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Spaces of Sections of Banach Algebra Bundles

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Spaces of sections of Banach algebra bundles

by

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Abstract

Suppose that B is a G-Banach algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, X is a finite dimensional compact metric space, $\zeta : P \to X$ is a standard principal G-bundle, and $A_\zeta = \Gamma(X, P \times_G B)$ is the associated algebra of sections. We produce a spectral sequence which converges to $\pi_*(\mathrm{GL}_\mathrm{o} A_\zeta)$ with

$$E^2_{-p,q} \cong \check{H}^p(X; \pi_q(\mathrm{GL}_\mathrm{o} B)).$$

A related spectral sequence converging to $K_{*+1}(A_\zeta)$ (the real or complex topological K-theory) allows us to conclude that if B is Bott-stable, (i.e., if $\pi_*(\mathrm{GL}_\mathrm{o} B) \to K_{*+1}(B)$ is an isomorphism for all $* > 0$) then so is A_ζ.

Key Words: general linear group of a Banach algebra, spectral sequences, localization, Bott-stable, K-theory for Banach algebras, unstable K-theory.

Mathematics Subject Classification 2010: 46L80, 46L85, 46M20, 55Q52, 55R20, 55T25.

Contents

1. Introduction

Suppose that X is a finite-dimensional compact metric space and $\zeta : P \to X$ is a principal G bundle for some topological group G that acts on a Banach algebra B[1] by algebra automorphisms. Let $E = P \times_G B \to X$ be the associated fibre bundle. Assume that the fibre bundle is *standard*.[2] Let

$$A_\xi = \Gamma(X, E)$$

denote the set of continuous sections of the bundle. This has a natural structure of a Banach algebra. If B is unital then A_ξ is also unital, with identity the canonical section that to each point $x \in X$ assigns the identity in E_x.

We are interested in $\mathrm{GL}\, A_\xi$, the group of invertible elements in A_ξ. (If A_ξ is not unital then we understand this to mean the kernel of the natural map $\mathrm{GL}(A_\xi^+) \to \mathrm{GL}(\mathbb{F})$.) This is a space of the homotopy type of a CW-complex, second countable if B is separable. It may have many (homeomorphic) path components; let $\mathrm{GL}_\mathrm{o}\, A_\xi$ denote the path component of the identity.

Let $\mathbb{P}$ denote a subring of the rational numbers. (We allow the cases $\mathbb{P} = \mathbb{Z}$ and $\mathbb{P} = \mathbb{Q}$ as well as intermediate rings.) Cohomology is reduced Čech cohomology. Our principal result is the following. It is proved in Sections 4 and 5.

Theorem A *Suppose that X is a finite-dimensional compact metric space and that $\zeta : E = P \times_G B \to X$ is a standard fibre bundle of Banach algebras over $\mathbb{F}$. Let A_ξ denote the associated section algebra. Then:*

1. *There is a second quadrant spectral sequence converging to $\pi_*(\mathrm{GL}_\mathrm{o}\, A_\xi) \otimes \mathbb{P}$ with*

$$E^2_{-p,q} \cong \check{H}^p(X; \pi_q(\mathrm{GL}_\mathrm{o}\, B) \otimes \mathbb{P})$$

and

$$d^r : E^r_{-p,q} \longrightarrow E^r_{-p-r,q+r-1}.$$

2. *If X has dimension at most n, then $E^{n+1} = E^\infty$.*

3. *The spectral sequence is natural with respect to pullback diagrams*

$$
\begin{array}{ccc}
f^*E & \longrightarrow & E \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\ f\ } & X
\end{array}
$$

---

[1] We consider Banach algebras over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and use K_* to denote real or complex topological K-theory, respectively.

[2] Being *standard* is a technical condition which is automatic if X has the homotopy type of a finite CW-complex. See Definition 5.1.

and associated map $f^* : A_\zeta \longrightarrow A_{f^*\zeta}$.

4. The spectral sequence is natural with respect to G-equivariant maps $\alpha : B \to B'$ of Banach algebras.

Note that the E^2-term of the spectral sequence is independent of the bundle ζ. Generally, this spectral sequence has several types of non-zero differentials, and some of these definitely *do* depend upon the bundle. There is one very special case in which the spectral sequence does collapse at E^2.

Corollary 1.1 ([14], [12]) *With the above notation, if A_ζ is a unital continuous-trace C^*-algebra over $\mathbb{C}$ and $\mathbb{P} = \mathbb{Q}$ then $E^2 = E^\infty$ and*

$$\pi_*(\mathrm{GL_o}\, A_\zeta) \otimes \mathbb{Q} \cong \check{H}^*(X; \pi_*(\mathrm{GL_o}\, B) \otimes \mathbb{Q}).$$

Note that the assumptions in Corollary 1.1 imply that the Dixmier-Douady invariant in $\check{H}^3(X; \mathbb{Q})$ is zero. Thus this corollary does not contradict the non-collapse results of Atiyah and Segal [2]. See Remark 8.4 for more discussion of differentials in the spectral sequence.

The Corollary was established for X compact metric, $B = M_n(\mathbb{C})$, $\mathbb{P} = \mathbb{Q}$ and ζ trivial, so that $A_\zeta = F(X, M_n(\mathbb{C}))$, by Lupton, Phillips, Schochet, and Smith [14] and in general by Klein, Schochet, and Smith [12]. The idea of using a spectral sequence to compute $\pi_*(F(X, Y))$ goes back to Federer [8], and note particularly the work of S. Smith [21]. A more abstract study of the homotopy groups of sections was undertaken by Legrand [13] and his spectral sequence would seem to overlap with ours in special cases.

This spectral sequence is analogous to the Atiyah-Hirzebruch spectral sequence. In fact, if the functor $\pi_*(\mathrm{GL_o}(-)) \otimes \mathbb{Q}$ is replaced by $K_*(-)$ then this has been studied by J. Rosenberg [20] in the context of continuous-trace C^*-algebras. Our Theorem B generalizes his result.

Note that in many cases of interest, for instance $B = M_n(\mathbb{C})$, the groups $\pi_*(\mathrm{GL_o}\, B)$ are unknown, and so the integral version of the spectral sequence cannot be used directly to compute $\pi_*(\mathrm{GL_o}\, A_\zeta)$. However, frequently the groups $\pi_*(\mathrm{GL_o}\, B) \otimes \mathbb{Q}$ *are* known and hence the rational form of the spectral sequence will be more tractable.

Our approach to this problem has two steps. First, assume that X is a finite complex, replace the space E by an appropriate diagram of spaces, localize the diagram, and then filter it. This is a consequence, noted in [8], of the basic Fibre Lemma of Bousfield and Kan [4]. Next, assume that X is a finite-dimensional compact metric space, write it as an inverse limit of finite CW-complexes, and use our first step results together with limit techniques of [12] to complete the argument.

Using the Banach algebra version of Bott periodicity established by R. Wood [24] and M. Karoubi [11] and taking limits of spectral sequences, we derive the following.

Theorem B *Suppose that X is a finite-dimensional compact metric space, B is a Banach algebra over $\mathbb{F}$, and $\zeta : E \to X$ is a standard fibre bundle of B-algebras. Then there is a second quadrant spectral sequence*

$$E^2_{-p,q}(A_{\zeta_\infty}) \cong \check{H}^p(X; K_{q+1}(B) \otimes \mathbb{P}) \Longrightarrow K_{*+1}(A_\zeta) \otimes \mathbb{P}$$

which is the direct limit of the corresponding spectral sequences converging to

$$\pi_*(\mathrm{GL}_\mathrm{o}(A_\zeta \otimes M_n(\mathbb{F})) \otimes \mathbb{P}.$$

If X has dimension at most n then $E^{n+1} = E^\infty$.

This result, established in Section 6, is due to J. Rosenberg [20] when A_ζ is a continuous-trace C^*-algebra over a finite complex X.

Next, compare the spectral sequences of Theorems A and B.

Definition 1.2 Let B be a Banach algebra over $\mathbb{F}$. Let us call B *Bott-stable* if the natural map

$$\pi_*(\mathrm{GL}_\mathrm{o} B) \to K_{*+1}(B)$$

is an isomorphism for every $* > 0$, or, equivalently, if the natural map

$$\mathrm{GL}_\mathrm{o} B \to \varinjlim \mathrm{GL}_\mathrm{o}(B \otimes M_n(\mathbb{F}))$$

is a homotopy equivalence. Similarly, let us call B *rationally Bott-stable* if the natural map

$$\pi_*(\mathrm{GL}_\mathrm{o} B) \otimes \mathbb{Q} \to K_{*+1}(B) \otimes \mathbb{Q}$$

is an isomorphism.

For example, the Calkin algebra $\mathcal{L}/\mathcal{K}$ is Bott-stable. The Bott Periodicity theorem for Banach algebras may be stated as follows: If the Banach algebra B is *stable* in the sense that the inclusion of a rank one projection induces an isomorphism $B \overset{\cong}{\longrightarrow} B \otimes \mathcal{K}$ (where $\mathcal{K}$ denotes the compact operators), then B is Bott-stable.

We introduce the term *Bott-stable* with some trepidation, but we have been unsuccessful in locating a prior use of this concept in the literature.

Bott stability is a bit weaker than K. Thomsen's use [23] of K-*stable*. He shows that many important complex C^*-algebras, such as $\mathcal{O}_n \otimes B$ ($\mathcal{O}_n$ the Cuntz algebra, B a C^*-algebra), infinite-dimensional simple AF algebras, properly infinite von

Neumann algebras and the corona algebras of σ-unital C^*-algebras are K-stable and hence Bott-stable.

The following theorem is proved at the end of Section 7.

Theorem C *Suppose that X is a finite-dimensional compact metric space, B is a Banach algebra over $\mathbb{F}$, and $\zeta : E \to X$ is a standard fibre bundle of B-algebras. If B is Bott-stable then A_ζ is Bott-stable. If B is rationally Bott-stable then A_ζ is rationally Bott-stable.*

We expect to apply these results in several directions. For example, we hope to shed light on differentials in the twisted K-theory spectral sequence by looking at them in the homotopy spectral sequence and then mapping over.

2. The Non-Unital Case

We pause to prove a technical result that allows us to handle the non-unital Banach algebra case uniformly with the unital case.

Suppose that B is a non-unital Banach algebra over $\mathbb{F}$ with a G action, $P \to X$ is a principal G bundle, and $A_\zeta = \Gamma(X, P \times_G B)$. There is a natural unitization B^+ of B and a canonical short exact sequence

$$0 \to B \longrightarrow B^+ \xrightarrow{\ p\ } \mathbb{F} \to 0$$

with a natural splitting $\mathbb{F} \to B^+$. The G-action extends from B to B^+ canonically with the group fixing the identity. Hence there is a split exact sequence of bundles

$$
\begin{array}{ccccc}
P \times_G B & \longrightarrow & P \times_G B^+ & \xrightarrow{\ p\ } & P \times_G \mathbb{F} \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\ 1\ } & X & \xrightarrow{\ 1\ } & X.
\end{array}
$$

Define

$$A_{\zeta^+} = \Gamma(X, P \times_G B^+).$$

The induced action of G on $\mathbb{F}$ must be the trivial action, and so $P \times_G \mathbb{F} \cong X \times \mathbb{F}$ is a trivial line bundle. Thus $\Gamma(X, P \times_G \mathbb{F}) \cong C(X, \mathbb{F})$ and the induced map

$$p_* : \mathrm{GL}_\mathrm{o}(A_{\xi+}) \longrightarrow \mathrm{GL}_\mathrm{o}(\Gamma(X, \mathbb{F} - 0)) \cong C(X, \mathbb{F} - 0)$$

splits canonically. Recall that the general linear group of the non-unital Banach algebra A_ξ is defined by

$$\mathrm{GL}(A_\xi) = Ker[\mathrm{GL}((A_\xi)^+) \longrightarrow \mathrm{GL}((A_\xi)^+/A_\xi) \cong \mathbb{F} - 0]$$

Proposition 2.1 *(N. C. Phillips [19]) With the above notation,*

1. *There is an isomorphism of topological groups*

$$\mathrm{GL}(A_\xi) \cong Ker\big[p_* : \mathrm{GL}(A_{\xi+}) \longrightarrow C(X, \mathbb{F} - 0)\big].$$

2. *There is an isomorphism of topological groups*

$$\mathrm{GL}(A_\xi) \cong \Gamma(X, P \times_G \mathrm{GL}B).$$

3. *There is a canonically split short exact sequence*

$$0 \to \pi_*(\mathrm{GL}(A_\xi)) \longrightarrow \pi_*(\mathrm{GL}(A_{\xi+})) \longrightarrow \pi_*(C(X, \mathbb{F} - 0)) \to 0.$$

Proof: (Phillips) If D is an ideal in any unital Banach algebra E then one can identify the unitization D^+ with the set

$$\{d + tI \in E \mid d \in D, t \in \mathbb{F}\}.$$

This identification is isometric on D and a homeomorphism on D^+. Taking $D = A_\xi$ and $E = A_{\xi+}$ implies that

$$A_\xi \cong Ker\big[A_{\xi+} \longrightarrow C(X, \mathbb{F})\big]$$

which yields the identification of $\mathrm{GL}(A_\xi)$.   □

This theorem allows us to write

$$\mathrm{GL}A_\xi \cong \Gamma(X, P \times_G \mathrm{GL}(B))$$

in the non-unital case (where it is not at all obvious) as well as in the unital case (where it is true essentially by definition) and hence develop the two cases at once.[3]

---

[3]The referee notes that for any algebra the group $\mathrm{GL}(A)$ has a description as

$$\mathrm{GL}(A) = \{a \in A \mid \exists b \in A : ab = ba = a + b\}$$

with multiplication $g \circ h := g + h - gh$ and unit 0. This identification makes the identification of $\mathrm{GL}A_\xi$ obvious.

For completeness, note that the groups $\pi_j(\mathrm{GL_o}(A_\zeta))$ and $\pi_j(\mathrm{GL_o}(A_{\zeta+}))$ contain essentially the same information, so that if one is known the other is determined, as follows.

The groups $\pi_*(C(X,\mathbb{C}-0))$ are well-known:

1. $\pi_0(C(X,\mathbb{C}-0)) = \pi_0(F(X,K(\mathbb{Z},1))) = \check{H}^1(X;\mathbb{Z})$ essentially by definition;

2. $\pi_1(C(X,\mathbb{C}-0)) = \check{H}^0(X;\mathbb{Z})$ by the early result of R. Thom [22], and

3. $\pi_j(C(X,\mathbb{C}-0)) = 0$ $j > 1$ also by Thom's result.

The real case is even simpler, since $\mathbb{R}-0 \simeq \pm 1$. Thus

$$C(X,\mathbb{R}-0) \cong C(X,\pm 1) = C(\pi_0(X),\pm 1)$$

which is a zero - dimensional set, finite if X has a finite number of path components.

To summarize:

Theorem 2.2 *Suppose that X is a compact metric space, $\zeta : P \to X$ is a principal G-bundle, B is a non-unital Banach algebra with a G-action, and A_ζ is the associated Banach algebra of sections. Then*

1. *There are naturally split short exact sequences*

$$0 \to \pi_j(\mathrm{GL}(A_\zeta)) \longrightarrow \pi_j(\mathrm{GL}(A_{\zeta+})) \longrightarrow \check{H}^{1-j}(X;\mathbb{Z}) \to 0$$

   for $j = 0,1$.

2. *For $j \geq 2$ (and $j \geq 1$ in the real case) there is a natural isomorphism*

$$\pi_j(\mathrm{GL_o}(A_\zeta)) \xrightarrow{\;\cong\;} \pi_j(\mathrm{GL_o}(A_{\zeta+})).$$

3. Diagrams and Localization

Let X denote a finite simplicial complex. Denote by $\mathcal{I}X$ the small category whose set of objects is the set of simplices of X and a morphism $\sigma \to \tau$ is a face inclusion of two simplices in the complex X. Thus there is at most one morphism between any two objects of $\mathcal{I}X$. The category $\mathcal{I}X$ serves as an indexing category for a diagram of spaces $\mathcal{D}X$, namely, a functor: $\mathcal{D}X : \mathcal{I}X \to Top$ of topological spaces that assigns to each simplex in the simplicial complex X, the underlying space of σ. So $\mathcal{D}X$ is a

diagram of contractible spaces, each homeomorphic to some $n-$ simplex, indexed by $\mathcal{I}X$.

Now given a fibration $F \to T \xrightarrow{p} X$, let us denote by $F_\sigma = p^{-1}(\sigma)$ the inverse images of a given simplex in the total space. Notice that for any inclusion of simplices $\sigma \subseteq \tau$ there is an inclusion of inverse images $F_\sigma \subseteq F_\tau$ and these inclusions are coherent. Thus the fibration (in fact any map to X) yields a diagram $\tilde{F}$ over $\mathcal{I}X$ made up of the various spaces F_σ, where σ runs over all the simplices of X.

We now present the space of sections as a homotopy limit using the following standard equivalence:

Lemma 3.1 *There is a natural equivalence*

$$\Gamma(X,T) \simeq holim_{\sigma \in \mathcal{I}X} F_\sigma.$$

Proof: Here is an outline of a proof: Let $\mathcal{D}X : \mathcal{I}X \to Top$ be the diagram as above. It is not hard to see that it is a free diagram and thus for any other diagram $W : \mathcal{I}X \to Top$ indexed by $\mathcal{I}X$, the homotopy limit of W is given by the space of "strictly coherent maps of diagrams"

$$holim_{\mathcal{I}X} W \cong Map_{\mathcal{I}X}(\mathcal{D}X, W).$$

In addition, the total space itself may be written as a homotopy colimit of the diagram (F_σ), $T \cong hocolim_{\mathcal{I}X} F_\sigma$ and of course for the identity $X \to X$ we get $X \cong hocolim_{\mathcal{I}X} \mathcal{D}X$.

We now notice that any section $X \to T$ gives a map of diagrams $\mathcal{D}X \to \tilde{T}$, since any simplex σ in X is mapped to the space $F\sigma$ over it and the various maps are coherent with respect to the inclusion of simplices. Moreover, every such map of diagrams gives, by gluing, a map $X \to T$ which is a section. Namely, given a map between the two diagrams of spaces $f : \mathcal{D}X \to \tilde{T}$ then $hocolim_{\mathcal{D}X} f$ is the desired section by the comment above, since in that case the homotopy colimit coincides with the strict direct limit, since both diagrams are free over $\mathcal{I}X$.

Thus the desired result follows

$$holim_{\mathcal{I}X} F_\sigma \cong holim_{\mathcal{I}X} \tilde{T} \cong Map_{\mathcal{I}X}(\mathcal{D}X, T) \cong \Gamma(X,T).$$

$\square$

Our interest lies in $GL_o A_\zeta$ and its localizations. Given $A_\zeta = \Gamma(X, P \times_G B)$, consider the fibre bundle $P \times_G GL_o B \longrightarrow X$. There is a natural comparison map:

$$c : \ (GL_o A_\zeta)_{\mathbb{P}} = \Gamma(X, P \times_G GL_o B)_{o\mathbb{P}} \subseteq \left[holim_{\mathcal{D}X} F_\sigma \right]_{\mathbb{P}} \longrightarrow holim_{\mathcal{D}X} \left([F_\sigma]_{\mathbb{P}} \right)$$

where we used Proposition 2.1 and Lemma 3.1 to produce the left maps and where the last map on the right is naturally associated to the localization. Notice that on the left hand side one first takes localization and then homotopy limit.

In order to facilitate the formulation of the following formula for the space of sections, let us denote by $holim_{\mathcal{D}X}[F_\sigma]_{\mathbb{P},o}$ the component of the localization of the homotopy limit that is hit via the above comparison map c by the component of the identity in the localization of space of sections. This is a shorthand for the cumbersome $\left(holim_{\mathcal{D}X}\left([F_\sigma]_{\mathbb{P}}\right)\right)_o$.

Theorem 3.2 *The map of connected spaces*

$$c : \quad (GL_o A_\zeta)_{\mathbb{P}} \longrightarrow holim_{\mathcal{D}X}[F_\sigma]_{\mathbb{P},o}$$

is a homotopy equivalence.

Proof: Recall that for any connected pointed space, if the action of its fundamental group on itself and all higher homotopy groups is trivial then the space is certainly *nilpotent*. Thus the underlying space of any connected topological group is nilpotent. Thus the diagram F_σ in our case is in fact a diagram of nilpotent spaces, since the fibres are all equivalent as spaces to the identity component GL_o of our group. Now we use a theorem about the localization of homotopy limits over certain diagrams of nilpotent spaces over finite dimensional indexing categories. Here we use the assumption that X is a finite simplicial complex. By the main theorem in ([8] Theorem 2.2 and Corollary. 2.3), the comparison map has a homotopically discrete fibre (possibly empty!) over each component of the range, so that the map c is, up to homotopy, a covering projection. Notice that localization in the domain of the map c, acts one component at a time. Thus the restriction of c to each component of its domain is a homotopy equivalence, in particular when one restricts to the component of the identity.

$\square$

Corollary 3.3 *There is a natural isomorphism of homotopy groups of the following connected spaces, taken with corresponding base points:*

$$\pi_j(GL_o A_\zeta) \otimes \mathbb{P} \cong \pi_j\left(holim_{\mathcal{D}X}[F_\sigma]_{\mathbb{P},o}\right)$$

for all $j \geq 0$.

4. The Spectral Sequence for Finite Complexes

In this section we construct the spectral sequence that is the main object of our work. Use the notation of the previous section and continue to assume that the base space

X is a finite simplicial complex. The simplicial complex X is filtered naturally by its skeleta and this induces a filtration $\mathcal{D}_p X$ of the diagram $\mathcal{D}X$.

The spectral sequence is a special case of the Bousfield-Kan spectral sequence for homotopy limits [4]. Here we give a presentation of it for the space of sections over a finite simplicial complex. Define

$$\mathcal{F}_p X = holim_{\mathcal{D}_p X}[F_\sigma]_{\mathbb{P},o}.$$

Form the associated exact couple. The appropriate grading turns out to be

$$D^1_{-p,q} \cong \pi_{q-p}(\mathcal{F}_p X) \otimes \mathbb{P}$$

and

$$E^1_{-p,q} = \pi_{q-p}((\mathcal{F}_{p-1}X)/(\mathcal{F}_p X)) \otimes \mathbb{P}$$

with differential

$$d^1 : E^1_{-p,q} \longrightarrow E^1_{-p-1,q}.$$

We may identify the E^1 term by noting that

$$E^1_{-p,q} \cong \pi_{q-p}(\Gamma(\vee_\alpha S^p, (\mathrm{GL_o} A_\zeta)|_{S^p}) \otimes \mathbb{P} \cong \pi_{q-p}(F_*(\vee_\alpha S^p, \mathrm{GL_o} B)) \otimes \mathbb{P} \cong$$

$$\cong \oplus_\alpha \pi_{q-p}(\Omega^p \mathrm{GL_o} B) \otimes \mathbb{P} \cong \oplus_\alpha \pi_q(\mathrm{GL_o} B) \otimes \mathbb{P} \cong C^p(X; \pi_q(\mathrm{GL_o} B) \otimes \mathbb{P}).$$

so that

$$E^1_{-p,q} \cong C^p(X; \pi_q(\mathrm{GL_o} B) \otimes \mathbb{P}),$$

the cellular cochains of X with coefficients in $\pi_q(\mathrm{GL_o} B) \otimes \mathbb{P}$. The d^1 differential is the usual cellular differential and so

$$E^2_{-p,q} \cong \check{H}^p(X; \pi_q(\mathrm{GL_o} B) \otimes \mathbb{P}).$$

The filtration is finite since X is a finite complex, and so the resulting spectral sequence converges to $\pi_*(\mathrm{GL_o} A) \otimes \mathbb{P}$. This proves Theorem A for the case X a finite simplicial complex.

This construction has two types of naturality associated with it.

Proposition 4.1   1. *The exact couple and resulting spectral sequence are natural with respect to pullbacks*

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\ f\ } & X \end{array}$$

where $f : X' \to X$ is a simplicial map of finite simplicial complexes and $f^*(E) \to X'$ is the pullback bundle.

2. *The spectral sequence is natural with respect to bundle maps*

$$
\begin{array}{ccc}
E & \longrightarrow & E' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;1\;} & X
\end{array}
$$

induced by a G-equivariant map $B \to B'$ on the fibres.

Proof: Both of these assertions are immediate from the construction. □

5. The Spectral Sequence for Compact Spaces

We move now to the general case, with finite-dimensional compact metric space X and principal G-bundle $\zeta : P \to X$. The group G acts on the Banach algebra B and may be regarded as a subgroup of $Aut(B)$, but this is a very large group in general.

Principal G-bundles are classified by maps $X \to BG$ and so there is a pullback diagram

$$
\begin{array}{ccc}
P \times_G B & \longrightarrow & EG \times_G B \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & BG
\end{array}
$$

but it is critical at this point that the space BG be of the homotopy type (and not just the weak homotopy type) of a CW complex.

Definition 5.1 The fibre bundle $E = P \times_G B \to X$ is *standard* if either X is of the homotopy type of a finite complex or (if not) then the structural group of the bundle reduces to some subgroup $G \subset Aut(B)$ which is of the homotopy type of a CW-complex, or equivalently, that the classifying space BG of the bundle has the homotopy type of a CW complex.[4]

The classical theorem of Eilenberg-Steenrod ([7] Theorem. X.10.1) gives us an inverse sequence of finite simplicial complexes X_j and simplicial maps together with a homeomorphism $X \cong \varprojlim X_j$. Let $h_j : X \to X_j$ denote the resulting structure maps. Then, as shown in [12], for j sufficiently large, there are maps[5]

$$
f_j : X_j \to BG
$$

---

[4]We note that non-standard bundles do exist. Take some topological group G that is not of the homotopy type of a CW-complex. Then BG also has this property and hence the universal principal G-bundle is not standard.

[5]The result in [12] depends upon work of Eilenberg-Steenrod which requires that the target space BG be of the homotopy type of a CW-complex. This is why standard bundles are required.

and a homotopy-commuting diagram of the form

$$
\begin{array}{ccccccc}
\mathrm{GL_o}\,E & \longrightarrow & \mathrm{GL_o}\,E_j & \longrightarrow & \mathrm{GL_o}\,E_{j-1} & \longrightarrow & EG \times_G \mathrm{GL_o}\,B \\
\downarrow & & \downarrow{\scriptstyle \zeta_j} & & \downarrow & & \downarrow \\
X & \xrightarrow{h_j} & X_j & \longrightarrow & X_{j-1} & \xrightarrow{f_{j-1}} & BG.
\end{array}
\tag{1}
$$

where each square is a pullback. Let $\zeta_j : \mathrm{GL_o}\,E_j \to X_j$ denote the associated bundles. Then there is a natural isomorphism

$$
\varinjlim \pi_*(\Gamma(\zeta_j)) \xrightarrow{\;\cong\;} \pi_*(\Gamma(\zeta)) \equiv \pi_*(\mathrm{GL_o}\,A_\zeta).
$$

For each finite simplicial complex X_j there is an exact couple

$$
(E^1_{*,*}(j), D^1_{*,*}(j))
$$

constructed in Section 4. The maps $X_j \to X_{j-1}$ are simplicial and hence induce morphisms of exact couples by Corollary 4.1. Define

$$
D^1_{*,*} = \varinjlim_j D^1_{*,*}(j))
$$

and

$$
E^1_{*,*} = \varinjlim_j (E^1_{*,*}(j)).
$$

This forms an exact couple, since filtered direct limits of exact sequences are exact. The resulting spectral sequence has

$$
E^2 \cong \varinjlim E^2(j) \cong \varinjlim \check{H}^*(X_j; \pi_*(\mathrm{GL_o}\,B) \otimes \mathbb{P}) \cong \check{H}^*(X; \pi_*(\mathrm{GL_o}\,B) \otimes \mathbb{P})
$$

and converges to

$$
\varinjlim \pi_*(\Gamma(\zeta_j)) \otimes \mathbb{P} \cong \pi_*(\mathrm{GL_o}\,A_\zeta) \otimes \mathbb{P}
$$

as required. (Note that Čech cohomology is required at this point since that is the cohomology theory that respects inverse limits of compact spaces.)

We turn next to the naturality claims.

Suppose given a pullback diagram

$$
\begin{array}{ccc}
f^*E & \longrightarrow & E \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f} & X.
\end{array}
$$

Then use the diagram (1) to construct the spectral sequence for $A_\zeta = \Gamma(X,E)$. Write $X = \lim X_j$ and $X' = \lim X'_j$.

Suppose given a bundle $E \to X$ as usual, with associated diagram

$$
\begin{array}{ccccccc}
\mathrm{GL}_\mathrm{o}\, E & \longrightarrow & \mathrm{GL}_\mathrm{o}\, E_j & \longrightarrow & \mathrm{GL}_\mathrm{o}\, E_{j-1} & \longrightarrow & EG \times_G \mathrm{GL}_\mathrm{o}\, B \\
\downarrow & & \downarrow{\scriptstyle \zeta_j} & & \downarrow & & \downarrow \\
X & \xrightarrow{\ h_j\ } & X_j & \longrightarrow & X_{j-1} & \longrightarrow & BG
\end{array}
\qquad (2)
$$

and with each square a pullback. Passing to cofinal subsequences and renumbering, expand diagram (2) to a homotopy-commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{GL}_\mathrm{o}\, E & \longrightarrow & \mathrm{GL}_\mathrm{o}\, E_j & \longrightarrow & \mathrm{GL}_\mathrm{o}\, E_{j-1} & \longrightarrow & EG \times_G \mathrm{GL}_\mathrm{o}\, B
\end{array}
\qquad (3)
$$

$$
\begin{array}{c}
X \xrightarrow{\ h_j\ } X_j \longrightarrow X_{j-1} \longrightarrow BG \\
X' \xrightarrow{\ h'_j\ } X'_j \longrightarrow X'_{j-1}
\end{array}
$$

with maps f, f_j, f_{j-1}.

and define bundles $f^*\zeta_j$ over each X'_j via pullback diagrams

$$
\begin{array}{ccc}
f_j^* E_j & \longrightarrow & E_j \\
\downarrow & & \downarrow \\
X'_j & \xrightarrow{\ f_j\ } & X_j.
\end{array}
$$

Proposition 4.1 then tells us that there is a morphism of spectral sequences

$$
E^*_{*,*}(A_{\zeta_j}) \longrightarrow E^*_{*,*}(A_{f^*\zeta_j})
$$

and taking direct limits of both sides there is a morphism of spectral sequences

$$
E^*_{*,*}(A_\zeta) \longrightarrow E^*_{*,*}(A_{f^*\zeta})
$$

as desired. This completes the proof of the naturality of the spectral sequence with respect to pullbacks of bundles.

Suppose that $\alpha : B \to B'$ is a map of Banach algebras over $\mathbb{F}$ that respects the group action G with corresponding bundles E and E' over X. Then there is a natural bundle map

$$
\begin{array}{ccc}
EG \times_G \mathrm{GL}_\mathrm{o}\, B & \longrightarrow & EG \times_G \mathrm{GL}_\mathrm{o}\, B' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ Id\ } & X
\end{array}
$$

and a coherent three-dimensional diagram showing the naturality of diagram (2) with respect to α. It is then routine to show that the various subsequent constructions carry this naturality along, essentially regarding B as coefficients.

This completes the proof of Theorem A.

6. Comparison Theorems

The spectral sequence of Theorem A is natural in several senses and these naturality results bring with them corresponding comparison theorems. These are routine consequences of the Comparison Theorem for spectral sequences.[6]

Theorem 6.1 *Suppose that X is a finite dimensional compact metric space, B and B' are Banach algebras over $\mathbb{F}$, and*

$$\zeta : E \to X \qquad and \qquad \zeta' : E' \to X$$

are two standard bundles of Banach algebras with fibres B and B' respectively. Suppose given a map of Banach bundles $f : E \to E'$. Then there is an induced natural map of Theorem A spectral sequences which induces the natural map on the E_2 and E_∞ levels. If the induced map

$$f_* : \pi_*(\mathrm{GL_o}\,B) \otimes \mathbb{P} \to \pi_*(\mathrm{GL_o}\,B') \otimes \mathbb{P}$$

is an isomorphism, then the induced map

$$f_* : \pi_*(\mathrm{GL_o}\,A_\zeta) \otimes \mathbb{P} \to \pi_*(\mathrm{GL_o}\,A_{\zeta'}) \otimes \mathbb{P}$$

is an isomorphism.

Proof: The fact that the first induced map is an isomorphism implies that the map of spectral sequences is an isomorphism at the E^2 level and "the standard comparison theorem" implies that the map of spectral sequences is an isomorphism at the E^∞ level, which implies that the second map is an isomorphism.   □

Theorem 6.2 *Suppose that X and X' are finite dimensional compact metric spaces. Suppose given a Banach algebra B over $\mathbb{F}$, a standard bundle $\zeta' : E' \to X'$ with fibre B and a continuous function $f : X \to X'$. Let $\zeta : E \to X$ denote the pullback*

---

[6]It turns out to be remarkably difficult to give proper credit for spectral sequence comparison theorems. Mac Lane [15] Theorem 11.1 is a fine reference, but he gives credit to J.C. Moore [18] in the Cartan Seminar 1954–5 (now available on line!) who gives credit to a paper of Eilenberg and Mac Lane. McCleary [16] gives credit to E. C. Zeeman, who credits Moore for the result that we use, more or less. None of these are exactly the theorem that we use here, but *hamaivin yavin* (a phrase found in Kabbalistic literature meaning "those who understand will understand.")

bundle. This induces a natural map of spectral sequence with the natural maps on E_2 and E_∞. If the induced map

$$f^* : \check{H}^*(X';\mathbb{P}) \to \check{H}^*(X;\mathbb{P})$$

is an isomorphism, then the induced map

$$f^* : \pi_*(\mathrm{GL_o}\, A_{\zeta'}) \otimes \mathbb{P} \to \pi_*(\mathrm{GL_o}\, A_{\zeta}) \otimes \mathbb{P}$$

is an isomorphism.

Proof: The fact that the map (1) is an isomorphism implies that the map of spectral sequences is an isomorphism at the E^2 level and "the standard comparison theorem" implies that the map of spectral sequences is an isomorphism at the E^∞ level, which implies that (2) is an isomorphism. □

7. The Stabilization Process

Suppose $\zeta : E = P \times_G B \to X$ is a bundle of Banach algebras over $\mathbb{F}$. Let G act trivially on $M_n(\mathbb{F})$. Form a new bundle

$$\zeta_n : P \times_G (B \otimes M_n(\mathbb{F})) \to X$$

(cf. [17] page 32) over X; the space of sections of this bundle is denoted

$$A_{\zeta_n} = \Gamma(X, P \times_G (B \otimes M_n(\mathbb{F}))) \cong A_\zeta \otimes M_n(\mathbb{F}).$$

It is elementary, then, to see that $\mathrm{GL}_n A_\zeta \equiv \mathrm{GL_o}(A_{\zeta_n})$ consists of sections

$$s : X \to E \otimes M_n(\mathbb{F})$$

with the property that $s(x) \in \mathrm{GL}_n(E_x)$ for all x and that the path component of the identity consists of those sections connected by a path of such sections to the identity section.

For any Banach algebra A over $\mathbb{F}$, there is a natural "upper left corner" homomorphism $\mathrm{GL}_n A \to \mathrm{GL}_{n+1} A$. We may form the associated homotopy groups and calculate $\varinjlim_n \pi_*(\mathrm{GL}_n A)$. Bott periodicity [3] and its extension to Banach algebras by Wood [24] and Karoubi [11] implies that these groups are periodic of period 2 (if $\mathbb{F} = \mathbb{C}$) or 8 (if $\mathbb{F} = \mathbb{R}$) and

$$\varinjlim_n \pi_*(\mathrm{GL}_n A) \xrightarrow{\cong} K_{*+1}(A).$$

Note that the left hand side is $\mathbb{Z}$-graded, with $\pi_* = 0$ for $* < 0$. The right hand side is typically seen as $\mathbb{Z}/8$-graded in the real case and $\mathbb{Z}/2$-graded in the complex case in recent literature, but it is better to think of $K_j(A)$ as defined for $j \geq 0$,[7] by the results of Bott, Wood, and Karoubi.

We want to study the passage to the limit in the last isomorphism. In general the matter can be complicated. For example, in the case $A = \mathbb{C}$ the groups $\pi_*(\mathrm{GL}_n\mathbb{C})$ have vast amounts of torsion, but the limit group $\varinjlim_n \pi_*(\mathrm{GL}_n\mathbb{C}) \cong K_{*+1}(\mathbb{C})$ is simply $\mathbb{Z}$ or 0 depending upon the parity of $*$.

Fix some standard bundle $\zeta : E = P \times_G B \to X$ of B-algebras over $\mathbb{F}$ and associated Banach algebra A_ζ. Following the procedure outlined above, produce the sequence of Banach algebras A_{ζ_n}. Then there is a spectral sequence

$$E^2_{-p,q}(A_\zeta) \cong \check{H}^p(X; \pi_q(\mathrm{GL}_\mathrm{o} B) \otimes \mathbb{P}) \Longrightarrow \pi_*(\mathrm{GL}_\mathrm{o} A_\zeta) \otimes \mathbb{P}$$

and, for each n, a spectral sequence

$$E^2_{-p,q}(A_{\zeta_n}) \cong \check{H}^p(X; \pi_q(\mathrm{GL}_n B) \otimes \mathbb{P}) \Longrightarrow \pi_*(\mathrm{GL}_\mathrm{o} A_{\zeta_n}) \otimes \mathbb{P}.$$

The natural maps $\mathrm{GL}_\mathrm{o} B \to \mathrm{GL}_n B \to \mathrm{GL}_{n+1} B$ induce morphisms of spectral sequences

$$E^r_{*,*}(A_\zeta) \longrightarrow E^r_{*,*}(A_{\zeta_n}) \longrightarrow E^r_{*,*}(A_{\zeta_{n+1}}).$$

The direct limit of spectral sequences is again a spectral sequence, since exactness is preserved under filtered direct limits, and hence form yet another spectral sequence

$$E^r_{*,*}(A_{\zeta_\infty}) \equiv \varinjlim_n E^r_{*,*}(A_{\zeta_n}).$$

(This is an abuse of notation since there is no bundle ζ_∞.) The E^2 term of this spectral sequence is readily identified:

$$E^2_{-p,q}(A_{\zeta_\infty}) \equiv \varinjlim_n E^2_{-p,q}(A_{\zeta_n}) = \varinjlim_n \check{H}^p(X; \pi_q(\mathrm{GL}_n B) \otimes \mathbb{P}) \cong$$

$$\cong \check{H}^p(X; \varinjlim_n \pi_q(\mathrm{GL}_n B) \otimes \mathbb{P}) \cong \check{H}^p(X; K_{q+1}(B) \otimes \mathbb{P}).$$

Similarly, the spectral sequence converges to

$$\varinjlim_n \pi_*(\mathrm{GL}_n A_\zeta) \otimes \mathbb{P} \cong K_{*+1}(A_\zeta) \otimes \mathbb{P}.$$

---

[7]J. F. Adams [1] page 619 notes that the Adams operations do not commute with periodicity and for that reason warns the reader "We shall be most careful not to identify $K_\mathbb{C}^{-n-2}(X, Y)$ with $K_\mathbb{C}^{-n}(X, Y)$ or $K_\mathbb{R}^{-n-8}(X, Y)$ with $K_\mathbb{R}^{-n}(X, Y)$. We therefore regard $K_\Lambda(X, Y)$ as graded over $\mathbb{Z}$, not over $\mathbb{Z}_2$ or $\mathbb{Z}_8$."

This completes the proof of Theorem B .

Let B be a Banach algebra over $\mathbb{F}$. Recall that B is *Bott-stable* if the natural map

$$\pi_*(\mathrm{GL_o}\,B) \to K_{*+1}(B)$$

is an isomorphism for every $* > 0$. We note in the Introduction that many important operator algebras have this property. Similarly, call B *rationally Bott-stable* if the natural map

$$\pi_*(\mathrm{GL_o}\,B) \otimes \mathbb{Q} \to K_{*+1}(B) \otimes \mathbb{Q}$$

is an isomorphism.

Here is the proof of Theorem C.

Proof: We prove the first statement, the second being similar. The natural map $\mathrm{GL_o}\,B \to \mathrm{GL}_n\,B$ induces a morphism of spectral sequences Φ_* from the spectral sequence

$$E^2_{-p,q}(A_\zeta) \cong \check{H}^p(X;\pi_q(\mathrm{GL_o}\,B)) \Longrightarrow \pi_*(\mathrm{GL_o}\,A_\zeta)$$

to the spectral sequence

$$E^2_{-p,q}(A_{\zeta_\infty}) \cong \check{H}^p(X;K_{q+1}(B)) \Longrightarrow K_{*+1}(A_\zeta).$$

At the E^2 level the map Φ_2 is an isomorphism, because B is Bott-stable. The standard comparison theorem implies that Φ_n is an isomorphism for each n and hence the map

$$\Phi_\infty : \pi_*(\mathrm{GL_o}\,A_\zeta) \longrightarrow K_{*+1}(A_\zeta)$$

is an isomorphism. This is just the statement that A_ζ is Bott-stable. $\qquad\square$

Note that the result does not imply anything about differentials in the spectral sequence; there is no reason to assume or conclude that $E^2 = E^\infty$.

8. Collapse Theorems

Recall that an *H-space structure* on a based space $(X,*)$ is a map $m\colon X \times X \to X$ whose restriction to $X \times *$ and $* \times X$ is homotopic to the identity as based maps. If an H-space structure on X is understood, we call X an *H-space*. One says that X is *homotopy associative* if the maps $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$ are homotopic. A *homotopy inverse* for X is a map $\iota\colon X \to X$ such that the composites $m \circ (\iota \times \mathrm{id})$ and $m \circ (\mathrm{id} \times \iota)$ are homotopic to the identity.

Definition 8.1 An H-space X is *group-like* if its multiplication is homotopy associative and has a homotopy inverse.

If X is a group-like H-space then the set of path components $\pi_0(X)$ acquires a group structure.

Theorem 8.2 *Suppose that X is a finite dimensional compact metric space, and suppose given a fibre bundle $\zeta : E = P \times_G B \to X$ of Banach algebras over $\mathbb{C}$. Assume further that the classifying space of $\mathrm{GL}_o B$ has the rational homotopy type of a group-like H-space. (This condition holds, for instance, if $\mathrm{GL}_o B$ has rationally periodic homotopy groups.) Then*

1. *$\mathrm{GL}_o A_\zeta$ is rationally H-equivalent to the function space $F(X, \mathrm{GL}_o B)$.*

2. *There is an isomorphism*

$$\pi_*(\mathrm{GL}_o A_\zeta) \otimes \mathbb{Q} \cong \check{H}^*(X; \mathbb{Q}) \otimes \pi_*(\mathrm{GL}_o B) \otimes \mathbb{Q}.$$

3. *The spectral sequence of Theorem A has $E^2 = E^\infty$.*

Note that an element $h \otimes x \in \check{H}^*(X; \mathbb{Q}) \otimes \pi_*(\mathrm{GL}_o B) \otimes \mathbb{Q}$ has total degree $|x| - |h|$ (corresponding to the fact that the spectral sequence lies in the second quadrant). Our convention is to discard all terms of non-positive degree.

Proof: Part (1) of the Theorem is established in ([12], page 264, Addendum E). Then the results of [14] on the rational homotopy of function spaces of this type imply Part (2). This of course implies that the spectral sequence must collapse. □

The Theorem immediately implies Corollary 1.1 of the Introduction.

Remark 8.3 Our results can then be contrasted with the various results of Dadarlat (cf. [5], [6]) who computes $\pi_*(\mathrm{GL}_o A_\zeta)$ in terms of K-theory under various stringent assumptions on B.

Remark 8.4 In general there are many sources of differentials in the spectral sequence. If the bundle is trivial then the problem reduces to the homotopy of function spaces, and this is a complicated situation even rationally. Smith's paper [21] gives many examples of non-trivial differentials in that case. In the infinite-dimensional continuous-trace situation explored by Atiyah and Segal [2], they produce a plethora of differentials in the spectral sequence after passing to de Rham cohomology, and so (in contrast to the classical Atiyah-Hirzebruch spectral sequence) this spectral sequence has differentials that do not vanish mod torsion.

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