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NECESSARY CONDITIONS IN NONSMOOTH MINIMIZATION VIA LOWER AND UPPER SUBGRADIENTS

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Abstract. The paper concerns first-order necessary optimality conditions for problems of *minimizing* nonsmooth functions under various constraints in infinite-dimensional spaces. Based on advanced tools of variational analysis and generalized differential calculus, we derive general results of two independent types called *lower subdifferential* and *upper subdifferential* optimality conditions. The former ones involve basic/limiting subgradients of cost functions, while the latter conditions are expressed via Fréchet/regular upper subgradients in fairly general settings. All the upper subdifferential and major lower subdifferential optimality conditions obtained in the paper are new even in finite dimensions. We give applications of general optimality conditions to mathematical programs with equilibrium constraints deriving new results for this important class of intrinsically nonsmooth optimization problems.

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1 Introduction

This paper is devoted to the study of necessary optimality conditions for *constrained minimization* problems in infinite-dimensional spaces. A general problem of this type with (non-specified) *geometric* constraints can be written as:

$$\text{minimize } \varphi_0(x) \text{ subject to } x \in \Omega \subset X, \tag{1.1}$$

where $\varphi_0 \colon X \to \overline{\mathbb{R}} := [-\infty, \infty]$ is an extended-real-valued function on a Banach space X finite at a reference point, and where Ω is an arbitrary nonempty subset of X. The constrained problem (1.1) is obviously equivalent to the *unconstrained* problem:

$$\text{minimize } \varphi_0(x) + \delta(x; \Omega), \quad x \in X,$$

where the indicator function $\delta(\cdot; \Omega)$ of Ω, defined by $\delta(x; \Omega) := 0$ if $x \in \Omega$ and $\delta(x; \Omega) := \infty$ otherwise, imposes an "infinite penalty" on the constraint violation. Denoting by

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* \Big| \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\} \tag{1.2}$$

the (lower) *Fréchet subdifferential*, known also as the regular or viscosity subdifferential, of $\varphi\colon X \to \overline{I\!R}$ at $\bar{x}$ with $|\varphi(\bar{x})| < \infty$, one can observe directly from the definition that the following *generalized Fermat rule* holds: if $\bar{x}$ a local minimizer of φ, then $0 \in \widehat{\partial}\varphi(\bar{x})$. Applying this to the unconstrained form of (1.1), we get

$$0 \in \widehat{\partial}(\varphi_0 + \delta(\cdot; \Omega))(\bar{x}) \tag{1.3}$$

provided that $\bar{x}$ gives a local minimum to $\varphi_0(x)$ subject to $x \in \Omega$. If φ_0 happens to be Fréchet differentiable at $\bar{x}$ with the derivative/gradient $\nabla\varphi_0(\bar{x})$, then (1.3) is equivalent to the inclusion

$$-\nabla\varphi_0(\bar{x}) \in \widehat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \, \middle| \, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}, \tag{1.4}$$

where $\widehat{N}(\bar{x}; \Omega) = \widehat{\partial}\delta(\bar{x}; \Omega)$ is the *Fréchet normal cone* to Ω at $\bar{x} \in \Omega$, and where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. Indeed, the equivalence between (1.3) and (1.4) follows from the simple sum rule for Fréchet subgradients:

$$\widehat{\partial}(\varphi_0 + \varphi_1)(\bar{x}) = \nabla\varphi_0(\bar{x}) + \widehat{\partial}\varphi_1(\bar{x})$$

held in Banach spaces for any function φ_0 Fréchet differentiable at $\bar{x}$ and an arbitrary function $\varphi_1\colon X \to \overline{I\!R}$ finite at $\bar{x}$.

If φ_0 is not Fréchet differentiable at $\bar{x}$, the above way does not lead to valuable optimality conditions, since Fréchet-like subgradients generally possess a poor calculus even for simple nonsmooth functions in finite dimensions. To be able to proceed further, one needs to employ more *robust* subdifferentials satisfying required calculus rules. In what follows we are going to develop such an approach based on our *basic/sequential limiting subgradients* of extended-real-valued functions and the corresponding *normal cone* and *coderivative* constructions for sets and set-valued mappings enjoying a number of useful calculus rules in arbitrary Banach spaces and fairly comprehensive calculi in the Asplund space setting; see below. In this way we derive general first-order optimality conditions of the *lower subdifferential type* for minimization problems with various constraints typically arising in applications. In particular, for problem (1.1), which is actually the simplest albeit general constrained optimization problem, the corresponding lower subdifferential optimality condition reads as

$$0 \in \partial\varphi_0(\bar{x}) + N(\bar{x}; \Omega) \tag{1.5}$$

provided that φ_0 is Lipschitz continuous around $\bar{x}$, as well as under more general qualification and normal compactness assumptions. We also derive lower subdifferential optimality conditions for minimization problems with many geometric constraints given by set intersections, with operator constraints defined by inverse images of set-valued mappings, with functional constraints given by equalities and inequalities, and with equilibrium constraints governed by parametric generalized equations and variational inequalities. For the latter class of minimization problems related to hierarchical optimization, *second-order subdifferential* constructions are useful in applications to first-order optimality conditions. Note that the realization of this approach in the case of infinite-dimensional spaces is based not only on calculus rules for subdifferentials and coderivatives, but also on calculus results ensuring the preservation of the so-called *sequential normal compactness*

properties for functions, sets, and set-valued mappings that are automatic in finite dimensions while playing a crucial role in infinite-dimensional optimization and variational analysis.

Along with lower subdifferential optimality conditions held for problems of minimizing general cost functions, we derive necessary optimality conditions of a new type that are especially efficient for special classes of functions under minimization and those describing inequality constraints, being often more powerful for these special classes than the former ones. Such *upper subdifferential* (or superdifferential) conditions, which seem to be rather surprising for *minimization* problems, involve the following *Fréchet upper subdifferential* construction for a given function $\varphi: X \to \overline{I\!R}$ finite at $\bar{x}$ defined by

$$\widehat{\partial}^+\varphi(\bar{x}) := -\widehat{\partial}(-\varphi)(\bar{x}) = \Big\{ x^* \in X^* \mid \limsup_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le 0 \Big\}. \tag{1.6}$$

Note that the upper subdifferential (1.6) is known also as the "superdifferential" being particularly useful in the theory of viscosity solutions for PDE problems; see, e.g., [5]. Following [25], we adopt the "upper" terminology, which seems to be more in accordance with the sense of such constructions.

It happens that Fréchet upper subgradients of extended-real-valued functions admit certain *smooth variational descriptions* allowing us to reduce, in particular, necessary optimality conditions for problem (1.1), given *each* $x^* \in \widehat{\partial}^+\varphi_0(\bar{x})$, to those for a counterpart of (1.1) with a Fréchet differentiable cost function whose derivative equals x^*. This leads to upper subdifferential conditions for (1.1) of the type

$$-\widehat{\partial}^+\varphi_0(\bar{x}) \subset \widehat{N}(\bar{x}; \Omega). \tag{1.7}$$

Such conditions carry nontrivial information for minimization problems with $\widehat{\partial}^+\varphi_0(\bar{x}) \ne \emptyset$, e.g., for problems of minimizing concave functions or, more generally, for nonsmooth functions with $\widehat{\partial}\varphi_0(\bar{x}) = \emptyset$. Note that the emptiness of $\widehat{\partial}^+\varphi_0(\bar{x})$ is itself is an easy checkable necessary condition for minimization of φ_0 that does not depend on constraints.

Upper subdifferential conditions are especially efficient for the class of functions that are *upper regular* at a local minimum point; see Sections 2 and 3. They are generally independent from lower subdifferential ones but may give essentially stronger results for some classes of minimization problems. In this paper we derive upper subdifferential conditions for minimization problems with the same types of general constraints as the lower subdifferential conditions discussed above. More specific results of the upper subdifferential type are obtained for minimization problems with inequality constraints.

The rest of the paper is organized as follows. Section 2 presents basic definitions and preliminaries from generalized differentiation and variational analysis widely used in what follows. In Section 3 we derive necessary optimality conditions of both lower and upper subdifferential types for constrained minimization problems in form (1.1) and also for problems with many geometric constraints given by set intersections. Section 4 deals with minimization problems that contain, together with geometric constraints, also constraints of operator and functional types given generally by inverse images of set-valued mappings and particularly by equalities and inequalities with real-valued functions. The final Section 5 is devoted to lower and upper subdifferential optimality conditions for general classes of mathematical programs with equilibrium constraints in infinite-dimensional spaces. Most of the results obtained seem to be new not only in the case of

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upper subdifferential conditions but for lower subdifferential ones as well, *even in finite dimensions.* They admit essential simplifications in finite-dimensional spaces when all the assumptions on the sequential normal compactness hold *automatically.*

Our notation is basically standard, with special symbols introduced where they are defined. Unless otherwise stated, all spaces considered are Banach whose norms are always denoted by $\|\cdot\|$. For any space X we consider its dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ means the canonical pairing. For multifunctions $F\colon X \rightrightarrows X^*$ the expression

$$\underset{x\to\bar{x}}{\mathrm{Lim\,sup}}\, F(x) := \{x^* \in X^* \mid \ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^*$$
$$\text{with } x_k^* \in F(x_k) \text{ for all } k \in I\!N\}$$

signifies the *sequential* Painlevé-Kuratowski upper/outer limit with respect to the norm topology in X and the weak* topology in X^*, where $I\!N := \{1, 2, \dots\}$.

2 Preliminaries

As mentioned in Section 1, for applications to necessary optimality conditions in this paper we need, along with the Fréchet-like constructions (1.2), (1.4), and (1.6), their robust counterparts defined as follows. The reader can find more details on these constructions and their history in the books [12, 25] and papers [3, 13, 18] in, respectively, finite and infinite dimensions.

Given a nonempty subset Ω of a Banach space X and a number $\varepsilon \geq 0$, we first define the ε-*enlargement* of the cone $\widehat{N}(\cdot;\Omega)$ in (1.4) by

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{x^* \in X^* \ \middle|\ \underset{u\xrightarrow{\Omega}x}{\mathrm{lim\,sup}}\, \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq \varepsilon\right\} \text{ for } x \in \Omega \tag{2.1}$$

and by $\widehat{N}_\varepsilon(x;\Omega) := \emptyset$ for $x \notin \Omega$. Then the *basic normal cone* to Ω at $\bar{x} \in \Omega$ is given by

$$N(\bar{x};\Omega) := \underset{\substack{x\to\bar{x}\\ \varepsilon\downarrow 0}}{\mathrm{Lim\,sup}}\, \widehat{N}_\varepsilon(x;\Omega) \tag{2.2}$$

as the sequential Painlevé-Kuratowski upper limit of ε-normals (2.1) at nearby points. When X is *Asplund* (i.e., its every separable subspace has a separable dual; see [24] for more information) and Ω is closed around $\bar{x}$, one can equivalently put $\varepsilon = 0$ in (2.2) and hence replace $\widehat{N}_\varepsilon(\cdot;\Omega)$ with the Fréchet normal cone; see [18, Theorem 2.9]. However, one cannot remove ε from (2.2) and the subsequent definitions without loss of crucial properties in general Banach spaces.

Given $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$, the (lower) *basic subdifferential* of φ at $\bar{x}$ can be defined geometrically by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\,\varphi)\}. \tag{2.3}$$

If φ is lower semicontinuous (l.s.c.) around $\bar{x}$ and if X is Asplund, construction (2.3) is equivalent to the analytic representation

$$\partial\varphi(\bar{x}) = \underset{x\xrightarrow{\varphi}\bar{x}}{\mathrm{Lim\,sup}}\, \widehat{\partial}\varphi(x),$$

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where $x \xrightarrow{\varphi} \bar{x}$ means that $x \to \bar{x}$ and $\varphi(x) \to \varphi(\bar{x})$. The basic *upper superdifferential* of φ at $\bar{x}$ is defined by $\partial^+\varphi(\bar{x}) := -\partial(-\varphi)(\bar{x})$ and can be represented via basic normals to the hypograph of φ as well as via sequential limits of Fréchet upper subgradients similarly to the basic subdifferential. Recall that φ is *lower regular* (resp. *upper regular*) at $\bar{x}$ if

$$\partial\varphi(\bar{x}) = \hat{\partial}\varphi(\bar{x}) \quad \left(\text{resp. } \partial^+\varphi(\bar{x}) = \hat{\partial}^+\varphi(\bar{x})\right). \tag{2.4}$$

Upper regular functions are of special interest for this paper in connection with upper subdifferential optimality conditions. Note that this class contains, in particular, all concave continuous functions and all functions strictly differentiable at $\bar{x}$, as well as other functions φ for which $-\varphi$ is lower regular at $\bar{x}$; cf. [12, 25]. Note that $\hat{\partial}^+\varphi(\bar{x}) \neq \emptyset$ if φ is upper regular at $\bar{x}$ and Lipschitz continuous around this point while X is Asplund. This follows from the fact that $\partial^+\varphi(\bar{x}) \neq \emptyset$ for any locally Lipschitzian function on an Asplund space; see [18, Corollary 3.9]. If φ is concave and continuous around $\bar{x}$, then $\hat{\partial}^+\varphi(\bar{x}) \neq \emptyset$ in any Banach space, which is well known in convex analysis. Observe also that

$$\bar{\partial}\varphi(\bar{x}) = \mathrm{cl}^*\hat{\partial}^+\varphi(\bar{x}) \tag{2.5}$$

for every function φ on an Asplund space that is upper regular at $\bar{x}$ and Lipschitz continuous around this point, where $\bar{\partial}\varphi(\bar{x})$ stands for the Clarke generalized gradient [4], and where cl^* denotes the topological closure of a set in the weak* topology of X^*. Moreover, the weak* closure is redundant in (2.5) if X is *weakly compactly generated* (WCG), in particular, it is either reflexive or separable. Indeed, by the symmetry property of the Clarke generalized gradient for locally Lipschitzian functions [4, Proposition 2.3.1], its representation through the basic subdifferential in Asplund spaces [18, Theorem 8.11], and the convexity of $\hat{\partial}^+\varphi(\bar{x})$ one has

$$\bar{\partial}\varphi(\bar{x}) = -\bar{\partial}(-\varphi)(\bar{x}) = -\mathrm{cl}^*\mathrm{co}\,\partial(-\varphi)(\bar{x}) = \mathrm{cl}^*\mathrm{co}\,\partial^+\varphi(\bar{x}) = \mathrm{cl}^*\hat{\partial}^+\varphi(\bar{x}),$$

where $\partial\varphi(\bar{x})$ and hence $\hat{\partial}^+\varphi(\bar{x})$ are weak* closed in WCG spaces due to [18, Theorem 9.2].

Given a set-valued mapping $F: X \rightrightarrows Y$ between Banach spaces, we define its *normal coderivative* $D_N^*F(\bar{x},\bar{y}): Y^* \rightrightarrows X^*$ at $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$ by

$$D_N^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x},\bar{y}); \mathrm{gph}\,F)\} \tag{2.6}$$

and the corresponding *mixed coderivative* by

$$D_M^*F(\bar{x},\bar{y})(y^*) := \left\{ x^* \in X^* \;\middle|\; \begin{array}{l} \exists \varepsilon_k \downarrow 0,\ (x_k, y_k) \to (\bar{x},\bar{y}),\ x_k^* \xrightarrow{w^*} x^*,\ y_k^* \xrightarrow{\|\cdot\|} y^*, \\ \text{with } (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \mathrm{gph}\,F),\ k \to \infty \end{array} \right\}, \tag{2.7}$$

where ε_k can be equivalently omitted if the graph of F is closed around $(\bar{x},\bar{y})$ and if both X and Y are Asplund. We also omit $\bar{y}$ in (2.6) and (2.7) when $F = f: X \to Y$ is single-valued and use the common coderivative symbol D^*F if both coderivatives agree. This happens, in particular, when Y is finite-dimensional, while the mixed coderivative may be strictly smaller (never bigger) than the normal coderivative even for single-valued Lipschitzian mappings into the Hilbert space $Y = \ell^2$. One has the *scalarization formulas*

$$D_M^*f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}), \quad D_N^*f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \tag{2.8}$$

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with $\langle y^*, f \rangle(x) := \langle y^*, f(x) \rangle$, where the first formula in (2.8) holds for every locally Lipschitzian mapping between Banach spaces, while the second scalarization formula is established in [18, Theorem 5.2] for the case of Asplund spaces X and *strictly Lipschitzian* mappings f. The latter subclass of Lipschitzian mappings is proved to reduce to compactly Lipschitzian mappings in the sense of Thibault; see [26] for more details. If, in particular, f is strictly differentiable at $\bar{x}$, then formulas (2.8) reduce both coderivatives to the adjoint derivative operator

$$D^*_M f(\bar{x})(y^*) = D^*_N f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad y^* \in Y^*,$$

in any Banach spaces X and Y. Using the coderivative $D^* E_\varphi(\bar{x})$ of the epigraphical multifunction $E_\varphi(x) := \{\mu \in I\!R | \ \mu \geq \varphi(x)\}$ associated with $\varphi \colon X \to \overline{I\!R}$, we get back to the basic subdifferential $\partial \varphi(\bar{x})$ and define the (lower) *singular subdifferential* $\partial^\infty \varphi(\bar{x})$ of φ at $\bar{x}$ by

$$\partial \varphi(\bar{x}) = D^* E_\varphi(\bar{x}, \varphi(\bar{x}))(1), \quad \partial^\infty \varphi(\bar{x}) := D^* E_\varphi(\bar{x}, \varphi(\bar{x}))(0). \tag{2.9}$$

It is easy to see that $\partial^\infty \varphi(\bar{x}) = \{0\}$ for locally Lipschitzian functions φ on arbitrary Banach spaces.

In this paper we also use the construction of the (normal) *second-order subdifferential* of $\varphi \colon X \to \overline{I\!R}$ at $\bar{x}$ relative to $\bar{y} \in \partial \varphi(\bar{x})$ defined by

$$\partial^2_N \varphi(\bar{x}, \bar{y})(u) := D^*_N(\partial \varphi)(\bar{x}, \bar{y})(u), \quad u \in X^{**}. \tag{2.10}$$

The mixed second-order subdifferential is defined similarly, but we do not need it in what follows. Note that for $\varphi \in \mathcal{C}^2$ one has

$$\partial^2_N \varphi(\bar{x})(u) = \{\nabla^2 \varphi(x)^* u\}, \quad u \in X^{**},$$

where $\nabla^2 \varphi(\bar{x})$ stands for the classical second-order derivative operator.

Next we recall certain *normal compactness* properties of sets from *products* of Banach spaces; see [19] and its references for the genesis of these and related properties and more discussions. A set $\Omega \subset X \times Y$ is *sequentially normally compact* (SNC) at $(\bar{x}, \bar{y}) \in \Omega$ if for any sequences $(\varepsilon_k, x_k, x^*_k, y^*_k) \in [0, \infty) \times \Omega \times X^* \times Y^*$ satisfying

$$\varepsilon_k \downarrow 0, \quad (x_k, y_k) \to (\bar{x}, \bar{y}), \quad (x^*_k, y^*_k) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \Omega) \tag{2.11}$$

one has the implication

$$(x^*_k, y^*_k) \xrightarrow{w^*} (0, 0) \implies \|(x^*_k, y^*_k)\| \to 0 \quad \text{as} \quad k \to \infty.$$

This set is *partially sequentially normally compact* (PSNC) at $(\bar{x}, \bar{y})$ with respect to X if for any above sequences satisfying (2.11) one has

$$\left[x^*_k \xrightarrow{w^*} 0 \ \text{and} \ \|y^*_k\| \to 0 \right] \implies \|x^*_k\| \to 0 \quad \text{as} \ k \to \infty.$$

Finally, Ω is *strongly PSNC* at $(\bar{x}, \bar{y})$ with respect to X if

$$\left[(x^*_k, y^*_k) \xrightarrow{w^*} (0, 0) \right] \implies \|x^*_k\| \to 0 \quad \text{as} \ k \to \infty.$$

for every sequences satisfying (2.11).

It follows from the definitions that

$$\text{SNC} \Longrightarrow \text{strong PSNC} \Longrightarrow \text{PSNC}$$

for any $(\bar{x}, \bar{y}) \in \Omega$ and that the above properties automatically hold in finite dimensions. Note that ε_k may be equivalently omitted in (2.1) if both spaces X, Y are Asplund and if Ω is locally closed around $(\bar{x}, \bar{y})$. Note also that the SNC property, in contrast to the other two, does not depend on the product structure on the Banach space in question. It is closely related to the *compactly epi-Lipschitzian* property of sets in the sense of [2], but the latter may be stronger in nonseparable Banach and Asplund spaces; see [8, 10] for recent comprehensive studies.

The corresponding SNC/PSNC properties of a set-valued mapping $F: X \rightrightarrows Y$ are defined via those for its graph at $(\bar{x}, \bar{y}) \in \text{gph}\, F$. We omit "with respect to X" when referring to the PSNC properties of mappings. Recall [13] that $F: X \rightrightarrows Y$ is PSNC at $(\bar{x}, \bar{y})$, for any Banach spaces X and Y, if it satisfies the *Aubin Lipschitz-like property* (known also as the "pseudo-Lipschitzian" property; see [1, 25]) around this point.

An extended-real-valued function $\varphi: X \to \overline{I\!R}$ is *sequentially normally epi-compact* (SNEC) at $\bar{x}$ if its epigraph is SNC at $(\bar{x}, \varphi(\bar{x}))$. Note that if $\varphi: X \to I\!R$ is locally Lipschitzian around $\bar{x}$, it is SNC and hence SNEC at this point.

3 Optimality Conditions under Geometric Constraints

First let us derive necessary optimality conditions, of both lower and upper subdifferential types, for the initial problem (1.1) with the only (abstract) geometric constraint given by an arbitrary set $\Omega \subset X$ in infinite dimensions.

Theorem 3.1 (upper and lower subdifferential conditions for problems with abstract geometric constraints). *Let $\bar{x}$ be a local optimal solution to the minimization problem (1.1) in a Banach space X with $|\varphi_0(\bar{x})| < \infty$. Then the following hold:*
(i) *For every $x^* \in \hat{\partial}^+ \varphi_0(\bar{x})$ one has $-x^* \in \hat{N}(\bar{x}; \Omega)$, i.e.,*

$$-\hat{\partial}^+ \varphi_0(\bar{x}) \subset \hat{N}(\bar{x}; \Omega) \quad and \quad -\hat{\partial}^+ \varphi_0(\bar{x}) \subset N(\bar{x}; \Omega). \tag{3.1}$$

(ii) *Suppose that X is Asplund, that φ_0 is l.s.c. around $\bar{x}$, and that Ω is locally closed around this point. Assume also that*

$$\partial^\infty \varphi_0(\bar{x}) \cap (-N(\bar{x}; \Omega)) = \{0\} \tag{3.2}$$

and that either Ω is SNC at $\bar{x}$ or φ_0 is SNEC at $\bar{x}$; all these assumptions are satisfied if φ_0 is locally Lipschitzian around $\bar{x}$. Then one has

$$\partial \varphi_0(\bar{x}) \cap (-N(\bar{x}; \Omega)) \neq \emptyset, \quad i.e., \quad 0 \in \partial \varphi_0(\bar{x}) + N(\bar{x}; \Omega). \tag{3.3}$$

Proof. Let us establish the upper subdifferential conditions in (i). Since $\hat{N}(\bar{x}; \Omega) \subset N(\bar{x}; \Omega)$, we just need to prove the first inclusion in (3.1). Take any $x^* \in \hat{\partial}^+ \varphi_0(\bar{x})$ and observe that, in an arbitrary Banach space X, there is a function $s: X \to I\!R$ with

$$s(\bar{x}) = \varphi_0(\bar{x}) \quad and \quad s(x) \geq \varphi_0(x) \quad whenever \quad x \in X$$

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such that $s(\cdot)$ is Fréchet differentiable at $\bar{x}$ with $\nabla s(\bar{x}) = x^*$. Indeed, it follows directly from definition (1.6) that the function

$$s(x) := \max\left\{\varphi_0(x), \varphi_0(\bar{x}) + \langle x^*, x - \bar{x}\rangle\right\}$$

enjoys all the above properties. One therefore has

$$s(\bar{x}) = \varphi_0(\bar{x}) \leq \varphi_0(x) \leq s(x) \quad \text{for all} \ x \in \Omega,$$

and thus $\bar{x}$ is a local optimal solution to the constrained minimization problem:

$$\text{minimize} \ s(x) \ \text{subject to} \ x \in \Omega$$

with a Fréchet differentiable objective. Applying now the necessary optimality condition (1.4) in the latter problem, we get

$$-x^* = -\nabla s(\bar{x}) \in \widehat{N}(\bar{x}; \Omega),$$

which justifies the upper subdifferential optimality conditions (3.1) in general Banach spaces.

Next let us prove the subdifferential optimality condition (3.3) under the assumptions made in (ii). As mentioned in Section 1, one has inclusion (1.3) by the generalized Fermat rule. This immediately yields

$$0 \in \partial(\varphi_0 + \delta(\cdot; \Omega))(\bar{x})$$

in terms of the basic subdifferential of the sum $\varphi_0 + \delta(\cdot; \Omega)$. Applying the subdifferential sum rule proved in [18, Theorem 4.1] to the latter sum and taking into account that

$$\partial\delta(\bar{x}; \Omega) = \partial^\infty\delta(\bar{x}; \Omega) = N(\bar{x}; \Omega),$$

we arrive at (3.3) under the assumptions made. As mentioned above, φ_0 is SNEC at $\bar{x}$ and $\partial^\infty\varphi_0(\bar{x}) = \{0\}$ (i.e., the qualification condition (3.2) automatically holds) if φ_0 is Lipschitz continuous around $\bar{x}$. This ends the proof of the theorem. $\qquad\square$

Note that the lower subdifferential optimality conditions in Theorem 3.1 apply to a very general class of extended-real-valued cost functions, while the upper subdifferential conditions in (i) is efficient only if $\widehat{\partial}^+\varphi_0(\bar{x}) \neq \emptyset$. Nevertheless, the upper subdifferential conditions may give an *essentially stronger* result for special important classes of nonsmooth problems. In particular, for concave continuous functions φ_0 one has

$$\partial\varphi_0(\bar{x}) \subset \partial^+\varphi_0(\bar{x}) = \widehat{\partial}^+\varphi_0(\bar{x}) \neq \emptyset.$$

Then comparing the second inclusion in (3.1) (which is even weaker than the first inclusion therein) with the one in (3.3), we see that the upper subdifferential necessary condition requires that *every* element x^* of the set $\widehat{\partial}^+\varphi_0(\bar{x})$ must belong to $-N(\bar{x}; \Omega)$, instead of that *some* element x^* from the *smaller* set $\partial\varphi_0(\bar{x})$ belongs to $-N(\bar{x}; \Omega)$ by the lower subdifferential one. This shows that the upper subdifferential necessary conditions for local minima may have sizeable advantages over the

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lower subdifferential conditions above when the former efficiently apply. For example, consider the following simple one-dimensional problem:

$$\text{minimize } \varphi_0(x) := -|x| \text{ subject to } x \in \Omega := [-1, 0] \subset \mathbb{R}.$$

Obviously $\bar{x} = 0$ is not an optimal solution to this problem. However, it cannot be eliminated by the lower subdifferential condition (3.3), which is satisfied:

$$\partial \varphi(0) = \{-1, 1\}, \quad N(0; \Omega) = [0, \infty), \quad \text{and} \quad -1 \in -N(0, \Omega).$$

On the other hand, the upper subdifferential conditions in (3.1), which are the same in this case, do not hold for $\bar{x} = 0$ giving

$$\hat{\partial}^+ \varphi(0) = [-1, 1] \quad \text{and} \quad [-1, 1] \not\subset N(0; \Omega).$$

Recall also that $\hat{\partial}^+ \varphi_0(\bar{x}) \neq \emptyset$ if φ_0 is locally Lipschitzian and *upper regular* at $\bar{x}$ while X is Asplund. Moreover, $\bar{\partial} \varphi_0(\bar{x}) = \hat{\partial}^+ \varphi_0(\bar{x})$ for the Clarke generalized gradient if in addition X is WCG; see (2.5). Thus in this case we have

$$-\bar{\partial} \varphi_0(\bar{x}) \subset \hat{N}(\bar{x}; \Omega) \subset N(\bar{x}; \Omega)$$

by (3.1) instead of $\bar{\partial} \varphi_0(\bar{x}) \cap \left(- \text{cl}^* \text{co} \, N(\bar{x}; \Omega) \right) \neq \emptyset$ by Clarke's counterpart; cf. [4].

Now let us consider minimization problems with finitely *many geometric constraints* that typically arise in applications. Having in mind particular applications in Sections 4 and 5 of this paper, we pay the main attention to problems with geometric constraints given by two set intersections:

$$\text{minimize } \varphi_0(z) \text{ subject to } z \in \Omega_1 \cap \Omega_2. \tag{3.4}$$

Most results for problems with finitely many geometric constraints can be reduced to the case of two constraint problems (3.4) by induction.

To derive more general and powerful results needed for subsequent applications, we consider problems (3.4) given in spaces with a *product structure* $X \times Y$ that particularly occurs in the framework of mathematical programs with equilibrium constraints; see Section 5. The next theorem gives both upper and lower subdifferential optimality conditions for such problems.

Theorem 3.2 (upper and lower subdifferential conditions for problems with many geometric constraints). *Let $\bar{z}$ be a local optimal solution to problem (3.4), where the sets $\Omega_1, \Omega_2 \subset X \times Y$ are locally closed around $\bar{z}$, and where both spaces X and Y are Asplund. The following hold:*

(i) Assume that the set system $\{\Omega_1, \Omega_2\}$ satisfies the limiting qualification condition at $\bar{z}$: for any sequences $z_{ik} \overset{\Omega_i}{\to} \bar{z}$ and $z_{ik}^* \overset{w^*}{\to} z_i^*$ as $k \to \infty$ with $z_{ik}^* \in \hat{N}(z_{ik}; \Omega_i)$, $i = 1, 2$, one has

$$\|z_{1k}^* + z_{2k}^*\| \to 0 \Longrightarrow z_1^* = z_2^* = 0. \tag{3.5}$$

Suppose also that either one of the sets Ω_i is SNC at $\bar{z}$, or Ω_1 is PSNC at $\bar{z}$ with respect to X while Ω_2 is strongly PSNC at this point with respect to Y. Then

$$-\hat{\partial}^+ \varphi_0(\bar{z}) \subset N(\bar{z}; \Omega_1) + N(\bar{z}; \Omega_2). \tag{3.6}$$

(ii) *In addition to the assumptions in* (i), *suppose that* φ_0 *is l.s.c. around* $\bar{z}$ *and SNEC at this point and that*

$$\left(-\partial^\infty \varphi_0(\bar{z})\right) \cap \left[N(\bar{z}; \Omega_1) + N(\bar{z}; \Omega_2)\right] = \{0\} \tag{3.7}$$

(all the additional assumptions are satisfied if φ_0 *is Lipschitz continuous around* $\bar{z}$*). Then one has*

$$0 \in \partial \varphi_0(\bar{z}) + N(\bar{z}; \Omega_1) + N(\bar{z}; \Omega_2). \tag{3.8}$$

(iii) *Assume that* φ_0 *is l.s.c. around* $\bar{z}$, *that both* Ω_1 *and* Ω_2 *are SNC at this point, and that the qualification condition*

$$\begin{bmatrix} z_0^* \in \partial^\infty \varphi_0(\bar{z}), & z_1^* \in N(\bar{z}; \Omega_1), & z_2^* \in N(\bar{z}; \Omega_2), \\ z_0^* + z_1^* + z_2^* = 0 \end{bmatrix} \implies z_0^* = z_1^* = z_2^* = 0 \tag{3.9}$$

holds. Then one has (3.8).

Proof. To prove (i), we base on the second upper subdifferential inclusion in Theorem 3.1(i) involving the basic normal cone to $\Omega := \Omega_1 \cap \Omega_2$. This gives

$$-\widehat{\partial}^+ \varphi_0(\bar{z}) \subset N(\bar{z}; \Omega_1 \cap \Omega_2).$$

Now we can use the intersection rule for the basic normal cone to $\Omega_1 \cap \Omega_2$ that is not available for Fréchet normals in (3.1). Employing the general result of [19, Theorem 4.2], one has

$$N(\bar{z}; \Omega_1 \cap \Omega_2) \subset N(\bar{z}; \Omega_1) + N(\bar{z}; \Omega_2) \tag{3.10}$$

under the limiting qualification condition and the SNC/PSNC assumptions made in (i). Thus we arrive at the upper subdifferential inclusion (3.6).

Assertion (ii) of the theorem follows from Theorem 3.1(ii) under the SNEC assumption on φ_0 and from the above intersection rule of [19, Theorem 4.2] by substituting (3.10) into (3.2) and (3.3).

It remains to prove (iii). Using Theorem 3.1(ii) in the case of SNC sets Ω, we need to express the SNC assumption on Ω and the other conditions of that theorem in terms of Ω_1, Ω_2, and φ_0. To proceed, one needs to employ the SNC preservation/calculus rules developed in [20]. In particular, corollary 3.6 of that paper ensures the SNC property of the intersection $\Omega_1 \cap \Omega_2$ at $\bar{z}$ provided that both Ω_i are SNC at this point and that the qualification condition

$$N(\bar{z}; \Omega_1) \cap \left(-N(\bar{z}; \Omega_2)\right) = \{0\} \tag{3.11}$$

is satisfied. These assumptions automatically guarantee the fulfillment of the intersection rule (3.10). It is easy to check that (3.9) implies both qualification conditions (3.2) at $\bar{z}$ and (3.11). Indeed, (3.11) follows right from (3.9) with $z_0^* = 0$. To get (3.2) at $\bar{z}$, we take $z_0^* \in N(\bar{z}; \Omega_1 \cap \Omega_2)$ with $-z_0^* \in \partial^\infty \varphi_0(\bar{z})$ and find $z_i^* \in N(\bar{z}; \Omega_i)$, $i = 1, 2$, such that $z_1^* + z_2^* = z_0^*$ by (3.10). Thus $z_0^* + z_1^* + z_2^* = 0$, which gives $z_0^* = 0$ by (3.9) and ends the proof of the theorem. $\qquad\square$

As observed, the *normal* qualification condition (3.11) implies the *limiting* one in Theorem 3.2. Indeed, the former corresponds to the replacement of the implication in (3.5) by

$$z_{1k}^* + z_{2k}^* \xrightarrow{w^*} 0 \implies z_1^* = z_2^* = 0.$$

We will see in Section 5 that, being applied to graphs of set-valued mappings, the limiting qualification condition of Theorem 3.2 has essential advantages in comparison with the normal qualification condition (3.11).

4 Optimality Conditions under Operator and Functional Constraints

In this section we derive necessary optimality conditions of both lower and upper subdifferential types for minimization problems that contain, along with geometric constraints, also constraints given by set-valued and single-valued mappings/operators between possibly infinite-dimensional spaces, as well as in more conventional forms involving real-valued functions. The general problem under consideration is as follows:

$$\text{minimize } \varphi_0(x) \text{ subject to } x \in F^{-1}(\Theta) \cap \Omega, \tag{4.1}$$

where $\varphi_0 \colon X \to \overline{I\!R}$, $F \colon X \rightrightarrows Y$, $\Omega \subset X$, $\Theta \subset Y$, and where

$$F^{-1}(\Theta) := \{x \in X \mid F(x) \cap \Theta \neq \emptyset\}$$

in the *inverse image* of the set Θ under the set-valued mapping F between Banach spaces. Model (4.1) covers many special classes of optimization problems, in particular, classical problems of nonlinear programming with equality and inequality constraints.

Observe that (4.1) reduces to the problem of constrained minimization admitting only geometric constraints given by the intersection of two sets: $\Omega_1 = F^{-1}(\Theta)$ and $\Omega_2 = \Omega$. Thus one can apply the results of the preceding subsection and then *calculus rules* for basic normals to inverse images and intersections as well as those preserving SNC properties. In this way we arrive at necessary optimality conditions of the lower and upper subdifferential types in both *normal* (Kuhn-Tucker) form under some constraint qualifications ensuring a nonzero multiplier associated with the cost function, as well as a *non-qualified* (Fritz John) form that does not impose constraint qualifications and does not ensure the nontriviality of the above multiplier. For brevity we present here only some results in the latter form.

Let us start with *upper subdifferential conditions*. Recall that a set-valued mapping $M \colon X \rightrightarrows Y$ is *inner semicompact* at $\bar{x}$ with $M(\bar{x}) \neq \emptyset$ if for every sequence $x_k \to \bar{x}$ with $M(x_k) \neq \emptyset$ there is a sequence $y_k \in M(x_k)$ that contains a convergent subsequence. We say that $M(\cdot)$ is inner semicompact *around* $\bar{x}$ if this property holds for every x in some neighborhood of $\bar{x}$. The latter property obviously holds for set-valued mappings that are locally compact (locally bounded when $\dim Y < \infty$) around the reference point.

Theorem 4.1 (upper subdifferential optimality conditions under operator constraints). *Given a local optimal solution $\bar{x}$ to problem (4.1), we have the following assertions:*

(i) *Assume that X and Y are Banach, that $\Omega = X$ and $\Theta = \{0\}$, and that $F = f \colon X \to Y$ is Fréchet differentiable at $\bar{x}$. Then there exists $\lambda_0 \geq 0$ such that for every $x_0^* \in \widehat{\partial}^+ \varphi_0(\bar{x})$ there is $y^* \in Y^*$ for which*

$$0 = \lambda_0 x_0^* + \nabla f(\bar{x})^* y^*, \quad (\lambda_0, y^*) \neq 0, \tag{4.2}$$

provided that either f is strictly differentiable at $\bar{x}$ or $\dim Y < \infty$.

(ii) *Assume that X is Asplund while Y is Banach, that $f \colon X \to Y$ is strictly differentiable at $\bar{x}$ with the surjective derivative $\nabla f(\bar{x})$, and that Ω is locally closed around $\bar{x}$. Then there exists $\lambda_0 \geq 0$ such that for every $x_0^* \in \widehat{\partial}^+ \varphi_0(\bar{x})$ there is $y^* \in N(f(\bar{x}); \Theta)$ for which*

$$-\lambda_0 x_0^* - \nabla f(\bar{x})^* y^* \in N(\bar{x}; \Omega), \quad (\lambda_0, y^*) \neq 0, \tag{4.3}$$

provided that either Ω is SNC at $\bar{x}$ or Θ is SNC at $f(\bar{x})$.

(iii) *Assume that both X and Y are Asplund, that Ω and Θ are closed, and that $M(\cdot) := F(\cdot) \cap \Theta$ is inner semicompact around $\bar{x}$. Then there exists $\lambda_0 \geq 0$ such that for every $x_0^* \in \widehat{\partial}^+ \varphi_0(\bar{x})$ there are $\bar{y} \in M(\bar{x})$ and dual elements $y^* \in N(\bar{y}; \Theta)$, $x_1^* \in D_N^* F(\bar{x}, \bar{y})(y^*)$, and $x_2^* \in N(\bar{x}; \Omega)$ satisfying*

$$0 = \lambda_0 x_0^* + x_1^* + x_2^*, \quad (\lambda_0, y^*, x_1^*) \neq 0, \tag{4.4}$$

provided that one of the following properties holds for every $\bar{y} \in M(\bar{x})$:

(a) Ω *is SNC at $\bar{x}$ and F^{-1} is PSNC at $(\bar{y}, \bar{x})$;*

(b) Ω *is SNC at $\bar{x}$ and Θ is SNC at $\bar{y}$;*

(c) F *is PSNC at $(\bar{x}, \bar{y})$ and Θ is SNC at $\bar{y}$;*

(d) F *is SNC at $(\bar{x}, \bar{y})$.*

Proof. To prove (i) in the general Banach space setting, we first assume that f is Fréchet differentiable at $\bar{x}$ with the surjective derivative $\nabla f(\bar{x})$. Then for any set $\Theta \subset Y$ with $f(\bar{x}) \in \Theta$ one has

$$\widehat{N}(\bar{x}; f^{-1}(\Theta)) = \nabla f(\bar{x})^* \widehat{N}(f(\bar{x}); \Theta), \tag{4.5}$$

which follows from [21, Theorem 3.1] and the Lyusternik-Graves theorem on metric regularity. Since the proof in [21] requires the metric regularity just *at* (but not around) the reference point, it ensures the fulfillment of (4.5) also in the case when f is merely Fréchet differentiable at $\bar{x}$ with the surjective derivative provided that Y is finite-dimensional. It can be done by using the Brouwer fixed point theorem instead of the Lyusternik-Graves result; cf. the arguments in [9, Proposition 7] establishing a somewhat different but related controllability property. Then substituting (4.5) into the first inclusion in (3.1) with $\Omega = f^{-1}(\Theta)$, we get

$$-\widehat{\partial}^+ \varphi_0(\bar{x}) \subset \nabla f(\bar{x})^* \widehat{N}(f(\bar{x}); \Theta).$$

For $\Theta = \{0\}$ the latter gives (4.2) with $\lambda_0 = 1$ under the surjectivity assumption on $\nabla f(\bar{x})$. If $\nabla f(\bar{x})$ is not surjective, then $\ker \nabla f(\bar{x})^* \neq \{0\}$, i.e., there is $0 \neq y^* \in Y^*$ such that $\nabla f(\bar{x})^* y^* = 0$. Thus we get (4.2) with $\lambda_0 = 0$ and $y^* \neq 0$.

To prove (ii) when X is Asplund (while Y may be arbitrarily Banach) and f is strictly differentiable at $\bar{x}$ with the surjective derivative, we apply assertion (i) of Theorem 3.2 with $\Omega_1 = f^{-1}(\Theta)$ and $\Omega_2 = \Omega$ assuming that either Ω or $f^{-1}(\Theta)$ is SNC at $\bar{x}$ and $f(\bar{x})$, respectively, and that

$$N(\bar{x}; f^{-1}(\Theta)) \cap (-N(\bar{x}; \Omega)) = \{0\}.$$

When Ω is SNC at $\bar{x}$, this yields

$$-\widehat{\partial}^+ \varphi_0(\bar{x}) \subset \nabla f(\bar{x})^* N(f(\bar{x}); \Theta) + N(\bar{x}; \Omega) \tag{4.6}$$

under the qualification condition

$$\nabla f(\bar{x})^* N(f(\bar{x}); \Theta) \cap (-N(\bar{x}; \Omega)) = \{0\}. \tag{4.7}$$

Indeed, it follows from the the basic normal cone counterpart of equality (4.5) established in [21, Corollary 3.9] for any mapping f between Banach spaces that is strictly differentiable at $\bar{x}$ with the

surjective derivative. The latter assumptions ensure by [21, Corollary 5.3] that the SNC property of $f^{-1}(\Theta)$ at $\bar{x}$ is equivalent to the one for Θ at $f(\bar{x})$. Thus (4.6) implies (4.3) with $\lambda_0 = 1$ under the qualification condition (4.7) and the assumptions made in (ii). The negation of (4.7) means that (4.3) holds with $\lambda_0 = 0$ and $y^* \neq 0$, which completes the proof of (ii).

It remains to prove (iii). Again applying the upper subdifferential assertion (i) of Theorem 3.2 with $\Omega_1 = F^{-1}(\Theta)$ and $\Omega_2 = \Omega$, we now are able to proceed with a general case of set-valued mappings F in the operator constraints of (4.1) having in hands the powerful tools of comprehensive calculus rules (including those for the preservation of SNC properties) in the Asplund space settings. First observe that the set $F^{-1}(\Theta)$ is locally closed around $\bar{x}$ due to the closedness and inner semicompactness assumptions made in (iii). Hence, by Theorem 3.2(i), one has

$$-\hat{\partial}^+ \varphi_0(\bar{x}) \subset N(\bar{x}; F^{-1}(\Theta)) + N(\bar{x}; \Omega) \tag{4.8}$$

provided the qualification condition

$$N(\bar{x}; F^{-1}(\Theta)) \cap (-N(\bar{x}; \Omega)) = \{0\} \tag{4.9}$$

and that either Ω or $F^{-1}(\Theta)$ is SNC at $\bar{x}$. The SNC calculus result of [20, Theorem 3.8] ensures the latter property of the inverse image $F^{-1}(\Theta)$ under the qualification condition

$$N(\bar{y}; \Theta) \cap \ker D_N^* F(\bar{x}, \bar{y}) = \{0\} \quad \text{for all} \ \ \bar{y} \in M(\bar{x}) \tag{4.10}$$

assuming also that either F is PSNC at $(\bar{x}, \bar{y})$ and Θ is SNC at $\bar{y}$, or F is SNC at $(\bar{x}, \bar{y})$ for every $\bar{y} \in M(\bar{x})$. Now we apply the calculus rule of [19, Theorem 4.4] providing the inclusion

$$N(\bar{x}; F^{-1}(\Theta)) \subset \bigcup \left[D_N^* F(\bar{x}, \bar{y})(y^*) \middle| \ \bar{y} \in M(\bar{x}), \ y^* \in N(\bar{y}; \Theta) \right] \tag{4.11}$$

under the qualification condition (4.10) and the assumptions that either F^{-1} is PSNC at $(\bar{y}, \bar{x})$ or Θ is SNC at $\bar{y}$ for all $\bar{y} \in M(\bar{x})$. Substituting (4.11) into (4.8) and (4.9) and combining the SNC/PSNC assumptions made on Ω, Θ, F, and F^{-1} above, we arrive at the upper subdifferential optimality condition

$$-\hat{\partial}^+ \varphi_0(\bar{x}) \subset \bigcup \left[D_N^* F(\bar{x}, \bar{y})(y^*) \middle| \ \bar{y} \in M(\bar{x}), \ y^* \in N(\bar{y}; \Theta) \right] + N(\bar{x}; \Omega) \tag{4.12}$$

under one of the assumptions (a)–(d) in (iii) and the constraint qualifications (4.10) and

$$\bigcup \left[D_N^* F(\bar{x}, \bar{y})(y^*) \middle| \ \bar{y} \in M(\bar{x}), \ y^* \in N(\bar{y}; \Theta) \right] \cap (-N(\bar{x}; \Omega)) = \{0\}, \tag{4.13}$$

which therefore ensure the result of (iii) in the normal form ($\lambda_0 = 1$).

If the above constraint qualifications are not satisfied, we have the optimality conditions in (iii) of the Fritz John type, i.e., with λ_0 in (4.2) possibly equal to zero but then either y^* or x_1^* is not. Indeed, when (4.10) is not satisfied, there are $\bar{y} \in M(\bar{x})$ and $0 \neq y^* \in N(\bar{y}; \Theta)$ such that $0 \in D_N^* F(\bar{x}, \bar{y})(y^*)$. This gives (4.2) with $\lambda_0 = 0$, $y^* \neq 0$, $x_1^* = x_2^* = 0$. If (4.13) is not satisfied, then there are $\bar{y} \in M(\bar{x})$ and $y^* \in N(\bar{y}; \Theta)$, $0 \neq x^* \in D_N^* F(\bar{x}, \bar{y})(y^*)$ such that $-x^* \in N(\bar{x}; \Omega)$. This gives (4.2) with $\lambda_0 = 0$, $x_1^* = -x_2^* = x^*$, which completes the proof of the theorem. $\qquad \square$

Next let us derive *lower subdifferential conditions* for problems (4.1) with general constraints. To furnish this, we are based on the subdifferential conditions of Theorem 3.2(ii) and the calculus

rules as in the proof of the previous theorem assuming for simplicity that φ_0 is Lipschitz continuous around the reference point. In this way one may also derive lower subdifferential conditions in (4.1) for problems with non-Lipschitzian cost functions based on the corresponding results of Theorem 3.2. For brevity we only present below a lower subdifferential counterpart of assertion (iii) in Theorem 4.1.

Theorem 4.2 (upper subdifferential optimality conditions under operator constraints).
Let $\bar{x}$ be a local optimal solution to problem (4.1). In addition to the assumptions of Theorem 4.1(iii), suppose that φ_0 is Lipschitz continuous around $\bar{x}$. Then there are $\lambda_0 \geq 0$, $x_0^* \in \partial \varphi_0(\bar{x})$, $\bar{y} \in M(\bar{x})$, $y^* \in N(\bar{y}; \Theta)$, $x_1^* \in D_N^* F(\bar{x}, \bar{y})(y^*)$, and $x_2^* \in N(\bar{x}; \Omega)$ such that (4.4) holds provided that one of the properties (a)–(d) in (iii) of Theorem 4.1 is fulfilled for every $\bar{y} \in M(\bar{x})$.

Proof. The proof is based on the lower subdifferential inclusion

$$0 \in \partial \varphi_0(\bar{x}) + N(\bar{z}; F^{-1}(\Theta)) + N(\bar{x}; \Omega)$$

from Theorem 3.2(ii) and the usage of calculus rules as in the proof of Theorem 4.1(iii). □

Both upper and lower subdifferential conditions obtained and Theorems 4.1 and 4.2 allow significant simplifications when the operator constraints in (4.1) are given by single-valued and *strictly Lipschitzian* mappings.

Corollary 4.3 (optimality conditions under strictly Lipschitzian constraints). *Let $\bar{x}$ be a local optimal solution to problem (4.1), where X and Y are Asplund, Ω and Θ are closed, and $F = f: X \to Y$ is single-valued and strictly Lipschitzian around $\bar{x}$. Then there exists $\lambda_0 \geq 0$ such that for every $x^* \in \widehat{\partial}^+ \varphi_0(\bar{x})$ there is $y^* \in N(f(\bar{x}); \Theta)$ satisfying*

$$-\lambda_0 x^* \in \partial \langle y^*, f \rangle(\bar{x}) + N(\bar{x}; \Omega), \quad (\lambda_0, y^*) \neq 0,$$

provided that one of the following properties is fulfilled:
  (a) *Ω is SNC at $\bar{x}$ and f^{-1} is PSNC at $(f(\bar{x}), \bar{x})$;*
  (b) *Θ is SNC at $f(\bar{x})$.*

If in addition φ_0 is Lipschitz continuous around $\bar{x}$, then there are $\lambda_0 \geq 0$ and $y^* \in N(f(\bar{x}); \Theta)$ satisfying

$$0 \in \lambda_0 \partial \varphi_0(\bar{x}) + \partial \langle y^*, f \rangle(\bar{x}) + N(\bar{x}; \Omega), \quad (\lambda_0, y^*) \neq 0,$$

provided that either (a) or (b) holds.

Proof. These results follow from Theorems 4.1(iii) and 4.2, respectively, due to the normal scalarization formula (2.8), which ensures that $x_1^* = 0$ if $y^* = 0$ in the conditions above. In this case the requirements in (b) and (c) of Theorem 4.1 reduce to the SNC property of Θ at $f(\bar{x})$, since f is automatically PSNC $\bar{x}$ due to its locally Lipschitz continuity. Let us mention that the SNC property of f in (d) of Theorem 4.1 is redundant for the case of strictly Lipschitzian mappings. Indeed, one can show by using the classical Josefson-Nissenzweig theorem on the weak* convergence to zero of some sequence of unit dual vectors in every infinite-dimensional Banach space (see, e.g., the proof

of [21, Theorem 5.1]) that a strictly Lipschitzian mapping $f: X \to Y$ is SNC at $\bar{x}$ *if and only if Y is finite-dimensional.* Thus properties (a)–(d) in Theorem 4.1 reduce to (a) and (b) in the corollary. $\square$

The lower subdifferential optimality conditions of Theorem 4.2 and Corollary 4.3 improve and extend previous results obtained for minimization problems in terms of basic normals and subgradients under consideration in finite-dimensional and Asplund space settings; see, in particular, the results and comments in [3, 12, 14, 25] and the references therein. The upper subdifferential results obtained above seem to be new in the optimization theory.

Let us consider a special class of problems (4.1) concerning *nondifferentiable programming* with finitely many *functional constraints* if equality and inequality types given by

$$\begin{cases} \text{minimize } \varphi_0(x) \text{ subject to } x \in \Omega, \\ \varphi_i(x) \leq 0, \quad i = 1, \ldots, m, \\ \varphi_i(x) = 0, \quad i = m+1, \ldots, m+r, \end{cases} \tag{4.14}$$

where $\varphi_i: X \to I\!\!R$ for $i = 0, \ldots, m+r$ and $\Omega \subset X$. The latter problem corresponds to (4.1) with the single-valued mapping $F = (\varphi_1, \ldots, \varphi_{m+r}): X \to I\!\!R^{m+r}$ and the closed convex cone

$$\Theta := \Big\{ (\alpha_1, \ldots, \alpha_{m+r}) \in I\!\!R^{m+r} \Big| \ \alpha_i \leq 0 \ \text{for} \ i = 1, \ldots, m \quad \text{and} \\ \alpha_i = 0 \ \text{for} \ i = m+1, \ldots, m+r \Big\}. \tag{4.15}$$

Taking into account relationships (2.8) and (2.9) between the coderivatives and subdifferentials, one can easily deduce from Theorems 4.1, 4.2 and Corollary 4.3 necessary optimality conditions for (4.14) involving (lower) basic and singular subgradients of the constraint functions; cf. [14, Theorem 5.1].

Now we present new necessary optimality conditions of the *upper subdifferential type* specific for problems (4.14), which involve Fréchet upper subgradients not only of cost functions but also of those describing *inequality constraints*. To proceed, we use *variational descriptions* of Fréchet subgradients in a subclass of Asplund spaces admitting *Lipschitzian C^1 bump functions*, which is automatic in Banach spaces with Fréchet differentiable renorms, in particular, in any reflexive space; see [7] and its references.

Theorem 4.4 (upper subdifferential conditions in nondifferentiable programming). *Let $\bar{x}$ be a local optimal solution to problem (4.14), where the set Ω is locally closed around $\bar{x}$ and the functions φ_i are continuous around this point for $i = m+1, \ldots, m+r$. Suppose also that X admits a Lipschitzian C^1 bump function and that either Ω or $f := (\varphi_{m+1}, \ldots, \varphi_{m+r})$ is SNC at $\bar{x}$. Then for any Fréchet supergradients $x_i^* \in \widehat{\partial}^+ \varphi_i(\bar{x})$, $i = 0, \ldots, m$, there are $(\lambda_0, \ldots, \lambda_{m+r}) \in I\!\!R^{m+r+1}$, $x^* \in D^* f(\bar{x})(\lambda_{m+1}, \ldots, \lambda_{m+r})$, and $\widetilde{x}^* \in N(\bar{x}; \Omega)$ satisfying the relations*

$$\lambda_i \geq 0 \ \text{for} \ i = 0, \ldots, m, \quad \lambda_i \varphi_i(\bar{x}) = 0 \ \text{for} \ i = 1, \ldots, m, \tag{4.16}$$

$$0 = \sum_{i=0}^{m} \lambda_i x_i^* + x^* + \widetilde{x}^*, \quad (\lambda_0, \ldots, \lambda_{m+r}, x^*) \neq 0. \tag{4.17}$$

If φ_i are Lipschitz continuous around $\bar{x}$ for $i = m+1, \ldots, m+r$, then in addition to (4.16) one has

$$-\sum_{i=0}^{m} \lambda_i x_i^* \in \partial \Big(\sum_{i=m+1}^{m+r} \lambda_i \varphi_i \Big)(\bar{x}) + N(\bar{x}; \Omega), \quad (\lambda_0, \ldots, \lambda_{m+r}) \neq 0, \tag{4.18}$$

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with no other assumptions on (φ_i, Ω) besides the local closedness of Ω.

Proof. Take arbitrary $x_i^* \in \widehat{\partial}^+ \varphi_i(\bar{x})$ for $i = 0, \ldots, m$ and apply the variational description from [7, Theorem 4.6(ii)] with $\mathcal{S} = \mathcal{LC}^1$ therein to the Fréchet subgradients $-x_i^* \in \widehat{\partial}(-\varphi_i)(\bar{x})$. In this way we find functions $s_i \colon X \to \mathbb{R}$ for $i = 0, \ldots, m$ satisfying $s_i(\bar{x}) = \varphi_i(\bar{x})$, $s_i(x) \geq \varphi_i(x)$ for all x from some neighborhood of $\bar{x}$, and such that each $s_i(x)$ is continuously differentiable around $\bar{x}$ with $\nabla s_i(\bar{x}) = x_i^*$. It is easy to check that $\bar{x}$ is a local solution to the following optimization problem of type (4.14), where the cost and inequality constraint functions are continuously differentiable around this point:

$$\begin{cases} \text{minimize} \quad s_0(x) \quad \text{subject to } x \in \Omega, \\ s_i(x) \leq 0, \quad i = 1, \ldots, m, \\ \varphi_i(x) = 0, \quad i = m+1, \ldots, m+r. \end{cases} \tag{4.19}$$

Apply now the necessary conditions of Theorem 4.1(iii) to problem (4.19), which corresponds to (4.1) with the single-valued mapping $F := (s_1, \ldots, s_m, \varphi_{m+1}, \ldots, \varphi_{m+r})$ and the set Θ defined in (4.15). Observe that

$$N((\varphi_1(\bar{x}), \ldots, \varphi_{m+r}(\bar{x})); \Theta) = \Big\{ (\lambda_1, \ldots, \lambda_{m+r}) \in \mathbb{R}^{m+r} \Big| \lambda_i \geq 0, \\ \lambda_i \varphi_i(\bar{x}) = 0 \text{ for } i = 1, \ldots, m \Big\}$$

with $s_i(\bar{x}) = \varphi_i(\bar{x})$, $i = 1, \ldots, m$, and that

$$F(x) = (s(x), 0) + (0, \varphi_{m+1}(x), \ldots, \varphi_{m+r}(x)) \tag{4.20}$$

for the above F, where $s := (s_1, \ldots, s_m) \colon X \to \mathbb{R}^m$ is continuously differentiable around $\bar{x}$. Thus the condition $y^* \in N(\bar{y}; \Theta)$ in Theorem 4.1(iii) with $y^* = (\lambda_1, \ldots, \lambda_{m+r})$ reduces to the sign and complementary slackness conditions in (4.16) as $i = 1, \ldots, m$. Since $Y = \mathbb{R}^{m+r}$ in Theorem 4.1(iii), one can directly check that the SNC and PSNC properties of F in (4.20) are equivalent to the SNC property of $f = (\varphi_{m+1}, \ldots, \varphi_{m+r})$. It is easy also to see that one of the requirements (a)–(d) in Theorem 4.1(iii) holds if and only if either Ω or f is SNC at $\bar{x}$. Using the smoothness of the function s in sum (4.20), we can show that relation (4.4) with $x_1^* \in D^* F(\bar{x}, \bar{y})(y^*)$ and $x_2^* \in N(\bar{x}; \Omega)$ therein is equivalent to

$$0 = \sum_{i=0}^m \lambda_i \nabla s_i(\bar{x}) + x^* + \widetilde{x}^*, \quad (\lambda_0, \ldots, \lambda_{m+r}, \widetilde{x}^*) \neq 0,$$

with $x^* \in D^* f(\bar{x})(\lambda_{m+1}, \ldots, \lambda_{m+r})$, $\widetilde{x}^* \in N(\bar{x}; \Omega)$, and $\lambda_0 \geq 0$. Recalling that $\nabla s_i(\bar{x}) = x_i^*$ for $i = 0, \ldots, m$, we arrive at (4.17). To derive (4.18) from (4.17) when φ_i are locally Lipschitzian for $i = m+1, \ldots, m+r$, it is sufficient to observe that f is automatically SNC at $\bar{x}$ in this case and then to apply the (common) scalarization formula in (2.8) to the coderivative $D^* f(\bar{x})$, which gives

$$D^* f(\bar{x})(\lambda_{m+1}, \ldots, \lambda_{m+r}) = \partial \Big(\sum_{i=m+1}^{m+r} \lambda_i \varphi_i \Big)(\bar{x})$$

and completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5 Mathematical Programs with Equilibrium Constraints

In this section we consider a special class of optimization problems known as *mathematical programs with equilibrium constraints* (MPEC). A characteristic feature of these problems is the presence, among other constraints, "equilibrium constraints" of the type $y \in S(x)$, where $S(x)$ usually represents the solution map to a "lower-level" problem of parametric optimization. MPEC naturally appear in various aspects of hierarchical optimization and equilibrium theory as well as in many practical applications, especially those related to mechanical and economic modeling. We refer the reader to the books [11, 23] for systematic expositions, examples, and applications of such problems in finite-dimensional spaces.

A general class of MPEC considered in this section is given in the following *abstract form*:

$$\text{minimize } \varphi(x,y) \text{ subject to } y \in S(x), \ x \in \Omega, \tag{5.1}$$

where $S: X \rightrightarrows Y$ be a set-valued mapping between Banach spaces, $\varphi: X \to \overline{\mathbb{R}}$, and $\Omega \subset X$. Note that this is an optimization problem with respect to both variables x and y although the constraints on them are given in different forms. The crux of the matter is the presence of the *equilibrium constraints* $y \in S(x)$ on the decision variable y, where the sets $S(x)$ typically describe the so-called *solution maps* to parametric variational inequalities and complementarity problems of various types.

Our main attention is paid to the case when the equilibrium map S is given in the form

$$S(x) := \{ y \in Y \mid 0 \in f(x,y) + Q(x,y) \} \tag{5.2}$$

with $f: X \times Y \to Z$ and $Q: X \times Y \rightrightarrows Z$, i.e., S describes solution maps to the *parametric variational systems/generalized equations* defined by

$$0 \in f(x,y) + Q(x,y).$$

Such a model covers solution maps to the classical variational inequalities and complementarity problems as well as to their various extensions and modifications. We refer the reader to [22, 27, 28] and the bibliographies therein for first-order necessary optimality conditions obtained for important special cases of finite-dimensional MPEC problems of type (5.1), (5.2) that particularly involve basic normals, subgradients, and coderivatives of the initial data. In what follows we derive new optimality conditions in both lower and upper subdifferential forms for general MPEC problems and some of their specifications.

Let us first consider problem (5.1). It can be reduced to the standard form (3.4) with *two* geometric constraints given in spaces with *product structures*. Based on Theorem 3.2, we derive the two types of subdifferential optimality conditions of the *normal* type under mild constraint qualifications involving the *mixed* coderivative of S. For simplicity we assume the Lipschitz continuity of the cost function φ in the case of lower subdifferential conditions. Note again that in all the presented results the SNC/PSNC assumptions are *automatic* if the spaces in question are finite-dimensional.

Theorem 5.1 (upper and lower subdifferential conditions for abstract MPEC). *Let $(\bar{x}, \bar{y})$ be a local optimal solution to problem (5.1). Assume that the spaces X and Y are Asplund and that*

the sets gph S *and* Ω *are locally closed around* $(\bar{x}, \bar{y})$ *and* $\bar{x}$, *respectively. Assume also that either* S *is PSNC at* $(\bar{x}, \bar{y})$ *or* Ω *is SNC at* $\bar{x}$, *and that the mixed qualification condition*

$$D_M^* S(\bar{x}, \bar{y})(0) \cap (-N(\bar{x}; \Omega)) = \{0\} \tag{5.3}$$

is fulfilled. Then one has

$$-x^* \in D_N^* S(\bar{x}, \bar{y})(y^*) + N(\bar{x}; \Omega) \tag{5.4}$$

for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$. *In in addition* φ *is supposed to be Lipschitz continuous around* $(\bar{x}, \bar{y})$, *then there is* $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$ *such that* (5.4) *holds.*

Proof. Observe that $\bar{z} = (\bar{x}, \bar{y})$ provides a local minimum to the function φ subject to the constraints $z = (x, y) \in \Omega_1 := \mathrm{gph}\, S$ and $z \in \Omega_2 := \Omega \times Y$ in the Asplund space $X \times Y$. Applying the upper subdifferential conditions of Theorem 3.2(i) to the latter problem, one can easily see that the PSNC property of Ω_1 at $\bar{z}$ with respect to X reduces to the PSNC property of the mapping S at this point, and that Ω_2 is always strongly PSNC at $\bar{z}$ with respect to Y being also SNC at this point if and only if Ω is SNC at $\bar{x}$. Moreover, the mixed qualification condition (5.3) clearly implies that the set system $\{\Omega_1, \Omega_2\}$ satisfies the limiting qualification condition (3.5) at $\bar{z}$. Thus we have, by Theorem 3.2(i), that

$$-\widehat{\partial}^+ \varphi(\bar{x}, \bar{y}) \subset N((\bar{x}, \bar{y}); \mathrm{gph}\, S) + N(\bar{x}; \Omega) \times \{0\},$$

which surely implies the upper subdifferential condition (5.4) for *every* $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$.

If φ is additionally assumed to be locally Lipschitzian around $(\bar{x}, \bar{y})$, it is automatically SNEC at this point and the qualification condition (3.7) holds. This we have

$$(0, 0) \in \partial \varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \mathrm{gph}\, S) + N(\bar{x}; \Omega) \times \{0\}$$

by Theorem 3.2(ii), which implies (5.4) with *some* $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$. □

Observe that, due to [13, Theorem 3.3], the equilibrium constraint map S is PSNC at $(\bar{x}, \bar{y})$ and the mixed qualification condition (5.3) automatically holds if S satisfies the Aubin *Lipschitz-like* (or "pseudo-Lipschitzian") property, which therefore is a *constraint qualification* ensuring the *normal form* of both lower and upper subdifferential optimality conditions for general MPEC. The reader can find efficient conditions for the Lipschitz-like property of variational systems (5.2) and their specifications in [16, 22, 27, 28] and the references therein.

Note also that the optimality conditions in the normal form of Theorem 5.1 easily imply the ones in the non-qualified (Fritz John) form with *no constraint qualification* (5.3). In the case of upper subdifferential conditions we have $\lambda \in \{0, 1\}$ such that for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there exist $x_1^* \in D_N^* S(\bar{x}, \bar{y})(y^*)$ and $x_2^* \in N(\bar{x}; \Omega)$ satisfying

$$\lambda x^* + x_1^* + x_2^* = 0, \quad (\lambda, x_1^*) \neq 0, \tag{5.5}$$

provided that either S is PSNC at $(\bar{x}, \bar{y})$ or Ω is SNC at $\bar{x}$. Indeed, (5.5) reduces to (5.4) with $\lambda = 1$ when the constraint qualification (5.3) is imposed. The negation of (5.3) implies (5.5) with $\lambda = 0$, since $D_M^* S(\bar{x}, \bar{y}) \subset D_N^* S(\bar{x}, \bar{y})$. Similarly, (5.5) gives a non-qualified lower subdifferential

condition with some $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ when φ is locally Lipschitzian. In what follows we present only normal/qualified conditions for MPEC problems.

Next let us consider a general class of MPEC problems with equilibrium constraints governed by *parameter-dependent variational systems* of type (5.2), i.e., the MPEC given by:

$$\text{minimize } \varphi(x, y) \text{ subject to } 0 \in f(x, y) + Q(x, y), \quad x \in \Omega. \tag{5.6}$$

Based on Theorem 5.1, we derive both upper and lower subdifferential conditions for problem (5.6) employing recent results of [16] on computing and estimating coderivatives of solution maps (5.2) together with SNC calculus rules in infinite dimensions.

Theorem 5.2 (upper and lower subdifferential conditions for MPEC with general variational constraints). *Let $(\bar{x}, \bar{y})$ be a local optimal solution to* (5.6), *where $f: X \times Y \to Z$ and $Q: X \times Y \rightrightarrows Z$ are mappings between Asplund spaces. Assume that f is continuous around $(\bar{x}, \bar{y})$, that Ω is locally closed around $\bar{x}$, and that the graph of Q is locally closed around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} := -f(\bar{x}, \bar{y})$. Suppose also that one of the following assumptions* (a)-(c) *holds:*

(a) *Ω and Q are SNC at $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{z})$, respectively, and the two qualification conditions*

$$\left[(x^*, 0) \in D_N^* f(\bar{x}, \bar{y})(z^*) + D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*), \quad -x^* \in N(\bar{x}; \Omega) \right] \implies x^* = 0, \tag{5.7}$$

$$\left[(x^*, y^*) \in D_N^* f(\bar{x}, \bar{y})(z^*) \cap \left(-D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*) \right) \right] \implies x^* = y^* = z^* = 0 \tag{5.8}$$

are satisfied; the latter is equivalent to

$$\left[0 \in \partial\langle z^*, f\rangle(\bar{x}, \bar{y}) + D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*) \right] \implies z^* = 0 \tag{5.9}$$

when f is strictly Lipschitzian around $(\bar{x}, \bar{y})$.

(b) *Ω is SNC at $\bar{x}$, $\dim Z < \infty$, f is Lipschitz continuous around $(\bar{x}, \bar{y})$, and the qualification conditions*

$$\left[(x^*, 0) \in \partial\langle z^*, f\rangle(\bar{x}, \bar{y}) + D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*), \quad -x^* \in N(\bar{x}; \Omega) \right] \implies x^* = 0$$

and (5.9) *are satisfied.*

(c) *Q is SNC at $(\bar{x}, \bar{y}, \bar{z})$, f is PSNC at $(\bar{x}, \bar{y})$ (which is automatic when it is Lipschitz continuous around this point), and the qualification conditions* (5.7) *and* (5.8) *hold.*

Then for every $(x^*, y^*) \in \widehat{\partial}^+\varphi(\bar{x}, \bar{y})$ there are $\widetilde{x}^* \in N(\bar{x}; \Omega)$ and $z^* \in Z^*$ such that

$$(-x^* - \widetilde{x}^*, -y^*) \in D_N^* f(\bar{x}, \bar{y})(z^*) + D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*). \tag{5.10}$$

If in addition φ is Lipschitz continuous around $(\bar{x}, \bar{y})$, then (5.10) *is satisfied for some $z^* \in Z^*$, $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$, and $\widetilde{x}^* \in N(\bar{x}; \Omega)$.*

Proof. Let us apply the upper subdifferential optimality conditions from Theorem 5.1 to problem (5.6), i.e., in the case when the equilibrium constraints $y \in S(x)$ are given in the variational/generalized equation form (5.2). It is easy to see that the continuity and closedness assumptions made on f and Q ensure the local closedness of S. To proceed further, we first assume

that Ω is SNC at $\bar{x}$ and use the coderivative upper estimate for such mappings S obtained in [16, Theorem 4.1]. This gives the inclusion

$$D_N^* S(\bar{x}, \bar{y})(y^*) \subset \left\{ x^* \in X^* \middle| \begin{array}{l} \exists z^* \in Z^* \text{ with} \\ (x^*, -y^*) \in D_N^* f(\bar{x}, \bar{y})(z^*) + D_N^* Q(\bar{x}, \bar{y}, \bar{z})(z^*) \end{array} \right\} \tag{5.11}$$

under the qualification condition (5.8) and the assumptions on either Q made in (a) or on f and Z made in (b). Then substituting (5.11) into (5.3) and (5.4), we arrive at at the upper subdifferential optimality conditions of the theorem under the assumptions made in either (a) or (b).

Now we consider the remaining case when S is PSNC in Theorem 5.1 and provide efficient conditions in terms of f and Q ensuring the latter (even SNC) property for the equilibrium map S given in (5.2). Observe that the graph of S is represented as the inverse image

$$\text{gph}\, S = g^{-1}(\text{gph}\, Q) \quad \text{with} \quad g(x, y) := (x, y, -f(x, y)). \tag{5.12}$$

Applying [20, Theorem 3.8] in this setting, we check that the graph of S is SNC at $(\bar{x}, \bar{y})$ if the qualification condition (5.8) is satisfied and the mapping g in (5.12) is PSNC at $(\bar{x}, \bar{y})$. Let us show that the latter is equivalent to the PSNC property of f at this point in the Asplund space setting. Indeed, taking sequences $(x_k^*, y_k^*) \in \widehat{D}^* g(x_k, y_k)(u_k^*, v_k^*, z_k^*)$ with $(x_k^*, y_k^*) \xrightarrow{w^*} (0, 0)$ and $\|(u_k^*, v_k^*, z_k^*)\| \to 0$, we get

$$(x_k^*, y_k^*) = (u_k^*, v_k^*) + (\hat{x}_k^*, \hat{y}_k^*) \quad \text{with} \quad (\hat{x}_k^*, \hat{y}_k^*) \in \widehat{D}^* f(x_k, y_k)(-z_k^*)$$

due to the representation

$$g(x, y) = (x, y, 0) + (0, 0, -f(x, y))$$

and the elementary equality rule for representing $\widehat{D}^* g(x_k, y_k)$ in the above sum. This implies that $(\hat{x}_k^*, \hat{y}_k^*) \xrightarrow{w^*} (0, 0)$, and hence $\|(\hat{x}_k^*, \hat{y}_k^*)\| \to 0$ by the PSNC property of f. Thus $\|(x_k^*, y_k^*)\| \to 0$ as well, i.e., g is PSNC at $(\bar{x}, \bar{y})$. This ends the proof of the upper subdifferential part in the theorem.

The last (lower subdifferential) statement of the theorem follows from the lower subdifferential result of Theorem 5.1 by the above arguments. $\qquad \square$

In MPEC problems most interesting for the theory and applications, equilibrium/variational constraints are usually defined via first-order *subdifferentials of extended-real-valued functions*. In particular, the classical equilibrium constraints given by parametric variational inequalities and complementarity conditions are naturally defined in terms of subgradients and normals for convex functions and sets. Let us consider a broader class of such MPEC with equilibrium constraints defined via the basic *subdifferential* (2.3) of *composite functions* with no convexity assumptions:

$$\text{minimize } \varphi(x, y) \text{ subject to } 0 \in f(x, y) + \partial(\psi \circ g)(x, y), \quad x \in \Omega, \tag{5.13}$$

where $f: X \times Y \to X^* \times Y^*$ and $g: X \times Y \to W$ are single-valued mappings between Banach spaces, and where $\psi: W \to \overline{I\!R}$ is an extended-real-valued function. The MPEC problem (5.13) is a special case of (5.6) with the subdifferential set-valued mapping $Q(x, y) = \partial(\psi \circ g)(x, y)$. Since coderivatives of first-order subdifferential mappings define second-order subdifferentials as in (2.10),

one can therefore deduce necessary optimality conditions for (5.13) from the ones for (5.6) obtained in Theorem 5.2 using *second-order subdifferential chain rules*. Let us present some upper and lower subdifferential optimality conditions obtained in this way for MPEC problems of type (5.13). First we consider the case of smooth and parameter-independent mappings $g: Y \to W$ in (5.13) with *surjective derivatives* in infinite-dimensional settings.

Theorem 5.3 (optimality conditions for MPEC governed by parameter-independent generalized variational inequalities). *Let $(\bar{x}, \bar{y})$ be a local optimal solution to problem* (5.13) *with $f: X \to Y$, $g: Y \to W$, and $\psi: W \to \overline{\mathbb{R}}$. Suppose that W is Banach, X is Asplund, Y is finite-dimensional and that the following assumptions hold:*

(a) *$f: X \times Y \to Y^*$ is strictly differentiable at $(\bar{x}, \bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x}, \bar{y}): X \to Y^*$.*

(b) *g is continuously differentiable around $\bar{y}$ with the surjective derivative $\nabla g(\bar{y}): Y \to W$, and the mapping $\nabla g: Y \to \mathcal{L}(Y, W)$ into the space of linear bounded operators from Y to W is strictly differentiable at $\bar{y}$.*

(c) *Ω is locally closed around $\bar{x}$ and the graph of $\partial \psi$ is locally closed around $(\bar{w}, \bar{v})$, where $\bar{w} := g(\bar{y})$ and where $\bar{v} \in W^*$ is a unique functional satisfying the relations*

$$-f(\bar{x}, \bar{y}) = \nabla g(\bar{y})^* \bar{v}, \quad \bar{v} \in \partial \psi(\bar{w}).$$

Then for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there is $u \in Y$ such that

$$
\begin{aligned}
&-x^* \in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega) \quad and \\
&-y^* \in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla^2 \langle \bar{v}, g \rangle(\bar{y})^* u + \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{y}) u)
\end{aligned}
\tag{5.14}
$$

provided that $u = 0$ is the only vector satisfying the system of inclusions

$$
\begin{cases}
0 \in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega), \\
0 \in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla^2 \langle \bar{v}, g \rangle(\bar{y})^* u + \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{y}) u).
\end{cases}
$$

In in addition φ is locally Lipschitzian around $(\bar{x}, \bar{y})$, then there are $u \in Y$ and $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$ satisfying (5.14).

Proof. To establish the upper subdifferential conditions of the theorem, we employ the results of Theorem 5.2 under the assumptions in (c) for $Q(y) := \partial(\psi \circ g)(y)$. Taking into account the strict differentiability of f at $(\bar{x}, \bar{y})$ with the surjectivity of $\nabla_x f(\bar{x}, \bar{y})$ and the parameter-independence of Q, one has the qualification condition (5.8) automatically fulfilled, while (5.7) reduced to

$$\left[0 \in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega), \; 0 \in \nabla_y f(\bar{x}, \bar{y})^* u + \partial^2(\psi \circ g)(\bar{y}, \bar{z})(u) \right] \implies u = 0$$

with $\bar{z} := -f(\bar{x}, \bar{y})$ provided that the mapping $\partial(\psi \circ g)(\cdot)$ is locally closed-graph around $(\bar{y}, \bar{z})$. Observe the SNC property of Q and PSNC property of f at the reference points follow immediately from the the finite dimensionality of Y and the strict differentiability of f. Then, by the superdifferential optimality condition of Theorem 5.2 applying to (5.13), for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there is $u \in Y$ such that

$$-x^* \in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega), \quad -y^* \in \nabla_y f(\bar{x}, \bar{y})^* u + \partial^2(\psi \circ g)(\bar{y}, \bar{z})(u).$$

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Using now the first-order subdifferential chain rule of [21, Corollary 3.11] in the case of inner mappings g with the surjective derivative at $\bar{y}$ (and hence at y near $\bar{y}$), we have the equality

$$\partial(\psi \circ g)(y) = \nabla g(y)^* \partial\psi(w)$$

for all y close to $\bar{y}$ and $w = g(y)$, which implies that the graph of $\partial(\psi \circ g)(\cdot)$ is locally closed around $(\bar{y}, \bar{z})$ if and only if the subdifferential mapping $\partial\psi(\cdot)$ is closed-graph around $(\bar{w}, \bar{v})$. Applying further the second-order subdifferential chain rule of [15, Theorem 4.1] to $\partial^2(\psi \circ g)(\bar{y}, \bar{z})$ and taking into account that $\nabla g(\bar{y})^{**} = \nabla g(\bar{y})$ under the assumptions made, one has

$$\partial_N^2(\psi \circ g)(\bar{y}, \bar{z})(u) = \nabla^2 \langle \bar{v}, g \rangle(\bar{y})^* u + \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{y})u).$$

Substituting this into the above relationships, we arrive at the upper subdifferential conditions stated in the theorem. If φ is locally Lipschitzian around $(\bar{x}, \bar{y})$, the lower subdifferential result of the theorem is deduced by a similar way from the one in Theorem 5.2. □

Note that the closed-graph assumption on $\partial\psi$ in the above theorem automatically holds for *continuous* functions ψ. It also holds for the so-called *amenable* functions, which play a major role in finite-dimensional variational analysis and optimization; see [25].

Recall that a function $\phi: X \to \overline{I\!R}$ is *amenable* at $\bar{x}$ if there is a neighborhood U of $\bar{x}$ on which ψ can be represented in the composition form $\phi = \psi \circ g$ with a C^1 mapping $g: U \to I\!R^m$ and a proper l.s.c. convex function $\psi: I\!R^m \to \overline{I\!R}$ satisfying the qualification condition

$$\partial\psi^\infty(g(\bar{x})) \cap \ker \nabla g(\bar{x})^* = \{0\}.$$

It is *strongly amenable* at $\bar{x}$ if such a representation exists with g not just C^1 but C^2. Our next theorem contains upper and lower subdifferential optimality conditions for MPEC (5.13) with *parameter-dependent potentials* $\phi(x, y) := (\psi \circ g)(x, y)$ given by strongly amenable functions.

Theorem 5.4 (optimality conditions for MPEC with parameter-dependent amenable potentials). *Let $(\bar{x}, \bar{y})$ be a local optimal solution to problem (5.13) in finite-dimensional spaces. Assume that $\Omega \subset I\!R^n$ is locally closed around $\bar{x}$, that $f: I\!R^n \times I\!R^m \to I\!R^n \times I\!R^m$ is continuous around $(\bar{x}, \bar{y})$, and that $\phi = \psi \circ g$ is strongly amenable at this point with $g: I\!R^n \times I\!R^m \to I\!R^l$. Denote $\bar{w} := g(\bar{x}, \bar{y})$, $\bar{z} := -f(\bar{x}, \bar{y}) \in \partial(\psi \circ g)(\bar{x}, \bar{y})$,*

$$M(\bar{x}, \bar{y}) := \{\bar{v} \in I\!R^l \mid v \in \partial\psi(\bar{w}), \ \nabla g(\bar{x}, \bar{y})^* \bar{v} = \bar{z}\}$$

and impose the following second-order qualification conditions:

$$\partial^2\psi(\bar{w}, \bar{v})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\} \text{ for all } \bar{v} \in M(\bar{x}, \bar{y}),$$

$$(p, q) \in \bigcup_{\bar{v} \in M(\bar{x}, \bar{y})} \left[\nabla^2 \langle \bar{v}, g \rangle(\bar{x}, \bar{y})u + \nabla g(\bar{x}, \bar{y})^* \partial^2\psi(\bar{w}, \bar{v})(\nabla g(\bar{x}, \bar{y})u) \right]$$
$$\cap \left[-D^* f(\bar{x}, \bar{y})(u) \right] \implies (p, q, u) = (0, 0, 0), \tag{5.15}$$

$$\left[(x^*, 0) \in D^* f(\bar{x}, \bar{y})(u) + \bigcup_{\bar{v} \in M(\bar{x}, \bar{y})} \left[\nabla^2 \langle \bar{v}, g \rangle(\bar{x}, \bar{y})(u) \right.\right.$$
$$\left.\left. + \nabla g(\bar{x}, \bar{y})^* \partial^2\psi(\bar{w}, \bar{v})(\nabla g(\bar{x}, \bar{y})u) \right], \ -x^* \in N(\bar{x}; \Omega) \right] \implies x^* = 0.$$

Then for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there is $u \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$
\begin{aligned}
(-x^*, -y^*) \quad \in D^* f(\bar{x}, \bar{y})(u) + \bigcup_{\bar{v} \in M(\bar{x}, \bar{y})} \Big[\nabla^2 \langle \bar{v}, g \rangle(\bar{x}, \bar{y})(u) \\
+ \nabla g(\bar{x}, \bar{y})^* \partial^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{x}, \bar{y})u) \Big] + N(\bar{x}; \Omega).
\end{aligned} \tag{5.16}
$$

If in addition φ is Lipschitz continuous around $(\bar{x}, \bar{y})$, then there are subgradients $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$ satisfying (5.16) with some vector $u \in \mathbb{R}^n \times \mathbb{R}^m$.

Proof. It is sufficient to justify the upper subdifferential part of the theorem, because the proof of the lower subdifferential part is similar. We apply Theorem 5.2 for $Q(x, y) = \partial(\psi \circ g)$ under the assumptions in (a) in the finite-dimensional setting. Since

$$
D^* Q(\bar{x}, \bar{y}, \bar{z}) = \partial^2(\psi \circ g)(\bar{x}, \bar{y}, \bar{z}), \tag{5.17}
$$

one may employ the second-order subdifferential chain rule for $\psi \circ g$ from [15, Theorem 4.2(ii)], which is available under the assumptions made (and even in more general infinite-dimensional settings). Using the cited theorem (actually its Corollary 4.3), we have the inclusion

$$
\partial^2(\psi \circ g)(\bar{x}, \bar{y}, \bar{z})(u) \subset \bigcup_{\bar{v} \in M(\bar{x}, \bar{y})} \Big[\nabla^2 \langle \bar{v}, g \rangle(\bar{x}, \bar{y})^* u + \nabla g(\bar{x}, \bar{y})^* \partial^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{x}, \bar{y})u) \Big].
$$

Substituting this into the corresponding relationships of Theorem 5.2 with the coderivative expression (5.17), we arrive at the conclusions of the theorem. $\qquad\square$

Observe that the qualification condition (5.15) reduces to

$$
0 \in \partial \langle u, f \rangle(\bar{x}, \bar{y}) + \bigcup_{\bar{v} \in M(\bar{x}, \bar{y})} \Big[\nabla^2 \langle \bar{v}, g \rangle(\bar{x}, \bar{y})^* u + \nabla g(\bar{x}, \bar{y})^* \partial^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{x}, \bar{y})u) \Big] \implies u = 0
$$

when f is locally Lipschitzian around $(\bar{x}, \bar{y})$. It holds automatically if $g = g(y)$ and f is strictly differentiable at $(\bar{x}, \bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x}, \bar{y})$.

Finally in this paper we consider a class of MPEC problems with equilibrium constraints involving another type of *subdifferential compositions*, namely:

$$
\text{minimize } \varphi(x, y) \text{ subject to } 0 \in f(x, y) + (\partial \psi \circ g)(x, y), \quad x \in \Omega, \tag{5.18}
$$

where $g: X \times Y \to W$, $\psi: W \to \overline{\mathbb{R}}$, and $f: X \times Y \to W^*$. One can see that the main difference between the two composite forms in (5.13) and (5.18) is that the former involves the first-order subdifferential of the composite potential $\psi \circ g$, while the generalized equation in (5.18) contains a composition in its *field*/set-valued part $Q(x, y) = (\partial \psi \circ g)(x, y)$. Systems of the latter type frequently arise, e.g., in the modeling of mechanical and economic equilibria and cover, in particular, parameter-dependent *implicit complementarity* problems [22]: given $x \in \mathbb{R}^n$, find $y \in \mathbb{R}^m$ satisfying

$$
f(x, y) \geq 0, \quad y - g(x, y) \geq 0, \quad \langle f(x, y), y - g(x, y) \rangle = 0. \tag{5.19}
$$

The standard nonlinear complementarity problem corresponds to (5.19) with $g = 0$. Our next theorem contains general necessary optimality conditions in the upper and lower subdifferential forms for infinite-dimensional MPEC of type (5.18).

Theorem 5.5 (optimality conditions for general MPEC with composite fields). *Let* $(\bar{x}, \bar{y})$ *be a local optimal solution to problem (5.18) with Ω closed around $\bar{x}$, $\bar{w} := g(\bar{x}, \bar{y})$, and $\bar{z} := -f(\bar{x}, \bar{y})$. The following assertions hold:*

(i) *Assume that X, Y are Asplund while W is Banach, that $g = g(y)$ is strictly differentiable at $\bar{y}$ with the surjective derivative $\nabla g(\bar{y})$, that f is strictly differentiable at $(\bar{x}, \bar{y})$ with the surjective partial derivative $\nabla_x f(\bar{x}, \bar{y})$, and that $u = 0 \in W^{**}$ is the only element satisfying*

$$0 \in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega), \quad 0 \in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{z})(u).$$

Then for every $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there is $u \in W^{**}$ such that

$$\begin{aligned} -x^* &\in \nabla_x f(\bar{x}, \bar{y})^* u + N(\bar{x}; \Omega), \\ -y^* &\in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{z})(u) \end{aligned} \tag{5.20}$$

provided that either Ω is SNC at $\bar{x}$ or $\partial \psi$ is SNC at $(\bar{w}, \bar{z})$.

(ii) *Assume that X, Y, W, W^* are Asplund, that f and g are continuous around $(\bar{x}, \bar{y})$, that the graph of $\partial \psi$ is norm-closed around $(\bar{w}, \bar{z})$, that*

$$\partial_N^2 \psi(\bar{w}, \bar{z})(0) \cap \ker D_N^* g(\bar{x}, \bar{y}) = \{0\},$$

that $x^* = 0$ is the only element satisfying

$$(x^*, 0) \in D_N^* f(\bar{x}, \bar{y})(u) + D_N^* g(\bar{x}, \bar{y}) \circ \partial_N^2 \psi(\bar{w}, \bar{z})(u), \quad -x^* \in N(\bar{x}; \Omega)$$

for some $u \in W^{**}$, and that $(x^*, y^*, u) = (0, 0, 0)$ is the only one satisfying

$$(x^*, y^*) \in D_N^* f(\bar{x}, \bar{y})(u) \cap \left(-D_N^* g(\bar{x}, \bar{y}) \circ \partial_N^2 \psi(\bar{w}, \bar{z})(u) \right).$$

Then for every supergradient $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there are $\tilde{x}^* \in N(\bar{x}; \Omega)$ and $u \in W^{**}$ such that

$$(-x^* - \tilde{x}^*, -y^*) \in D_N^* f(\bar{x}, \bar{y})(u) + D_N^* g(\bar{x}, \bar{y}) \circ \partial_N^2 \psi(\bar{w}, \bar{z})(u) \tag{5.21}$$

provided that either f is Lipschitz continuous around $(\bar{x}, \bar{y})$ and $\dim W < \infty$, or g is PSNC at $(\bar{x}, \bar{y})$ and $\partial \psi$ is SNC at $(\bar{w}, \bar{z})$, or g is SNC at $(\bar{x}, \bar{y})$ and $\partial \psi^{-1}$ is PSNC at $(\bar{z}, \bar{w})$.

(iii) *Assume that φ is Lipschitz continuous around $(\bar{x}, \bar{y})$ in addition to the assumptions in either (i) or (ii). Then there are, respectively, $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$ and $u \in W^{**}$ satisfying (5.20) and $(x^*, y^*) \in \partial \varphi(\bar{x}, \bar{y})$, $\tilde{x}^* \in N(\bar{x}; \Omega)$, $u \in W^{**}$ satisfying (5.21).*

Proof. To justify (i), we employ the upper subdifferential result of Theorem 5.1 with the equilibrium constraints given by

$$S(x) := \{y \in Y \mid 0 \in f(x, y) + (\partial \psi \circ g)(x, y)\}.$$

Since one obviously has

$$\operatorname{gph} S = \{(x, y) \in X \times Y \mid h(x, y) \in \operatorname{gph}(\partial \psi \circ g)\} \quad \text{with} \quad h(x, y) := (y, -f(x, y))$$

and $\nabla h(\bar{x}, \bar{y})$ is surjective if and only if $\nabla_x f(\bar{x}, \bar{y})$ is, it follows from [21, Corollary 3.9] that

$$\begin{aligned} D_N^* S(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid \ &\exists u \in W^{**} \text{ with } x^* = \nabla_x f(\bar{x}, \bar{y})^* u, \\ &-y^* \in \nabla_y f(\bar{x}, \bar{y})^* u + D_N^* (\partial \psi \circ g)(\bar{y}, \bar{z})(u)\} \end{aligned} \tag{5.22}$$

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for any Banach spaces X, Y, W. Moreover, Corollary 5.3 of [21] implies that the SNC property of S at $(\bar{x}, \bar{y})$ is equivalent to the one for $\partial\psi \circ g$ at $(\bar{y}), \bar{z})$. In turn, the latter is equivalent to the SNC property of $\partial\psi$ at $(\bar{w}, \bar{z})$ by [21, Corollary 5.4], since $\nabla g(\bar{y})$ is assumed to be surjective. To complete the proof of (i), it is sufficient to employ the chain rule

$$D_N^*(\partial\psi \circ g)(\bar{y}, \bar{z})(u) = \nabla g(\bar{y})^* \partial_N^2 \psi(\bar{w}, \bar{z})(u)$$

from [21, Theorem 3.10] and substitute it into (5.22).

The proof of assertion (ii) in the Asplund space setting is based on the application of Theorem 5.2 with $Q(x, y) = (\partial\psi \circ g)(x, y)$. The sufficient conditions for the SNC property of the composition $\partial\psi \circ g$ are derived from [20, Theorem 3.8] similarly to the proof of Theorem 5.2. The lower subdifferential conditions in (iii) under the assumptions made follow from Theorems 5.2 and 5.3 by employing the above arguments. $\square$

Our final result concerns optimality conditions for MPEC (5.18) in the case of strictly differentiable mappings f and g with possible *non-surjective* derivatives when the relations of Theorem 5.5 admit essential simplifications.

Corollary 5.6 (optimality conditions for special MPEC with composite fields). *Let $(\bar{x}, \bar{y})$ be a local optimal solution to problem (5.18) with $f: X \times Y \to \mathbb{R}^m$ and $g: X \times Y \to \mathbb{R}^m$ strictly differentiable at $(\bar{x}, \bar{y})$ and with $\Omega \subset X$ closed around $\bar{x}$. Assume that X and Y are Asplund, that $\text{gph } \partial\psi$ is closed around $(\bar{w}, \bar{z})$ (which is automatic for continuous and amenable functions), that*

$$\partial^2\psi(\bar{w}, \bar{z})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\}.$$

and that the system of inclusions

$$\begin{cases} x^* \in \nabla_x f(\bar{x}, \bar{y})^* u + \nabla_x g(\bar{x}, \bar{y})^* \partial^2\psi(\bar{w}, \bar{z})(u), & -x^* \in N(\bar{x}; \Omega), \\ 0 \in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla_y g(\bar{x}, \bar{y})^* \partial^2\psi(\bar{w}, \bar{z})(u) \end{cases}$$

has only the trivial solution $x^* = u = 0$. Then for every upper subgradient $(x^*, y^*) \in \widehat{\partial}^+ \varphi(\bar{x}, \bar{y})$ there is a vector $u \in \mathbb{R}^m$ such that

$$\begin{aligned} -x^* &\in \nabla_x f(\bar{x}, \bar{y})^* u + \nabla_x g(\bar{x}, \bar{y})^* \partial^2\psi(\bar{w}, \bar{z})(u) + N(\bar{x}; \Omega), \\ -y^* &\in \nabla_y f(\bar{x}, \bar{y})^* u + \nabla_y g(\bar{y})^* \partial^2\psi(\bar{w}, \bar{z})(u). \end{aligned} \tag{5.23}$$

If in addition the cost function φ is Lipschitz continuous around $(\bar{x}, \bar{y})$, then there are a lower subgradient $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ and a vector $u \in \mathbb{R}^m$ satisfying (5.23).

Proof. This easily follows from assertions (ii) and (iii) of Theorem 5.5 due to the coderivative representation for strictly differentiable mappings; see Section 2. $\square$

In the case of finite-dimensional spaces X and Y the lower subdifferential result of Corollary 5.6 is strongly related to the necessary optimality conditions from [22, Theorem 3.1] obtained for a composite MPEC problem of type (5.18) with $\partial\psi$ replaced by a set-valued mapping of closed graph and with geometric constraints on both x and y.

To conclude this paper, we observe that MPEC problems are *intrinsically nonsmooth*, even in the simplest settings of equilibrium constraints governed by parameter-dependent variational inequalities and complementarity conditions. For models (5.13) and (5.18) this relates to the non-smoothness of the potential ψ, which is actually the indicator (extended-real-valued) function of a convex set for the case of complementarity and standard variational inequality constraints. Practical implementations of the optimality conditions obtained in Theorems 5.3–5.5 require therefore computing/estimating the second-order subdifferentials for attractive classes of nonsmooth functions ψ in (5.13) and (5.18). Efficient calculations of second-order subdifferentials and their applications to special MPEC and related problems are given in [6, 17, 22] and the references therein. Such calculations and the results obtained above allow us to extend classes of MPEC that can be efficiently handled by generalized differential methods of variational analysis.

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