1}^2 - h_{e-1}^2), \quad \gamma_e = \frac{1}{3} \cot \theta_e |\tau| \quad (3.3)$$

and $\phi_I \in P_1(\tau)$ is the linear interpolation of ϕ on τ .

LEMMA 3.1. *Let m_e denote t_e or n_e . Assume that \mathcal{M}_h satisfy *Condition* $(\alpha, \sigma, \delta/2)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$. For any interior edge $e \in \mathcal{M}_h$ and two elements $\tau, \tau' \subset \Omega_e$, we have*

$$|\beta_e| + |\beta'_e| \lesssim h_e^2, \quad |\gamma_e| + |\gamma'_e| \lesssim h_e^2, \quad \forall e \in \mathcal{E}_h; \quad (3.4)$$

$$|\beta_e - \beta'_e| \lesssim h_e^{2+\alpha} / r_e^{\alpha+\delta(1-\alpha)/2}, \quad |\gamma_e - \gamma'_e| \lesssim h_e^{2+\alpha} / r_e^{\alpha+\delta(1-\alpha)/2}, \quad \forall e \in \mathcal{E}_{1,h}; \quad (3.5)$$

$$\int_e \frac{\partial^2 \phi}{\partial t_e \partial m_e} \frac{\partial v_h}{\partial t_e} \lesssim |\phi|_{W^{2,\infty}(e)} \|\nabla v_h\|_{L^2(\tau)}; \quad (3.6)$$

$$\int_e \frac{\partial^2(\phi - \phi_Q)}{\partial t_e \partial m_e} \frac{\partial v_h}{\partial t_e} \lesssim |\phi|_{H^3(\tau)} \|\nabla v_h\|_{L^2(\tau)}. \quad (3.7)$$

Proof. The arguments for (3.4), (3.5), and (3.6) are trivial, and for (3.7) follows from the trace theorem and the standard error estimate $|\phi - \phi_Q|_{H^2(\tau)} \lesssim h_\tau |\phi|_{H^3(\tau)}$. \square

To deal with the singularity at the origin O we introduce the following lemma. Recall that v is the singular part of the decomposition $u = v + w$.

LEMMA 3.2. *Let $\mathcal{M}^O = \{\tau \in \mathcal{M}_h : \text{the origin } O \in \partial\tau\}$ be the set of elements with one vertex at O .*

$$\|\nabla v - \nabla v_I\|_{L^2(\tau)} \lesssim h_\tau^\delta, \quad \forall \tau \in \mathcal{M}^O.$$

where $v_I = I_h^1 v$ is the linear interpolation of v .

Proof.

$$\|\nabla v - \nabla v_I\|_{L^2(\tau)} \lesssim \|\nabla v\|_{L^2(\tau)} + \|\nabla v_I\|_{L^2(\tau)}, \quad (3.8)$$

It follows from (1.3) that

$$\|\nabla v\|_{L^2(\tau)} \lesssim \left(\int_\tau |\nabla v|^2 \right)^{1/2} \lesssim \left(\int_\tau r^{2\delta-2} \right)^{1/2} \lesssim \left(\int_0^{h_\tau} r^{2\delta-2} r \, dr \right)^{1/2} \lesssim h_\tau^\delta. \quad (3.9)$$

Since $\nabla C = 0$, for any constant C , we have,

$$\begin{aligned} \|\nabla v_I\|_{L^2(\tau)} &= \|\nabla(v_I - v(O))\|_{L^2(\tau)} \lesssim h_\tau \max_{z \in \mathcal{N}_h \cap \tau} |\nabla(v_I - v(O))(z)| \\ &\lesssim h_\tau \frac{1}{h_\tau} \max_{z \in \mathcal{N}_h \cap \tau} |v(z) - v(O)| \\ &= \max_{z \in \mathcal{N}_h \cap \tau} \left| \int_0^1 \frac{d}{dt} v(z t) dt \right| = \max_{z \in \mathcal{N}_h \cap \tau} \left| \int_0^1 z \cdot \nabla v(z t) dt \right|. \end{aligned}$$

Noting that $|z| \lesssim h_\tau$ for $\tau \in \mathcal{M}^O$, it follows from the assumption (1.3) that

$$\|\nabla v_I\|_{L^2(\tau)} \lesssim \int_0^1 h_\tau \cdot (h_\tau t)^{\delta-1} dt \lesssim h_\tau^\delta. \quad (3.10)$$

The proof is completed by combining (3.8)–(3.10). \square

LEMMA 3.3. *Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/2)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then for any $v_h \in \mathring{V}_h^1$*

$$\left| \int_\Omega \nabla(u - u_I) \cdot \nabla v_h \right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right), \quad (3.11)$$

where $u_I = I_h^1 u \in V_h^1$ is the piecewise linear interpolation of u .

Proof. From the decomposition $u = v + w$,

$$\int_\Omega \nabla(u - u_I) \cdot \nabla v_h = \int_\Omega \nabla(v - v_I) \cdot \nabla v_h + \int_\Omega \nabla(w - w_I) \cdot \nabla v_h, \quad (3.12)$$

where $v = I_h^1 v$ and $w = I_h^1 w$ are the linear interpolations of v and w , respectively.

We first estimate $\int_{\Omega} \nabla(v - v_I) \cdot \nabla v_h$. Let $\mathcal{E}^O = \{e \in \mathcal{E}_h : e \subset \partial\tau \text{ the origin } O \in \tau\}$ and $\partial\mathcal{E}^O = \{e \in \mathcal{E}^O : O \notin e\}$. Recall that \mathcal{M}^O is the set of elements with one vertex at O . Applying (3.2),

$$\begin{aligned} \int_{\Omega} \nabla(v - v_I) \cdot \nabla v_h &= \sum_{\tau \in \mathcal{M}_h} \int_{\tau} \nabla(v - v_I) \cdot \nabla v_h = \sum_{\tau \in \mathcal{M}^O} \int_{\tau} \nabla(v - v_I) \cdot \nabla v_h \\ &\quad + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^O} \sum_{e \subset \partial\tau} \left(\beta_e \int_e \frac{\partial^2 v_Q}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 v_Q}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right) \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} I_j &= \sum_{e \in \mathcal{E}_{j,h} \setminus \mathcal{E}^O} \left[(\beta_e - \beta'_e) \int_e \frac{\partial^2 v}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + (\gamma_e - \gamma'_e) \int_e \frac{\partial^2 v}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right. \\ &\quad + \beta_e \int_e \frac{\partial^2 (v_Q - v)}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 (v_Q - v)}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \\ &\quad \left. + \beta'_e \int_e \frac{\partial^2 (v - v_Q)}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma'_e \int_e \frac{\partial^2 (v - v_Q)}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right], \quad j = 1, 2, \\ I_3 &= \sum_{\tau \in \mathcal{M}^O} \int_{\tau} \nabla(v - v_I) \cdot \nabla v_h, \\ I_4 &= \sum_{e \in \partial\mathcal{E}^O} \left(\beta_e \int_e \frac{\partial^2 v_Q}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 v_Q}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right). \end{aligned}$$

First, I_3 can be estimated by Lemma 3.2 and the fact that $h_{\tau} \approx \underline{h}$ for $\tau \in \mathcal{M}^O$:

$$|I_3| \lesssim \underline{h}^{\delta} \sum_{\tau \in \mathcal{M}^O} \|\nabla v_h\|_{L^2(\tau)} \lesssim \underline{h}^{\delta} \|\nabla v_h\|_{L^2(\Omega)}. \quad (3.14)$$

Secondly, I_4 can be estimated by Lemma 3.1, the assumption (1.3), and the fact that $h_e \approx r_e \approx \underline{h}$ for $e \in \partial\mathcal{E}^O$.

$$\begin{aligned} |I_4| &\lesssim \sum_{e \in \partial\mathcal{E}^O} h_e^2 (|v|_{W^{2,\infty}(e)} + |v|_{H^3(\tau: \tau \in \Omega_e, \tau \notin \mathcal{M}^O)}) \|\nabla v_h\|_{L^2(\tau: \tau \in \Omega_e, \tau \notin \mathcal{M}^O)} \\ &\lesssim \sum_{e \in \partial\mathcal{E}^O} h_e^2 (r_e^{\delta-2} + h_e r_e^{\delta-3}) \|\nabla v_h\|_{L^2(\tau: \tau \in \Omega_e, \tau \notin \mathcal{M}^O)} \\ &\lesssim \underline{h}^{\delta} \sum_{e \in \partial\mathcal{E}^O} \|\nabla v_h\|_{L^2(\tau: \tau \in \Omega_e, \tau \notin \mathcal{M}^O)} \lesssim \underline{h}^{\delta} \|\nabla v_h\|_{L^2(\Omega)}. \end{aligned} \quad (3.15)$$

Next we estimate I_1 . Notice that $h_e \approx h_{\tau}$ and $r_e \approx r_{\tau}$ for $\tau \subset \Omega_e$ and $e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O$.

It follows from Lemma 3.1 and the assumption (1.3) that

$$\begin{aligned}
|I_1| &\lesssim \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} \left[\frac{h_e^{2+\alpha}}{r_e^{\alpha+\delta(1-\alpha)/2}} r_e^{\delta-2} + h_e^2 h_\tau r_\tau^{\delta-3} \right] \|\nabla v_h\|_{L^2(\tau; \tau \in \Omega_e)} \\
&\lesssim \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} \left[h_e^{2+\alpha} r_e^{\delta-2-\alpha-\delta(1-\alpha)/2} + h_e^3 r_e^{\delta-3} \right] \|\nabla v_h\|_{L^2(\tau; \tau \in \Omega_e)} \\
&\lesssim \left\{ \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} \left[h_e^2 h_e^{2+2\alpha} r_e^{2\delta-4-2\alpha-\delta(1-\alpha)} + h_e^2 h_e^4 r_e^{2\delta-6} \right] \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \left\{ \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} \left[h_e^2 \underline{h}^{\delta(1+\alpha)} r_e^{(2-\delta)(1+\alpha)} r_e^{2\delta-4-2\alpha-\delta(1-\alpha)} + h_e^2 \underline{h}^{2\delta} r_e^{4-2\delta} r_e^{2\delta-6} \right] \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}$$

Here we have used $h_e \approx r_e^{1-\delta/2} \underline{h}^{\delta/2}$ to derive the last inequality. Therefore

$$\begin{aligned}
I_1 &\lesssim \left\{ \underline{h}^{\delta(1+\alpha)} \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} h_e^2 r_e^{-2} \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \left\{ \underline{h}^{\delta(1+\alpha)} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^O} h_\tau^2 r_\tau^{-2} \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \left\{ \underline{h}^{\delta(1+\alpha)} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^O} \int_\tau r^{-2} \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \left\{ \underline{h}^{\delta(1+\alpha)} \int_{\underline{h}}^1 r^{-1} dr \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \underline{h}^{\delta(1+\alpha)/2} (|\ln \underline{h}|^{1/2}) \|\nabla v_h\|_{L^2(\Omega)}. \tag{3.16}
\end{aligned}$$

Finally, we estimate I_2 . Notice that $h_e \lesssim r_e$ for $e \notin \mathcal{E}^O$. It follows from Lemma 3.1 and the assumption (1.3) that

$$\begin{aligned}
|I_2| &\lesssim \sum_{e \in \mathcal{E}_{2,h} \setminus \mathcal{E}^O} \left[h_e^2 r_e^{\delta-2} + h_e^2 h_\tau r_\tau^{\delta-3} \right] \|v_h\|_{L^2(\tau; \tau \in \Omega_e)} \lesssim \sum_{e \in \mathcal{E}_{2,h} \setminus \mathcal{E}^O} h_e^2 r_e^{\delta-2} \|v_h\|_{L^2(\tau; \tau \in \Omega_e)} \\
&\lesssim \left\{ \sum_{e \in \mathcal{E}_{2,h} \setminus \mathcal{E}^O} h_e^4 r_e^{2\delta-4} \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \underline{h}^\delta \left\{ \sum_{e \in \mathcal{E}_{2,h} \setminus \mathcal{E}^O} 1 \right\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}$$

Here we have used $h_e \approx r_e^{1-\delta/2} \underline{h}^{\delta/2}$ to derive the last inequality. Therefore

$$|I_2| \lesssim \underline{h}^\delta \{\#\mathcal{E}_{2,h}\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)} \lesssim \underline{h}^\delta \{N^\sigma\}^{1/2} \|\nabla v_h\|_{L^2(\Omega)}. \tag{3.17}$$

From Lemma 2.1, $\underline{h}^\delta \approx 1/N$, $|\ln \underline{h}| \approx \ln N$. Combining (3.13)–(3.17) we have

$$\begin{aligned}
\left| \int_\Omega \nabla(v - v_I) \cdot \nabla v_h \right| &\lesssim \left(\underline{h}^{\delta(1+\alpha)/2} (|\ln \underline{h}|^{1/2}) + \underline{h}^\delta \{N^\sigma\}^{1/2} \right) \|\nabla v_h\|_{L^2(\Omega)} \\
&\lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right).
\end{aligned} \tag{3.18}$$

Now we turn to the estimate for $\int_\Omega \nabla(w - w_I) \cdot \nabla v_h$. Since w is smooth, we do not separate the point O . From (3.2),

$$\int_\Omega \nabla(w - w_I) \cdot \nabla v_h = \sum_{\tau \in \mathcal{M}_h} \int_\tau \nabla(w - w_I) \cdot \nabla v_h = J_1 + J_2, \tag{3.19}$$

where

$$\begin{aligned}
J_j = \sum_{e \in \mathcal{E}_{j,h}} & \left[(\beta_e - \beta'_e) \int_e \frac{\partial^2 w}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + (\gamma_e - \gamma'_e) \int_e \frac{\partial^2 w}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right. \\
& + \beta_e \int_e \frac{\partial^2 (w_Q - w)}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma_e \int_e \frac{\partial^2 (w_Q - w)}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \\
& \left. + \beta'_e \int_e \frac{\partial^2 (w - w_Q)}{\partial t_e^2} \frac{\partial v_h}{\partial t_e} + \gamma'_e \int_e \frac{\partial^2 (w - w_Q)}{\partial t_e \partial \mathbf{n}_e} \frac{\partial v_h}{\partial t_e} \right], \quad j = 1, 2.
\end{aligned}$$

By a similar argument as for I_1 and I_2 we can prove that

$$\left| \int_{\Omega} \nabla(w - w_I) \cdot \nabla v_h \right| \lesssim \frac{1}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}. \quad (3.20)$$

Now, the proof of the lemma follows from (3.12), (3.18), and (3.20). \square

Applying Lemma 3.3 we obtain the following superconvergence result between the finite element solution u_h and the linear interpolation u_I of the solution of the problem (1.1)

THEOREM 3.4. *Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/2)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$ and that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u_h - u_I)\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (3.21)$$

Proof. Taking $v_h = u_h - u_I$ in Lemma 3.3 we have

$$\begin{aligned}
\|\nabla(u_h - u_I)\|_{L^2(\Omega)}^2 &= A(u_h - u_I, v_h) = A(u - u_I, v_h) \\
&= \int_{\Omega} \nabla(u - u_I) \cdot \nabla v_h \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}} \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}$$

The proof of the theorem is completed by canceling $\|\nabla v_h\|_{L^2(\Omega)}$ on both sides of the inequality. \square

4. Superconvergence between the finite element solution and quadratic interpolation.

Most parts of the proofs are similar to those for linear elements and is omitted. We only point out the different parts. In this section u_h is the solution of (1.4) with $k = 2$, that is, the quadratic finite element approximation of u .

We first introduce some estimates over triangles from [9]. Recall that $\phi_Q = \Pi_Q \phi$ is the quadratic interpolations defined in (3.1) based on the moment conditions.

LEMMA 4.1. *Assume that $\phi \in H^4(\tau)$, then there holds*

$$\begin{aligned}
\int_{\tau} \nabla(\phi - \Pi_Q \phi) \cdot \nabla v_h &= \sum_{e \subset \partial \tau} \sum_{s=0}^3 \left(a_e^s(\tau) \frac{|\tau|}{h_e} + b_e^s(\tau) \right) \int_e \frac{\partial^3 \phi}{\partial \mathbf{n}_e^s \partial t_e^{3-s}} \frac{\partial^2 v_h}{\partial t_e^2} \\
&+ O(h_\tau^3) |\phi|_{H^4(\tau)} \|v_h\|_{H^1(\tau)}, \quad \forall v_h \in P_2(\tau),
\end{aligned} \quad (4.1)$$

where for $s = 0, 1, 2, 3$,

$$|a_e^s(\tau)| + |a_e^s(\tau')| \lesssim h_e^3, \quad |b_e^s(\tau)| + |b_e^s(\tau')| \lesssim h_e^4, \quad \text{if } e \in \mathcal{E}_h; \quad (4.2)$$

$$|a_e^s(\tau)| |\tau| - a_e^s(\tau') |\tau'| \lesssim h_e^{5+\alpha} / r_e^{\alpha+\delta(1-\alpha)/3}, \quad |b_e^s(\tau) - b_e^s(\tau')| \lesssim h_e^{4+\alpha} / r_e^{\alpha+\delta(1-\alpha)/3}, \quad (4.3)$$

if \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/3)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and $e \in \mathcal{E}_{1,h}$.

To obtain the superconvergence of $\|\nabla(u_h - I_h^2 u)\|_{L^2(\Omega)}$, we estimate the difference between two quadratic interpolation operators Π_Q and I_h^2 . It easy to check that [15]

$$\Pi_Q p - I_h^2 p = 0, \quad \forall p \in P_3.$$

From the Bramble-Hilbert lemma, we have

$$\int_{\tau} (\nabla \Pi_Q \phi - \nabla I_h^2 \phi) \cdot \nabla v_h \lesssim h_{\tau}^3 |\phi|_{H^4(\tau)} \|\nabla v_h\|_{L^2(\tau)}.$$

Therefor we have the following lemma from (4.1).

LEMMA 4.2. Assume that $\phi \in H^4(\tau)$, then there holds

$$\begin{aligned} \int_{\tau} \nabla(\phi - I_h^2 \phi) \cdot \nabla v_h &= \sum_{e \subset \partial \tau} \sum_{s=0}^3 \left(a_e^s(\tau) \frac{|\tau|}{h_e} + b_e^s(\tau) \right) \int_e \frac{\partial^3 \phi}{\partial \mathbf{n}_e^2 \partial t_e^{3-s}} \frac{\partial^2 v_h}{\partial t_e^2} \\ &+ O(h_{\tau}^3) |\phi|_{H^4(\tau)} \|v_h\|_{H^1(\tau)}, \quad \text{for } v_h \in P_2(\tau), \end{aligned} \quad (4.4)$$

Recall from Lemma 2.1, in the quadratic case, if $h_{\tau} \approx r_{\tau}^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$, then the degree of freedoms N of the finite element equation (1.4) satisfies

$$N \approx \frac{1}{\underline{h}^{2\delta/3}}. \quad (4.5)$$

The following lemma is analogous to Lemma 3.2. We omit the proof.

LEMMA 4.3. For v in the decomposition (1.2),

$$\|\nabla v - \nabla I_h^2 v\|_{L^2(\tau)} \lesssim h_{\tau}^{\delta}, \quad \forall \tau \in \mathcal{M}^{\circ}.$$

The following lemma is similar to Lemma 3.3

LEMMA 4.4. Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/3)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_{\tau} \approx r_{\tau}^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then for any $v_h \in \mathring{V}_h^2$

$$\left| \int_{\Omega} \nabla(u - I_h^2 u) \cdot \nabla v_h \right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (4.6)$$

Proof. From the decomposition $u = v + w$,

$$\int_{\Omega} \nabla(u - I_h^2 u) \cdot \nabla v_h = \int_{\Omega} \nabla(v - I_h^2 v) \cdot \nabla v_h + \int_{\Omega} \nabla(w - I_h^2 w) \cdot \nabla v_h, \quad (4.7)$$

We first estimate the term $\int_{\Omega} \nabla(v - I_h^2 v) \cdot \nabla v_h$. It follows from Lemma 4.2 that

$$\int_{\Omega} \nabla(v - I_h^2 v) \cdot \nabla v_h = \sum_{\tau \in \mathcal{M}_h} \int_{\tau} \nabla(v - I_h^2 v) \cdot \nabla v_h = I_1 + I_2 + I_3 + I_4, \quad (4.8)$$

where

$$\begin{aligned} I_j &= \sum_{e=\tau \cap \tau' \in \mathcal{E}_{j,h} \setminus \mathcal{E}^{\circ}} \sum_{s=0}^3 \left\{ \frac{a_e^s(\tau) |\tau| - a_e^s(\tau') |\tau'|}{h_e} + [b_e^s(\tau) - b_e^s(\tau')] \right\} \int_e \frac{\partial^3 v}{\partial \mathbf{n}_e^2 \partial t_e^{3-s}} \frac{\partial^2 v_h}{\partial t_e^2} \\ &+ O(h_e^3) |v|_{H^4(\Omega_e)} \|v_h\|_{H^1(\Omega_e)}, \quad j = 1, 2, \end{aligned}$$

$$I_3 = \sum_{\tau \in \mathcal{M}^O} \int_{\tau} \nabla(v - I_h^2 v) \cdot \nabla v_h,$$

$$I_4 = \sum_{e \in \partial \mathcal{E}^O} \sum_{s=0}^3 \left(a_e^s(\tau) \frac{|\tau|}{h_e} + b_e^s(\tau) \right) \int_e \frac{\partial^3 v}{\partial \mathbf{n}_e^s \partial t_e^{3-s}} \frac{\partial^2 v_h}{\partial t_e^2} + O(h_\tau^3) |v|_{H^4(\tau)} \|v_h\|_{H^1(\tau)}.$$

Notice that the τ in I_4 is not in \mathcal{M}^O .

Form Lemma 4.3,

$$|I_3| \lesssim \underline{h}^\delta \|\nabla v_h\|_{L^2(\Omega)}. \quad (4.9)$$

It follows from (4.2) and the assumption (1.3) that

$$\begin{aligned} |I_4| &\lesssim \sum_{e \in \partial \mathcal{E}^O} h_e^5 r_e^{\delta-3} |v_h|_{W^{2,\infty}(\tau)} + h_\tau^3 h_\tau r_e^{\delta-4} \|v_h\|_{H^1(\tau)} \\ &\lesssim \sum_{e \in \partial \mathcal{E}^O} h_e^3 r_e^{\delta-3} \|v_h\|_{H^1(\tau)} + h_e^4 r_e^{\delta-4} \|v_h\|_{H^1(\tau)} \lesssim \underline{h}^\delta \|v_h\|_{H^1(\Omega)}. \end{aligned} \quad (4.10)$$

Here we have used the inverse estimate $|v_h|_{W^{2,\infty}(\tau)} \lesssim h_e^{-2} \|v_h\|_{H^1(\tau)}$ and the fact that $h_e \approx r_e \approx \underline{h}$ for $e \in \partial \mathcal{E}^O$.

Next we estimate I_1 . It follows from Lemma 4.1 and the assumption (1.3) that

$$\begin{aligned} |I_1| &\lesssim \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} \left[\frac{h_e^{5+\alpha}}{r_e^{\alpha+\delta(1-\alpha)/3}} r_e^{\delta-3} |v_h|_{W^{2,\infty}(\tau)} + h_e^3 h_\tau r_\tau^{\delta-4} \|v_h\|_{H^1(\Omega_e)} \right] \\ &\lesssim \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} [h_e^{3+\alpha} r_e^{\delta-3-\alpha-\delta(1-\alpha)/3} + h_e^4 r_e^{\delta-4}] \|v_h\|_{H^1(\Omega_e)} \\ &\lesssim \left\{ \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} [h_e^2 h_e^{4+2\alpha} r_e^{2\delta-6-2\alpha-2\delta(1-\alpha)/3} + h_e^2 h_e^6 r_e^{2\delta-8}] \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \\ &\lesssim \left\{ \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} [h_e^2 \underline{h}^{2\delta(2+\alpha)/3} r_e^{(4+2\alpha)(1-\delta)/3} r_e^{2\delta-6-2\alpha-2\delta(1-\alpha)/3} \right. \\ &\quad \left. + h_e^2 \underline{h}^{2\delta} r_e^{6-2\delta} r_e^{2\delta-8}] \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \end{aligned}$$

Here we have used $h_e \approx r_e^{1-\delta/3} \underline{h}^{\delta/3}$ to derive the last inequality. Therefore

$$\begin{aligned} I_1 &\lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{e \in \mathcal{E}_{1,h} \setminus \mathcal{E}^O} h_e^2 r_e^{-2} \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^O} h_\tau^2 r_\tau^{-2} \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \\ &\lesssim \left\{ \underline{h}^{2\delta(2+\alpha)/3} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^O} \int_{\tau} r^{-2} \right\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \underline{h}^{\delta(2+\alpha)/3} |\ln \underline{h}|^{1/2} \|v_h\|_{H^1(\Omega)}. \end{aligned} \quad (4.11)$$

By a similar argument for (3.17) we can show that

$$|I_2| \lesssim \underline{h}^\delta \{\#\mathcal{E}_{2,h}\}^{1/2} \|v_h\|_{H^1(\Omega)} \lesssim \underline{h}^\delta \{N^\sigma\}^{1/2} \|v_h\|_{H^1(\Omega)}. \quad (4.12)$$

Notice that $\|v_h\|_{H^1(\Omega)} \lesssim \|\nabla v_h\|_{L^2(\Omega)}$ from the Poincaré's inequality. Combining (4.8)–(4.12) we have

$$\begin{aligned} \left| \int_{\Omega} \nabla(v - v_I) \cdot \nabla v_h \right| &\lesssim \left(\underline{h}^{\delta(2+\alpha)/3} |\ln \underline{h}|^{1/2} + \underline{h}^{\delta} \{N^{\sigma}\}^{1/2} \right) \|\nabla v_h\|_{L^2(\Omega)} \\ &\lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}} \|\nabla v_h\|_{L^2(\Omega)}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \end{aligned} \quad (4.13)$$

The estimate for the term $\int_{\Omega} \nabla(w - I_h^2 w) \cdot \nabla v_h$ is similar as above. It follows from Lemma 4.2 that

$$\int_{\Omega} \nabla(w - I_h^2 w) \cdot \nabla v_h = \sum_{\tau \in \mathcal{M}_h} \int_{\tau} \nabla(w - I_h^2 w) \cdot \nabla v_h = J_1 + J_2, \quad (4.14)$$

where

$$\begin{aligned} J_j = \sum_{e=\tau \cap \tau' \in \mathcal{E}_{j,h}} \sum_{s=0}^3 \left\{ \frac{a_e^s(\tau) |\tau| - a_e^s(\tau') |\tau'|}{h_e} + [b_e^s(\tau) - b_e^s(\tau')] \right\} \int_e \frac{\partial^3 w}{\partial n_e^s \partial t_e^{3-s}} \frac{\partial^2 v_h}{\partial t_e^2} \\ + O(\underline{h}_e^3) |w|_{H^4(\Omega_e)} \|v_h\|_{H^1(\Omega_e)}, \quad j = 1, 2. \end{aligned}$$

There holds

$$\left| \int_{\Omega} \nabla(w - w_I) \cdot \nabla v_h \right| \lesssim \frac{1}{N^{1+\rho}} \|\nabla v_h\|_{L^2(\Omega)}. \quad (4.15)$$

Now, the proof of the lemma follows from (4.7), (4.13), and (4.15). \square

Applying Lemma 4.4 we can obtain the following superconvergence result between the quadratic finite element solution u_h and the quadratic interpolation $I_h^2 u$ of the solution of the problem (1.1).

THEOREM 4.5. *Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/3)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_{\tau} \approx r_{\tau}^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u_h - I_h^2 u)\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (4.16)$$

5. The asymptotically exact a posteriori error estimators. In this section, we apply a newly developed gradient recovery operator, called polynomial preserving recovery (PPR) [12, 14, 16], to define an *a posteriori* error estimator. We further prove some superconvergence property of the recovery operator. As a consequence, the error estimator based on PPR is asymptotically exact under a mesh density assumption.

5.1. The gradient recovery operator G_h and its superconvergence. Given a node $z \in \mathcal{N}_h$, we select $n \geq m = (k+2)(k+3)/2$ sampling points z_j , $j = 1, 2, \dots, n$, in an element patch ω_z containing z (z is one of z_j), and fit a polynomial of degree $k+1$, in the least squares sense, with values of u_h at those sampling points. In other words, we are looking for $p_{k+1} \in \mathcal{P}_{k+1}$ such that

$$\sum_{j=1}^n (p_{k+1} - u_h)^2(z_j) = \min_{q \in \mathcal{P}_{k+1}} \sum_{j=1}^n (q - u_h)^2(z_j). \quad (5.1)$$

The recovered gradient in the neighborhood of z is then defined as

$$G_h u_h = \nabla p_{k+1}. \quad (5.2)$$

It was proved in [12] that the above least squares fitting procedure has a unique solution as long as those n sampling points are not on the same conic curve. Furthermore, the gradient recovery operator $G_h : C(\Omega) \mapsto V_h^k \times V_h^k$, $k = 1$ or 2 , has the following properties.

- (i) $\|G_h v_h\|_{L^2(\Omega)} \lesssim \|\nabla v_h\|_{L^2(\Omega)}$, $\forall v_h \in V_h^k$;
- (ii) For any nodal point z , $(G_h p)(z) = \nabla p(z)$ if $p \in P_{k+1}(\omega_z)$;
- (iii) $|(G_h \phi)(z)| \lesssim \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \omega_z} |\phi(z')|$ for any node $z \in \mathcal{N}_h$;
- (iv) $G_h \phi = G_h I_h^k \phi$.

Since $I_h^k \phi$ and ϕ have the same nodal values and G_h uses only nodal values, so (iv) is clear. The polynomial preserving property (ii) can be established easily by the least squares procedure [16]. A key observation is that G_h provides a finite difference scheme at each node $z \in \mathcal{N}_h$, therefore, (iii) is obvious. Under a very mild mesh condition, "the sum of any two adjacent angles in \mathcal{M}_h is at most π ", the bounded-ness property (i) can be proved, though not trivial. A reader is referred to [12, 14, 16] for more details.

We first consider the case of linear finite elements and then state the corresponding results for quadratic elements since the proofs are similar. We have from (i) and (iv),

$$\begin{aligned} \|G_h u_h - \nabla u\|_{L^2(\Omega)} &\leq \|G_h u_h - G_h u_I\|_{L^2(\Omega)} + \|G_h u - \nabla u\|_{L^2(\Omega)} \\ &\lesssim \|\nabla(u_h - u_I)\|_{L^2(\Omega)} + \|G_h u_I - \nabla u\|_{L^2(\Omega)}. \end{aligned} \quad (5.3)$$

Here u_I is the linear interpolation of u . The estimate for the first term of the right hand side of the inequality (5.3) is given in Theorem 3.4. To estimate the second term we need the following lemma.

LEMMA 5.1. *Under the conditions (ii)–(iii), for any element $\tau \in \mathcal{M}_h$ and any function $\phi \in W^{3,\infty}(\tilde{\tau})$,*

$$\|G_h \phi_I - \nabla \phi\|_{L^2(\tau)} \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde{\tau})},$$

where $\tilde{\tau} = \bigcup \{\omega_z : z \in \mathcal{N}_h \cap \tau\}$ and ϕ_I is the linear interpolation of ϕ .

Proof. Let $(\nabla \phi)_I$ be the linear interpolation of $\nabla \phi$. Then

$$\|G_h \phi_I - \nabla \phi\|_{L^2(\tau)} \leq \|G_h \phi_I - (\nabla \phi)_I\|_{L^2(\tau)} + \|(\nabla \phi)_I - \nabla \phi\|_{L^2(\tau)}. \quad (5.4)$$

The standard theory of finite element interpolation estimates says that [4]

$$\|(\nabla \phi)_I - \nabla \phi\|_{L^2(\tau)} \lesssim h_\tau^2 |\phi|_{H^3(\tau)} \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde{\tau})}. \quad (5.5)$$

For a node $z \in \tau$, let $\phi_2(x, y)$ be the 2nd-degree Taylor expansion of ϕ at the point z . It is clear that

$$|\phi(x, y) - \phi_2(x, y)| \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde{\tau})}, \quad \forall (x, y) \in \tilde{\tau}.$$

Form Condition (ii) and (iii),

$$\begin{aligned} |(G_h \phi_I - (\nabla \phi)_I)(z)| &= |(G_h \phi_I - \nabla \phi)(z)| = |(G_h(\phi_I - \phi_2) - (\nabla \phi - \nabla \phi_2))(z)| \\ &= |(G_h(\phi_I - \phi_2))(z)| \lesssim \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \omega_z} |(\phi - \phi_2)(z')| \\ &\lesssim h_\tau^2 |\phi|_{W^{3,\infty}(\omega_z)}. \end{aligned}$$

Therefore

$$\|G_h \phi_I - (\nabla \phi)_I\|_{L^2(\tau)} \lesssim h_\tau \max_{z \in \mathcal{N}_h \cap \tau} |(G_h \phi_I - (\nabla \phi)_I)(z)| \lesssim h_\tau^3 |\phi|_{W^{3,\infty}(\tilde{\tau})}. \quad (5.6)$$

The proof of the lemma is completed by combining (5.4)–(5.6). \square

The following theorem is devoted to the estimate of the second term of (5.3).

THEOREM 5.2. *Assume that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_I - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N}. \quad (5.7)$$

Proof. Recall the decomposition $u = v + w$.

$$\|G_h u_I - \nabla u\|_{L^2(\Omega)} \leq \|G_h v_I - \nabla v\|_{L^2(\Omega)} + \|G_h w_I - \nabla w\|_{L^2(\Omega)}, \quad (5.8)$$

where $v_I = I_h^1 v$ and $w_I = I_h^1 w$ are the linear interpolations of v and w respectively.

We first estimate the singular part $\|G_h v_I - \nabla v\|_{L^2(\Omega)}$. Introduce the set of triangles $\mathcal{M}^{\tilde{O}} = \{\tau \in \mathcal{M}_h : \text{the origin } O \in \tilde{\tau}\}$. For any $\tau \in \mathcal{M}^{\tilde{O}}$,

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \leq \|G_h v_I\|_{L^2(\tau)} + \|\nabla v\|_{L^2(\tau)}. \quad (5.9)$$

From Condition (ii), $G_h C = 0$, for any constant C . Thus, from Condition (iii),

$$\begin{aligned} \|G_h v_I\|_{L^2(\tau)} &= \|G_h(v_I - v(O))\|_{L^2(\tau)} \lesssim h_\tau \max_{z \in \mathcal{N}_h \cap \tau} |G_h(v_I - v(O))(z)| \\ &\lesssim h_\tau \frac{1}{h_\tau} \max_{z' \in \mathcal{N}_h \cap \tilde{\tau}} |v(z') - v(O)| \\ &= \max_{z' \in \mathcal{N}_h \cap \tilde{\tau}} \left| \int_0^1 \frac{d}{dt} v(z't) dt \right| = \max_{z' \in \mathcal{N}_h \cap \tilde{\tau}} \left| \int_0^1 z' \cdot \nabla v(z't) dt \right|. \end{aligned}$$

Since $\tau \in \mathcal{M}^{\tilde{O}}$, $|z'| \lesssim \underline{h}$. It follows from the assumption (1.3) that

$$\|G_h v_I\|_{L^2(\tau)} \lesssim \int_0^1 \underline{h} \cdot (\underline{h}t)^{\delta-1} dt \lesssim \underline{h}^\delta. \quad (5.10)$$

On the other hand,

$$\|\nabla v\|_{L^2(\tau)} \lesssim \left(\int_\tau |\nabla v|^2 \right)^{1/2} \lesssim \left(\int_\tau r^{2\delta-2} \right)^{1/2} \lesssim \left(\int_0^{ch} r^{2\delta-2} r dr \right)^{1/2} \lesssim \underline{h}^\delta. \quad (5.11)$$

Here ch is the diameter of $\tilde{\tau}$. Combining (5.9), (5.10), and (5.11), we obtain

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \lesssim \underline{h}^\delta, \quad \text{for } \tau \in \mathcal{M}^{\tilde{O}}. \quad (5.12)$$

It follows from Lemma 5.1 and (1.3) that

$$\|G_h v_I - \nabla v\|_{L^2(\tau)} \lesssim h_\tau^3 |v|_{W^{3,\infty}(\tilde{\tau})} \lesssim h_\tau^3 r_\tau^{\delta-3}, \quad \text{for } \tau \in \mathcal{M}_h \setminus \mathcal{M}^{\tilde{O}}, \quad (5.13)$$

where r_τ is the distance from O to the barycenter of τ . Therefore from $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$,

$$\begin{aligned} \|G_h v_I - \nabla v\|_{L^2(\Omega)}^2 &= \sum_{\tau \in \mathcal{M}_h} \|G_h v_I - \nabla v\|_{L^2(\tau)}^2 \lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\bar{0}}} h_\tau^6 r_\tau^{2\delta-6} \\ &\lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\bar{0}}} h_\tau^2 r_\tau^{4-2\delta} \underline{h}^{2\delta} r_\tau^{2\delta-6} \lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\bar{0}}} \underline{h}^{2\delta} h_\tau^2 r_\tau^{-2} \\ &\lesssim \underline{h}^{2\delta} + \underline{h}^{2\delta} \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^{\bar{0}}} \int_\tau r^{-2} \lesssim \underline{h}^{2\delta} + \underline{h}^{2\delta} \int_{\underline{h}}^1 r^{-1} dr \lesssim \underline{h}^{2\delta} + \underline{h}^{2\delta} |\ln \underline{h}|. \end{aligned}$$

Therefore Lemma 2.1 implies that

$$\|G_h v_I - \nabla v\|_{L^2(\Omega)} \lesssim \underline{h}^\delta (1 + |\ln \underline{h}|^{1/2}) \lesssim \frac{1 + (\ln N)^{1/2}}{N} \quad (5.14)$$

Next we turn to estimate the term $\|G_h w_I - \nabla w\|_{L^2(\Omega)}$ in (5.8). Since w is smooth, we do not have to divide \mathcal{M}_h into two parts as above. From Lemma 5.1 and the assumption (1.3),

$$\begin{aligned} \|G_h w_I - \nabla w\|_{L^2(\Omega)} &\lesssim \left(\sum_{\tau \in \mathcal{M}_h} \|G_h w_I - \nabla w\|_{L^2(\tau)}^2 \right)^{1/2} \lesssim \left(\sum_{\tau \in \mathcal{M}_h} h_\tau^6 \right)^{1/2} \\ &\lesssim \left(\sum_{\tau \in \mathcal{M}_h} h_\tau^2 r_\tau^{4-2\delta} \underline{h}^{2\delta} \right)^{1/2} \lesssim \underline{h}^\delta \left(\int_\Omega r^{4-2\delta} \right)^{1/2} \lesssim \underline{h}^\delta \lesssim \frac{1}{N}. \end{aligned} \quad (5.15)$$

The proof of the theorem is completed by inserting the estimates (5.14) and (5.15) into the inequality (5.8). \square

The following superconvergence result of the gradient operator recovery G_h can be proved by combining (5.3), Theorem 3.4, and 5.2.

THEOREM 5.3. *Let u_h is the linear finite element approximation of u . Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/2)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_h - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1/2+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (5.16)$$

We remark that the result of Theorem 5.3 is a superconvergence result since the asymptotically optimal convergence rate of $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is $O(1/N^{1/2})$

Next we state the results for quadratic finite elements. The following theorem provides the the estimate for the gradient recovery operator G_h

THEOREM 5.4. *Assume that $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h I_h^2 u - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (5.17)$$

The superconvergence of the gradient recovery operator G_h is presented in the following theorem which is parallel to Theorem 5.3.

THEOREM 5.5. *Let u_h be the quadratic finite element approximation of u . Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/3)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$ and that $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|G_h u_h - \nabla u\|_{L^2(\Omega)} \lesssim \frac{1 + (\ln N)^{1/2}}{N^{1+\rho}}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (5.18)$$

5.2. The a posteriori error estimators. With preparation in the previous sections, it is now straightforward to prove the asymptotic exactness of error estimators based on the recovery operator G_h . The global error estimator is naturally defined by

$$\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}. \quad (5.19)$$

THEOREM 5.6. *Let u_h be the linear finite element approximation of u . Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/2)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$, and that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Furthermore, assume that*

$$\frac{1}{N^{1/2}} \lesssim \|\nabla(u - u_h)\|_{L^2(\Omega)}. \quad (5.20)$$

Then

$$\left| \frac{\eta_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1 \right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^\rho}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (5.21)$$

The following lemma says that $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is the asymptotically optimal on the meshes \mathcal{M}_h satisfying $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ as the degree of freedoms $N \rightarrow \infty$.

LEMMA 5.7. *Let u_h be the linear finite element approximation of u . Assume that $h_\tau \approx r_\tau^{1-\delta/2} \underline{h}^{\delta/2}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u - u_I)\|_{L^2(\Omega)} \lesssim \frac{1}{N^{1/2}} \text{ and hence } \|\nabla(u - u_h)\|_{L^2(\Omega)} \lesssim \frac{1}{N^{1/2}}.$$

Proof. Recall u is decomposed as $u = v + w$ satisfying (1.3). Noticing that

$$\|\nabla(v - v_I)\|_{L^2(\tau)} \lesssim h_\tau |v|_{H^2(\tau)} \lesssim h_\tau^2 r_\tau^{\delta-2}, \quad \forall \tau \in \mathcal{M}_h \setminus \mathcal{M}^0,$$

and that

$$\|\nabla(w - w_I)\|_{L^2(\tau)} \lesssim h_\tau |w|_{H^2(\tau)} \lesssim h_\tau^2, \quad \forall \tau \in \mathcal{M}_h,$$

From Lemma 3.2 we have

$$\begin{aligned} \|\nabla(u - u_I)\|_{L^2(\Omega)}^2 &\lesssim \|\nabla(v - v_I)\|_{L^2(\Omega)}^2 + \|\nabla(w - w_I)\|_{L^2(\Omega)}^2 \\ &= \sum_{\tau \in \mathcal{M}_h} (\|\nabla(v - v_I)\|_{L^2(\tau)}^2 + \|\nabla(w - w_I)\|_{L^2(\tau)}^2) \\ &\lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^0} h_\tau^4 r_\tau^{2\delta-4} \lesssim \underline{h}^{2\delta} + \sum_{\tau \in \mathcal{M}_h \setminus \mathcal{M}^0} h_\tau^2 r_\tau^{2-\delta} \underline{h}^\delta r_\tau^{2\delta-4} \\ &\lesssim \underline{h}^{2\delta} + \underline{h}^\delta \int_\Omega r^{\delta-2} \lesssim \underline{h}^{2\delta} + \underline{h}^\delta. \end{aligned}$$

From Lemma 2.1,

$$\|\nabla(u - u_I)\|_{L^2(\Omega)}^2 \lesssim \frac{1}{N^2} + \frac{1}{N},$$

which completes the proof of the lemma. \square

The following lemma says that, for the the quadratic finite element approximation u_h , $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is asymptotically optimal on the meshes \mathcal{M}_h satisfying $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ as the degree of freedoms $N \rightarrow \infty$.

LEMMA 5.8. *Let u_h be the quadratic finite element approximation of u . Assume that $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Then*

$$\|\nabla(u - I_h^2 u)\|_{L^2(\Omega)} \lesssim \frac{1}{N} \text{ and hence } \|\nabla(u - u_h)\|_{L^2(\Omega)} \lesssim \frac{1}{N}.$$

From Theorem 5.5, we can prove the asymptotic exactness of error estimators based on the recovery operator G_h for quadratic elements.

THEOREM 5.9. *Let u_h be the quadratic finite element approximation of u . Assume that \mathcal{M}_h satisfy Condition $(\alpha, \sigma, \delta/3)$ with $0 < \alpha \leq 1$ and $0 \leq \sigma < 1$ and that $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3}$ for any $\tau \in \mathcal{M}_h$. Furthermore, assume that*

$$\frac{1}{N} \lesssim \|\nabla(u - u_h)\|_{L^2(\Omega)}. \quad (5.22)$$

Then

$$\left| \frac{\eta_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1 \right| \lesssim \frac{1 + (\ln N)^{1/2}}{N^\rho}, \quad \rho = \min\left(\frac{\alpha}{2}, \frac{1-\sigma}{2}\right). \quad (5.23)$$

6. Implementation and numerical examples. In this section we present some examples to verify the asymptotic exactness of error estimators η_h based on the recovery operator G_h using quadratic finite elements. For the examples on linear elements we refer to [7].

The implementation of the adaptive algorithm in this section is based on the FEMLAB. We define the local a posteriori error estimator on element τ as follows:

$$\eta_\tau = \|G_h u_h - \nabla u_h\|_{L^2(\tau)}.$$

Then the global error estimator

$$\eta_h = \left(\sum_{\tau \in \mathcal{M}_h} \eta_\tau^2 \right)^{1/2}$$

Now we describe the adaptive algorithm we have used in this paper.

Algorithm. Given tolerance $\text{TOL} > 0$.

- Generate an initial mesh \mathcal{M}_h over Ω ;
- While $\eta_h > \text{TOL}$ do
 - Choose a set of elements $\widehat{\mathcal{M}}_h \subset \mathcal{M}_h$ such that

$$\left(\sum_{\tau \in \widehat{\mathcal{M}}_h} \eta_\tau^2 \right)^{1/2} > 0.7 \left(\sum_{\tau \in \mathcal{M}_h} \eta_\tau^2 \right)^{1/2},$$

then refine the elements in $\widehat{\mathcal{M}}_h$. Denote the new mesh by \mathcal{M}_h also.

- solve the discrete problem (1.4) on \mathcal{M}_h
- compute error estimators on \mathcal{M}_h

end while

Example 1. The Laplace equation on the L-shaped domain of Figure 6.1 with the Dirichlet boundary condition so chosen that the true solution is $r^{2/3} \sin(2\theta/3)$ in polar coordinates.

Figure 6.1 plots the initial mesh and the adaptively refined mesh of 3565 elements after 15 adaptive iterations. Figure 6.2 show the asymptotic exactness of error estimators $\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$ for the Laplace equation on L-shaped domain. It is shown that

$$\|\nabla u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1}), \quad \|G_h u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1.2}),$$

and

$$\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \approx 1 + O(N^{-0.5}).$$

Notice that the decay of $\|\nabla u_h - \nabla u\|_{L^2(\Omega)}$ is quasi-optimal, $\|G_h u_h - \nabla u\|_{L^2(\Omega)}$ is super-convergent by an order of $O(N^{-1.2})$, and $\eta_h / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ approaches 1 at the rate of $O(N^{-0.5})$ which is faster than expect rate $O(N^{-0.2})$ indicated by Theorem 5.9. In this paper, the L^2 norms are calculated by using the six points Gauss quadrature rule over triangles.

Let us have a close look at the mesh density assumption $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3} = r_\tau^{7/9} \underline{h}^{2/9}$ for $\delta = 2/3$. We shall verify this on the final mesh, which has 112880 elements, after 24 adaptive iterations. We choose $\underline{h} = \min_{\tau \in \mathcal{M}_h} h_\tau \approx 5.96 \times 10^{-8}$ and have

$$0.44 \leq \frac{h_\tau}{r_\tau^{7/9} \underline{h}^{2/9}} \leq 2.35$$

for all element $\tau \in \mathcal{M}_h$. Note that the ratio between the upper and lower bounds is less than 6. this fact indicates that all elements in the final mesh satisfy the mesh density assumption.

Example 2. Let $\Omega = \{(x_1, x_2) : |x_1|, |x_2| < 0.5\} \setminus \{(x_1, x_2) : 0 \leq x_1 < 0.5\}$ be the domain with a crack. We consider the Poisson equation

$$-\Delta u = 1$$

with Dirichlet boundary condition so chosen that the true solution is $r^{1/2} \sin(\theta/2) - \frac{1}{4}r^2$ in polar coordinates.

Figure 6.3 plots the initial mesh and the adaptively refined mesh of 3353 elements after 16 adaptive iterations. Figure 6.4 show the asymptotic exactness of error estimators $\eta_h = \|G_h u_h - \nabla u_h\|_{L^2(\Omega)}$ for the crack problem. It is shown that

$$\|\nabla u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1}), \quad \|G_h u_h - \nabla u\|_{L^2(\Omega)} \approx O(N^{-1.1}),$$

and

$$\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \approx 1 + O(N^{-0.3}).$$

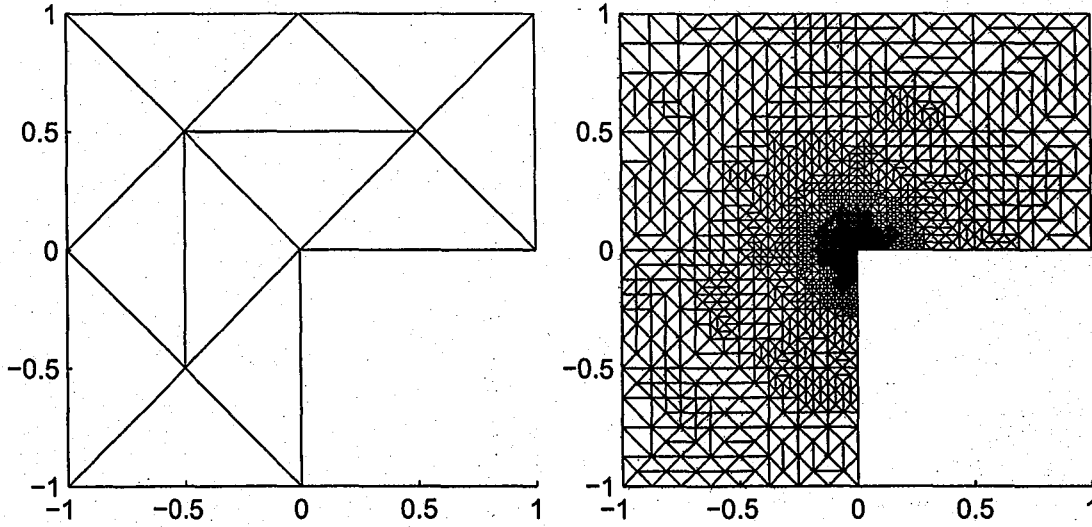


FIG. 6.1. The initial mesh (left) and the adaptively refined mesh (right) of 3565 elements after 15 adaptive iterations for the Laplace equation on L-shaped domain

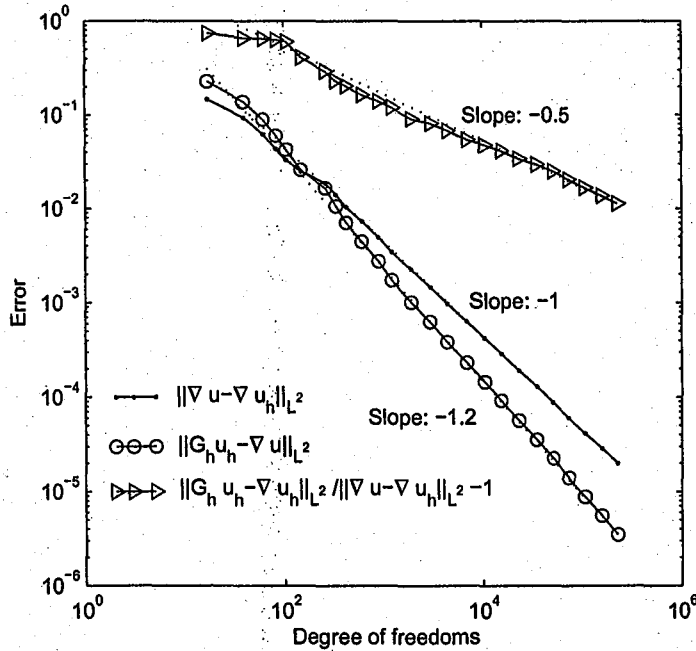


FIG. 6.2. $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)} - 1$, $\|\nabla u - \nabla u_h\|_{L^2(\Omega)}$, and $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$ versus the degree of freedoms for the Laplace equation on L-shaped domain. Dotted lines give reference slopes.

Notice that the decay of $\|\nabla u_h - \nabla u\|_{L^2(\Omega)}$ is quasi-optimal, $\|G_h u_h - \nabla u\|_{L^2(\Omega)}$ is super-convergent by an order of $O(N^{-1.1})$, and $\eta_h / \|\nabla u - \nabla u_h\|_{L^2(\Omega)}$ approaches 1 at the rate of $O(N^{-0.3})$ which is faster than expect rate $O(N^{-0.1})$ indicated by Theorem 5.9.

Let us have a close look at the mesh density assumption $h_\tau \approx r_\tau^{1-\delta/3} \underline{h}^{\delta/3} = r_\tau^{5/6} \underline{h}^{1/6}$ for $\delta = 1/2$. We shall verify this on the final mesh, which has 110790 elements, after 27

adaptive iterations. We choose $\underline{h} = \min_{\tau \in \mathcal{M}_h} h_\tau \approx 3.73 \times 10^{-9}$ and have

$$0.32 < \frac{h_\tau}{r_\tau^{5/6} \underline{h}^{1/6}} < 1.92$$

for all element $\tau \in \mathcal{M}_h$. Note that the ratio between the upper and lower bounds is 6. this fact indicates that all elements in the final mesh satisfy the mesh density assumption.

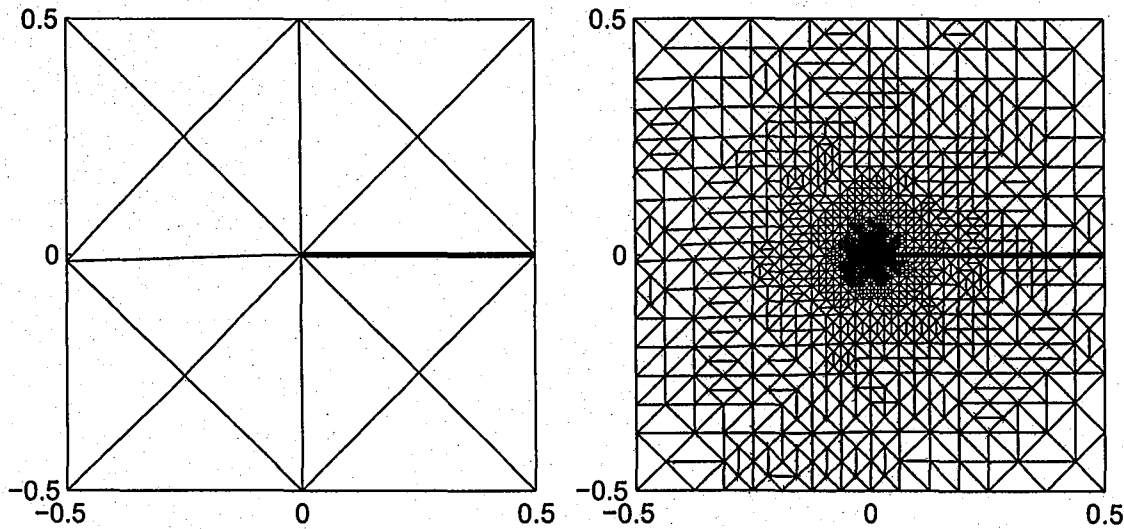


FIG. 6.3. The initial mesh (left) and the adaptively refined mesh (right) of 3353 elements after 16 adaptive iterations for the crack problem

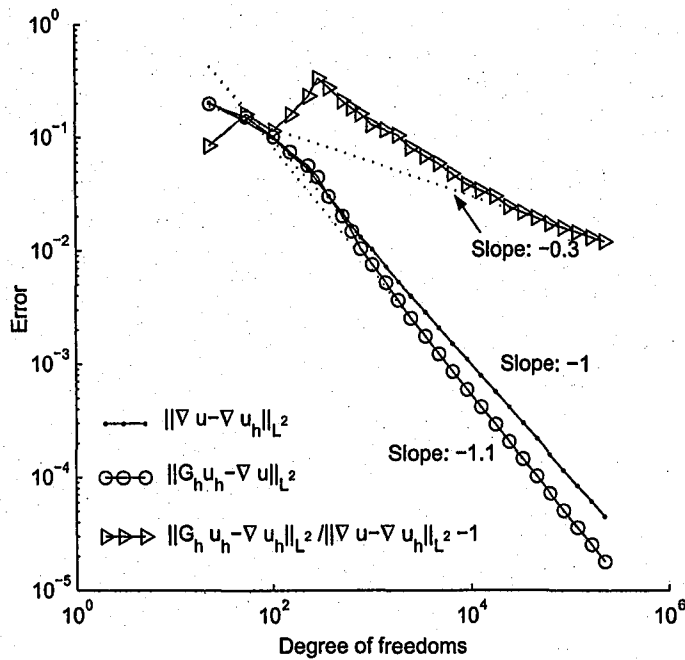


FIG. 6.4. $\|G_h u_h - \nabla u_h\|_{L^2(\Omega)} / \|\nabla u - \nabla u_h\|_{L^2(\Omega)} - 1$, $\|\nabla u - \nabla u_h\|_{L^2(\Omega)}$, and $\|\nabla u - G_h u_h\|_{L^2(\Omega)}$ versus the degree of freedoms for the crack problem. Dotted lines give reference slopes.

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