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DISCRETE APPROXIMATIONS AND NECESSARY OPTIMALITY CONDITIONS FOR FUNCTIONAL-DIFFERENTIAL INCLUSIONS OF NEUTRAL TYPE

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Abstract— **This paper deals with necessary optimality conditions for optimal control systems governed by constrained functional-differential inclusions of neutral type. While some results are available for smooth control systems governed by neutral functional-differential equations, we are not familiar with any results for neutral functional-differential inclusions, even with smooth cost functionals in the absence of endpoint constraints. Developing the method of discrete approximations and employing advanced tools of generalized differentiation, we conduct a variational analysis of neutral functional-differential inclusions and obtain new necessary optimality conditions of both Euler-Lagrange and Hamiltonian types.**

I. INTRODUCTION

This paper concerns the study of optimal control problems for the so-called neutral functional-differential inclusions, which contain time-delays in both state and velocity variables. Such inclusions belong to the broad class of hereditary systems known also as systems with memory or aftereffect. They have been investigated in the form of controlled functional-differential equations being important for various practical applications, particularly to problems of automatic control, economic dynamics, modeling of ecological, biological, and chemical processes, etc.; see examples and discussions in [2], [3], [8], [11], [12], [15] and their references.

In this paper we consider the following dynamic optimization (generalized optimal control) problem (P), which is to minimize

$$J[x] := \varphi(x(a), x(b)) + \int_a^b f(x(t), x(t-\Delta), t)\, dt \quad (1)$$

over feasible arcs $x : [a-\Delta, b] \to I\!R^n$, which are continuous on $[a-\Delta, a)$ and $[a, b]$ (with a possible jump at $t = a$) and such that the combination $x(t) - Ax(t-\Delta)$ is absolutely continuous on $[a, b]$, satisfying the neutral functional-differential inclusion

$$\begin{cases} \frac{d}{dt}[x(t) - Ax(t-\Delta)] \\ \qquad \in F(x(t), x(t-\Delta), t) \quad \text{a.e. } t \in [a, b], \\ x(t) = c(t), \quad t \in [a-\Delta, a), \end{cases} \quad (2)$$

with the endpoint constraints

$$(x(a), x(b)) \in \Omega \subset I\!R^{2n}. \quad (3)$$

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We always assume that $F : I\!R^n \times I\!R^n \times [a, b] \rightrightarrows I\!R^n$ is a set-valued mapping of closed graph, Ω is a closed set, $\Delta > 0$ is a constant delay, and A is a constant $n \times n$ matrix. Note that the neutral-type operator in the left-hand side of (2) is given in the Hale form [8]

For nondelayed systems governed by differential inclusions ($\Delta = 0,\ A = 0$) necessary optimality conditions have been studied intensively during recent years; see [4], [9], [17], [25], [26], [27], [29] and the references therein. Some results are known for delay-differential (or differential-difference) inclusions corresponding to $A = 0$ in (2); see [5], [6], [13], [19], [20].

Observe that neutral-type systems are essentially different from their counterparts with $A = 0$. In particular, it is well known that an analog of the Pontryagin maximum principle does not generally hold for neutral systems, even in the classical smooth framework with no convexity assumptions. In a sense, neutral-type systems combine properties of continuous-time and discrete-time control systems; indeed, they can be treated as discrete-time systems regarding velocity variables. On the other hand, neutral systems have some similarities with the so-called hybrid and algebraic-differential equations important in engineering control applications.

In this paper we derive necessary optimality conditions for the neutral-type control problem (P) under natural assumptions on its initial data involving nonsmooth functions and nonconvex sets. These conditions are obtained in extended Euler-Lagrange and Hamiltonian forms involving advanced generalized differential constructions of variational analysis.

Our approach is based on the method of discrete approximations, in the line developed in [15], [17] for nondelayed differential inclusions and in [19], [20] for delay-differential systems with $A = 0$. This method, which is certainly of independent interest from both qualitative and numerical viewpoints, allows us to construct a well-posed parametric family of optimal control problems for approximating systems governed by discrete-time analogs of neutral functional-differential inclusions. A crucial issue is to establish stability of such approximations that ensures an appropriate strong convergence of optimal solutions. Convergence analysis of this method and its application to necessary optimality conditions for neutral systems are essentially more involved in comparison with the cases of differential and delay-differential inclusions.

The approximating discrete-time control problems can be reduced to special problems of nonsmooth programming with an increasing number of geometric constraints that may have empty interiors. To handle such problems, we use suitable generalized differential tools of variational analysis satisfying a comprehensive calculus that allows us to derive general necessary optimality conditions for finite-difference analogs of neutral functional-differential inclusions. Then passing to the limit from well-posed discrete approximations with the strong convergence of optimal solutions and employing generalized differential calculus, we obtain necessary optimality conditions for (P).

The rest of the paper is organized as follows. In Section II we show that some combination built upon a given admissible trajectory of the neutral inclusion (2) can be strongly approximated by the corresponding combination built upon admissible trajectories of discrete-time systems. The convergence analysis is conducted in Section III for a sequence of well-posed discrete approximations to (P) involving an appropriate perturbation of the endpoint constraints (3) that is consistent with the step of discretization. The required strong convergence of optimal solutions is established under an intrinsic property of (P) called relaxation stability. Section IV contains the basic constructions and required material on generalized differentiation needed for performing a variational analysis of discrete-time and continuous-time optimal control problems in the subsequent sections. These constructions and calculus rules are used in Section V for deriving general necessary optimality conditions for nonconvex discrete-time inclusions arising in discrete approximations of (P). The main results on the extended Euler-Lagrange and Hamiltonian conditions for neutral functional-differential inclusions are derived in Section VI via passing to the limit from discrete approximations.

Our notation is basically standard. The transposed matrix of A is denoted by A^*. $I\!B$ is always the closed unit ball of $I\!R^n$. $\text{haus}(\Omega_1,\Omega_2)$ is denoted the Hausdorff distance between two compact sets Ω_1 and Ω_2 in $I\!R^n$. Given a multifunction $F\colon X \rightrightarrows Y$ between finite-dimensional spaces, the Painlevé-Kuratowski upper/outer limit of $F(x)$ as $x \to \bar{x}$ is defined by

$$\operatorname{Lim\,sup}_{x\to\bar{x}} F(x) := \big\{y \in Y \mid \exists\, x_k \to \bar{x},\ \exists\, y_k \to y \text{ with } y_k \in F(x_k) \quad \text{for all} \quad k \in I\!N\big\},$$

where $I\!N$ stands for the collection of all natural numbers. We refer the reader to [17] and [24] for additional material. The full version of this paper appears in [21].

II. Discrete Approximations of Neutral Inclusions

This section concerns the study of discrete approximations of an arbitrary admissible trajectory to the neutral functional-differential inclusion (2). Let $\bar{x}(t)$ be an admissible trajectory in (P), i.e., it is continuous on $[a-\Delta, a)$ and $[a,b]$ (with a possible jump at $t=a$), the combination $x(t)-Ax(t-\Delta)$ is absolutely continuous on $[a,b]$, and (2) is satisfied. Note that the endpoint constraints (3) may not hold for $\bar{x}(t)$; if they do hold, $\bar{x}(t)$ is feasible to (P). The following standing assumptions are imposed throughout the paper:

(H1) There are an open set $U \subset I\!R^n$ and two positive numbers ℓ_F and m_F such that $\bar{x}(t) \in U$ for all $t \in [a-\Delta, b]$, the sets $F(x,y,t)$ are closed, and

$$F(x,y,t) \subset m_F I\!B,$$
$$F(x_1,y_1,t) \subset F(x_2,y_2,t) + \ell_F(|x_1-x_2| + |y_1-y_2|)I\!B$$

for all $(x,y),(x_1,y_1),(x_2,y_2) \in U\times U$ and $t\in[a,b]$.

(H2) $F(x,y,t)$ is Hausdorff continuous for a.e. $t\in[a,b]$ uniformly in $(x,y)\in U\times U$.

(H3) The function $c(t)$ is continuous on $[a-\Delta, a]$.

Following [7], we consider the so-called averaged modulus of continuity for the multifunction $F(x,y,t)$ with $(x,y)\in U\times U$ and $t\in[a,b]$ that is defined by

$$\tau(F;h) := \int_a^b \sigma(F;t,h)\,dt,$$

where $\sigma(F;t,h) := \sup\big\{\vartheta(F;x,y,t,h) \big| \, (x,y)\in U\times U\big\}$ with

$$\vartheta(F;x,y,t,h) := \sup\{\text{haus}(F(x,y,t_1),F(x,y,t_2)) : (t_1,t_2)\in[t-h/2,t+h/2]\cap[a,b]\}.$$

It is proved in [7] that $\tau(F;h)\to 0$ as $h\to 0$ under the assumption (H2).

To construct a sequence of discrete approximations of the given neutral-differential inclusion, we replace the derivative in (2) by the Euler finite difference

$$d[x(t)-Ax(t-\Delta)]/dt \approx [x(t+h)-Ax(t+h-\Delta)-x(t)+Ax(t-\Delta)]/h.$$

For any $N\in I\!N$ we consider the step of discretization $h_N := \Delta/N$ and define the discrete partition $t_j := a+jh_N$ as $j=-N,\ldots,k$ and $t_{k+1}:=b$, where k is a natural number determined from $a+kh_N \le b < a+(k+1)h_N$. Then the corresponding neutral functional-difference inclusions associated with (2) are given by

$$\begin{cases} x_N(t_{j+1}) - Ax_N(t_{j+1}-\Delta) \in x_N(t_j) \\ -Ax_N(t_j-\Delta) + h_N F(x_N(t_j), x_N(t_j-\Delta), t_j) \\ \qquad\qquad \text{for} \quad j=0,\ldots,k; \\ x_N(t_j) = c(t_j) \quad \text{for } j=-N,\ldots,-1. \end{cases} \tag{4}$$

A collection of vectors $\{x_N(t_j)|\ j=-N,\ldots,k+1\}$ satisfying (4) is a discrete trajectory and the corresponding collection $\{[x_N(t_{j+1})-Ax_N(t_{j+1}-\Delta)-x_N(t_j)+Ax_N(t_j-\Delta)]/h_N : j=0,\ldots,k\}$ is a combined discrete velocity for (4). We consider extensions $x_N(t)$ of discrete trajectories to the continuous-time interval $[a-\Delta,b]$ defined piecewise-linearly on $[a,b]$ and piecewise-constantly, continuously from the right on $[a-\Delta,a)$. We also define piecewise-constant extensions of combined discrete velocities on $[a,b]$ by $v_N(t) := [x_N(t_{j+1})-Ax_N(t_{j+1}-\Delta)-x_N(t_j)+Ax_N(t_j-\Delta)]/h_N$, $\quad t\in[t_j,t_{j+1})$, $j=0,\ldots,k$.

Let $W^{1,2}[a,b]$ be a standard Sobolev space of absolutely continuous functions $x\colon [a,b] \to I\!R^n$ with the norm

$$\|x(\cdot)\|_{W^{1,2}} := \max_{t\in[a,b]} |x(t)| + \Big(\int_a^b |\dot{x}(t)|^2 dt\Big)^{1/2}.$$

The following theorem establishes a strong approximation of any admissible trajectory for the given neutral functional-differential inclusion by corresponding solutions to discrete approximations (4).

Theorem 1: Let $\bar{x}(t)$ be an admissible trajectory for (2) under hypotheses (H1)–(H3). Then there is a sequence $\{z_N(t_j) \mid j = -N,\dots,k+1\}$, of solutions to (4) such that $z_N(t_0) = \bar{x}(a)$ for all $N \in I\!N$, the extended discrete trajectories $z_N(t)$, $a - \Delta \le t \le b$, converge uniformly to $\bar{x}(t)$ on $[a-\Delta, b]$, and their extended combinations $z_N(t) - Az_N(t-\Delta)$ converge to $\bar{x}(t) - A\bar{x}(t-\Delta)$ in the $W^{1,2}$-norm on $[a,b]$ as $N \to \infty$. In particular, some subsequence of $\{d[z_N(t) - Az_N(t-\Delta)]/dt\}$ converges pointwisely to $d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt$ for a.e. $t \in [a,b]$.

Proof. Following [17], we first find a sequence $\{\omega_N(t)\}$ such that $\omega_N(t)$ is constant in the interval $[t_j, t_{j+1})$ and $\omega_N(t) \to d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt$ strongly in $L^1[a,b]$. Using this sequence, we construct the desired discrete trajectories $\{z_N\}$ via the proximal algorithm. Finally we show that the extended discrete trajectories $z_N(t)$, $a - \Delta \le t \le b$, have all the properties listed in the theorem.

III. Strong Convergence of Discrete Optimal Solutions

In this section we construct a sequence of well-posed discrete approximations of the problem (P) such that optimal solutions to discrete approximation problems strongly converge, in the sense described below, to a given optimal solution $\bar{x}(t)$ to (P).

Given $\bar{x}(t)$, $a - \Delta \le t \le b$, take its approximation $z_N(t)$ from Theorem 1 and denote $\eta_N := |z_N(t_{k+1}) - \bar{x}(b)|$. For any natural number N we consider the following discrete-time dynamic optimization problem (P_N), which is to minimize

$$\begin{aligned} J_N[x_N] &= \varphi(x_N(t_0), x_N(t_{k+1})) + |x_N(t_0) - \bar{x}(a)|^2 \\ &+ h_N \sum_{j=0}^{k} f(x_N(t_j), x_N(t_j - \Delta), t_j) \\ &+ \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \Big| \frac{d}{dt}[\bar{x}(t) - A\bar{x}(t-\Delta)] - [x_N(t_{j+1}) \\ &- Ax_N(t_{j+1-N}) - x_N(t_j) + Ax_N(t_{j-N})]/h_N \Big|^2 dt \end{aligned}$$

subject to the dynamic constraints governed by neutral functional-difference inclusions (4), the endpoint constraints

$$(x_N(t_0), x_N(t_{k+1})) \in \Omega_N := \Omega + \eta_N I\!B, \tag{5}$$

which are η_N-perturbations of the original endpoint constraints (3), and the auxiliary constraints

$$|x_N(t_j) - \bar{x}(t_j)| \le \varepsilon, \quad j = 1,\dots,k+1, \tag{6}$$

with some $\varepsilon > 0$. The latter auxiliary constraints are needed to guarantee the existence of optimal solutions in (P_N) and can be ignored in the derivation of necessary optimality conditions; see below.

In what follows we select $\varepsilon > 0$ in (6) such that $\bar{x}(t) + \varepsilon I\!B \subset U$ for all $t \in [a-\Delta, b]$ and take sufficiently large N ensuring that $\eta_N < \varepsilon$. Note that problems (P_N) have feasible solutions, since the trajectories z_N from Theorem 1 satisfy all the constraints (4), (5), and (6). Therefore, by the classical Weierstrass theorem in finite dimensions, each (P_N) admits an optimal solution $\bar{x}_N(t)$ under the following assumption imposed in addition to (H1)–(H3).

(H4) φ is continuous on $U \times U$, $f(x,y,t)$ is continuous for a.e. $t \in [a,b]$ uniformly in $(x,y) \in U \times U$ and continuous on $U \times U$ uniformly in $t \in [a,b]$, and Ω is locally closed around $(\bar{x}(a), \bar{x}(b))$.

To prove the strong convergence of optimal solutions to (P_N), we need to involve an important intrinsic property of (P) called the relaxation stability. Following the line originated by Jack Warga in optimal control theory (see [28] and its references), we consider the relaxed problem (R) of minimizing the cost functional (1) on admissible trajectories of the convexified neutral functional-differential inclusion

$$\begin{cases} \frac{d}{dt}[x(t) - Ax(t-\Delta)] \\ \quad \in \mathrm{co}\, F(x(t), x(t-\Delta), t), \ \text{a.e. } t \in [a,b], \\ x(t) = c(t), \ t \in [a-\Delta, a) \end{cases} \tag{7}$$

with the endpoint constraints (3). Any admissible trajectory for (7) is called a relaxed trajectory for (2).

Definition 2: Problem (P) is said to be stable with respect to relaxation if

$$\inf\,(P) = \inf\,(R).$$

This property, which obviously holds under the convexity assumption on the sets $F(x,y,t)$, goes far beyond the convexity. General sufficient conditions for the relaxation stability of the neutral-type problem (P) follows from [11]. We also refer the reader to [1], [17], [19], [28] for more detailed discussions on the validity of the relaxation stability.

Now we are ready to establish the following strong convergence theorem for optimal solutions to discrete approximations, which makes a bridge between optimal control problems governed by neutral functional-differential and functional-difference inclusions.

Theorem 3: Let $\bar{x}(t)$ be an optimal solution to problem (P), which is assumed to be stable with respect to relaxation. Suppose also that hypotheses (H1)–(H4) hold. Then any sequence $\{\bar{x}_N(t)\}$, $N \in I\!N$, of optimal solutions to (P_N) extended to to the continuous interval $[a-\Delta, b]$ converges uniformly to $\bar{x}(t)$ on $[a-\Delta, b]$, and the sequence of their combinations $\bar{x}_N(t) - A\bar{x}_N(t-\Delta)$ converges *to $\bar{x}(t) - A\bar{x}(t-\Delta)$ in the $W^{1,2}$-norm on $[a,b]$ as $N \to \infty$.*

Proof. Since the trajectories z_N built in Theorem 1 are feasible solutions to (P_N), one has $J_N[\bar{x}_N] \le J_N[z_N]$. Noting that $J_N[z_N] \to J[\bar{x}]$ as $N \to \infty$, we conclude that

$$\limsup_{N\to\infty} J_N[\bar{x}_N] \le J[\bar{x}].$$

Then we show, following the line in [17] and employing the relaxation stability of (P), that

$$\left|\bar{x}_N(a) - \bar{x}(a)\right|^2 + \int_a^b \Big|\frac{d}{dt}[\bar{x}_N(t) - A\bar{x}_N(t-\Delta)] - \frac{d}{dt}[\bar{x}(t) - A\bar{x}(t-\Delta)]\Big|^2 dt \to 0$$

as $N \to \infty$, which completes the proof of the theorem.

IV. Tools of Generalized Differentiation

Observe that problems (P_N) are essentially nonsmooth, even in the case of smooth functions φ and f in the cost functional and the absence of endpoint constraints. The main source of nonsmoothness comes from the increasing number of geometric constraints in (4), which reflect the discrete dynamics and may have empty interiors. To conduct a variational analysis of such problems, we use appropriate tools of generalized differentiation introduced in [14] and then developed and applied in many publications; see, in particular, the books [15], [24] for detailed treatments and further references.

Recall the the basic/limiting normal cone to the set $\Omega \subset I\!R^n$ at the point $\bar{x} \in \Omega$ is

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{x\to\bar{x},\, x\in\Omega} \widehat{N}(x;\Omega), \tag{8}$$

and where

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^* \in I\!R^n \Big|\ \limsup_{x\to\bar{x},\, x\in\Omega} \frac{\langle x^*, x-\bar{x}\rangle}{|x-\bar{x}|} \le 0\Big\} \tag{9}$$

is the cone of Fréchet (or regular) normals to Ω at $\bar{x}$. For convex sets Ω both cones $N(\bar{x};\Omega)$ and $\widehat{N}(\bar{x};\Omega)$ reduce to the normal cone of convex analysis. Note that the basic normal cone (8) is often nonconvex while satisfying a comprehensive calculus, in contrast to (9).

Given an extended-real-valued function $\varphi\colon I\!R^n \to \overline{I\!R} := [-\infty,\infty]$ finite at $\bar{x}$, the subdifferential of φ at $\bar{x}$ is defined geometrically

$$\partial\varphi(\bar{x}) := \big\{x^* \in I\!R^n \big|\ (x^*,-1) \in N((\bar{x},\varphi(\bar{x}));\mathrm{epi}\,\varphi)\big\} \tag{10}$$

via basic normals to the epigraph $\mathrm{epi}\,\varphi := \{(x,\mu) \in I\!R^{n+1} |\ \mu \ge \varphi(x)\}$.

Given a set-valued mapping $F\colon I\!R^n \rightrightarrows I\!R^m$ with the graph $\mathrm{gph}\,F := \{(x,y) \in I\!R^n \times I\!R^m |\ y \in F(x)\}$, the coderivative $D^*F(\bar{x},\bar{y})\colon I\!R^m \rightrightarrows I\!R^n$ of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \{x^* \in I\!R^n \big|\ (x^*,-y^*) \in N((\bar{x},\bar{y});\mathrm{gph}\,F)\}. \tag{11}$$

The following two results are crucial in what follows. The first one gives a complete coderivative characterization of the classical local Lipschitzian property of multifunctions imposed in our standing assumption (H1); cf. [16, Theorem 5.11] and [24, Theorem 9.40].

Proposition 4: Let $F\colon I\!R^n \rightrightarrows I\!R^m$ be a closed-graph multifunction locally bounded around $\bar{x}$. Then the following conditions are equivalent:
(i) *F is locally Lipschitzian around $\bar{x}$.*
(ii) *There exist a neighborhood U of $\bar{x}$ and a number $\ell > 0$ such that*

$$\sup\{|x^*| : x^* \in D^*F(x,y)(y^*)\} \le \ell|y^*|,\ \forall\, x \in U,\ y \in F(x),\ y^* \in I\!R^m.$$

The next result taken from [15, Corollary 7.5] provides necessary optimality conditions for a general problem (MP) of nonsmooth mathematical programming with many geometric constraints:

$$\begin{cases} \text{minimize } \phi_0(z) \quad \text{subject to} \\ \phi_j(z) \le 0, \quad j = 1,\ldots,r, \\ g_j(z) = 0, \quad j = 0,\ldots,m, \\ z \in \Lambda_j, \quad j = 0,\ldots,l, \end{cases}$$

where $\phi_j\colon I\!R^d \to I\!R$, $g_j\colon I\!R^d \to I\!R^n$, and $\Lambda_j \subset I\!R^d$.

Proposition 5: Let $\bar{z}$ be an optimal solution to (MP). Assume that all ϕ_i are Lipschitz continuous, that g_j are continuously differentiable, and that Λ_j are locally closed near $\bar{z}$. Then there exist real numbers $\{\mu_j |\ j = 0,\ldots,r\}$ as well as vectors $\{\psi_j \in I\!R^n |\ j = 0,\ldots,m\}$ and $\{z_j^* \in I\!R^d |\ j = 0,\ldots,l\}$, n t all zero, such that

$$\mu_j \ge 0 \quad for \quad j = 0,\ldots,r, \tag{12}$$

$$\mu_j\phi_j(\bar{z}) = 0 \quad for \quad j = 1,\ldots,r, \tag{13}$$

$$z_j^* \in N(\bar{z};\Lambda_j) \quad for \quad j = 0,\ldots,l, \tag{14}$$

$$-\sum_{j=0}^{l} z_j^* \in \partial\Big(\sum_{j=0}^{r}\mu_j\phi_j\Big)(\bar{z}) + \sum_{j=0}^{m}\nabla g_j(\bar{z})^*\psi_j. \tag{15}$$

For applications in this paper we need the following modifications of the basic constructions (8), (10), and (11) for sets, functions, and set-valued mappings depending on a parameter t from a topological space T (in our case $T = [a,b]$).

Given $\Omega\colon T \rightrightarrows I\!R^n$ and $\bar{x} \in \Omega(\bar{t})$, we define the extended normal cone to $\Omega(\bar{t})$ at $\bar{x}$ by

$$\widetilde{N}(\bar{x};\Omega(\bar{t})) := \mathop{\mathrm{Limsup}}_{(t,x)\overset{\mathrm{gph}\Omega}{\to}(\bar{t},\bar{x})} \widehat{N}(x;\Omega(t)). \tag{16}$$

For $\varphi\colon I\!R^n \times T \to \overline{I\!R}$ finite at $(\bar{x},\bar{t})$ and for $F\colon I\!R^n \times T \rightrightarrows I\!R^m$ with $\bar{y} \in F(\bar{x},\bar{t})$, the extended subdifferential of φ at $(\bar{x},\bar{t})$ and the extended coderivative of F at $(\bar{x},\bar{y},\bar{t})$

with respect to x are given, respectively, by

$$\widetilde{\partial}_x\varphi(\bar{x},\bar{t}) := \{x^* \in \mathbb{R}^n \,|\, (x^*,-1) \in \widetilde{N}((\bar{x},\varphi(\bar{x},\bar{t})); \operatorname{epi}\varphi(\cdot,\bar{t}))\} \tag{17}$$

and, whenever $y^* \in \mathbb{R}^m$, by

$$\widetilde{D}_x^*F(\bar{x},\bar{y},\bar{t})(y^*) := \{x^* \in \mathbb{R}^n \,|\, (x^*,-y^*) \in \widetilde{N}((\bar{x},\bar{y}); \operatorname{gph} F(\cdot,\bar{t}))\} \tag{18}$$

Note that the sets (16)–(18) may be bigger in some situations than the corresponding sets $N(\bar{x};\Omega(\bar{t}))$, $\partial_x\varphi(\bar{x},\bar{t})$, and $D_x^*F(\bar{x},\bar{y},\bar{t})(y^*)$, where the latter two sets stand for the subdifferential (10) of $\varphi(\cdot,\bar{t})$ at $\bar{x}$ and the coderivative (11) of $F(\cdot,\bar{t})$ at $(\bar{x},\bar{y},\bar{t})$, respectively. Efficient conditions ensuring equalities for these sets are discussed in [17], [18], [20].

It is not hard to check that the extended constructions (16)–(18) are robust with respect to their variables, which is important for performing limiting procedures in what follows. In particular,

$$\widetilde{N}(\bar{x};\Omega(\bar{t})) = \operatorname*{Limsup}_{(t,x)\overset{\operatorname{gph}\Omega}{\to}(\bar{t},\bar{x})} \widetilde{N}(x;\Omega(t)).$$

V. Necessary Optimality Conditions for Discrete Approximations

This section concerns necessary optimality conditions for discrete approximation problems (P_N). We derive such conditions in the extended Euler-Lagrange form by reducing (P_N) to nonsmooth mathematical programs and employing generalized differential calculus for the basic constructions (8), (10), and (11).

Let us reduce the dynamic optimization problem (P_N) for each $N \in \mathbb{N}$ to the mathematical programming problem (MP) considered in Section IV with the decision vector

$$z := (x_0^N, x_1^N, \ldots, x_{k+1}^N, v_0^N, v_1^N, \ldots, v_k^N) \in \mathbb{R}^{n(2k+3)}$$

and the following data:

$$\phi_0(z) := \varphi(x_0^N, x_{k+1}^N) + |x_0^N - \bar{x}(a)|^2 + h_N \sum_{j=0}^{k} f(x_j^N, x_{j-N}^N, t_j) + \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} |v_j^N - d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt|^2\, dt,$$

$$\phi_j(z) := |x_j^N - \bar{x}(t_j)| - \varepsilon, \quad j = 1,\ldots,k+1,$$
$$\Lambda_j := \{(x_0^N,\ldots,v_k^N) \,|\, v_j^N \in F(x_j^N, x_{j-N}^N, t_j)\}, \quad j = 0,\ldots,k,$$
$$\Lambda_{k+1} := \{(x_0^N,\ldots,v_k^N) \,|\, (x_0^N, x_{k+1}^N) \in \Omega_N\},$$
$$g_j(z) := x_{j+1}^N - Ax_{j+1-N}^N - x_j^N + Ax_{j-N}^N - h_N v_j^N, \quad j = 0,\ldots,k.$$

where $x_j^N := c(t_j)$ for $j < 0$.

Let $\bar{z}^N = (\bar{x}_0^N,\ldots,\bar{x}_{k+1}^N, \bar{v}_0^N,\ldots,\bar{v}_k^N)$ be an optimal solution to (MP). Applying Proposition 5, we find real numbers μ_j^N and vectors $z_j^* \in \mathbb{R}^{n(2k+3)}$ for $j = 0,\ldots,k+1$ as well as vectors $\psi_j^N \in \mathbb{R}^n$ for $j = 0,\ldots,k$, not all zero, such that conditions (12)–(15) are satisfied.

Taking $z_j^* = (x_{0,j}^*,\ldots,x_{k+1,j}^*, v_{0,j}^*,\ldots,v_{k,j}^*) \in N(\bar{z}^N;\Lambda_j)$ for $j = 0,\ldots,k$, we observe that all but one components of z_j^* are zero and the remaining one satisfies

$$(x_{j,j}^*, x_{j-N,j}^*, v_{j,j}^*) \in N((\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N); \operatorname{gph} F(\cdot,\cdot,t_j))$$

for $j = 0,\ldots,k$. Similarly, the condition $z_{k+1}^* \in N(\bar{z}^N;\Lambda_{k+1})$ is equivalent to

$$(x_{0,k+1}^*, x_{k+1,k+1}^*) \in N((\bar{x}_0^N, \bar{x}_{k+1}^N); \Omega_N)$$

with all the other components of z_{k+1}^* equal to zero. Employing Theorem 2 on the convergence of discrete approximations, we have $\phi_j(\bar{z}^N) < 0$ for $j = 1,\ldots,k+1$ whenever N is sufficiently large. Thus $\mu_j^N = 0$ for these indexes due to the complementary slackness conditions (13). Let $\lambda^N := \mu_0^N \geq 0$. Therefore, Proposition 5 is equivalent to the following relationships:

$$-x_{0,0}^* - x_{0,N}^* - x_{0,k+1}^* = \lambda^N u_0^N + \lambda^N h_N \vartheta_0^N + \lambda^N h_N \kappa_0^N + 2\lambda^N(\bar{x}_0^N - \bar{x}(a)) - \psi_0^N - A^*(\psi_{N-1}^N - \psi_N^N),$$

$$-x_{j,j}^* - x_{j,j+N}^* = \lambda^N h_N \kappa_j^N + \lambda^N h_N \vartheta_j^N + \psi_{j-1}^N - \psi_j^N - A^*(\psi_{j+N-1}^N - \psi_{j+N}^N), \quad j = 1,\ldots,k-N,$$

$$-x_{k-N+1,k-N+1}^* = \lambda^N h_N v_{k-N+1}^N + \psi_{k-N}^N - \psi_{k-N+1}^N + A^*\psi_k^N,$$

$$-x_{j,j}^* = \lambda^N h_N \vartheta_j^N + \psi_{j-1}^N - \psi_j^N, \quad j = k-N+2,\ldots,k,$$

$$-x_{k+1,k+1}^* = \lambda^N u_{k+1}^N + \psi_k^N,$$

$$-v_{j,j}^* = \lambda^N \theta_j^N - h_N \psi_j^N, \quad j = 0,\ldots k$$

with the notation

$$(u_0^N, u_{k+1}^N) \in \partial\varphi(\bar{x}_0^N, \bar{x}_{k+1}^N),$$
$$(\vartheta_j^N, \kappa_{j-N}^N) \in \partial f(\bar{x}_j^N, \bar{x}_{j-N}^N, t_j),$$
$$\theta_j^N := -2\int_{t_j}^{t_{j+1}} \left(d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt - \bar{v}_j^N\right) dt.$$

Based on the above relationships, we arrive at the following necessary optimality conditions for discrete-time problems (P_N), where $f_j(\cdot,\cdot) := f(\cdot,\cdot,t_j)$ and $F_j(\cdot,\cdot) := F(\cdot,\cdot,t_j)$.

Theorem 6: Let $\bar{z}^N$ be an optimal solution to problem (P_N). Assume that the sets Ω and $\operatorname{gph} F_j$ are closed and that the functions φ and f_j are Lipschitz continuous around the points $(\bar{x}_0^N, \bar{x}_{k+1}^N)$ and $(\bar{x}_j^N, \bar{x}_{j-N}^N)$, respectively, for all $j = 0,\ldots,k$. Then there exist $\lambda^N \geq 0$, p_j^N $(j = 0,\ldots,k+N+1)$, and q_j^N $(j = -N,\ldots,k+1)$, not all zero, such that

$$p_j^N = 0, \quad j = k+2,\ldots,k+N+1, \tag{19}$$

$$q_j^N = 0, \quad j = k - N + 1, \dots, k + 1, \tag{20}$$

$$\begin{aligned}(p_0^N + q_0^N, -p_{k+1}^N) &\in \lambda^N \partial\varphi(\bar{x}_0^N, \bar{x}_{k+1}^N)\\ &+N((\bar{x}_0^N, \bar{x}_{k+1}^N); \Omega_N),\end{aligned} \tag{21}$$

$$\begin{aligned}&([P_{j+1}^N - P_j^N]/h_N, [Q_{j-N+1}^N - Q_{j-N}^N]/h_N,\\ &\quad -\lambda^N\theta_j^N/h_N + p_{j+1}^N + q_{j+1}^N)\\ &\quad \in \lambda^N\big(\partial f_j(\bar{x}_j^N, \bar{x}_{j-N}^N), 0\big)\\ &+N((\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N); \operatorname{gph} F_j), \quad j = 0, \dots, k,\end{aligned} \tag{22}$$

with the notation

$$P_j^N := p_j^N - A^* p_{j+N}^N, \quad Q_j^N := q_j^N - A^* q_{j+N}^N,$$

$$\bar{v}_j^N := [(\bar{x}_{j+1}^N - \bar{x}_j^N) + A(x_{j-N}^N - x_{j-N+1}^N)]/h_N.$$

Proof. Most of the proof has been actually done above, and we just need to change notation in the relationships formulated right before the theorem. Let first

$$\widetilde{p}_j^N := \begin{cases}\psi_{j-1}^N & \text{for } j = 1, \dots, k+1,\\ 0 & \text{for } j = k+2, \dots, k+N+1.\end{cases}$$

$$\widetilde{q}_j^N := \begin{cases}\lambda^N\kappa_j^N + x_{j,j+N}^*/h_N, & \text{for } j = 0, \dots, k-N,\\ 0, & \text{for } j = k-N+1, \dots, k+1,\end{cases}$$

and then define q_j^N for $j = -N, \dots, k+1$ by the recurrent formula

$$q_j^N := q_{j+1}^N - A^*(q_{j+N+1}^N + q_{j+N}^N) - h_N\widetilde{q}_j^N,$$

and let

$$\begin{aligned}p_0^N &:= \lambda^N u_0^N + x_{0,k+1}^* - q_0^N,\\ p_j^N &:= \widetilde{p}_j^N - q_j^N \quad \text{for } j = 1, \dots, k+N+1,\end{aligned}$$

It is easy to check that all the relationships (19)–(22) hold.

Corollary 7: In addition to the assumptions of Theorem 3, suppose that the mapping F_j is bounded and Lipschitz continuous around $(\bar{x}_j^N, \bar{x}_{j-N}^N)$ for each $j = 0, \dots, k$. Then conditions $\lambda^N \geq 0$ and (19)–(22) hold with $(\lambda^N, p_{k+1}^N) \neq 0$, i.e., one can let

$$(\lambda^N)^2 + |p_{k+1}^N|^2 = 1.$$

Proof. If $\lambda^N = 0$, then (22) together with (19) and (20) imply that

$$\Big(\frac{p_{k+1}^N - p_k^N}{h_N}, \frac{-q_{k-N}^N}{h_N}\Big) \in D^*F_k(\bar{x}_k^N, \bar{x}_{k-N}^N, \bar{v}_k^N)(-p_{k+1}^N).$$

Assuming now that $p_{k+1}^N = 0$, we get

$$\Big(\frac{-p_k^N}{h_N}, \frac{-q_{k-N}^N}{h_N}\Big) \in D^*F_k(\bar{x}_k^N, \bar{x}_{k-N}^N, \bar{v}_k^N)(0),$$

which implies $p_k^N = q_{k-N}^N = 0$. Repeating the above procedure, we arrive at contradiction with the nontriviality assertion in Theorem 6.

VI. Optimality Conditions for Functional-Differential Inclusions

In this section we obtain the main results of the paper providing necessary optimality conditions for the original dynamic optimization problem (P) in both extended Euler-Lagrange and Hamiltonian forms involving generalized differential constructions of Section IV. Our major theorem establishes the following conditions of the Euler-Lagrange type derived by the limiting procedure from discrete approximations.

Theorem 8: Let $\bar{x}(t)$ be an optimal solution to problem (P) under hypotheses (H1)–(H4), where φ and $f(\cdot,\cdot,t)$ are assumed to be Lipschitz continuous instead of the plain continuity. Suppose also that (P) is stable with respect to relaxation. Then there exist a number $\lambda \geq 0$ and continuous functions $p\colon [a, b+\Delta] \to I\!R^n$ and $q\colon [a-\Delta, b] \to I\!R^n$ such that $p(t) - A^*p(t+\Delta)$ and $q(t-\Delta) - A^*q(t)$ are absolutely continuous on $[a,b]$ and the following conditions hold:

$$\lambda + |p(b)| = 1, \tag{23}$$

$$p(t) = 0 \ \textit{for} \ t \in (b, b+\Delta], \tag{24}$$

$$q(t) = 0 \ \textit{for} \ t \in (b-\Delta, b], \tag{25}$$

$$\begin{aligned}(p(a) + q(a), -p(b)) &\in \lambda\partial\varphi(\bar{x}(a), \bar{x}(b))\\ &+N((\bar{x}(a), \bar{x}(b)); \Omega),\end{aligned} \tag{26}$$

$$\begin{aligned}&(d[p(t) - A^*p(t+\Delta)]/dt,\\ &d[q(t-\Delta) - A^*q(t)]/dt) \in \operatorname{co}\{(u, w,\\ &p(t) + q(t)) \in \lambda(\widetilde{\partial} f(\bar{x}(t), \bar{x}(t-\Delta), t), 0) +\\ &\widetilde{N}\big((\bar{x}(t), \bar{x}(t-\Delta), d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt);\\ &\operatorname{gph} F(t)\big)\} \quad \textit{a.e.} \ t \in [a,b].\end{aligned} \tag{27}$$

Proof. In what follows we use the notation $\bar{x}^N(t)$, $p^N(t)$, $q^N(t-\Delta)$, $P^N(t)$, and $Q^N(t-\Delta)$ for piecewise linear extensions of the corresponding discrete functions on $[a,b]$; $\dot{P}^N(t)$ and $\dot{Q}^N(t-\Delta)$ are the piecewise constant derivatives. First we estimate $(P^N(t), Q^N(t-\Delta))$ when N is sufficient large. Using (19) and (20), we derive from (22) that

$$\begin{aligned}&\Big(\frac{p_{j+1}^N - p_j^N}{h_N} - \lambda^N\vartheta_j^N, \frac{q_{j-N+1}^N - q_{j-N}^N}{h_N} - \lambda^N\kappa_{j-N}^N,\\ &-\lambda^N\theta_j^N/h_N + p_{j+1}^N\Big) \in N((\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N); \operatorname{gph} F_j)\end{aligned}$$

with some $(\vartheta_j^N, \kappa_{j-N}^N) \in \partial f_j(\bar{x}_j^N, \bar{x}_{j-N}^N)$.

For $j = k - N + 2, \dots, k + N + 1$, by definition of the coderivative (11) and Proposition 4, we have

$$\begin{aligned}&\Big|\Big(\frac{p_{j+1}^N - p_j^N}{h_N} - \lambda^N\vartheta_j^N, \frac{q_{j-N+1}^N - q_{j-N}^N}{h_N} - \lambda^N\kappa_{j-N}^N\Big)\Big|\\ &\qquad \leq \ell_F|\lambda^N\theta_j^N/h_N - p_{j+1}^N|.\end{aligned}$$

Since $|(\vartheta_j^N, \kappa_{j-N}^N)| \le \ell_f$ due to the Lipschitz continuity of f with modulus ℓ_f, we derive from the above that

$$\begin{aligned}|(p_j^N, q_{j-N}^N)| &\le \ell_F|\theta_j^N| + (\ell_F + 1)h_N\ell_f \\ &+(\ell_F h_N + 1)|(p_{j+1}^N, q_{j-N+1}^N)| \\ &\le \exp[\ell_F(b-a)](1 + \ell_f(\ell_F+1)/\ell_F + \ell_F\nu_N),\end{aligned}$$

which implies the uniform boundedness of $\{(p_j^N, q_{j-N}^N)\colon j = k-N+2,\dots,k+N+1\}$ and hence of $(p^N(t), q^N(t-\Delta))$ on $[b-\Delta, b]$.

Next we consider $j = k-2N+2,\dots,k+1$ and derive from (22) that

$$\begin{aligned}&\Big|\Big(\frac{p_{j+1}^N - p_j^N}{h_N} - \lambda^N\vartheta_j^N, \frac{q_{j-N+1}^N - q_{j-N}^N}{h_N} - \lambda^N\kappa_{j-N}^N\Big)\Big| \\ &\le \ell_F|\lambda^N\theta_j^N/h_N - p_{j+1}^N - q_{j+1}^N| \\ &+\Big|\Big(\frac{A^*p_{j+N+1}^N - A^*p_{j+N}^N}{h_N}, \frac{A^*q_{j+1}^N - A^*q_j^N}{h_N}\Big)\Big|.\end{aligned}$$

This implies due to Proposition 4 and the uniform boundedness of p_{j+N}^N and q_j^N by some constant $\alpha > 0$ for such j that

$$\begin{aligned}&\Big|\Big(\frac{p_{j+1}^N - p_j^N}{h_N} - \lambda^N\vartheta_j^N, \frac{q_{j-N+1}^N - q_{j-N}^N}{h_N} - \lambda^N\kappa_{j-N}^N\Big)\Big| \\ &\le \ell_F|\lambda^N\theta_j^N/h_N - p_{j+1}^N - q_{j+1}^N| + \alpha/h_N.\end{aligned}$$

Therefore

$$\begin{aligned}|(p_j^N, q_{j-N}^N)| &\le \ell_F|\theta_j^N| + (\ell_F + 1)h_N\ell_f \\ &+(\ell_F h_N + 1)|(p_{j+1}^N, q_{j-N+1}^N)| + (\ell_F h_N + 1)\alpha \\ &\le \exp[\ell_F(b-a)](1 + (\ell_f + \alpha)(\ell_F+1)/\ell_F + \ell_F\nu_N)\end{aligned}$$

for $j = k-2N+2,\dots,k+1$. This shows that p_j^N and q_{j-N}^N are uniformly bounded for $j = k-2N+2,\dots,k+1$, and hence the sequence$\{p^N(t), q^N(t-\Delta)\}$ is uniformly bounded on $[b-2\Delta, b-\Delta]$. Repeating the above procedure, we conclude that both sequences $\{p^N(t), q^N(t-\Delta)\}$ and $\{P^N(t), Q^N(t-\Delta)\}$ are uniformly bounded on the whole interval $[a,b]$.

Next we estimate $(\dot P^N(t), \dot Q^N(t-\Delta))$ on $[a,b]$. (22) and Proposition 4 yield, for $t_j \le t < t_{j+1}$ with $j = 0,\dots,k$, that

$$\begin{aligned}&|(\dot P^N(t), \dot Q^N(t-\Delta))| \\ &= \Big|\Big(\frac{P_{j+1}^N - P_j^N}{h_N}, \frac{Q_{j-N+1}^N - Q_{j-N}^N}{h_N}\Big)\Big| \\ &\le \ell_F|\theta_j^N| + \ell_F|p_{j+1}^N| + \ell_F|q_{j+1}^N| + \ell_f.\end{aligned}$$

Thus the sequence $\{\dot P^N(t), \dot Q^N(t-\Delta)\}$ is weakly compact in $L^1[a,b]$. We find two absolutely continuous functions $P(t)$ and $Q(t-\Delta)$ on $[a,b]$ such that $\dot P^N(t) \to \dot P(t)$,$\dot Q^N(t-\Delta) \to \dot Q(t-\Delta)$ weakly in $L^1[a,b]$ and $P^N(t) \to P(t)$, $Q^N(t-\Delta) \to Q(t-\Delta)$ uniformly on $[a,b]$ as $N \to \infty$. Since $p^N(t)$ and $q^N(t-\Delta)$ are uniformly bounded on $[a, b+\Delta]$, they surely converge to some functions $p(t)$ and $q(t-\Delta)$ weakly in $L^1[a, b+\Delta]$. Thus, $p(t)$, $q(t)$ satisfy (24) and

$$\begin{aligned}P(t) &= p(t) - A^*p(t+\Delta), \\ Q(t-\Delta) &= q(t-\Delta) - A^*q(t).\end{aligned}$$

Note that $p(t)$ and $q(t)$ are continuous on $[a, b+\Delta]$ and $[a-\Delta, b]$, respectively. Conditions (23) and (26) follow by passing to the limit from (7) and (21), respectively, taking into account the robustness of the basic subdifferential (10) and the normal cone (8).

To justify the Euler-Lagrange inclusion (27), we rewrite the discrete Euler-Lagrange inclusion (22) in the form

$$\begin{aligned}&(\dot P^N(t), \dot Q^N(t-\Delta)) \\ &\in \big\{(u,w)\big|\ \big(u, w, p^N(t_{j+1}) + q^N(t_{j+1}) - \lambda^N\theta_j^N/h_N\big) \\ &\quad\in \lambda^N\big(\partial f(\bar x(t_j), \bar x(t_j - \Delta), t_j), 0\big) \\ &\quad+ \big(N(\bar x_j^N, \bar x_{j-N}^N, \bar v_j^N); \operatorname{gph} F_j)\big\}\end{aligned}$$

for $t \in [t_j, t_{j+1}]$ with $j = 0,\dots,k$. By the classical Mazur theorem there is a sequence of convex combinations of the functions $(\dot P^N(t), \dot Q^N(t-\Delta))$ that converges to $(\dot P(t), \dot Q(t-\Delta))$ for a.e. $t \in [a,b]$. Passing the limit in above inclusion, we arrive at (27) and complete the proof of the theorem.

Observe that for the Mayer problem (P_M), which is (1)–(3) with $f = 0$, the generalized Euler-Lagrange inclusions (27) is equivalently expressed in terms of the extended coderivative (18) with respect to the first two variables of $F = F(x,y,t)$, i.e., in the form

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t) - A^*p(t+\Delta)], \frac{d}{dt}[q(t-\Delta) - A^*q(t)]\Big) \\ &\in \operatorname{co} \widetilde D_{x,y}^* F(\bar x(t), \bar x(t-\Delta), \\ &d[\bar x(t) - A\bar x(t-\Delta)]/dt, t)(-p(t) - q(t))\end{aligned} \tag{28}$$

for a.e. $t \in [a,b]$.

It turns out that the extended Euler-Lagrange inclusion obtained above implies, under the relaxation stability of the original problems, two other principal optimality conditions expressed in terms of the Hamiltonian function built upon the mapping F in (2). The first condition called the extended Hamiltonian inclusion is given below in terms of a partial convexification of the basic subdifferential (10) for the Hamiltonian function. The second one is an analog of the classical Pontryagin maximum principle for neutral functional-differential inclusions. Recall that an analog of the maximum principle does not generally hold even in the case of optimal control problems governed by smooth functional-differential equations of neutral type.

The following relationships between the extended Euler-Lagrange and Hamiltonian inclusions are based on Rockafellar's dualization theorem [23] that concerns subgradients of abstract Lagrangian and Hamiltonian associated with set-valued mappings regardless the dynamics in (2). For simplicity we consider the case of the Mayer problem (P_M) for autonomous functional-differential inclusions of neutral type. Then the Hamiltonian function for F in (2) is defined by

$$H(x,y,p) := \sup\left\{\langle p,v\rangle \middle|\ v \in F(x,y)\right\}.$$

Corollary 9: Let $\bar{x}(\cdot)$ be an optimal solution to the Mayer problem (P_M) for the autonomous neutral functional-differential inclusion (2) under the assumptions of Theorem 4. Then there exist a number $\lambda \geq 0$ and continuous functions $p\colon [a,b+\Delta] \to I\!R^n$ and $q\colon [a-\Delta,b] \to I\!R^n$ such that $p(t) - A^*p(t+\Delta)$ and $q(t-\Delta) - A^*q(t)$ are absolutely continuous on $[a,b]$ and, besides (23)–(27), we have the following:
the extended Hamiltonian inclusion

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t) - A^*p(t+\Delta)],\ \frac{d}{dt}[q(t-\Delta) - A^*q(t)]\Big)\\ &\in \mathrm{co}\,\{(u,w) \mid (-u,-w,d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt)\\ &\qquad \in \partial H\big(\bar{x}(t),\bar{x}(t-\Delta),p(t)+q(t)\big)\}\end{aligned} \tag{29}$$

and the maximum condition

$$\begin{aligned}&\langle p(t)+q(t), \tfrac{d}{dt}[\bar{x}(t) - A\bar{x}(t-\Delta)]\rangle\\ &= H\big(\bar{x}(t),\bar{x}(t-\Delta),p(t)+q(t)\big)\end{aligned} \tag{30}$$

for a.e. $t \in [a,b]$. Moreover, if F is convex-valued around $(\bar{x}(t),\bar{x}(t-\Delta))$, then (29) is equivalent to the Euler-Lagrange inclusion

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t) - A^*p(t+\Delta)],\ \frac{d}{dt}[q(t-\Delta) - A^*q(t)]\Big)\\ &\in \mathrm{co}\, D^*F\big(\bar{x}(t),\bar{x}(t-\Delta),\\ &\qquad d[\bar{x}(t) - A\bar{x}(t-\Delta)]/dt\big)(-p(t)-q(t))\end{aligned} \tag{31}$$

for a.e. $t \in [a,b]$, which automatically implies the maximum condition (30) in this case.

Proof. Since (P_M) is stable with respect to relaxation, $\bar{x}(t)$ is an optimal solution to the relaxed problem (R_M) whose only difference from (P_M) is that the neutral functional-differential inclusion (2) is replaced by its convexification (7). By Theorem 8 the optimal solution $\bar{x}(t)$ satisfies conditions (23)–(27) and the relaxed counterpart of (28), which is the same as (31) in this case with F replaced by $\mathrm{co}\, F$. According to [23, Theorem 3.3] one has

$$\begin{aligned}&\mathrm{co}\left\{(u,v)\middle|\ (u,w,p) \in N((x,y,v);\mathrm{gph}(\mathrm{co}\, F)\right\}\\ &= \mathrm{co}\left\{(u,w)\middle|\ (-u,-w,v) \in \partial H_R(x,y,p)\right\},\end{aligned}$$

where H_R stands for the Hamiltonian (29) of the relaxed system. It is easy to check that $H_R = H$. Thus the extended Euler-Lagrange inclusion for the relaxed system implies the extended Hamiltonian inclusion (29), which surely yields the maximum condition (30). When F is convex-valued, (29) and (31) are equivalent due to the mentioned result of [23]. This completes the proof of the corollary.

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