6-1-2006

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Department of Mathematics
Research Report

2006 Series
#6

This research was partly supported by the National Science Foundation and the Australian Research Council.

METHODS OF VARIATIONAL ANALYSIS IN MULTIOBJECTIVE OPTIMIZATION [1]

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The paper concerns new applications of advanced methods of variational analysis and generalized differentiation to constrained problems of multiobjective/vector optimization. We pay the main attention to general notions of optimal solutions for multiobjective problems that are induced by geometric concepts of extremality in variational analysis while covering various notions of Pareto and other type of optimality/efficiency conventional in multiobjective optimization. Based on the extremal principles in variational analysis and on appropriate tools of generalized differentiation with well-developed calculus rules, we derive necessary optimality conditions for broad classes of constrained multiobjective problems in the framework of infinite-dimensional spaces. Applications of variational techniques in infinite dimensions require certain "normal compactness" properties of sets and set-valued mappings, which play a crucial rôle in deriving the main results of this paper.

Keywords: Multiobjective and vector optimization; Variational analysis; Generalized differentiation; Subdifferential calculus; Optimality conditions; Normal compactness; Banach and Asplund spaces

Mathematical Subject Classification 2000: Primary: 49J52, 49J53; Secondary: 90C29

1 Introduction

Variational analysis has been recognized as a fruitful and rapidly developing area in mathematics that mainly concerns optimization, equilibrium, and related problems while applying variational principles and perturbation/approximation techniques to a broad spectrum of problems, which may not be of optimization nature. We refer the reader to the now-classical monograph by Rockafellar and Wets [19] devoted to the key issues of variational analysis in finite dimensions and to the recent mutually complementary books by Borwein and Zhu [3] and by Mordukhovich [12, 13] concerning basic theory and numerous applications of variational analysis in both finite-dimensional and infinite-dimensional settings.

This paper addresses general classes of problems in *multiobjective/vector optimization*, which are important from both viewpoints of optimization theory and its various applications, especially to economic modeling, operations research, etc. It is partly based on the author's plenary talk at the Eighth International Conference on Parametric Optimization and Related Topics (Cairo, Egypt, November-December 2005) and widely uses the methods and results developed in [12, 13], being nevertheless basically self-contained.

[1]Research was partially supported by the National Science Foundation under grant DMS-0304989 and by the Australian Research Council under grant DP-0451168.

We pay the main attention to considering two concepts of (vector) optimality in multiobjective problems that are in fact generated by certain geometric notions of *local extremal points* for systems of sets and set-valued mappings, respectively, which play a fundamental role in variational analysis and its applications via the so-called *extremal principles*; see [12, 13]. These concepts of optimal solutions to multiobjective problems happen to extend standard and generalized notions of Pareto-like and other types of multiobjective optimality/efficiency that are conventional in vector optimization theory and applications.

An efficient study of constrained multiobjective optimization problems with respect to the afore-mentioned concepts of vector optimality requires the usage of appropriate *robust* tools of *generalized differentiation* for sets, set-valued mappings, and extended-real-valued functions satisfying comprehensive calculus rules ("full calculus"). Furthermore, variational analysis of these problems in infinite-dimensional spaces unavoidable requires certain "normal compactness" properties of sets and mappings, which allow us to conduct limiting procedures under perturbation/approximation techniques lying at the heart of variational methods. We employ the weakest sequential versions of such properties (called *SNC–sequential normal compactness*), which enjoy a full "SNC calculus," i.e., comprehensive rules ensuring their preservation under various operations.

Using these tools and associated machinery, we establish in this paper first-order necessary optimality conditions in general constrained multiobjective problems and their specifications with *no* using any *scalarization* techniques typical in vector optimization theory and its applications; see, e.g., the recent book by Jahn [6] and the references therein.

The paper is organized as follows. In Section 2 we define the *notions of optimal solutions* to constrained problems of multiobjective optimization studied below. They are deduced from geometric concepts of extremality for systems of sets and set-valued mappings and are compared with conventional notions of efficiency/optimality in vector optimization.

In Section 3 we overview *dual-space* constructions of *generalized differentiation* (normals, coderivatives, and subgradients) used throughout the paper. It is complemented by several versions of *sequential normal compactness* for sets and set-valued mappings in infinite-dimensional spaces. This material provides basic tools of variational analysis used for deriving necessary optimality conditions in the multiobjective problems under consideration.

Section 4 is devoted to the study of constrained multiobjective problems, where the notion of vector optimality for a "cost" mapping $f\colon X \to Y$ between Banach spaces is defined via a certain *generalized order* on Y. Employing the full power of the *exact extremal principle* of variational analysis in the *product* space $X \times Y$ under minimal *partial SNC* requirements, we derive extended necessary conditions for generalized order optimality in constrained multiobjective problems and discuss some of their implementations and specifications. Besides the extremal principle, extended rules of the afore-mentioned generalized differential and SNC calculi, which hold for our basic constructions involved, play a crucial role in the derivation of these results.

In Section 5 we study multiobjective optimization problems, where the notion of vector optimality is defined by a general *closed preference relation*, which must satisfy certain "local satiation" and "almost transitivity" requirements motivated by the possibility to employ a version of the *extremal principle* for systems of *set-valued mappings* (or *moving* *sets*). Involving somewhat different limiting generalized differential constructions and SNC properties in comparison with those from Section 4, we derive generally independent necessary conditions for constrained multiobjective optimization with respect to closed preference relations satisfying the afore-mentioned properties.

Throughout the paper we use the standard notation of variational analysis; see [12, 13]. Unless otherwise stated, all the spaces under consideration are Banach, with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between the space X in question and its topological dual X^*; the weak* topology on X^* is denoted by w^*. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, the *sequential Kuratowski-Painlevé upper/outer limit* of f as $x \to \bar{x}$ is

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\Big\}, \tag{1.1}$$

where $I\!N := \{1,2,\ldots\}$. Recall that $x \xrightarrow{\Omega} \bar{x}$ indicates the convergence of $x \to \bar{x}$ with $x \in \Omega$. Finally, $I\!B_X$ signifies the closed unit ball of X, where the subindex "X" may be dropped if no confusion arises.

2 Extremal Points and Optimal Solutions to Constrained Multiobjective Problems

In this section we introduce two major notions of optimal solutions to constrained problems of multiobjective optimization and discuss their relationships with other solution notions in vector optimization and with geometric concepts of extremality in variational analysis. Let us start with the notion of "generalized order optimality" under arbitrary geometric constraints defined in the vein of [11, 13].

Definition 2.1 (constrained generalized order optimality). *Given a cost mapping $f\colon X \to Y$ between Banach spaces, an ordering set $\Theta \subset Y$ with $0 \in \Theta$, and a constraint set $\Omega \subset X$ we say that a point $\bar{x} \in \Omega$ is* LOCALLY (f,Θ)-OPTIMAL *subject to the abstract/geometric constraints $x \in \Omega$ if there are a neighborhood U of $\bar{x}$ and a sequence of $y_k \in Y$ with $\|y_k\| \to 0$ as $k \to \infty$ such that*

$$f(x) - f(\bar{x}) \notin \Theta - y_k \ \text{ for all } \ x \in \Omega \cap U \ \text{ as } \ k \in I\!N. \tag{2.1}$$

The set Θ in Definition 2.1 can be viewed as a generator of an extended *order/preference relation* between $y_1, y_2 \in Y$ defined by $y_1 - y_2 \in \Theta$. In the scalar case of $Y = I\!R$ and $\Theta = I\!R_-$, the above notion clearly reduces to the standard *minimization* of the cost function f.

Note that we do *not* assume, as in many other abstract notions of vector optimality (see, e.g., Neustadt [16] and Pallaschke and Rolewicz [17]) that the ordering set Θ is either convex or of nonempty interior. If it is a convex subcone of Y with $\operatorname{ri}\Theta \neq \emptyset$, then the concept of Definition 2.1 encompasses a *Pareto-type optimality/efficiency* requiring that

$$\text{there is no } \ x \in \Omega \cap U \ \text{ with } \ f(x) - f(\bar{x}) \in \operatorname{ri}\Theta.$$

To see this, just let $y_k := -y_0/k$, $k \in I\!N$, with some $y_0 \in \mathrm{ri}\,\Theta$. The standard *weak Pareto* efficiency corresponds to the more restrictive relation $f(x) - f(\bar{x}) \in \mathrm{int}\,\Theta$, while the *Pareto* efficiency means that there is no $x \in \Omega \cap U$ for which $f(x) - f(\bar{x}) \in \Theta$ and $f(\bar{x}) - f(x) \notin \Theta$; compare, e.g., the book by Jahn [6] and its references.

The general notion of vector optimality from Definition 2.1 essentially goes back to the early work by Kruger [8] and Mordukhovich [10] motivated the the concept of *set extremality* introduced in their paper [9].

Definition 2.2 (extremal points of sets). *Given nonempty subsets $\Omega_1, \Omega_2 \subset X$ of a norm space, we say that $\bar{x}$ is* LOCAL EXTREMAL POINT *of the set system $\{\Omega_1, \Omega_2\}$ if $\bar{x} \in \Omega_1 \cap \Omega_2$ and if there are a sequence $\{a_k\} \subset X$ with $\|a_k\| \to 0$ as $k \to \infty$ and a neighborhood U of $\bar{x}$ such that*

$$(\Omega_1 - a_k) \cap \Omega_2 \cap U = \emptyset \quad \textit{for all} \quad k \in I\!N. \tag{2.2}$$

Let us now demonstrate that the notion of (f, Θ)-optimality subject to $x \in \Omega$ can be easily reduced to the above extremality of some set system in $X \times Y$ built in what follows. Given (f, Θ, Ω) as in Definition 2.1, define the set

$$\mathcal{E}(f, \Omega, \Theta) := \big\{(x, y) \in X \times Y \big|\ f(x) - y \in \Theta,\ x \in \Omega\big\}, \tag{2.3}$$

which is called the *generalized epigraph* of the mapping f on Ω with respect to Θ. In particular, if $f = (\varphi_1, \ldots, \varphi_m)\colon X \to I\!R^m$ on the whole space $\Omega = X$ and if $\Theta = I\!R^m_-$ is the nonnegative orthant of $Y = I\!R^m$, then set (2.3) is the epigraph of f with respect to the standard order on $I\!R^m$. For $\Theta = \{0\}$, set (2.3) is obviously the usual graph of f.

Let $\bar{x}$ be a local (f, Θ)-optimal solution subject to the set constraint $x \in \Omega$, and let U be the corresponding neighborhood of $\bar{x}$ from Definition 2.1. Suppose for convenience that $f(\bar{x}) = 0$. Consider the system of sets $\{\Omega_1, \Omega_2\}$ in the space $X \times Y$ defined by

$$\Omega_1 := \mathcal{E}(f, \Theta, \Omega) \quad \text{and} \quad \Omega_2 := \mathrm{cl}\, U \times \{0\} \tag{2.4}$$

and observe that $(\bar{x}, 0)$ is an extremal point of this system. Indeed, one obviously has $(\bar{x}, 0) \in \Omega_1 \times \Omega_2$. Furthermore, taking the sequence $\{y_k\} \subset Y$ from Definition 2.1 and letting $a_k := -y_k$, we directly arrive at (2.2) for the set system (2.4) with no neighborhood needed therein: this can be easily checked by contradiction due to the structures of these sets. Note that we use the closure of U in (2.4) for the subsequent application of the extremal principle, where the closedness of the sets in question is essential.

Next let us discuss another general concept of vector optimality, which was introduced in the following form by Mordukhovich, Treiman and Zhu [14] while having various predecessors; see [14] and the book [13] for comments, discussions, and references.

Given a Banach space Y and a subset $R \subset Y \times Y$, we say that y_1 is *preferred to* y_2 (notation $y_1 \prec y_2$) if $(y_1, y_2) \in R$. In what follows, we consider *nonreflexive* preference relations, i.e., such that the preference set R does not contain the diagonal $(y, y) \in Y^2$.

Definition 2.3 (closed preference relations). *Let*

$$\mathcal{L}(y) := \big\{v \in y \big|\ v \prec y\big\}$$

be a LEVEL SET *at $y \in Y$ with respect to the given preference $\prec$. We say that the preference $\prec$ is* LOCALLY SATIATED *around $\bar{y}$ if $y \in \operatorname{cl}\mathcal{L}(y)$ for all y in some neighborhood of $\bar{y}$. Furthermore, the preference $\prec$ is* ALMOST TRANSITIVE *on Y provided that for all $v \prec y$ and $u \in \operatorname{cl}\mathcal{L}(v)$ one has $u \prec y$. The preference relation $\prec$ is called* CLOSED *around $\bar{y}$ if it is locally satiated and almost transitive simultaneously.*

Observe that, while the local satiation property definitely holds for any reasonable preference, the almost transitivity requirement may be violated for some natural preferences significant in applications that are particularly covered by the notion of generalized order optimality from Definition 2.1. To illustrate it, consider the so-called *generalized Pareto* preference, which is an important special case of generalized order optimality, induced by a closed subcone $\Theta \subset Y$ such that $y_1 \prec y_2$ if and only if $y_1 - y_2 \in \Theta$ and $y_1 \neq y_2$. As proved in the dissertation by Eisenhart [4] (see also [13, Proposition 5.56]), the generalized Pareto preference is almost transitive *if and only if* the cone Θ is *convex and pointed*; the latter means that $\Theta \cap (-\Theta) = \{0\}$. A specific example of the generalized Pareto while "not-almost-transitive" behavior is provided by the preference described by the *lexicographical order* on $\mathbb{R}^m$; see [13, Example 5.57].

Note that the principal difference between the preference concepts from Definition 2.3 and Definition 2.1 is that, instead of the *linear translation* of sets in the extremal system induced by generalized order optimality, preference relations of Definition 2.3 involve *nonlinear transformations* of set-valued mappings/moving sets. The latter is closely interrelated with the following notion of local *extremal points* of *set-valued mappings*, which was also introduced in [14]. For simplicity, consider the case of two mappings in the extremal system only needed in this paper.

Definition 2.4 (extremal points of set-valued mappings). *Let $S_i\colon M_i \rightrightarrows X$, $i = 1, 2$, be set-valued mappings from metric spaces (M_i, d_i) into a normed space X. We say that $\bar{x}$ is a* LOCAL EXTREMAL POINT *of the system $\{S_1, S_2\}$ at $(\bar{s}_1, \bar{s}_2)$ provided that $\bar{x} \in S_1(\bar{s}_1) \cap S_2(\bar{s}_2)$ and there exists a neighborhood U of $\bar{x}$ such that for every $\varepsilon > 0$ there are s_i with $S_i(s_i) \neq \emptyset$ satisfying the relationships*

$$d(s_i, \bar{s}_i) \le \varepsilon, \quad \operatorname{dist}\big(\bar{x}; S_i(s_i)\big) \le \varepsilon \quad \textit{as} \quad i = 1, 2, \quad \textit{and}$$

$$S_1(s_1) \cap S_2(s_2) \cap U = \emptyset. \tag{2.5}$$

Let us show that optimal solutions to constrained multiobjective problems defined by closed preference relations reduce to extremal points of set-valued mappings.

Proposition 2.5 (optimal solutions via extremal points in multiobjective optimization with closed preferences). *Let $\bar{x}$ be an optimal solution to the constrained multiobjective problem:*

$$\textit{minimize} \quad f(x) \quad \textit{subject to} \quad x \in \Omega \subset X,$$

where "minimizing" the mapping $f\colon X \to Y$ between Banach spaces is induced by the closed preference relation $\prec$ with the level set $\mathcal{L}(y)$ from Definition 2.3. *Then $(\bar{x}, f(\bar{x}))$ is a local extremal point for the system of set-valued mappings $S_i\colon M_i \rightrightarrows X \times Y$ defined by*

$$S_1(s_1) := \Omega \times \mathrm{cl}\,\mathcal{L}(s_1) \;\; \textit{with} \;\; M_1 := \mathcal{L}\big(f(\bar{x})\big) \cup \big\{f(\bar{x})\big\},$$

$$S_2(s_2) = S_2 := \big\{\big(x, f(x)\big) \big|\, x \in X\big\} \;\; \textit{with} \;\; M_2 := \{0\}$$

at $f(\bar{x}) \in M_1$ and $0 \in M_2$, respectively.

Proof. It is easy to observe that $(\bar{x}, f(\bar{x})) \in S_1(f(\bar{x})) \cap S_2(0)$ due to the *local satiation* property of $\prec$. To establish (2.5), assume the contrary and find, given an arbitrary neighborhood U of $(\bar{x}, f(\bar{x}))$, a point $s_1 \in \mathcal{L}\big(f(\bar{x})\big)$ close to $f(\bar{x})$ while not equal to the latter by the preference *nonreflexivity* such that

$$S_1(s_1) \cap S_2 \cap U \neq \emptyset.$$

This yields the existence of x near $\bar{x}$ satisfying

$$\big(x, f(x)\big) \in S_1(s_1) = \Omega \times \mathrm{cl}\,\mathcal{L}(s_1).$$

Hence $x \in \Omega$ and $f(x) \prec f(\bar{x})$ by the *almost transitivity* property of $\prec$. This contradicts the local optimality of $\bar{x}$ in the multiobjective problem under consideration. $\triangle$

Our primary goal in what follows is to derive efficient *necessary optimality conditions* for general constrained multiobjective problems described above and for their important specification. To proceed, we overview in the next section certain basic tools of generalized differentiation and appropriate sequential normal compactness properties needed to conduct variational analysis in Sections 4 and 5.

3 Tools of Variational Analysis

Let us begin with the basic constructions of *generalized differentiation* used in the sequel. We follow the author's book [12], where the reader can find more details, comments, and references. The main framework of this paper is a major collection of Banach spaces known as the class of *Asplund spaces.* Recall that a Banach space X is Asplund if its any separable subspace has a separable dual. This class includes all spaces admitting an equivalent renorm Fréchet differentiable off the origin, particularly every *reflexive* space. On the other hand, there are Asplund spaces that fail to admit even a Gâteaux differentiable renorm at nonzero points. We refer the reader to the book by Phelps [18] for more information on Asplund spaces and their various applications. Since the generalized differential constructions reviewed below are used in this paper only in the Asplund space framework, we present their simplified definitions and needed properties held in the this setting. The interested reader can consult with the afore-mentioned book [12] for the corresponding constructions and results in more general settings.

We start with generalized normals to (locally) closed *sets* $\Omega \subset X$. Given $\bar{x} \in \Omega$, the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} \widehat{N}(x;\Omega), \tag{3.1}$$

where "Lim sup" stands for the *sequential* upper/outer limit (1.1) of the *Fréchet normal cone* (or the *prenormal cone*) to Ω at $x \in \Omega$ given by

$$\widehat{N}(x;\Omega) := \Big\{x^* \in X^* \Big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \le 0\Big\} \tag{3.2}$$

with $\widehat{N}(x;\Omega) := \emptyset$ for $x \notin \Omega$.

Given a *set-valued mapping* $F\colon X \rightrightarrows Y$ of closed graph

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\},$$

define its *normal coderivative* and *Fréchet coderivative* at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ by, respectively,

$$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\}, \tag{3.3}$$

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in \widehat{N}\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\}. \tag{3.4}$$

If $F = f\colon X \to Y$ is *strictly differentiable* at $\bar{x}$ (in particular, if $f \in C^1$), then

$$D^*f(\bar{x})(y^*) = \widehat{D}^*f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}, \quad y^* \in Y^*,$$

i.e., both coderivatives (3.3) and (3.4) are positively homogeneous extensions of the classical *adjoint* derivative operator to nonsmooth and set-valued mappings.

Finally, consider a real-valued *function* $\varphi\colon X \to I\!R$ *locally Lipschitzian* around $\bar{x}$; in this paper we do not use more general functions. Then the (basic, limiting) *subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} \widehat{\partial}\varphi(x), \tag{3.5}$$

where the sequential limit (1.1) of the *Fréchet subdifferential* mapping $\widehat{\partial}\varphi(\cdot)$ is given by

$$\widehat{\partial}\varphi(x) := \Big\{x^* \in X^* \Big|\ \frac{\varphi(u)-\varphi(x)-\langle x^*, u-x\rangle}{\|u-x\|} \ge 0\Big\}.$$

In this section, we are not going to review appropriate properties of the above generalized differential constructions used in Sections 4 and 5: these properties will be invoked with the exact references to [12] in the corresponding places of the proofs in the subsequent sections. Just mention here that our *basic/limiting* constructions (3.1), (3.3), and (3.5) enjoy comprehensive calculus rules (*full calculus*) in the framework of Asplund spaces under consideration, which are based on the *extremal principle* of variational analysis.

Next we recall "normal compactness" properties of sets and mappings that are automatic in finite dimensions while playing a crucial role in infinite-dimensional variational analysis and its applications; see [12, 13] for more details and references. Since these properties are employed in the paper only in the Asplund space setting, we give simplified definitions equivalent to the general ones [12] for the cases under consideration.

A locally closed set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences of $(x_k, x_k^*) \in X \times X^*$ satisfying

$$x_k \stackrel{\Omega}{\to} \bar{x} \ \text{ and } \ x_k^* \stackrel{w^*}{\to} 0 \ \text{ with } \ x_k^* \in \widehat{N}(x_k; \Omega), \quad k \in I\!N,$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. Besides finite dimensions, this property always holds when Ω is *compactly epi-Lipschitzian* (CEL) around $\bar{x}$ in the sense of Borwein and Strójwas [2], i.e., there are a compact set $C \subset X$, a neighborhood U of $\bar{x}$, a neighborhood O of the origin in X, and a number $\gamma > 0$ such that

$$\Omega \cap U + tO \subset \Omega + tC \ \text{ for all } \ t \in (0, \gamma).$$

In general (CEL)$\Rightarrow$(SNC), where the implication may be *strict* even for convex cones in Asplund spaces; see Fabian and Mordukhovich [5] for a detail study of relationships between SNC and CEL properties of sets in Banach spaces.

In what follows we also need more subtle *partial* modifications of sequential normal compactness, which take into account the *product structure* of the spaces in question. Let us present an Asplund space adaptation of the general properties of this type from [12, Definition 3.3] used in this paper for products of two and three Asplund spaces; note that products of Asplund spaces are also Asplund [18].

Definition 3.1 (partial SNC properties in product spaces). *Let Ω belong to the product $\prod_{j=1}^m X_j$ of Asplund spaces, and let $J \subset \{1, \ldots, m\}$. Assuming the local closedness of Ω around $\bar{x} \in \Omega$, we say that:*

(i) *Ω is* PARTIALLY SEQUENTIALLY NORMALLY COMPACT (PSNC) *at $\bar{x}$ with respect to $\{X_j | \, j \in J\}$ (i.e., with respect to the product $\prod_{j \in J} X_j$) if for any sequences $x_k \stackrel{\Omega}{\to} \bar{x}$ and $x_k^* = (x_{1k}^*, \ldots, x_{mk}^*) \in \widehat{N}(x_k; \Omega)$ one has the implication*

$$\Big[x_{jk}^* \stackrel{w^*}{\to} 0, \ j \in J \quad \& \quad \|x_{jk}^*\| \to 0, \ j \in \{1, \ldots, m\} \setminus J\Big] \Longrightarrow \|x_{jk}^*\| \to 0, \ j \in J.$$

(ii) *Ω is* STRONGLY PSNC *at $\bar{x}$ with respect to $\{X_j | \, j \in J\}$ if for any sequences $x_k \stackrel{\Omega}{\to} \bar{x}$ and $(x_{1k}^*, \ldots, x_{mk}^*) \in \widehat{N}(x_k; \Omega)$ one has*

$$\Big[x_{jk}^* \stackrel{w^*}{\to} 0, \ j = 1, \ldots, m\Big] \Longrightarrow \|x_{jk}^*\| \to 0, \ j \in J.$$

It is worth mentioning the two extreme cases in Definition 3.1:

(a) $J = \emptyset$ when any closed set Ω satisfies both properties in (i) and (ii), and

(b) $J = \{1, \ldots, m\}$ when both properties (i) and (ii) do not depend on the product structure and reduce to the SNC property of sets defined above.

Note that (set-valued and single-valued) *mappings* $F \colon X \rightrightarrows Y$ are naturally associated with the product structure of the *graphs* $\operatorname{gph} F \subset X \times Y$. In this case, the above properties of sets induce the corresponding properties of mappings via their graphs. Observe [12, Proposition 1.68] that the graph of F is PSNC at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with respect to X provided that F satisfies the so-called *Lipschitz-like* (Aubin's "pseudo-Lipschitz" [1]) property around $(\bar{x}, \bar{y})$, which means that there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell\|x - u\|I\!B \quad \text{whenever} \quad x, u \in U \tag{3.6}$$

with some modulus $\ell > 0$. When $V = Y$ in (3.6), this property reduces to the classical (Hausdorff) Lipschitz continuity of F around $\bar{x}$. Furthermore, the Lipschitz-like property of F is known to be equivalent to the *metric regularity* and *linear openness* properties of the inverse F^{-1} around $(\bar{y}, \bar{x})$.

Let us also observe that the graph of $F\colon X \rightrightarrows Y$ is *strongly PSNC* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with respect to Y provided that F is *partially CEL* around $(\bar{x}, \bar{y})$ in the sense of Jourani and Thibault [7]; see [12, Theorem 1.75].

Finally, we emphasize a crucial fact for the theory and applications of the SNC properties under consideration: they enjoy a strong/full *SNC calculus* (in the sense of their preservation under a variety operations upon sets and mappings), which is mainly based on the *extremal principle*; see [12] for more details.

4 Generalized Order Optimality in Multiobjective Problems

In this section we derive verifiable necessary conditions for optimal solutions to constrained multiobjective problems, where the generalized order optimality is understood in the sense of Definition 2.1, and discuss some of their specifications. The mail tools of our variational analysis involves the (exact) *extremal principle* for systems of sets in *product spaces* as well as powerful results of the generalized differential and SNC *calculi* available for the constructions used in the framework of Asplund spaces.

The following version of the exact/pointwise extremal principle for systems of closed sets in Asplund spaces (see particularly [13, Lemma 5.58] and the related material of [12, Chapter 2]) plays a crucial role in the variational analysis conducted in this section.

Lemma 4.1 (extremal principle for set systems in product spaces). *Let $\bar{x} \in \Omega_1 \cap \Omega_2$ be a local extremal point of the sets $\Omega_1, \Omega_2 \subset X_1 \times X_2$ that are supposed to be locally closed around $\bar{x}$, and let*

$$J_1, J_2 \subset \{1, 2\} \quad \textit{with} \quad J_1 \cup J_2 = \{1, 2\}.$$

Assume that both spaces X_1 and X_2 are Asplund, and that Ω_1 is PSNC at $\bar{x}$ with respect to J_1 while Ω_2 is strongly PSNC at $\bar{x}$ with respect to J_2. Then there is $x^* \neq 0$ satisfying

$$x^* \in N(\bar{x}; \Omega_1) \cap \big(-N(\bar{x}; \Omega_2)\big).$$

The next theorem provides major necessary conditions for generalized order optimality of Definition 2.1. Besides the extremal principle of Lemma 4.1, the proof of this theorem uses the full strength of generalized differential and SNC calculi, particularly the basic *intersection rules* for sets; see below.

Theorem 4.2 (necessary conditions for constrained generalized order optimality). *Let $\bar{x}$ be a local (f,Θ)-optimal solution subject to $x \in \Omega$, where $f\colon X \to Y$ is a mapping between Asplund spaces continuous around $\bar{x}$ relative to Ω, and where the sets $\Omega \subset X$ and $\Theta \subset Y$ are locally closed around $\bar{x} \in \Omega$ and $0 \in \Theta$, respectively. Assume furthermore that either Θ is SNC at 0, or the set*

$$\Xi := \big\{(x,y) \in X \times Y \big|\, x \in \Omega,\ y = f(x)\big\} \tag{4.1}$$

is PSNC at $(\bar{x}, f(\bar{x}))$ with respect to Y. Then there is $y^* \in Y^*$ satisfying

$$0 \neq y^* \in N(0;\Theta) \cap \big[\ker D^* f_\Omega(\bar{x})\big], \tag{4.2}$$

where $f_\Omega(x)$ denotes the restriction of f to Ω equal $f(x)$ for $x \in \Omega$ and $\emptyset$ otherwise.

Proof. Let $\bar{x}$ be locally (f,Θ)-optimal subject to $x \in \Omega$ in the sense of Definition 2.1, where we assume for simplicity that $f(\bar{x}) = 0$. Then, as shown in Section 2, the point $(\bar{x}, 0)$ is *locally extremal* for the set system $\{\Omega_1, \Omega_2\}$ defined in (2.4). It is easy to see that the set $\Omega_1 = \mathcal{E}(f,\Theta,\Omega)$ from (2.3) is locally closed around $(\bar{x},0)$ under the continuity and closedness assumptions imposed on f and on Θ and Ω, respectively. Note that these assumptions may be significantly relaxed to ensure the closedness of $\mathcal{E}(f,\Theta,\Omega)$ in more specific situations. In particular, for the standard vector optimization setting of $f = (\varphi_1,\ldots,\varphi_m)\colon X \to I\!R^m$ and $\Theta = I\!R^m_-$, it is sufficient to assume merely the *lower semicontinuity* of φ_i around $\bar{x}$ to guarantee the required closedness of the generalized epigraph (2.3).

We intend to apply the *extremal principle* of Lemma 4.1 to the local extremal point $(\bar{x},0)$ of the closed set system (2.4) in the *product space* $X \times Y$. To proceed, we need to designate index sets $J_1, J_2 \subset \{1,2\}$ with $J_1 \cup J_2 = \{1,2\}$ such that Ω_1 is PSNC at $(\bar{x},0)$ with respect to J_1 while Ω_2 is strongly PSNC at $(\bar{x},0)$ with respect to J_2.

Let us take $J_1 = \{2\}$ and $J_2 = \{1\}$, i.e., $X_1 = Y$ and $X_2 = X$ in the framework of Lemma 4.1. It is easy to see that Ω_2 in (2.4) is *strongly* PSNC at $(\bar{x},0)$ with respect to X, since U is a neighborhood of $\bar{x}$. Observe that this set is SNC at $(\bar{x},0)$ *if and only if* $\dim Y < \infty$, i.e., the product structure in (2.4) is very essential in what follows. Thus it remains to show that the generalized epigraph $\Omega_1 = \mathcal{E}(f,\Omega,\Theta)$ from (2.3) is PSNC at $(\bar{x},0)$ with respect to Y under the assumptions imposed in the theorem. The desired property means that, given arbitrary sequences $(x_k,y_k) \to (\bar{x},0)$ with $x_k \in \Omega$ and $f(x_k) - y_k \in \Theta$ and given generalized normals $(x_k^*, y_k^*) \in \widehat{N}\big((x_k,y_k);\mathcal{E}(f,\Omega,\Theta)\big)$, we have

$$\Big[\|x_k^*\| \to 0, \quad y_k^* \overset{w^*}{\to} 0\Big] \Longrightarrow \|y_k^*\| \to 0. \tag{4.3}$$

Observe from the structure of $\mathcal{E}(f,\Theta,\Omega)$ in (2.3) that the above sequences satisfy

$$(x_k^*, y_k^*, 0) \in \widehat{N}((x_k,y_k,v_k);\Lambda_1 \cap \Lambda_2) \ \text{ for all } \ k \in I\!N, \tag{4.4}$$

where $v_k := f(x_k) - y_k$ and where the sets $\Lambda_1, \Lambda_2 \subset X \times Y \times Y$ are defined by

$$\Lambda_1 := \operatorname{gph} h \ \text{ with } \ h(x,y) := f_\Omega(x) - y \ \text{ and } \ \Lambda_2 := X \times Y \times \Theta. \tag{4.5}$$

We are going to justify, using the full strength of Theorem 3.79 from [12] on the *PSNC property of set intersections* in the three space product $X \times Y \times Y = X_1 \times X_2 \times X_3$ (which is one of the major results of the *SNC calculus* developed in [12] on the base of the *extremal principle*), that the set $\Lambda_1 \cap \Lambda_2$ is PSNC at $(\bar{x}, 0, 0)$ with respect to $Y = X_2$.

Consider first **Case 1** when the set Θ is *SNC* at 0 in the alternative assumptions of the theorem. To ensure by [12, Theorem 3.79] the PSNC property of the set intersection $\Lambda_1 \cap \Lambda_2$ with respect to $Y = X_2$ at the point $(\bar{x}, 0, 0)$, we need to designate some index sets $J_1, J_2 \subset \{1, 2, 3\}$ such that $J_1 \cap J_2 = \{2\}$, $J_1 \cup J_2 = \{1, 2, 3\}$, the set Λ_1 is PSNC at $(\bar{x}, 0, 0)$ with respect to $J_1 = \{2\}$, the set Λ_2 is strongly PSNC property at $(\bar{x}, 0, 0)$ with respect to $J_2 \setminus J_1$, and the "mixed qualification condition" of [12, Definition 3.78] holds for the set system $\{\Lambda_1, \Lambda_2\}$ at $(\bar{x}, 0, 0)$ with respect to $(J_1 \setminus J_2) \cup (J_2 \setminus J_1)$. Letting

$$J_1 := \{2\} \text{ and } J_2 := \{1, 2, 3\}, \tag{4.6}$$

note that the set Λ_2 in (4.5) is SNC at $(\bar{x}, 0, 0)$, i.e., its strong PSNC property at $(\bar{x}, 0, 0)$ with respect to $J_2 \setminus J_1 = \{1, 3\}$ is automatic. Further, the required PSNC property of the other set Λ_1 at the point $(\bar{x}, 0, 0)$ with respect to $J_1 = \{2\}$ means that for any sequences $(x_k, y_k, v_k) \to (\bar{x}, 0, 0)$ and $(x_k^*, y_k^*, v_k^*) \in \widehat{N}\big((x_k, y_k, v_k); \operatorname{gph} h\big)$ one has

$$\Big[\|(x_k^*, v_k^*)\| \to 0, \quad y_k^* \xrightarrow{w^*} 0\Big] \Longrightarrow \|y_k^*\| \to 0 \text{ as } k \to \infty. \tag{4.7}$$

To justify (4.7), observe from the coderivative definition (3.4) that

$$(x_k^*, y_k^*, v_k^*) \in \widehat{N}\big(((x_k, y_k, v_k); \operatorname{gph} h\big) \Longleftrightarrow (x_k^*, y_k^*) \in \widehat{D}^* h(x_k, y_k)(-v_k^*).$$

The latter implies, by taking into account the structure of h in (4.5) and using the coderivative sum rule from [12, Theorem 1.62(i)], that

$$x_k^* \in \widehat{D}^* f_\Omega(x_k)(-v_k^*) \quad \text{and} \quad y_k^* = v_k^* \text{ for all } k \in I\!N. \tag{4.8}$$

This gives $\|y_k^*\| = \|v_k^*\| \to 0$ as $k \to \infty$ and thus (4.7), which justifies the PSNC property of Λ_1 at $(\bar{x}, 0, 0)$ with respect to $J_1 = \{2\}$.

To establish the required PSNC property of the intersection $\Lambda_1 \cap \Lambda_2$ with respect to $Y = X_2$, it remains—by the choice of J_1 and J_2 in (4.6)—to check that the system $\{\Lambda_1, \Lambda_2\}$ satisfies the *mixed qualification condition* at $(\bar{x}, 0, 0)$ with respect to

$$(J_1 \setminus J_2) \cup (J_2 \setminus J_1) = \{1, 3\}.$$

The latter means, by [12, Definition 3.78] and the structures of Λ_1 and Λ_2 in (4.5), that

$$\Big[(x_k, y_k, v_k, u_k) \to (\bar{x}, 0, 0, 0), \ \|x_k^*\| \to 0, \ y_k^* \xrightarrow{w^*} 0, \ (u_k^*, v_k^*) \xrightarrow{w^*} (u^*, -u^*)\Big]$$
$$\Longrightarrow u^* = 0 \text{ whenever } u^* \in N(0; \Theta) \tag{4.9}$$

for any sequences $(x_k^*, y_k^*, v_k^*) \in \widehat{N}\big((x_k, y_k, v_k); \operatorname{gph} h\big)$ and $u_k^* \in \widehat{N}(u_k; \Theta)$ as $k \to \infty$. It follows from (4.8) that (4.9) is satisfied if

$$N(0; \Theta) \cap \big[\ker D^* f_\Omega(\bar{x})\big] = \{0\}. \tag{4.10}$$

Without loss of generality, we may always assume that (4.10) holds. Indeed, otherwise we immediately arrive at the optimality condition (4.2) of the theorem. Therefore, all the assumptions of [12, Theorem 3.79] are satisfied ensuring that $\Lambda_1 \cap \Lambda_2$ is PSNC at $(\bar{x}, 0, 0)$ with respect to $X_2 = Y$. The latter gives (4.3), which allows us to apply the extremal principle from Lemma 4.1 to the set system (2.4) at $(\bar{x}, 0)$.

Thus using Lemma 4.1 and taking into account the structures of the sets Ω_1 and Ω_2 in (2.4), we find $y^* \in Y^*$ satisfying

$$(0, -y^*) \in N\big((\bar{x}, f(\bar{x})); \mathcal{E}(f, \Omega, \Theta)\big), \quad y^* \neq 0. \tag{4.11}$$

This implies the optimality condition (4.2) by [13, Lemma 5.23(ii)] and hence completes the proof of the theorem in Case 1 under consideration.

It remains to consider **Case 2**, where the set (4.1) is assumed to be *PSNC* at $(\bar{x}, f(\bar{x}))$ *with respect to* Y. Let us show that in this case the intersection $\Lambda_1 \cap \Lambda_2$ of the sets Λ_i in (4.5) is also PSNC at $(\bar{x}, 0, 0)$ with respect to $Y = X_2$ in the product of the three Asplund spaces $X \times Y \times Y = X_1 \times X_2 \times X_3$. To accomplish this, we apply again the PSNC intersection rule of [12, Theorem 3.79] with a different arrangement of the index sets J_1 and J_2 therein in comparison with (4.6) in Case 1. Namely, let

$$J_1 := \{2, 3\} \quad \text{and} \quad J_2 := \{1, 2\}. \tag{4.12}$$

Then $J_2 \setminus J_1 = \{1\}$, and the set Λ_2 is obviously *strongly* PSNC at $(\bar{x}, 0, 0)$ with respect to J_2. We now check that Λ_1 is PSNC at $(\bar{x}, 0, 0)$ with respect to J_1 under the PSNC assumption on set (4.1) imposed in the theorem. Indeed, the required PSNC property of Λ_1 means that for any sequences $(x_k, y_k, v_k) \to (\bar{x}, 0, 0)$ and $(x_k^*, y_k^*, v_k^*) \in \widehat{N}\big((x_k, y_k, v_k); \text{gph}\, h\big)$ one has

$$\Big[\|x_k^*\| \to 0, \quad (y_k^*, v_k^*) \xrightarrow{w^*} (0, 0)\Big] \Longrightarrow \|(y_k^*, v_k^*)\| \to 0 \text{ as } k \to \infty. \tag{4.13}$$

By arguments similar to those in Case 1, we get that (4.13) is equivalent to

$$\Big[x_k^* \in \widehat{D}^* f_\Omega(x_k)(y_k^*), \; x_k \to \bar{x}, \; \|x_k^*\| \to 0, \; y_k^* \xrightarrow{w^*} 0\Big] \Longrightarrow \|y_k^*\| \to 0$$

as $k \to \infty$, which is obviously equivalent to the assumed PSNC property of set (4.1).

To be able applying [12, Theorem 3.79] to the set intersection $\Lambda_1 \cap \Lambda_2$ with the index sets $\{J_1, J_2\}$ chosen in (4.12), we have also to check that the mixed classification condition from [12, Definition 3.78] holds for $\{\Lambda_1, \Lambda_2\}$ at $(\bar{x}, 0, 0)$ with respect to $(J_1 \setminus J_2) \cup (J_2 \setminus J_1) = \{1, 3\}$, which happens to be exactly the same as in Case 1. Thus we conclude that $\Lambda_1 \cap \Lambda_2$ is PSNC at $(\bar{x}, 0, 0)$ with respect to $X_2 = Y$. The latter allows us to apply the extremal principle from Lemma 4.1 to the set system (2.4) and thus to get (4.11), which yields (4.2) and completes the proof of the theorem. △

Note that the SNC/PSNC assumptions imposed on the sets Θ and (4.1) in Theorem 4.2 are *automatic* when the image space Y is *finite-dimensional*. Furthermore, in this case the optimality condition (4.2) can be equivalently represented in the *subdifferential form* as the existence of y^* satisfying

$$0 \in \partial\langle y^*, f_\Omega\rangle(\bar{x}) \quad y^* \in N(0; \Theta) \setminus \{0\} \tag{4.14}$$

provided that f is *locally Lipschitzian* around $\bar{x}$ relative to the constraint set Ω. This is due to the "scalarization" relationships between basic subgradients and coderivatives of single-valued Lipschitzian mappings that hold also for mappings with into infinite-dimensional spaces under additional assumptions; see [12] for more details and references.

Observe that the restriction f_Ω of f to Ω can be represented as the sum

$$f_\Omega(x) = f(x) + \Delta(x;\Omega), \quad x \in \Omega,$$

via the *indicator mapping* $\Delta(x;\Omega)$ of Ω (relative to the image space Y) equal $0 \in Y$ for $x \in \Omega$ and $\emptyset$ otherwise. Using this observation and employing well-developed coderivative and subdifferential *sum rules* in (4.2) and (4.14), we easily arrive at (generally more restrictive) necessary conditions for generalized order optimality expressed *separately* via the basic coderivative of f (or the subdifferential of its scalarization) and the normal cone to Ω.

Theorem 4.2 establishes necessary conditions for generalized order optimality in a broad class of multiobjective problems with arbitrary geometric constraints. Since these conditions are expressed in terms of generalized differential constructions and sequential normal compactness properties that enjoy *full calculi* [12], they can be implemented and applied to more specific problems with various constraints of operator, functional, equilibrium, and other types. We refer the reader to [13, Chapter 5] for a number of results in this direction (mainly for single-objective problems), which can be further developed and extended to multiobjective optimization based on Theorem 4.2 and comprehensive calculus rules.

As an example of the implementation and specification of the general results of Theorem 4.2, we present necessary optimality conditions in the following *minimax problem* under arbitrary geometric constraints:

$$\text{minimize } \varphi(x) := \sup\big\{\langle v^*, f(x)\rangle \big|\, v^* \in V^*\big\} \text{ subject to } x \in \Omega \subset X, \tag{4.15}$$

where V^* is a nonempty subset of the dual space Y^* with cl^* signifying the weak* topological closure. Note that optimal solutions to (4.15) are understood in the standard single-objective sense, while they can be naturally treated from the viewpoint of *multiobjective optimization*; see the proof below. Observe that such a reduction allows us, in particular, to *avoid* conventional assumptions on the *compactness* of the set $V^* \subset Y^*$ in (4.15), where the supremum may *not* be thus realized.

Theorem 4.3 (necessary conditions for constrained minimax problems). *Let $\bar{x}$ be a local optimal solution to the constrained minimax problem* (4.15) *with $|\varphi(\bar{x})| < \infty$, where $f\colon X \to Y$ is a mapping between Asplund spaces continuous around $\bar{x}$ relative to Ω. Suppose that Ω is locally closed around $\bar{x}$ and that V^* is an nonempty subset of Y^* for which there is $y_0 \in Y$ satisfying $\langle v^*, y_0\rangle = 1$ whenever $v^* \in V^*$. Assume also that either the cone*

$$\Lambda := \big\{y \in Y \big|\, \langle v^*, y\rangle \le 0 \ \textit{ whenever } \ v^* \in V^*\big\} \tag{4.16}$$

is SNC at the point $f(\bar{x}) - \varphi(\bar{x})y_0$, or the graphical set $\Xi \subset X \times Y$ from (4.1) *is PSNC at $(\bar{x}, f(\bar{x}))$ with respect to Y. Then there is $y^* \in Y^*$ satisfying the inclusions*

$$0 \in D^*_N f_\Omega(\bar{x})(y^*) \ \textit{ with } \ y^* \neq 0, \tag{4.17}$$

$$y^* \in \operatorname{cl}^*\Big\{\sum_{i=1}^{n}\alpha_i v_i^*\Big|\ \alpha_i \ge 0,\ v_i^* \in V^*,\ n \in I\!N\Big\} \tag{4.18}$$

and the complementary slackness condition

$$\langle y^*, f(\bar{x}) - \varphi(\bar{x})y_0\rangle = 0. \tag{4.19}$$

Proof. Given Λ in (4.16) and y_0 from the assumptions of the theorem, define the set

$$\Theta := \big(\varphi(\bar{x})y_0 - f(\bar{x})\big) + \Lambda, \tag{4.20}$$

which is always closed and convex in Y, regardless of V^* in (4.16). Let us show that $\bar{x}$ is locally (f,Θ)-optimal subject to $x \in \Omega$, in the sense of Definition 2.1, with the given mapping f in (4.15) and the ordering set Θ from (4.20). Indeed,

$$0 \in \Theta \quad \text{due to} \quad f(\bar{x}) - \varphi(\bar{x})y_0 \in \Lambda.$$

Construct further the sequence of $y_k := y_0/k$, $k \in I\!N$, and check that condition (2.1) holds along this sequence. Assuming the contrary, we find $x \in U$ from a neighborhood U of $\bar{x}$ such that $x \in \Omega$ and one has

$$\langle v^*, f(x)\rangle - \varphi(\bar{x}) = \langle v^*, f(x)\rangle - \varphi(\bar{x})\langle v^*, y_0\rangle \le -\frac{1}{k}\langle v^*, y_0\rangle < 0$$

whenever $v^* \in V^*$ as $k \to \infty$, which contradicts the local optimality of $\bar{x}$ in the minimax problem (4.15). Employing Theorem 4.2 in this setting and taking into account the *convexity* of the ordering set Θ in (4.20), we find $y^* \in Y^*$ satisfying (4.17) and such that

$$\big\langle y^*, y - \big(f(\bar{x}) - \varphi(\bar{x})y_0\big)\big\rangle \le 0 \ \text{ for all } \ y \in \Lambda; \tag{4.21}$$

the latter is equivalent to the optimality condition $y^* \in N(0;\Theta)$ in (4.2) due to the structure of Θ in (4.20) and by the fact that the basic normal cone (3.1) to convex sets reduces to the normal cone of convex analysis. It remains to show that (4.21) implies both conditions (4.18) and (4.19) in the theorem.

To proceed, we observe that (4.21) and the *conic structure* of Λ ensure the inequality

$$\big\langle y^*, \alpha y - \big(f(\bar{x}) - \varphi(\bar{x})y_0\big)\big\rangle \le 0 \ \text{ whenever } \ \alpha > 0 \ \text{ and } \ y \in \Lambda.$$

It implies, by passing to the limit as $\alpha \to \infty$, that $\langle y^*, y\rangle \le 0$ for all $y \in \Lambda$. This gives

$$y^* \in \operatorname{cl}^*\operatorname{co}\big[\operatorname{cone} V^*\big]$$

by the duality with Λ in (4.16), which is equivalent to (4.18). Moreover, we have

$$\langle y^*, f(\bar{x}) - \varphi(\bar{x})y_0\rangle \le 0,$$

since $f(\bar{x}) - \varphi(\bar{x})y_0 \in \Lambda$. The opposite inequality follows from (4.21) with $y = 0$. Thus we arrive at (4.19), which completes the proof of the theorem. △

5 Multiobjective Problems Defined by Closed Preferences

The vector optimization problem of our study in this section is as follows. Given a *closed* preference relation $\prec$ from Definition 2.3, a cost mapping $f\colon X \to Y$, and a constraint set $\Omega \subset X$, we consider the multiobjective problem:

$$\text{minimize } f(x) \text{ with respect to } \prec \text{ subject to } x \in \Omega, \tag{5.1}$$

where "minimization" of f is understood in the sense of the given preference $y_1 \prec y_2$ on Y. As seen in Proposition 2.5, local optimal solutions to problem (5.1) can be reduced to local extremal points of systems of *moving* sets, i.e., set-valued mappings. The main tool of studying such extremal points is provided by the *extremal principle for moving sets* derived in [14]; see also [13, Subsection 5.3.3]. To formulate this result, recall the notion of the *extended normal cone* to $S\colon Z \rightrightarrows X$ at $(\bar{z},\bar{x}) \in \operatorname{gph} S$ defined by

$$N_+\big(\bar{x}; S(\bar{z})\big) := \operatorname*{Lim\,sup}_{(z,x)\overset{\operatorname{gph} S}{\to}(\bar{z},\bar{x})} \widehat{N}\big(x; S(z)\big); \tag{5.2}$$

we refer the reader to [13, 14] for various properties of this construction; in particular, for conditions ensuring that $N_+\big(\bar{x}; S(\bar{z})\big) = N\big(\bar{x}; S(\bar{z})\big)$.

We also need recalling an appropriate modification of the SNC property for moving sets involving their *images* but not graphs as in the basic SNC property for set-valued mappings and its modifications presented in Section 3. Given $S\colon Z \rightrightarrows X$, we say that it is *imagely SNC* (abbr. ISNC) at $(\bar{z},\bar{x}) \in \operatorname{gph} S$ if

$$\Big[x_k^* \in \widehat{N}\big(x_k; S(z_k)\big),\quad (z_k,x_k) \overset{\operatorname{gph} S}{\to} (\bar{z},\bar{x}),\quad x_k^* \overset{w^*}{\to} 0\Big] \Longrightarrow \|x_k^*\| \to 0 \text{ as } k \to \infty.$$

Similarly to the case of fixed sets, there are strong relationships between the above ISNC property and the corresponding counterparts of the CEL property for moving sets. In particular, a mapping $S\colon Z \rightrightarrows X$ is ISNC at $(\bar{z},\bar{x})$ if there are numbers $\alpha,\eta > 0$ and a compact set $C \subset X$ such that

$$\widehat{N}\big(x; S(z)\big) \subset \Big\{x^* \in X^* \Big|\ \eta\|x^*\| \le \max_{c\in C} |\langle x^*, c\rangle|\Big\}$$

whenever $(z,x) \in \operatorname{gph} S \cap \big((\bar{z},\bar{x}) + \eta I\!B_{Z\times X}\big)$. The latter surely holds if S is *uniformly CEL* around $(\bar{x},\bar{z})$ in the sense that there are a compact set $C \subset Z$, neighborhoods $V \times U$ of $(\bar{x},\bar{z})$ and O of the origin in Z, and a number $\gamma > 0$ such that

$$S(x) \cap U + tO \subset S(x) + \gamma C \text{ for all } x \in U \text{ and } t \in (0,\gamma).$$

It is important to emphasize that the extended normal cone (5.2) and the ISNC property of moving sets, as well their mapping/function counterparts and partial analogs, enjoy *full calculi* similar to those for our basic constructions and SNC properties considered above; see [15] for various results and discussions in this direction.

Now we are ready to formulate the afore-mentioned exact/pointwise extremal principle for two set-valued mappings/moving sets; cf. [13, Theorem 5.72].

Lemma 5.1 (extremal principle for moving sets). *Let $S_i\colon M \rightrightarrows X$, $i = 1,2$, be set-valued mappings from metric spaces M_i to an Asplund space X. Assume that $\bar{x}$ is a local extremal point of the system $\{S_1, S_2\}$ at $(\bar{s}_1, \bar{s}_2)$ with $\bar{x} \in S_i(\bar{s}_i)$ as $i = 1,2$, where each mapping S_i is closed-valued around $\bar{s}_i$ and one of them are ISNC at the corresponding point $(\bar{s}_i, \bar{x})$. Then there is $x^* \in X^*$ satisfying*

$$0 \neq x^* \in N_+\big(\bar{x}; S_1(\bar{s}_1)\big) \cap \big[-N_+\big(\bar{x}; S_2(\bar{s}_2)\big)\big].$$

The next result, based on the extremal principle for moving sets from Lemma 5.1, employs the notion of "strict" Lipschitzian behavior that goes back to Thibault [20] who introduced an equivalent "compactly Lipschitzian" property of single-valued mappings; see [12] for more details and comments. Recall that $f\colon X \to Y$ is *strictly Lipschitzian* at $\bar{x}$ if it is locally Lipschitzian around this point and if there is a neighborhood V of the origin in X such that the sequence

$$\Big\{\frac{f(x_k + t_k v) - f(x_k)}{t_k}\Big\}, \quad k \in I\!N,$$

contains a norm convergent subsequence whenever $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$ as $k \to \infty$.

Obviously, this property reduces to the standard local Lipschitz continuity if Y is finite-dimensional, while in general it is more restrictive. We refer the reader to [12, 13] (especially to Subsection 3.1.3 of [12]) for a comprehensive study and applications of this class of mappings with values in infinite-dimensional spaces.

Theorem 5.2 (necessary conditions for constrained problems of multiobjective optimization defined by closed preferences). *Let $\bar{x}$ be a local optimal solution to problem* (5.1) *with a closed preference relation $\prec$ on Y, where $f\colon X \to Y$ is a mapping between Asplund spaces continuous around $\bar{x}$ and where the set $\Omega \subset X$ is locally closed around this point. Assume that:*

(a) *either the mapping $\mathrm{cl}\,\mathcal{L}\colon Y \rightrightarrows Y$ generated by the level set of the preference $\prec$ is ISNC at $\big(f(\bar{x}), f(\bar{x})\big)$ and the set Ω is SNC at $\bar{x}$,*

(b) *or the mapping f is SNC at $\bar{x}$.*

Then there is a pair $0 \neq (x^*, y^*) \in X^* \times Y^*$ satisfying

$$x^* \in D^* f(\bar{x})(y^*) \cap \big(-N(\bar{x}; \Omega)\big) \quad \text{and} \quad y^* \in N_+\big(f(\bar{x}); \mathrm{cl}\,\mathcal{L}(f(\bar{x}))\big). \tag{5.3}$$

If in addition f is strictly Lipschitzian at $\bar{x}$, then (5.3) *is equivalent to*

$$0 \in \partial\langle y^*, f\rangle(\bar{x}) + N(\bar{x}; \Omega) \quad \text{with} \quad y^* \in N_+\big(f(\bar{x}); \mathrm{cl}\,\mathcal{L}(f(\bar{x}))\big) \setminus \{0\}. \tag{5.4}$$

Proof. As proved in Proposition 2.5 above, the point $\big(\bar{x}, f(\bar{x})\big)$ happens to be *local extremal* at $\big(f(\bar{x}), 0\big)$ for the system of set-valued mappings $S_i\colon M_i \rightrightarrows X \times Y$, $i = 1,2$, defined by

$$\begin{cases} S_1(s_1) := \Omega \times \mathrm{cl}\,\mathcal{L}(s_1) \ \text{ with } \ M_1 := \mathcal{L}\big(f(\bar{x})\big) \cup \big\{f(\bar{x})\big\}, \\[2ex] S_2(s_2) = S_2 := \big\{\big(x, f(x)\big) \big|\, x \in X\big\} \ \text{ with } \ M_2 := \{0\} \end{cases} \tag{5.5}$$

under the general assumptions imposed in the theorem. Observe that the product space $X \times Y$ is Asplund, since both spaces X and Y are.

It easily follows from the above definition of the ISNC property and the structures of S_i in (5.5) that one of the mappings $\{S_1, S_2\}$ enjoys this property if either one of the assumptions (a) and (b) of the theorem holds. Furthermore, we check that

$$N_+\big((\bar{x}, f(\bar{x})); S_1(f(\bar{x}))\big) = N(\bar{x}; \Omega) \times N_+\big(f(\bar{x}); \mathrm{cl}\, \mathcal{L}(f(\bar{x}))\big) \quad \text{and}$$

$$N_+\big((\bar{x}, f(\bar{x})); S_2\big) = \big\{(x^*, y^*) \in X^* \times Y^* \big|\, x^* \in D^*f(\bar{x})(-y^*)\big\}.$$

Thus applying the extremal principle of Lemma 5.1 to the system $\{S_1, S_2\}$ in (5.5), we arrive at the optimality conditions (5.3).

To get (5.4) from (5.3), it is sufficient to observe that

$$D^*f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \quad \text{whenever} \quad y^* \in Y^*$$

provided that f is strictly Lipschitzian at $\bar{x}$; see [12, Theorem 3.28]. $\triangle$

Applying comprehensive generalized differential and SNC calculus rules for the constructions involved, we can derived from (5.3) and (5.4) various consequences of the results obtained for more specific types of constraints and preferences. Some results in this direction are presented in [13, Chapter 5]; see also [11, 14, 21] for previous developments in finite-dimensional settings.

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