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Optimization and Equilibrium Problems with Equilibrium Constraints

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OPTIMIZATION AND EQUILIBRIUM PROBLEMS WITH EQUILIBRIUM CONSTRAINTS[1]

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Abstract

The paper concerns optimization and equilibrium problems with the so-called equilibrium constraints (MPEC and EPEC), which frequently appear in applications to operations research. These classes of problems can be naturally unified in the framework of multiobjective optimization with constraints governed by parametric variational systems (generalized equations, variational inequalities, complementarity problems, etc.). We focus on necessary conditions for optimal solutions to MPECs and EPECs under general assumptions in finite-dimensional spaces. Since such problems are intrinsically nonsmooth, we use advanced tools of generalized differentiation to study optimal solutions by methods of modern variational analysis. The general results obtained are concretized for special classes of MPECs and EPECs important in applications.

Keywords: multiobjective optimization, equilibrium constraints, optimality conditions, variational analysis, generalized differentiation.

1   Introduction

This paper is devoted to the study of some classes of optimization and equilibrium problems that are particularly important for various applications in operations research, engineering, mechanics, economics, and other theoretical and practical areas. One class of such problems is known as *Mathematical Programs with Equilibrium Constraints* (MPECs). This class consists of *minimizing* real-valued functions subject to constraints given by some parametric *variational systems* (variational inequalities, complementarity problems, and the like) that often describe a certain kind of equilibrium given often (but far from always) as parametric solution sets to lower-level optimization problems. Classical representatives of such problems include bilevel programs and Stackelberg games. We refer the reader to the seminal book [2] and the recent papers [1, 13] for many results, practical examples, and discussions on MPECs, which have drawn an increasing attention of both researchers and practitioners. Another class of problems of increasing interest, known as *Equilibrium Problems with Equilibrium Constraints* (EPECs), focus on finding some *equilibrium* (rather than minimum) points subject to constraints described by parametric variational systems.

In this paper we study both classes of MPECs and EPECs from a unified viewpoint of *multiobjective optimization* with equilibrium constraints, which reduces to MPECs in the case of real-valued objective functions and gives EPECs when a (vector) objective means to find some kind of equilibrium. Our main goal is the derive necessary optimality conditions for such problems that are *intrinsically nonsmooth* and hence require generalized differentiation for their variational analysis.

In Section 2 we define and discuss the basic generalized differential constructions of our study: normals to arbitrary sets, coderivatives for set-valued mappings/multifunctions, and subgradients for extended-real-valued functions in finite-dimensional spaces. We review some of their properties important for applications in this paper.

The main Section 3 concerns multiobjective problems with equilibrium constraints, where optimal solutions are understood in the sense of 'minimization" of a vector function with respect to a certain *generalized order* defined by a given subset (may be nonconic and nonconvex) of the range space. Such a generalized order optimality covers, in particular, many conventional concepts in multiobjective optimization and equilibrium. We obtain optimality conditions for multiobjective problems of this type (including those for EPECs) using the the above tools of generalized differentiation. We also briefly consider multiobjective problems, where optimization of vector functions is conducted with respect to general nonreflexive *preference relations* satisfying certain local satiation and almost transitivity requirements.

Throughout the paper we use the standard notation; see, e.g., [12]. Recall that given a set-valued mapping $F: I\!R^n \rightrightarrows I\!R^m$, its *Painlevé-Kuratowski upper/outer limit* at $\bar{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \{y \in I\!R^m |\ \exists x_k \to \bar{x},\ \exists y_k \to y\ \text{with}\ y_k \in F(x_k)\ \text{as}\ k \to \infty\}.$$

2 Tools of Variational Analysis

Let us describe the basic generalized differential constructions employed in this paper, which were introduced in [3] and then were developed and applied in many publications; see, e.g., [4, 5, 12] for more details and references. Using a geometric approach, we start with normals to sets.

Given $\Omega \subset I\!R^n$ and $\bar{x} \in \Omega$, the (basic, limiting) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x}} \Big[\operatorname{cone}(x - \Pi(x; \Omega))\Big], \tag{2.1}$$

where "cone" stands for the conic hull of a set and where $\Pi(\cdot; \Omega)$ denoted the *Euclidean projector* of x to the closure $\operatorname{cl}\Omega$, i.e.,

$$\Pi(x; \Omega) := \Big\{w \in \operatorname{cl}\Omega\Big|\ \|x - w\| = \operatorname{dist}(x; \Omega)\Big\}.$$

For convex sets this cone reduces to the normal cone of convex analysis, but it is generally nonconvex even in simple settings, e.g., for the epigraphical and graphical sets associated with nonsmooth real functions as $\Omega = \operatorname{epi}(-|x|)$ and $\Omega = \operatorname{gph}|x|$. Note that the well-known Clarke normal cone to Ω at $\bar{x}$ agrees with the convex closure to $N(\bar{x}; \Omega)$.

Given a set-valued mapping $F: I\!R^n \rightrightarrows I\!R^m$ and a point $(\bar{x}, \bar{y})$ from its graph $\operatorname{gph} F := \{(x, y)|\ y \in F(x)\}$, the *coderivative* $D^*F(\bar{x}, \bar{y}): I\!R^m \rightrightarrows I\!R^n$ of F at $(\bar{x}, \bar{y})$ is defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in I\!R^n |\ (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)\}. \tag{2.2}$$

In general, $D^*F(\bar{x}, \bar{y})(\cdot)$ is a positively homogeneous mapping that reduces to the adjoint Jacobian

$$D^*F(\bar{x})(y^*) = \{\nabla F(\bar{x})^* y^*\}, \quad \bar{y} = F(\bar{x}), \quad y^* \in I\!R^m,$$

when F is single-valued and strictly differentiable at $\bar{x}$ (in particular, C^1).

Given an extended-real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$ finite at $\bar{x}$, we define its *basic subdifferential* ∂ and *singular subdifferential* ∂^∞ at this point by

$$\partial\varphi(\bar{x}) := D^*E_\varphi(\bar{x}, \varphi(\bar{x}))(1) \quad \text{and} \quad \partial^\infty\varphi(\bar{x}) := D^*E_\varphi(\bar{x}, \varphi(\bar{x}))(0) \tag{2.3}$$

via the coderivative of the epigraphical multifunction $E_\varphi(x) := \{\mu \in \mathbb{R}| \ \mu \geq \varphi(\bar{x})\}$. There are various equivalent descriptions of the constructions (2.1)–2.3) that can be found in [4, 5, 12]. Note that $\partial^\infty\varphi(\bar{x}) = \{0\}$ and $\partial\varphi(\bar{x}) \neq \emptyset$ if φ is Lipschitz continuous around $\bar{x}$, and that

$$D^*F(\bar{x})(y^*) = \partial\langle y^*, F\rangle(\bar{x}) \neq \emptyset \quad \text{for all} \quad y^* \in \mathbb{R}^m \tag{2.4}$$

when the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is single-valued and locally Lipschitzian around this point.

Among the most important advantages of our basic tools (2.1)–(2.3) in comparison with other generalized differential constructions of nonsmooth variational analysis, let us mention a comprehensive *calculus* allowing one to compute and estimate normals, subgradients, and coderivatives of various compositions, and also complete *coderivative characterizations* of metric regularity, openness, and Lipschitzian properties of multifunctions that play a fundamental role in many aspects of nonlinear analysis, especially those related to optimization, stability, and equilibrium; see the books [4, 12] and the references therein. An adequate machinery to deal with nonconvex sets and associated functional objects is provided by the so-called *extremal principle*, which can be viewed as a proper variational counterpart of the classical separation principle in nonconvex settings; see the survey paper [6] for detailed discussions and various applications.

3 Multiobjective Optimization with Equilibrium Constraints

Let us first consider multiobjective problems whose optimal solutions are understood with respect to following concept of *generalized order optimality* that particularly includes conventional notions of efficiency and equilibrium in various problems of vector optimization.

Definition 3.1 (generalized order optimality). *Given a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^d$ and a set $\Theta \subset \mathbb{R}^d$ containing the origin, we say that a point $\bar{x} \in X$ is* LOCALLY (f, Θ)-OPTIMAL *if there are a neighborhood U of $\bar{x}$ and a sequence $\{z_k\} \subset \mathbb{R}^d$ with $\|z_k\| \to 0$ as $k \to \infty$ such that*

$$f(x) - f(\bar{x}) \notin \Theta - z_k \quad \text{for all} \quad x \in U \quad \text{and} \quad k \in \mathbb{N} := \{1, 2, \ldots\}. \tag{3.1}$$

The set Θ in Definition 3.1 generates an *order/preference relation* between $z_1, z_2 \in \mathbb{R}^d$ defined via $z_1 - z_2 \in \Theta$. In the scalar case of $d = 1$ and $\Theta = \mathbb{R}_-$ the above optimality notion clearly reduces to the standard local optimality. Note that we don't generally assume that Θ is either convex or its interior is nonempty. If Θ is a convex subcone of $\mathbb{R}^d$, then the above optimality concept covers the conventional notions of *Pareto-type* optimality (equilibrium, efficiency) and the like requiring that there is no $z \in U$ with $f(x) - f(\bar{x}) \in \mathrm{ri}\,\Theta$. To see this, it suffices to take $z_k := -z_0/k$ for $k \in \mathbb{N}$ in (3.1) with some $z_0 \in \mathrm{ri}\,\Theta$. In classical cases it can be expressed via utility functions.

Let us first consider local (f, Θ)-optimal points of bivariate vector functions $f(x, y)$ subject to *abstract equilibrium constraints* in the form $y \in S(x)$, where $S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is an arbitrary set-valued mapping. In this context $S(x)$ may be a collection of equilibrium points (or optimal solutions to a

lower-level problem) depending on the parameter x, while y is a decision variable in the upper-level optimization/equilibrium problem over $y \in S(x)$. The following theorem gives necessary conditions for (f, Θ)-optimal solutions $(\bar{x}, \bar{y})$ under abstract equilibrium constraints.

Theorem 3.2 (generalized order optimality subject to abstract equilibrium constraints).
Let $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ be locally (f, Θ)-optimal subject to $y \in S(x)$, where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ with $\bar{z} := f(\bar{x}, \bar{y})$, where $\Theta \subset \mathbb{R}^d$ with $0 \in \Theta$, and where $S: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Then there is $z^* \in \mathbb{R}^d$ satisfying

$$(0, -z^*) \in N((\bar{x}, \bar{y}, \bar{z}); \mathcal{E}(f, S, \Theta)), \quad z^* \in N(0, \Theta) \setminus \{0\} \tag{3.2}$$

provided that the "generalized epigraphical" set

$$\mathcal{E}(f, S, \Theta) := \{(x, y, z) \in X \times Y \times Z \mid f(x, y) - z \in \Theta, \ y \in S(x)\}$$

is closed around $(\bar{x}, \bar{y}, \bar{z})$. The latter implies

$$0 \in D^* f(\bar{x}, \bar{y})(z^*) + N((\bar{x}, \bar{y}); \operatorname{gph} S), \quad z^* \in N(0, \Theta) \setminus \{0\} \tag{3.3}$$

if f is continuous around $(\bar{x}, \bar{y})$, Θ is closed around 0, and the qualification condition

$$\left[(x^*, y^*) \in D^* f(\bar{x}, \bar{y})(0), \quad -x^* \in D^* S(\bar{x}, \bar{y})(y^*)\right] \implies x^* = y^* = 0 \tag{3.4}$$

is fulfilled. Moreover, (3.4) holds automatically and (3.2) is equivalent to

$$0 \in \partial \langle z^*, f \rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \operatorname{gph} S), \quad z^* \in N(0, \Theta) \setminus \{0\} $$

if f is Lipschitz continuous around $(\bar{x}, \bar{y})$.

Sketch of the Proof. The EPEC under consideration is equivalent to the following multiobjective optimization problem under geometric constraints: find a local (f, Θ)-optimal point $(\bar{x}, \bar{y})$ subject to $(\bar{x}, \bar{y}) \in \operatorname{gph} S$. One can check that $(\bar{x}, \bar{y}, \bar{z})$ is an *extremal point* [5] for the system of closed sets $\{\Omega_1, \Omega_2\}$ in the space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d$, where $\Omega_1 := \mathcal{E}(f, S, \Theta)$ and $\Omega_2 := \operatorname{cl} U \times \{\bar{z}\}$, and where U is from (3.1). Using the extremal principle from [5, Theorem 3.2], we arrive at (3.2). Since

$$\mathcal{E}(f, S, \Theta) = g^{-1}(\Theta) \quad \text{with} \quad g(x, y, z) := f(x, y) + \Delta((x, y); \operatorname{gph} S) - z,$$

where $\Delta(u; \Omega) = 0 \in \mathbb{R}^d$ for $u \in \Omega \subset X \times Y$ and $\Delta(u; \Omega) = \emptyset$ otherwise, we derive (3.3) from (3.2) under the qualification condition (3.4) by the calculus rules of [5, Corollaries 4.5 and 5.5]. The last statement of the theorem follows from the scalarization formula (2.4). $\square$

Next let us consider "real" equilibrium constraints governed by the parametric *variational systems/generalized equations*

$$0 \in q(x, y) + Q(x, y), \tag{3.5}$$

where $q: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and $Q: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ are, respectively, single-valued and set-valued mappings both depending on the parameter x. It is well known that model (3.5) covers a vast

majority of variational systems important in applications. In particular, (3.5) reduces to the parametric *variational inequality*

$$\text{find } y \in \Omega \text{ such that } \langle q(x,y), u - y \rangle \geq 0 \text{ for all } u \in \Omega$$

when $Q(y) = N(y; \Omega)$ is the normal cone mapping generated by a convex set $\Omega \subset \mathbb{R}^m$. This gives the classical *nonlinear complementarity* problem when $\Omega = \mathbb{R}^m_+$.

The next theorem provides necessary conditions for generalized order optimality subject to the equilibrium constraints (3.5). For simplicity we present results only in the case of locally Lipschitzian mappings f and q.

Theorem 3.3 (optimality conditions for EPECs governed by generalized equations).
Let $(\bar{x}, \bar{y})$ be locally (f, Θ)-optimal subject to the equilibrium constraints (3.5), where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ and $q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is Lipschitz continuous around $(\bar{x}, \bar{y})$ with $\bar{p} := -q(\bar{x}, \bar{y})$, and where $\Theta \subset \mathbb{R}^d$ and $\operatorname{gph} Q \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ are closed around $0 \in \Theta$ and $(\bar{x}, \bar{y}, \bar{p})$, respectively. Assume that the adjoint generalized equation

$$0 \in \partial \langle p^*, q \rangle(\bar{x}, \bar{y}) + D^* Q(\bar{x}, \bar{y}, \bar{p})(p^*) \tag{3.6}$$

has only the trivial solution $p^* = 0$. Then there are $z^* \in N(0; \Theta) \setminus \{0\}$ and $p^* \in \mathbb{R}^p$ such that

$$0 \in \partial \langle z^*, f \rangle(\bar{x}, \bar{y}) + \partial \langle p^*, q \rangle(\bar{x}, \bar{y}) + D^* Q(\bar{x}, \bar{y}, \bar{p})(p^*).$$

Proof. This follows from Theorem 3.2 with

$$S(x) := \{ y \in \mathbb{R}^m \mid 0 \in q(x,y) + Q(x,y) \}$$

due to the coderivative inclusion

$$D^* S(\bar{x}, \bar{y})(y^*) \subset \left\{ x^* \in \mathbb{R}^n \,\middle|\, \exists p^* \in \mathbb{R}^p \text{ with } (x^*, -y^*) \in \partial \langle p^*, q \rangle(\bar{x}, \bar{y}) + D^* Q(\bar{x}, \bar{y}, \bar{p})(p^*) \right\}$$

established in [8, Theorem 4.1] assuming that (3.6) has only the trivial solution. $\qquad\square$

In EPECs and MPECs most interesting for the theory and applications, equilibrium/variational constraints are usually defined via *first-order subdifferentials* of extended-real-valued functions; see, e.g., the above cases of variational inequalities and complementarity problems. Let us consider a broad class of multiobjective optimization problems with equilibrium constraints, where the multivalued part of the generalized equation (3.5) is given by the *basic subdifferential* (2.3) *of the composition* $\partial(\psi \circ g)$ involving an extended-real-valued function ψ and a mapping g. Following mechanical terminology, we call the function $\phi := \psi \circ g$ under the subdifferential operator in the generalized equation by *potential*.

To study such problems, *second-order* generalized differential constructions happen to be useful. Given $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} \partial \varphi$, define the *second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$ as the coderivative of the first-order subdifferential mapping:

$$\partial^2 \varphi(\bar{x}, \bar{y})(u) := D^* (\partial \varphi)(\bar{x}, \bar{y})(u), \quad u \in \mathbb{R}^n. \tag{3.7}$$

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Observe that for C^2 functions φ the second-order construction (3.7) reduces to the Hessian matrix

$$\partial^2 \varphi(\bar{x})(u) = \{\nabla^2 \varphi(\bar{x})u\} \quad \text{with} \quad \nabla^2 \varphi(\bar{x})^* = \nabla^2 \varphi(\bar{x}).$$

We refer the reader to [7, 10] for more results and discussions on second-order subdifferentials and their calculus. Let us now present two results on necessary optimality conditions for multiobjective problems governed by generalized equations with *composite potentials* $\psi \circ g$. The first result concerns the case of parameter-independent potentials $(\psi \circ g)(y)$ involving arbitrary functions $\psi \colon I\!\!R^n \to \overline{I\!\!R}$. Such systems relate to (generalized) *hemivariational inequalities* labelled by HVI.

Theorem 3.4 (optimality conditions for EPECs governed by HVI). *Let $(\bar{x}, \bar{y})$ be locally (f, Θ)-optimal subject to*

$$0 \in q(x, y) + \partial(\psi \circ g)(y),$$

where $f \colon I\!\!R^n \times I\!\!R^m \to I\!\!R^d$ is Lipschitz continuous around $(\bar{x}, \bar{y})$, Θ is closed around $0 \in \Theta$, $q \colon I\!\!R^n \times I\!\!R^m \to I\!\!R^m$ is strictly differentiable at $(\bar{x}, \bar{y})$ with the partial Jacobian $\nabla_x q(\bar{x}, \bar{y})$ of full rank, $g \colon I\!\!R^m \to I\!\!R^s$ is C^2 around $\bar{y}$ with the Jacobian $\nabla g(\bar{y})$ of full rank, and $\psi \colon I\!\!R^s \to \overline{I\!\!R}$. Suppose that $\mathrm{gph}\, \partial\psi$ is closed around $(\bar{w}, \bar{v})$, where $\bar{w} := g(\bar{y})$ and $\bar{v} \in I\!\!R^s$ is a unique vector satisfying

$$-q(\bar{x}, \bar{y}) = \nabla g(\bar{y})^* \bar{v}, \quad \bar{v} \in \partial\psi(\bar{w});$$

the latter assumption is automatic if ψ is either convex or continuous around $\bar{w}$. Then there are $z^* \in N(0; \Theta) \setminus \{0\}$ and $u \in I\!\!R^m$ such that

$$0 \in \partial\langle z^*, f\rangle(\bar{x}, \bar{y}) + \nabla q(\bar{x}, \bar{y})^* u + \left(0, \nabla^2\langle \bar{v}, g\rangle(\bar{y})^* u + \nabla g(\bar{y})^* \partial^2 \psi(\bar{w}, \bar{v})(\nabla g(\bar{y})u)\right).$$

Proof. This follows from Theorem 3.3 with $Q(y) = \partial(\psi \circ g)(y)$ by computing

$$D^* Q(\bar{y}, \bar{p})(u) = \partial^2(\psi \circ g)(\bar{y}, \bar{p})(u) \quad \text{with} \quad \bar{p} := -q(\bar{x}, \bar{y})$$

using the second-order subdifferential chain rule from [10, Theorem 3.4(i)]. □

The next result concerns EPECs of the above type, but with *parameter-dependent* potentials that belong to a class of functions especially important in composite optimization. Recall [12] that $\varphi \colon I\!\!R^n \to \overline{I\!\!R}$ is *strongly amenable* at $\bar{x}$ if there is a neighborhood U of $\bar{x}$ on which φ can be represented in the composition form $\varphi = \psi \circ g$ with a C^2 mapping $g \colon U \to I\!\!R^m$ and a proper lower semicontinuous convex function $\psi \colon I\!\!R^m \to \overline{I\!\!R}$ satisfying the first-order qualification condition $\partial\psi(g(\bar{x})) \cap \ker \nabla g(\bar{x})^* = \{0\}$.

Theorem 3.5 (optimality conditions for EPECs with amenable potentials). *Let $(\bar{x}, \bar{y})$ be locally (f, Θ)-optimal subject to*

$$0 \in q(x, y) + \partial(\psi \circ g)(x, y), \tag{3.8}$$

where $f \colon I\!\!R^n \times I\!\!R^m \to I\!\!R^d$ and $\Theta \subset I\!\!R^d$ are the same as in the previous theorem, where $q \colon I\!\!R^n \times I\!\!R^m \to I\!\!R^n \times I\!\!R^m$ is Lipschitz continuous around $(\bar{x}, \bar{y})$, and where the potential in (3.8) is strongly amenable at $(\bar{x}, \bar{y})$. Denote $\bar{p} := -q(\bar{x}, \bar{y}) \in \partial(\psi \circ g)$, $\bar{w} := g(\bar{x}, \bar{y})$,

$$M(\bar{x}, \bar{y}) := \{\bar{v} \in W^* | \ \bar{v} \in \partial\psi(\bar{w}), \quad \nabla g(\bar{x}, \bar{y})^* \bar{v} = \bar{p}\}$$

and impose the second-order qualification conditions:

$$\partial^2\psi(\bar{w},\bar{v})(0) \cap \ker \nabla g(\bar{x},\bar{y})^* = \{0\} \ \textit{ for all } \ \bar{v} \in M(\bar{x},\bar{y}) \ \textit{ and}$$

$$\left[0 \in \partial\langle u,q\rangle(\bar{x},\bar{y}) + \bigcup_{\bar{v}\in M(\bar{x},\bar{y})} \left[\nabla^2\langle \bar{v},g\rangle(\bar{x},\bar{y})(u) + \nabla g(\bar{x},\bar{y})^*\partial^2\psi(\bar{w},\bar{v})(\nabla g(\bar{x},\bar{y})u)\right]\right] \Longrightarrow u = 0.$$

Then there are $z^* \in N(0;\Theta) \setminus \{0\}$ *and* $u \in I\!\!R^n \times I\!\!R^m$ *satisfying*

$$0 \in \partial\langle z^*,f\rangle(\bar{x},\bar{y}) + \partial\langle u,q\rangle(\bar{x},\bar{y}) + \bigcup_{\bar{v}\in M(\bar{x},\bar{y})} \left[\nabla^2\langle \bar{v},g\rangle(\bar{x},\bar{y})(u) + \nabla g(\bar{x},\bar{y})^*\partial^2\psi(\bar{w},\bar{v})(\nabla g(\bar{x},\bar{y})u)\right].$$

Proof. It can be obtained from Theorem 3.3 with $Q(x,y) = \partial(\psi \circ g)(x,y)$ by using the second-order subdifferential chain rule for amenable functions that follows from [7, Corollary 4.3]. $\quad\square$

Let us mention another class of EPECs important for applications, where equilibrium constraints are given in the form

$$0 \in q(x,y) + (\partial\psi \circ g)(x,y).$$

The latter includes, in particular, *implicit complementarity problems*. Necessary optimality conditions for such EPECs can be derived from Theorem 3.3 and generalized differential calculus similarly to the case of MPECs in [9]. Note that the results obtained above for multiobjective optimization problems with equilibrium constraints directly imply optimality conditions for MPECs that involve minimization of real-valued functions. In the latter case, however, some special results are obtained in [9], which don't have multiobjective counterparts.

In conclusion of the paper we briefly consider multiobjective problems with equilibrium constraints, where "minimization" of vector functions $f: I\!\!R^n \times \times I\!\!R^m \to I\!\!R^d$ is conducted with respect to nonreflexive *preference relations* $\prec$ satisfying the local satiation and almost transitivity requirements formulated in [11]. Such preferences are called *closed*. Given a closed preference $\prec$ on $I\!\!R^d$, define its (moving) *level set* at $z \in I\!\!R^d$ by

$$\mathcal{L}(z) := \{w \in I\!\!R^d | \ w \prec z\},$$

which is a set-valued mapping $\mathcal{L}: I\!\!R^d \rightrightarrows I\!\!R^d$. To formulate optimality conditions for EPECs with respect to closed preferences, we need the construction of the *extended normal cone* $\widetilde{N}(\bar{z};\Omega(\bar{x}))$ to a moving set $\Omega: I\!\!R^n \rightrightarrows I\!\!R^d$ at $(\bar{x},\bar{z}) \in \mathrm{gph}\,\Omega$ given in [11, Definition 4.3].

Theorem 3.6 (optimality conditions for EPECs with closed preferences). *Let* $(\bar{x},\bar{y})$ *be a local optimal solution to the multiobjective problem:*

$$\textit{minimize } f(x,y) \textit{ with respect to } \prec \textit{ subject to } y \in S(x),$$

where $f: I\!\!R^n \times I\!\!R^m \to I\!\!R^d$ *is Lipschitz continuous around* $(\bar{x},\bar{y})$ *with* $\bar{z} := f(\bar{x},\bar{y})$, *where* $S: I\!\!R^n \rightrightarrows I\!\!R^m$ *is closed-graph around* $(\bar{x},\bar{y})$, *and where the preference* $\prec$ *is closed. Then one has*

$$0 \in \partial\langle z^*,f\rangle(\bar{x},\bar{y}) + N((\bar{x},y);\mathrm{gph}\,S) \textit{ for some } z^* \in \widetilde{N}(\bar{z};\mathrm{cl}\,\mathcal{L}(\bar{z})) \setminus \{0\}.$$

Sketch of the Proof. First we check that $(\bar{x}, \bar{y}, \bar{z})$ is a locally extremal point of the system $\{S_1, S_2\}$ in the sense of [11, Definition 3.3], where

$$S_1(z) := \operatorname{gph} S \times \operatorname{cl} \mathcal{L}(z) \quad \text{and} \quad S_2 := \operatorname{gph} f.$$

Then applying the limiting extremal principle from [11, Theorem 4.7] and the scalarization formula (2.4), we arrive at the desired necessary conditions. $\qquad \square$

Similarly to Theorems 3.3–3.5, one can derive from Theorem 3.6 necessary optimality conditions for multiobjective optimization problems, particularly for EPECs, with respect to closed preferences and the equilibrium constraints considered therein.

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