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The estimation of two parameters of the Kumaraswamy distribution is considered under Type II progressive censoring with random removals, where the number of units removed at each failure time has a binomial distribution. The *MLE* was used to obtain the estimators of the unknown parameters, and the asymptotic variance - covariance matrix was also obtained. The formula to compute the expected test time was derived. A numerical study was carried out for different combinations of model parameters. Different censoring schemes were used for the estimation, and performance of these schemes was compared.

Keywords: Expected test time, maximum likelihood estimation, progressive censoring, random removals

Introduction

Life tests are often one of the main research topics in many experimental designs. There are several situations in life testing, in reliability experiments and survival analysis in which units are lost or removed from the experiments while they are still alive. The loss may occur out of control or be reassigned. The out of control case can happen when an individual under study (testing) drops out. The other case may occur because of limitation of funds or to save time. For more details Balakrishnan and Aggarwala (2000) provide a comprehensive reference on the subject of progressive censoring and its applications. There are several types of censoring schemes; the Type II censoring scheme is one of the most common for consideration. In a Type II censoring, a total of n units are placed on test, but

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instead of containing until all n units have failed, the test is terminated at time of the m_{th} ($1 \leq m \leq n$) unit failure. Type II censoring with different failure time distributions has been studied by many authors including Mann (1971), Meeker and Escobar (1991), and Lawless (2003).

If an experimenter desires to remove live units at points other than the n^{th} termination point of the life test, the above described scheme will not be of use to the experimenter. Type II censoring does not allow for units to be lost or removed from the test at other than the n_{th} termination point. This allowance may be desirable, as in the case of accidental breakage of experimental units, in which the loss of test units at points other than the termination point may be unavoidable. Intermediate removal may also be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought. These reasons lead directly into the area of progressive censoring; see Balakrishnan and Agarwala (2000).

A generalization of Type II censoring is progressive Type II censoring. The progressive Type II censored life test is described as follows. Firstly, the experimenter places n units on a test at time zero, with m failures to be observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed. At the second observed failure, r_{th} of the surviving units are randomly selected and removed. This experiment terminates at the time when the m_{th} failure is observed and the remaining $r_m = n - r_1 - ... - r_{m-1} - m$ surviving units are all removed. The statistical inference on the parameters of failure time distributions under progressive Type II censoring has been studied by several authors, such as Cohen (1963), Mann (1971), Viveros and Balakrishnan (1974), Balakrishnan and Aggarwala (2000), Ng et al. (2002), Chan and Balakrishnan (2004), Soliman (2008) and Raqab et al. (2010) (and the references therein). Note that, in this scheme, r_1, r_2, ..., r_m are all pre-fixed. However, in some practical situations, these numbers may occur at random. Yuen and Tse (1996) indicated that, for example, the number of patients who drop out from a clinical test at each stage is random and cannot be pre-determined. In some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the pattern of removal at each failure is random. Suppose that any test unit being dropped out from the life test is independent of the others but with the same removal probability p. Then, Tse et al. (2004) indicated that the number of test units removed at each failure time has a binomial distribution. The main purpose of this article is to assess the required time to complete a life test under progressive Type II censored data with random removal (*PCR*). Assume that the lifetime follows the Kumaraswamy distribution. The number of units removed at each failure time follows a binomial distribution with parameters *n* and *p*.

The Model

The maximum likelihood estimators for the parameters of the Kumaraswamy distribution are derived based on progressive Type II censoring. Let random variable X have a Kumaraswamy distribution with two positive shape parameters α and θ. The probability density function of X is given by

$$f\left(x,\alpha,\theta\right)=\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1},0<x<1,\alpha,\theta>0 \tag{1}$$

while the cumulative distribution function is given by

$$F\left(x,\alpha,\theta\right)=1-\left(1-x^{\alpha}\right)^{\theta},0<x<1,\alpha,\theta>0 \tag{2}$$

Kumaraswamy (1980) was interested in distributions for hydrological random variables and actually proposed a mixture of a probability mass, at zero and density (1) over (0,1).

The corresponding survival function of random variable X is

$$\bar{F}(x,\alpha,\theta)=(1-x^{\alpha})^{\theta},0<x<1,\alpha,\theta>0 \tag{3}$$

and the failure (hazard) rate function takes the following form

$$h(x)=\frac{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}}{(1-x^{\alpha})^{\theta}}=\frac{\alpha\theta x^{\alpha-1}}{(1-x^{\alpha})} \tag{4}$$

For $X \geq 0$, let $X_1 < X_2 < \ldots < X_m$ be the m ordered failure times out of n randomly selected items, where m is predetermined before testing. At the i_{th} failure, r_i items are removed from the test. For progressive Type II censored sample with predetermined number of removals, say $R_1=r_1,R_2=r_2,\ldots\ldots,R_{m-1}=r_{m-1}$, where $R=$ $R_1=r_1,R_2=r_2,\ldots\ldots,R_{m-1}=r_{m-1}$.

Let (X_1, X_2, ..., X_m) denote a progressive Type II censored sample. Then the joint probability density function of all m progressive Type II censored order statistic is given by

$$
\begin{aligned}
f_{X_1,X_2,\ldots,X_m}(x_1,x_2,\ldots,x_m) &= C\Pi_{i=1}^{m} f(x_i)\left[1-F(x_i)\right]^{r_i} \\
&\quad ,x_1 < x_2 < \ldots, < x_m \\
\textbf{where} \quad C &= n\left(n-r_1-1\right)\left(n-r_1-r_2-1\right) \\
&\quad \ldots\left(n-r_1-r_2-r_3-\ldots r_{m-1}-m+1\right)
\end{aligned}
\tag{5}
$$

Thus for progressive Type II censoring with pre-determined number of removals $R = r$, the conditional likelihood function can be written as (Cohen, 1963)

$$
\begin{aligned}
L_1(x,\alpha,\theta|\mathrm{R}=\mathrm{r}) = C\Pi_{i=1}^{m} f(x_i)\left[1-F(x_i)\right]^{r_i} &= C\Pi_{i=1}^{m}\left[\alpha\theta x_{(i)}^{\alpha-1}\left(1-x_{(i)}^{\alpha}\right)^{\theta-1}\right] \\
&\left[\left(1-x_{(i)}^{\alpha}\right)\right]^{\theta r_i}
\end{aligned}
\tag{6}
$$

Equation (6) is derived conditional on r_i, where r_i can be of any integer value between 0 and $n - m - (r_1 + r_2 + \ldots + r_{m-1})$. The main difference between Type II progressive censoring and *PCR* is that the *R* are pre-determined in the former case while they are random in the latter case. Note that m is predetermined in both cases. Under *PCR*, the r_i terms are random. In particular, assume that each r_i follows a binomial distribution, such that

$$
P\left(R_1 = r_1\right) = \binom{n-m}{r_1} p^{r_1}\left(1-p\right)^{n-m-r_1}
\tag{7}
$$

and

$$
\begin{aligned}
P\left(R_i = r_i, R_{i-1} = r_{i-1}, \ldots, R_1 = r_1\right) &= \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} \\
&\quad \times p^{r_i}\left(1-p\right)^{n-m-\sum_{j=1}^{i-1} r_j}
\end{aligned}
\tag{8}
$$

where $0 \le r_i \le n - m - \left(r_1 + r_2 + \ldots + r_{m-1}\right)$.

Furthermore, assume that R_i independent of X_i for all i. Then the likelihood function can be found as

$$L\left(x,\alpha,\theta,p \mid R=r\right) = L_1\left(x,\alpha,\theta \mid R=r\right)P\left(R,p\right) \tag{9}$$

where $P(R, p)$ is the is the probability distribution of the R terms ($R = r_1, r_2, \ldots, r_m$) and, in particular, results in

$$\begin{aligned} P\left(R,p\right) = {} & P\left(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2},\ldots,R_1 = r_1\right) \\ & \times P\left(R_{m-2} = r_{m-2} \mid R_{m-3} = r_{m-3},\ldots,R_1 = r_1\right) \\ & \times P\left(R_2 = r_2 \mid R_1 = r_1\right)P\left(R_1 = r_1\right) \end{aligned} \tag{10}$$

Substituting (4) and (5) into (7) results in

$$\begin{aligned} P\left(R,p\right) = {} & \frac{\left(n-m\right)!}{\Pi_{i=1}^{m-1} r_i!\left(n-m-\sum_{j=1}^{i-1} r_j\right)!} p^{\sum_{i=1}^{m-1} r_i} \\ & \times\left(1-p\right)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \log P\left(R,p\right) = {} & C + \sum_{i=1}^{m-1} r_i \log p \\ & + \left[(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i\right]\log\left(1-p\right) \end{aligned} \tag{12}$$

Maximum Likelihood Estimation

The maximum likelihood estimators of the parameters α, θ, and p are derived based on progressive Type II censored data with binomial removals. Both point and interval estimations of the parameters are derived.

Point Estimations

Because $P(R, p)$ does not depend on the parameters α and θ, the maximum likelihood estimators (MLEs) of α and θ can be derived by maximizing (6) directly. Similarly, because $L_1(x, \alpha, \theta \mid R = r)$ does not involve the binomial parameter p, then the MLE of p can be found by maximizing $P(R, p)$ directly. The log likelihood function of (9) is given by

$$\log L\left(x,\alpha,\theta,p \mid R=r\right)=\log L_1\left(x,\alpha,\theta \mid R=r\right)+\log P\left(R,p\right)$$

where

$$\log L_1\left(x,\alpha,\theta \mid R=r\right)=\log C+m\log\alpha+m\log\theta+\left(\alpha-1\right)\sum_{i=1}^{m}\log x_{(i)}+\left(\theta-1\right)\sum_{i=1}^{m}\log\left(1-x_{(i)}^{\alpha}\right)+\theta\sum_{i=1}^{m}r_i\log\left(1-x_{(i)}^{\alpha}\right) \tag{13}$$

Take the partial derivative of $\log L_1(x, \alpha, \theta \mid R = r)$ with respect to α and θ and let them be zero

$$\frac{\partial\log\left(L_1\right)}{\partial\alpha}=\frac{m}{\alpha}+m\log\theta+\sum_{i=1}^{m}\log x_{(i)}-\left(\theta-1\right)\sum_{i=1}^{m}\frac{x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)}-\theta\sum_{i=1}^{m}\frac{r_i x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)}=0 \tag{14}$$

and $$\frac{\partial\log(L_1)}{\partial\theta}=\frac{m}{\theta}+\sum_{i=1}^{m}\log\left(1-x_{(i)}^{\alpha}\right)+\sum_{i=1}^{m}r_i\log\left(1-x_{(i)}^{\alpha}\right)=0$$

$\hat{\theta}=\dfrac{-m}{\sum_{i=1}^{m}\left(r_i+1\right)\log\left(1-x_{(i)}^{\alpha}\right)}$ is given by $\hat{\theta}$ of θ. Thus the MLE $\hat{\alpha}$ of α and the MLE is the numerical solution of equation (14).

It is observed from (14) that the MLE of the parameter α cannot be obtained in closed form. It can be obtained by solving a one dimensional optimization problem. A simple fixed point iteration algorithm can be used to solve this optimization problem. Firstly, the parameter θ in log-likelihood (13) has been replaced by its *MLE* the $\widehat{\theta}$ resultant log-likelihood becomes

$$\log L_1(x,\alpha,\theta|\mathrm{R}=\mathrm{r}) = \log C + m\log\alpha + m\log\left\{\frac{-m}{\sum_{i=1}^{m}\left(r_i+1\right)\log\left(1-x_{(i)}^{\alpha}\right)}\right\}$$
$$+(\alpha-1)\sum_{i=1}^{m}\log x_{(i)} + \left\{\frac{-m}{\sum_{i=1}^{m}\left(r_i+1\right)\log\left(1-x_{(i)}^{\alpha}\right)}-1\right\}\sum_{i=1}^{m}\log\left(1-x_{(i)}^{\alpha}\right)$$
$$+\left\{\frac{-m}{\sum_{i=1}^{m}\left(r_i+1\right)\log(1-x^{\alpha}{}_{(i)})}\right\}\sum_{i=1}^{m}r_i\log\left(1-x_{(i)}^{\alpha}\right)$$

After some simplification it can be presented as

$$\begin{aligned}\log L_1(x,\alpha\,|\,R=r) \propto m\log\alpha + m\log\left\{\sum_{i=1}^{m}\left(r_i+1\right)\log\left(1-x_{(i)}^{\alpha}\right)\right\} \\ +(\alpha-1)\sum_{i=1}^{m}\log x_{(i)} - \sum_{i=1}^{m}\left(r_i+1\right)\log\left(1-x_{(i)}^{\alpha}\right)\end{aligned} \tag{15}$$

MLE of α can be obtained by maximizing (15) with respect to α and it is unique. Most of the standard iterative process can be used for finding the *MLE*. The following simple algorithm is proposed: If $\hat{\alpha}$ is the MLE of α, then it is obvious from $l'\left(\alpha\right)=\frac{\partial\log L_1(x,\alpha\,|\,R=r)}{\partial\alpha}=0$ that $\hat{\alpha}$ satisfies the following fixed point type equation; $g\left(\alpha\right)=\alpha$

$$\frac{\partial \log L_1(x,\alpha \mid R=r)}{\partial \alpha} = \frac{m}{\alpha} - \frac{m\sum_{i=1}^{m}\frac{(r_i+1)x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)}}{\sum_{i=1}^{m}(r_i+1)\log\left(1-x_{(i)}^{\alpha}\right)} + \sum_{i=1}^{m}\log x_{(i)} - \sum_{i=1}^{m}\frac{(r_i+1)x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)} = 0 \tag{16}$$

where

$$\alpha = g(\alpha) = \left[\frac{1}{m}\left\{\sum_{i=1}^{m}\frac{(r_i+1)x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)} - \sum_{i=1}^{m}\log x_{(i)}\right\} + \frac{\sum_{i=1}^{m}\frac{(r_i+1)x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)}}{\sum_{i=1}^{m}(r_i+1)\log\left(1-x_{(i)}^{\alpha}\right)}\right]^{-1}$$

The iterated result of the above function has been considered as an *MLE* of α and denoted by $\hat{\alpha}$. Now the approximate *MLE* of α has been incorporated in (14) to obtain the *MLE* of β.

Similarly, from (12) the partial derivative of log $P(R, p)$ with respect to binomial parameter p can be obtained by solving the following equation

$$\frac{\partial \log P(p,R)}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}{1-p} = 0$$

thus the *MLE* of $\hat{p}$ of p is given by

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i}$$

Interval Estimations

The approximate confidence intervals of the parameters based on the asymptotic distributions of the *MLE* of the parameters α, θ and p are derived in this subsection. The elements of the Fisher information matrix for the parameters of the Kumaraswamy distribution based on progressive censored samples have been derived explicitly. The Fisher information matrix can be defined as

$$I(\alpha,\theta,p) = -\mathrm{E}\begin{bmatrix} \frac{\partial^2 \log(L)}{\partial \alpha^2} & \frac{\partial^2 \log(L)}{\partial \alpha \, \partial \theta} & 0 \\ \frac{\partial^2 \log(L)}{\partial \theta \partial \alpha} & \frac{\partial^2 \log(L)}{\partial \theta^2} & 0 \\ 0 & 0 & \frac{\partial^2 \log(L)}{\partial p^2} \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\theta} & 0 \\ I_{\theta\alpha} & I_{\theta\theta} & 0 \\ 0 & 0 & I_{pp} \end{bmatrix}$$

For the information matrix for α, θ and p, find

$$\frac{-\partial^2 \log(L)}{\partial \alpha^2} = \frac{m}{\alpha^2} + \sum_{i=1}^{m} \frac{x_{(i)}^{\alpha}\left(\log x_{(i)}\right)^2}{\left(1-x_{(i)}^{\alpha}\right)^2}\left(\theta + \theta r_i - 1\right),$$

$$\frac{-\partial^2 \log(L)}{\partial \theta^2} = \frac{m}{\theta^2},$$

$$\frac{-\partial^2 \log(L)}{\partial \alpha \, \partial \theta} = \sum_{i=1}^{m} \frac{x_{(i)}^{\alpha} \log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)} + \sum_{i=1}^{m} \frac{r_i x_{(i)}^{\alpha} \log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)},$$

$$\frac{-\partial^2 \log(L)}{\partial P^2} = \frac{\sum_{I=}^{m-1} r_i}{p^2} + \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}{(1-p)^2},$$

and

$$\frac{-\partial^2 \log(L)}{\partial \alpha \, \partial p} = \frac{-\partial^2 \log(L)}{\partial \theta \, \partial p} = 0$$

In order to derive the expressions for $I_{\alpha\alpha}, I_{\alpha\theta}, I_{\theta\theta}$ **and** I_{pp} the distribution of the i^{th} order statistics from the Kumaraswamy distribution is required, which can be written as

$$g\left(x_{(i)}\right) = C_{n,i}\alpha\theta x_{(i)}^{\alpha-1}\left(1-x_{(i)}^{\alpha}\right)^{\theta-1} \times\left\{1-\left(1-x_{(i)}^{\alpha}\right)^{\theta}\right\}^{i-1}\left(1-x_{(i)}^{\alpha}\right)^{-\theta(n-i)}, \quad 0 < x_{(i)} < 1$$

where $C_{n,i} = \dfrac{n!}{(i-1)!(n-i)!}$

Here, the expectations necessary to derive the elements of the Fisher information matrix are

$$\begin{aligned}
\mathrm{E}\left[\frac{x_{(i)}^{\alpha}\log x_{(i)}}{\left(1-x_{(i)}^{\alpha}\right)}\right] &= C_{n,i}\alpha\theta\int_0^1 \log x_{(i)} x_{(i)}^{2\alpha-1}\left(1-x_{(i)}^{\alpha}\right)^{\theta(n+1-i)-2}\left\{1-\left(1-x_{(i)}^{\alpha}\right)^{\theta}\right\}^{i-1} dx_{(i)} \\
&= C_{n,i}\alpha\theta\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\int_0^1 \log x_{(i)} x_{(i)}^{2\alpha-1}\left(1-x_{(i)}^{\alpha}\right)^{\theta(n+1-i+j)-2} dx_{(i)} \\
&= C_{n,i}\alpha\theta\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\frac{1-HarmanicNumber\left[\theta(n+1-i+j)\right]}{4\alpha^2\theta(n+1-i+j)\left\{\theta(n+1-i+j)-1\right\}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[\frac{x_{(i)}^{\alpha}\left(\log x_{(i)}\right)^2}{\left(1-x_{(i)}^{\alpha}\right)^2}\right] &= C_{n,i}\alpha\theta\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\int_0^1\left(\log x_{(i)}\right)^2 x_{(i)}^{2\alpha-1}\left(1-x_{(i)}^{\alpha}\right)^{\theta(n+1-i+j)-3} dx_{(i)} \\
&= C_{n,i}\alpha\theta\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\frac{\begin{array}{l}6(\gamma-2)\gamma \\ +\pi^2 \\ +6\psi\left(0,\theta(n+1-i+j)\right)\left\{2\gamma+\psi\left(0,\theta(n+1-i+j)\right)-2\right\} \\ -6\psi\left(1,\theta(n+1-i+j)\right)\end{array}}{48\alpha^3\left\{\theta(n+1-i+j)-1\right\}\left\{\theta(n+1-i+j)-2\right\}}
\end{aligned}$$

where γ **and** $\psi(a,b)$ are Euler gamma and Poly gamma functions respectively. Using these results, the Fisher information matrix can be obtained, which can further be used to derive the elements of the approximate variance-covariance matrix as

$$V=\begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\theta} & 0 \\ V_{\theta\alpha} & V_{\theta\theta} & 0 \\ 0 & 0 & V_{pp} \end{pmatrix}=\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\theta} & 0 \\ I_{\theta\alpha} & I_{\theta\theta} & 0 \\ 0 & 0 & I_{pp} \end{pmatrix}^{-1}$$

$\hat{\alpha},\hat{\theta},\hat{p}$. It is known that the asymptotic distribution of the *MLE* is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{p} \end{pmatrix} \approx N\left[\begin{pmatrix} \alpha \\ \theta \\ p \end{pmatrix}, \begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\theta} & 0 \\ V_{\theta\alpha} & V_{\theta\theta} & 0 \\ 0 & 0 & V_{pp} \end{pmatrix}\right] \tag{18}$$

Because V involves the parameters α, θ and p, replace the parameters by the corresponding maximum likelihood estimators in order to obtain an estimate of V, which is denoted by $\hat{V}$. Using (18), approximate $100(\gamma)\%$ confidence intervals for α, θ and p are determined respectively as $\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\alpha\alpha}}$, $\hat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\theta\theta}}$, $\hat{p} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{pp}}$, where z_γ is the upper $100(\gamma)\%$ percentile of the standard normal distribution.

The Expected Time Test

In practical applications, it is often useful to have an idea of the test time of the whole test. For progressive Type II censoring sampling plan with random or binomial removals, the expected test time for the experiment is given by the expectation of the m^{th} order statistic $X_{(m)}$. From Balakrishnan and Aggarwala (2000), the conditional expectation of $X_{(m)}$ for a fixed set of $R = R_1 = r_1$, $R_2 = r_2$, $R_{m-1} = r_{m-1}$ is given by

$$E\left(X_{(m)} \mid R=r\right)=C(r)\sum_{l_1}^{r_1}\cdots\sum_{l_m}^{r_m}(-1)^{A}\frac{\binom{r_1}{l_1}\cdots\binom{r_m}{l_m}}{\Pi_{i=1}^{m-1} h(l_i)}\left[\int_0^1 xf(x)F^{h(l_m)-1}(x)\,dx\right] \tag{19}$$

where

$$A=\sum_{i=1}^{m} l_i, C(r)=n(n-r_1-1)(n-r_1-r_2-2)\ldots\left(n-\sum_{i=1}^{m-1}(r_i+1)\right), h(l)=l_1+l_2+l_i+i$$

and 'i' is the number of live units removed from experimentation (or number of failure units). Substituting (1) and (2) into (19) results in the following

$$E\left(X_{(m)} \mid R=r\right)=C(r)\sum_{l_1}^{r_1}\ldots\sum_{l_m}^{r_m}\left(-1\right)^{A}\frac{\binom{r_1}{l_1}\cdots\binom{r_m}{l_m}}{\Pi_{i=1}^{m-1} h(l_i)}\left[\int_0^1 x\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\right] \times\left[1-(1-x^{\alpha})^{\theta}\right]^{h(l_m)-1} dx \quad (20)$$

Let

$$\begin{aligned} S &= \left[\int_0^1 x\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1}\right]\left[1-\left(1-x^{\alpha}\right)^{\theta}\right]^{h(l_m)-1} dx \\ &= \int_0^1 x\alpha\theta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\theta-1}\sum_{k=0}^{h(l_m)-1}\left(-1\right)^k\binom{h(l_m)-1}{k}\left(1-x^{\alpha}\right)^{\theta k} dx \\ &= \theta\alpha\sum_{k=0}^{h(l_m)-1}(-1)^k\binom{h(l_m)-1}{k}\int_0^1 x^{\alpha}\left(1-x^{\alpha}\right)^{\theta k+\theta-1} dx \\ &= \theta\sum_{k=0}^{h(l_m)-1}\left(-1\right)^k\binom{h(l_m)-1}{k}\mathrm{B}\left(1+\frac{1}{\alpha},\theta\left(k+1\right)\right) \end{aligned}$$

Plugging this quantity into the right hand side of equation (20), the expected test time of progressive Type II censoring with fixed number of removal will be

$$E(X_{(m)} \mid R=r)=\theta C(r)\sum_{l_1}^{r_1}\ldots\sum_{l_m}^{r_m}\left(-1\right)^{A}\frac{\binom{r_1}{l_1}\cdots\binom{r_m}{l_m}}{\Pi_{i=1}^{m-1} h(l_i)} \times\sum_{k=0}^{h(l_m)-1}\left(-1\right)^k\binom{h(l_m)-1}{k}\mathrm{B}\left(1+\frac{1}{\alpha},\theta\left(k+1\right)\right) \quad (21)$$

Also, the expected time under the Type II censoring scheme without removal is defined by the expected value of the m_{th} failure time, denoted by $X^*(m)$ where

$$E\left(X^{*}(m)\right) = m\theta\binom{n}{m}\sum_{k=0}^{m-1}\left(-1\right)^{k}\binom{m-1}{k}\mathrm{B}\left(1+\frac{1}{\alpha},\theta\left(k+1\right)\right) \tag{22}$$

Because $r_i = 0$ for all $i = 1, 2, \ldots, m - 1$. Similarly, the expected value of $X_{(m)}$ for complete sample can be derived from (22) by setting $m = n$ and $r_i = 0$ as

$$E\left(X^{**}(m)\right) = n\theta\sum_{k=0}^{m-1}(-1)^{k}\binom{n-1}{k}\mathrm{B}\left(1+\frac{1}{\alpha},\theta\left(k+1\right)\right) \tag{23}$$

Under *PCR*, the *R* terms are random. The expected time to complete an experiment under *PCR* is given by taking the expectation of both sides equation (21) with respect to the *R* terms. That is

$$E\left[(X_{(m)})\right] = E_{R}E\left[(X_{(m)}) \mid R=r\right] = \sum_{r_1=0}^{g(r_1)}\sum_{r_2=0}^{g(r_2)}\cdots\sum_{r_{m-1}=0}^{g(r_{m-1})}P(R)E\left[(X_{(m)}) \mid R=r\right] \tag{24}$$

where $g(r_i) = n - m - (r_1 + r_2 + \ldots + r_{i-1})$ and $P(R)$ is given in (10). Thus equation (24) gives an expression to compute the expected time for given values of m and n. To see how much time is saved under Type II progressive censoring, compare equations (23) and (24) where the ratio of the expected test time for Type I progressive censoring sample with binomial removals (*PCR*) with respect to the expected time for complete sample , that is

$$REET_1 = \frac{E\left[\left(X_{(m)}\right)\right] \text{ under PCR for a sample size } n}{E\left(X^{**}_{(m)}\right) \text{ under complete sampling for a sample size } n} \tag{25}$$

If replacing the numerator by the expected test time under Type II progressive censoring with random removals (*PCR*), this ratio is defined by $REET_{2.}$ Notice that the ratios $REET_1$ and $REET_2$ provide important information in determining the shortest experiment time significantly if the sample size n is large. When $REET_1$ and $REET_2$ are closer to one, the test time under respective censoring scheme is closer to the complete sample. The influence of the binomial probability removals p on the expected time can be studied by analyzing $REET_1$ for various values of p. The comparisons between the three expected times will be made in order to reward some information about m and n on the duration of the experiment. As it seems, analytical comparisons between these three expected times is difficult. Therefore, these comparisons can be made numerically for various values of m, n, α, and θ.

Numerical Study

The *MLE*s, their variances and 95% confidence intervals for parameters of the Kumaraswamy distribution using progressively censored data with random removals are now computed. The computations were made for different censoring schemes including various choices of m and n. the parametric space includes

$$(\alpha,\theta)\in\left\{\begin{matrix}(0.50,0.75),(1.00,0.80),(2.00,3.00),\\(2.50,1.50),(5.00,3.00),(4.00,2.50)\end{matrix}\right\}.$$

The censoring schemes are framed as follows:

Scheme 1:
$n = 20$, $m = 15$,
$r_1 = \ldots = r_{14} = 0$,
$r_{15} = 5$

Scheme 2:
$n = 20$, $m = 15$,
$r_1 = \ldots = r_7 = r_9 = \ldots = r_{15} = 0$,
$r_8 = 5$

Scheme 3:
$n = 20$, $m = 15$,
$r_2 = \ldots = r_{15} = 0$,
$r_1 = 5$

Scheme 4:
$n = 20$, $m = 18$,
$r_1 = \ldots = r_{17} = 0$,
$r_{18} = 2$

Scheme 5:
$n = 20$, $m = 18$,
$r_1 = \ldots = r_8 = r_{11} = \ldots = r_{18} = 0$,
$r_9 = r_{10} = 1$

Scheme 6:
$n = 20$, $m = 18$,
$r_2 = \ldots = r_{18} = 0$,
$r_1 = 2$

Scheme 7:
$n = 30$, $m = 20$,
$r_1 = \ldots = r_{19} = 0$,
$r_{20} = 10$

Scheme 8:
$n = 30$, $m = 20$,
$r_1 = \ldots = r_{10} = r_{13} = \ldots = r_{20} = 0$,
$r_{11} = r_{12} = 5$

Scheme 9:
$n = 30$, $m = 20$,
$r_2 = \ldots = r_{20} = 0$,
$r_1 = 10$

Scheme 10:
$n = 30$, $m = 25$,
$r_1 = \ldots = r_{19} = 0$,
$r_{20} = 5$

Scheme 11:
$n = 30$, $m = 25$,
$r_1 = \ldots = r_{10} = r_{13} = \ldots = r_{20} = 0$,
$r_{11} = 2$, $r_{12} = 3$

Scheme 12:
$n = 30$, $m = 25$,
$r_2 = \ldots = r_{20} = 0$,
$r_1 = 5$

Scheme 13:
$n = 40$, $m = 30$,
$r_1 = \ldots = r_{29} = 0$,
$r_{30} = 10$

Scheme 14:
$n = 40$, $m = 30$,
$r_1 = \ldots = r_{14} = r_{17} = \ldots = r_{30} = 0$,
$r_{15} = r_{16} = 5$

Scheme 15:
$n = 40$, $m = 30$,
$r_2 = \ldots = r_{30} = 0$,
$r_1 = 10$

Scheme 16:
$n = 40$, $m = 36$,
$r_1 = \ldots = r_{35} = 0$,
$r_{36} = 4$

Scheme 17:
$n = 40$, $m = 36$,
$r_1 = \ldots = r_{17} = r_{20} = \ldots = r_{36} = 0$,
$r_{18} = r_{19} = 2$

Scheme 18:
$n = 40$, $m = 36$,
$r_2 = \ldots = r_{36} = 0$,
$r_1 = 4$

The notations used in the tables are

$V(\hat{\alpha})$: Variance of the estimator

$LL(\hat{\alpha})$: Lower limit of the confidence interval

$UL(\hat{\alpha})$: Upper limit of the confidence interval

Table 1. *MLE*s, their variances and 95% confidence intervals for parameters using $\alpha = 0.50$, $\theta = 0.75$

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	0.674492	0.941582	0.088685	0.059105	0.090802	1.258182	0.465076	1.418088
2	0.675560	0.955438	0.090411	0.060857	0.086219	1.264902	0.471920	1.438956
3	0.689900	0.988406	0.096476	0.065130	0.081111	1.298689	0.488204	1.488608
4	0.640946	0.917092	0.087820	0.046725	0.060112	1.221780	0.493417	1.340766
5	0.640955	0.926610	0.088027	0.047700	0.059434	1.222475	0.498538	1.354682
6	0.649568	0.934912	0.088397	0.048559	0.066829	1.232306	0.503005	1.366819
7	0.615257	0.909485	0.073949	0.041358	0.082264	1.148250	0.510886	1.308085
8	0.620614	0.912566	0.080909	0.041639	0.063103	1.178126	0.512616	1.312516
9	0.632944	0.915290	0.084524	0.041888	0.063112	1.202776	0.514146	1.316433
10	0.581573	0.836671	0.071226	0.028001	0.058485	1.104661	0.508696	1.164646
11	0.584405	0.854518	0.072029	0.029208	0.058375	1.110434	0.519547	1.189490
12	0.602121	0.896171	0.073575	0.032125	0.070475	1.133767	0.544872	1.247470
13	0.546138	0.809125	0.058654	0.021823	0.071455	1.020821	0.519583	1.098667
14	0.546197	0.816111	0.059933	0.022201	0.066365	1.026028	0.524069	1.108152
15	0.553892	0.823240	0.064746	0.022591	0.055164	1.052621	0.528647	1.117833
16	0.510985	0.774778	0.038303	0.016674	0.127392	0.894579	0.521684	1.027872
17	0.532987	0.781340	0.042096	0.016958	0.130846	0.935128	0.526102	1.036577
18	0.536114	0.796164	0.048681	0.017608	0.103665	0.968562	0.536084	1.056245

Table 2. *MLE*s, their variances and 95% confidence intervals for parameters using $\alpha = 1.00$, $\theta = 0.80$

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	1.232107	1.026742	0.103913	0.070280	0.600290	1.863925	0.507139	1.546344
2	1.238510	1.026802	0.105153	0.070288	0.602936	1.874084	0.507168	1.546435
3	1.242170	1.027064	0.106155	0.070324	0.603573	1.880767	0.507298	1.546830
4	1.218273	0.985327	0.099675	0.053937	0.599476	1.837070	0.530129	1.440525
5	1.220254	0.987677	0.099778	0.054195	0.601136	1.839372	0.531393	1.443960
6	1.220848	1.008086	0.103164	0.056458	0.591313	1.850382	0.542374	1.473797
7	1.174262	0.958474	0.095742	0.045934	0.567794	1.780731	0.538404	1.378543
8	1.190808	0.961994	0.097742	0.046272	0.578039	1.803578	0.540381	1.383606
9	1.215313	0.969807	0.098676	0.047026	0.599623	1.831002	0.544770	1.394844
10	1.132875	0.937474	0.083493	0.035154	0.566531	1.699219	0.569984	1.304964
11	1.141405	0.941570	0.089372	0.035462	0.555460	1.727350	0.572475	1.310666
12	1.163471	0.942070	0.092902	0.035500	0.566065	1.760876	0.572779	1.311362
13	1.113024	0.878943	0.080758	0.025751	0.556034	1.670015	0.564417	1.193469
14	1.128151	0.928314	0.081152	0.028726	0.569802	1.686501	0.596121	1.260507
15	1.129390	0.936065	0.081241	0.029207	0.570737	1.688044	0.601098	1.271031
16	1.110531	0.853722	0.076051	0.020246	0.570015	1.651048	0.574840	1.132605
17	1.111389	0.856941	0.078882	0.020399	0.560905	1.661872	0.577007	1.136875
18	1.111504	0.865018	0.080522	0.020785	0.555328	1.667680	0.582445	1.147590

Table 3. *MLE*s, their variances and 95% confidence intervals for parameters using α = 2.00, θ = 3.00

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	2.405773	3.331680	0.173545	0.740006	1.589262	3.222285	1.645617	5.017742
2	2.420848	3.341618	0.175239	0.744427	1.600362	3.241334	1.650526	5.032709
3	2.424628	3.349518	0.180851	0.747951	1.591108	3.258149	1.654428	5.044608
4	2.363294	3.323579	0.153386	0.613677	1.595669	3.130919	1.788164	4.858994
5	2.390266	3.328627	0.161481	0.615542	1.602645	3.177887	1.790880	4.866374
6	2.395203	3.331073	0.172947	0.616447	1.580101	3.210306	1.792196	4.869950
7	2.309179	3.258572	0.131193	0.530914	1.599256	3.019103	1.830440	4.686704
8	2.313648	3.285888	0.133527	0.539853	1.597437	3.029859	1.845784	4.725992
9	2.325209	3.321773	0.135464	0.551709	1.603822	3.046596	1.865942	4.777604
10	2.254230	3.191529	0.110980	0.407434	1.601281	2.907178	1.940450	4.442609
11	2.255714	3.220759	0.121270	0.414931	1.573166	2.938263	1.958221	4.483296
12	2.278655	3.255155	0.125297	0.423841	1.584868	2.972441	1.979134	4.531176
13	2.180235	3.120957	0.102417	0.324679	1.552983	2.807487	2.004137	4.237778
14	2.184516	3.135531	0.102438	0.327719	1.557201	2.811832	2.013496	4.257567
15	2.253760	3.140167	0.103276	0.328688	1.623882	2.883637	2.016473	4.263862
16	2.104538	3.112679	0.082326	0.269133	1.542166	2.666911	2.095871	4.129487
17	2.133270	3.117755	0.099195	0.270011	1.515963	2.750577	2.099288	4.136221
18	2.177042	3.117986	0.100672	0.270051	1.555156	2.798928	2.099444	4.136528

Table 4. *MLE*s, their variances and 95% confidence intervals for parameters using $\alpha = 2.50$, $\theta = 1.50$

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	2.747904	1.730365	0.239122	0.199611	1.789462	3.706346	0.854679	2.606050
2	2.757010	1.732250	0.240154	0.200046	1.796503	3.717517	0.855610	2.608889
3	2.792415	1.733128	0.241268	0.200249	1.829681	3.755148	0.856044	2.610212
4	2.708039	1.681290	0.227382	0.157041	1.773420	3.642658	0.904574	2.458007
5	2.735921	1.713229	0.228650	0.163064	1.798701	3.673141	0.921757	2.504700
6	2.739513	1.728698	0.237295	0.166022	1.784740	3.694287	0.930080	2.527315
7	2.704854	1.647548	0.191089	0.135721	1.848065	3.561643	0.925478	2.369617
8	2.705733	1.665756	0.214797	0.138737	1.797348	3.614119	0.935706	2.395806
9	2.705802	1.678866	0.221835	0.140930	1.782654	3.628950	0.943071	2.414662
10	2.677496	1.612695	0.171758	0.104031	1.865201	3.489791	0.980518	2.244871
11	2.691698	1.616866	0.182896	0.104570	1.853477	3.529919	0.983055	2.250678
12	2.700314	1.624152	0.183897	0.105515	1.859804	3.540824	0.987484	2.260819
13	2.601492	1.542551	0.139221	0.079315	1.870170	3.332814	0.990556	2.094546
14	2.628847	1.554005	0.165617	0.080498	1.831203	3.426491	0.997912	2.110099
15	2.652432	1.581927	0.167527	0.083416	1.850203	3.454662	1.015842	2.148012
16	2.564242	1.518715	0.118527	0.064069	1.889457	3.239027	1.022602	2.014829
17	2.585654	1.532091	0.126751	0.065203	1.887853	3.283454	1.031608	2.032574
18	2.590843	1.538084	0.128757	0.065714	1.887542	3.294144	1.035643	2.040524

Table 5. *MLE*s, their variances and 95% confidence intervals for parameters using α = 3.00, θ = 5.00

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	5.516209	3.339823	0.350879	0.743628	4.355202	6.677216	1.649639	5.030007
2	5.532574	3.379539	0.352004	0.761419	4.369708	6.695441	1.669256	5.089822
3	5.540173	3.381127	0.356985	0.762135	4.369107	6.711238	1.670041	5.092213
4	5.484345	3.318767	0.332414	0.611901	4.354300	6.614389	1.785575	4.851959
5	5.505484	3.325015	0.337346	0.614207	4.367087	6.643880	1.788936	4.861093
6	5.509684	3.333368	0.343636	0.617297	4.360723	6.658645	1.793431	4.873306
7	5.312807	3.283075	0.317312	0.538929	4.208731	6.416883	1.844204	4.721946
8	5.462259	3.311419	0.321464	0.548275	4.350982	6.573536	1.860126	4.762713
9	5.466648	3.313941	0.322346	0.549110	4.353848	6.579448	1.861542	4.766339
10	5.241934	3.232154	0.287640	0.417873	4.190746	6.293123	1.965150	4.499159
11	5.252454	3.250657	0.292829	0.422671	4.191826	6.313081	1.976400	4.524915
12	5.271842	3.252974	0.308544	0.423274	4.183126	6.360557	1.977808	4.528140
13	5.189160	3.208891	0.269755	0.343233	4.171176	6.207144	2.060604	4.357177
14	5.203086	3.220785	0.276195	0.345782	4.173022	6.233150	2.068242	4.373328
15	5.221027	3.226427	0.277078	0.346994	4.189318	6.252735	2.071865	4.380989
16	5.160790	3.144943	0.251366	0.274741	4.178117	6.143463	2.117595	4.172291
17	5.185568	3.147283	0.265617	0.275150	4.175422	6.195714	2.119170	4.175395
18	5.187746	3.196740	0.266302	0.283865	4.176298	6.199194	2.152472	4.241009

Table 6. *MLE*s, their variances and 95% confidence intervals for parameters using α = 4.00, θ = 2.50

Schemes	$\hat{\alpha}$	$\hat{\theta}$	$V(\hat{\alpha})$	$V(\hat{\theta})$	$LL(\hat{\alpha})$	$UL(\hat{\alpha})$	$LL(\hat{\theta})$	$UL(\hat{\theta})$
1	4.388710	2.724225	0.315652	0.494760	3.287524	5.489896	1.345577	4.102873
2	4.419342	2.761146	0.318780	0.508262	3.312714	5.525969	1.363813	4.158478
3	4.444269	2.786841	0.325157	0.517766	3.326626	5.561911	1.376505	4.197178
4	4.362571	2.681995	0.292876	0.399617	3.301859	5.423284	1.442977	3.921014
5	4.366521	2.690268	0.295321	0.402086	3.301390	5.431652	1.447428	3.933109
6	4.380645	2.721175	0.309534	0.411378	3.290184	5.471106	1.464056	3.978294
7	4.274244	2.667550	0.270601	0.355791	3.254665	5.293822	1.498445	3.836655
8	4.315312	2.668255	0.274710	0.355979	3.288022	5.342602	1.498840	3.837669
9	4.347334	2.671510	0.281012	0.356848	3.308326	5.386341	1.500669	3.842352
10	4.212643	2.589071	0.267819	0.268132	3.198320	5.226967	1.574155	3.603987
11	4.242698	2.638366	0.268059	0.278439	3.227919	5.257478	1.604127	3.672606
12	4.254647	2.644628	0.270255	0.279762	3.235719	5.273574	1.607934	3.681322
13	4.160297	2.574998	0.252870	0.221021	3.174688	5.145906	1.653547	3.496450
14	4.198490	2.579545	0.259322	0.221802	3.200386	5.196594	1.656467	3.502624
15	4.205975	2.580193	0.266779	0.221913	3.193622	5.218328	1.656883	3.503503
16	4.097057	2.552605	0.234649	0.180994	3.147622	5.046492	1.718754	3.386456
17	4.106227	2.554844	0.237447	0.181312	3.151148	5.061306	1.720262	3.389426
18	4.159811	2.558519	0.239472	0.181834	3.200668	5.118953	1.722736	3.394302

Tables 1-6 include the maximum likelihood estimates (*MLE*s), the variances of *MLE*s, and 95% confidence intervals for the parameters of the Kumaraswamy distribution under progressively Type II censored samples using different parametric values for various censoring schemes. It has been observed that by increasing the sample size (keeping censoring rate fixed), the estimated value of the parameter become closer to the true value, the variances of the *MLE*s decrease and widths of 95% confidence intervals tend to be lesser. This is an indication that the estimators are consistent in nature. It can further be assessed that the censoring schemes, concerned with survivals from the right, result in more precise results than their counterparts. As expected, the increase in true parametric values leads to the slower convergence of the estimates along with larger variances of the estimates which lean to increase the widths of the confidence intervals. The increase in censoring rate, that is, the smaller values of '*m*' has the same natural consequences. However, these negative impacts can be protected by employing larger ($n > 30$) sample sizes.

Conclusion

This study addressed the problem of estimation of parameters of the Kumaraswamy distribution under progressive censoring based on random removals. The maximum likelihood estimation was used to serve the purpose. The findings of the study indicate that the proposed estimators are consistent in nature. It is interesting to note that the removal of items from the right leads to the most efficient results.

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