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Interval Estimation For The Scale Parameter Of Burr Type X Distribution Based On Grouped Data

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The application of some bootstrap type intervals for the scale parameter of the Burr type X distribution with grouped data is proposed. The general asymptotic confidence interval procedure (Chen & Mi, 2001) is studied. The performance of these intervals is investigated and compared. Some of the bootstrap intervals give better performance for situations of small sample size and heavy censoring.

Key words: Bootstrap, Burr type X distribution, grouped data, interval estimation

Introduction

In many applications, individual observations are very naturally categorized into mutually exclusive and exhaustive groups; such type of data is often called grouped data. Grouped data arise frequently in life testing experiments when inspecting the test units intermittently for failure, this procedure is frequently used because it requires less testing effort than continuous inspection. The data obtained from intermittent inspection consists only of the number of failures in each inspection interval. Other examples of natural occurrences of grouped data are given in Pettitt and Stephens (1977). In this article, a different computer intensive confidence interval is obtained based on grouped data for the scale parameter of Burr Type X Distribution “ $BurrX(\nu, \theta)$ ” whose distribution function is given by

$$F(x, \nu, \theta) = \left[1 - e^{-\left(\frac{x}{\theta}\right)^2} \right]^\nu, \quad x > 0, \theta > 0, \nu > 0 \quad (1)$$

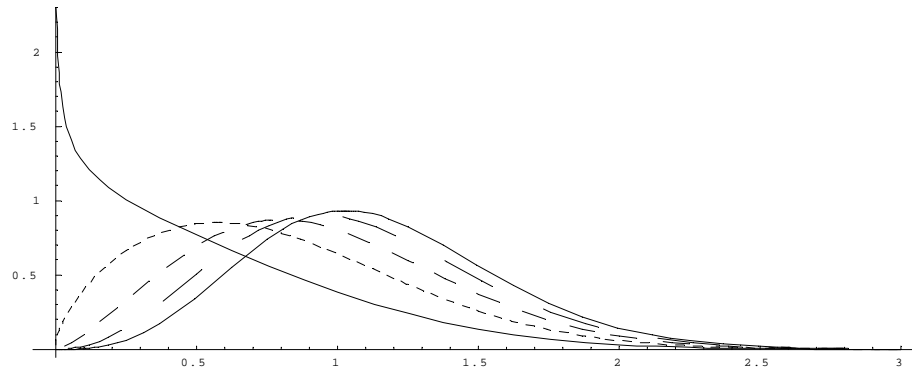
By taking the first derivative on (1), the density function of the BurrX distribution can be obtained as:

$$f(x, \nu, \theta) = \frac{2\nu x}{\theta^2} \left(e^{-\left(\frac{x}{\theta}\right)^2} \right) \left(1 - e^{-\left(\frac{x}{\theta}\right)^2} \right)^{\nu-1}$$

This density function introduced by Burr (1942) providing new model of life time data. The applications of the Burr distribution may be found in the literature for the different twelve types of this distribution. Various authors, considered BurrX in different aspects (e. g., Mudholkar et al., 1995; Mudholkar & Hutson, 1996; Jaheen, 1995, 1996; Surles & Padgett, 1998, 2001; Ahmad et al., 1997). This distribution is a generalized Rayleigh distribution and also it is considered as a special case of exponentiated Weibull distribution that introduced by Mudholkar and Sirvastava (1993). The shape of this distribution depends on the parameter ν , by increasing its value the more symmetry of the distribution. Figure 1 represents the BurrX density function with $\nu = 0.4, 0.8, 1.2, 1.6$ and 2 with unity scale:

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Figure 1. BurrX density with unity scale and different shaper parameters values.



In this article, bootstrap methods are used to construct confidence intervals for the scale parameter of BurrX distribution, θ . Many authors consider bootstrap methods in a vast range of domains (e. g., Davison et al., 2003; Noreen, 1989; Hall, 1988, 1992; Mooney & Duval, 1989). However, the widely used methods in constructing the confidence intervals consist of the second order accurate bootstrap confidence intervals, namely; “Bootstrap-t (BST)” which give good theoretical coverage probabilities but not reliable and the “Bias Corrected and Accelerated (BCa)” which is the second improved version of the Percentile intervals (PRC). These methods may provide good approximate confidence intervals and better than the usual standard intervals (Efron, 2003). We applied these methods, in addition to, the first improved version of PRC which is the Bias-corrected (BC). Other methods are used, Jackknife Intervals (JAC) and Intervals Based on the Bootstrap Standard Deviation (BSD) Intervals.

We review the approximated confidence interval proposed by Chen and Mi (2001). Next, different bootstrap confidence interval approaches are considered. Then, Monte Carlo evidence on the numerical performance of the bootstrap is presented. Finally, make recommendations.

Approximated Confidence Interval

This confidence interval was proposed by Chen and Mi (2001). They introduced an approximate confidence intervals of certain parameters for distribution function on $(0, \infty)$ which satisfy some monotonic conditions using grouped data. To explain their method, let X_1, X_2, \dots, X_n be a random sample from a distribution $F(X, \theta)$ of a continuous type having probability density on $(0, \infty)$, where θ is unknown parameter. Assume that there are K (groups) inspection times $0 < t_1 < t_2 < \dots < t_k < \infty$, and the probability of an observation fail in the i th group is $p_i = P(t_{i-1} < X \leq t_i)$, $i = 1, 2, \dots, K+1$ where $t_0 = 0$ and $t_{k+1} = \infty$. Let $r_i, i = 1, 2, \dots, K+1$ is the number of observations fail in the i^{th} interval $[t_{i-1}, t_i)$. Based on the upper endpoint approach they define the random variable

$$\zeta_n = \sum_{i=1}^k r_i t_i + r_{k+1} t_k \quad (2)$$

it follows that from the Central Limit Theorem

$$\frac{\zeta_n - ng(\theta)}{\sqrt{n}\sigma(\theta)} \rightarrow N(0,1) \text{ as } n \rightarrow \infty$$

where $g(\theta) = \sum_{i=1}^k t_i p_i + t_k p_{k+1}$,

and

$$\sigma^2(\theta) = \left(\sum_{i=1}^k t_i^2 p_i + t_k^2 p_{k+1} \right) - \left(\sum_{i=1}^k t_i p_i + t_k p_{k+1} \right)^2.$$

An approximate confidence interval for θ can be obtained based on the following estimates;

$$\hat{p}_i = \frac{r_i}{n}, \quad i = 1, 2, \dots, K + 1$$

and

$$s_n^2 = \left(\sum_{i=1}^k t_i \hat{p}_i + t_k \hat{p}_{k+1} \right) - \left(\sum_{i=1}^k t_i \hat{p}_i + t_k \hat{p}_{k+1} \right)^2$$

for p_i and $\sigma^2(\theta)$, respectively. It follows that, asymptotically:

$$P\left(\frac{\zeta_n}{n} - z_{\alpha/2} \frac{s_n}{\sqrt{n}} < g(\theta) < \frac{\zeta_n}{n} + z_{\alpha/2} \frac{s_n}{\sqrt{n}} \right) = 1 - \alpha.$$

When the function $g(\theta)$ is monotone, an approximate $(1 - \alpha)\%$ confidence interval for θ , call it the CM interval, can be obtained as:

$$\left[g^{-1}\left(\frac{\zeta_n}{n} - z_{\alpha/2} \frac{s_n}{\sqrt{n}} \right), g^{-1}\left(\frac{\zeta_n}{n} + z_{\alpha/2} \frac{s_n}{\sqrt{n}} \right) \right] \quad (3)$$

However, the above interval possesses exact coverage probabilities and symmetry probabilities only for sufficiently large sample sizes.

Methodology

The bootstrap is a nonparametric technique introduced by Efron in 1979. In this study, we consider six bootstrap methods to construct confidence intervals for the scale parameter of BurrX distribution, θ . For each one of the methods described below, the random variable

ζ_n as defined in (2) calculated from the original data. We generate B bootstrap series $x^{*1}, x^{*2}, \dots, x^{*B}$ and then ζ_n^* be calculated from the bootstrap sample for each series. The standard normal cumulative distribution function is denoted by $\Phi(\cdot)$, and z_α is the α percentile of the standard normal distribution.

t Interval (BTS Intervals)

BST is very similar to confidence intervals based on the t-Student distribution. To construct this interval let z_α^* be the α percentile of the empirical distribution of $Z^* = \frac{(\zeta_n^* - \zeta_n)}{s_n^*}$, where s_n^* is estimated standard error of ζ_n^* calculated from the bootstrap sample. Then BST interval for θ is given by

$$\left[g^{-1}\left(\zeta_n - z_{\alpha/2}^* s_n^* \right), g^{-1}\left(\zeta_n + z_{1-\alpha/2}^* s_n^* \right) \right] \quad (4)$$

The Percentile Interval (PRC Interval)

An alternative bootstrap interval has been proposed in Efron (1979), and is discussed in detail in Efron and Tibshirani (1993). Let \hat{g} be the cumulative distribution function of ζ_n^* , then the $1 - \alpha$ PRC interval is given by

$$\left[\hat{g}^{-1}\left(\frac{\alpha}{2} \right), \hat{g}^{-1}\left(1 - \frac{\alpha}{2} \right) \right] \quad (5)$$

The Bias Corrected Interval (Interval)

The bias corrected interval (Efron, 1982) is calculated using the corrected percentiles of the bootstrap distribution of ζ_n^* . The determination of the appropriate percentiles depends on a number $(\hat{z}_0 = \Phi^{-1}\left(\frac{\#\{\zeta_n^* < \zeta_n\}}{B} \right))$ which measure the median bias and called the bias correction. The $1 - \alpha$ BC interval is given by

$$\left[\hat{g}^{-1}(\alpha_1), \hat{g}^{-1}(\alpha_2) \right] \quad (6)$$

where

$$\alpha_1 = \Phi\left(2\hat{z}_0 + z_{\alpha/2}\right),$$

$$\alpha_2 = \Phi\left(2\hat{z}_0 + z_{1-\alpha/2}\right).$$

Bias Corrected and Accelerated Interval (BCa Interval)

In this method, we calculate the bias correction \hat{z}_0 as before. We need to calculate the acceleration:

$$\hat{a} = \frac{\sum_{i=1}^n (\zeta_n(\cdot) - \zeta_n(i))^3}{6 \left\{ \sum_{i=1}^n (\zeta_n(\cdot) - \zeta_n(i))^2 \right\}^{3/2}}.$$

Thus in the same way we calculated the BC interval; the $1 - \alpha$ BCa interval is given by

$$\left[\hat{g}^{-1}(\alpha_1), \hat{g}^{-1}(\alpha_2) \right] \quad (7)$$

where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{\alpha/2})}\right),$$

$$\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{1-\alpha/2})}\right);$$

and $\zeta_n(i)$ is calculated using the original data excluding the i -th observation and

$$\zeta_n(\cdot) = \frac{\sum_{i=1}^n \zeta_n(i)}{n}.$$

Jackknife Intervals (JAC Intervals)

An interval based on the jackknife (Efron & Tibshirani, 1993) can be constructed as follows;

$$\left[\zeta_n(\cdot) - z_{\alpha/2} s\hat{e}, \zeta_n(\cdot) + z_{1-\alpha/2} s\hat{e} \right] \quad (8)$$

where

$$s\hat{e}^2 = \frac{n-1}{n} \sum_{i=1}^n (\zeta_n(\cdot) - \zeta_n(i))^2$$

is the jackknife estimate of the variance of ζ_n .

Bootstrap Standard Deviation (BSD Intervals)

An interval similar in form to the based on the jackknife can be constructed as follows;

$$\left[\zeta_n - z_{\alpha/2} s\tilde{e}, \zeta_n + z_{1-\alpha/2} s\tilde{e} \right] \quad (9)$$

where

$$s\tilde{e}^2 = \frac{1}{B-1} \sum_{i=1}^B (\zeta_{i,n}^* - \bar{\zeta}_n^*)^2,$$

$$\bar{\zeta}_n^* = \frac{1}{B} \sum_{i=1}^B \zeta_{i,n}^*$$

is the bootstrap estimate of the variance of ζ_n .

Results

A simulation study is conducted to investigate the performance of the interval methods. The 95% confidence intervals for θ was constructed using the seven methods proposed in (3)- (9). The criterion of attainment of lower and upper error probabilities (Jennings, 1987) which are both taken equal to 0.025 was used. In order to compare the performance of the bootstrap estimates, 2000 samples were generated from the BurrX distribution with $\theta = 1$ and $\nu = 0.4, 0.8, 1.2, 1.6$ and 2 ; with different sizes $n = 20, 30, 50$ and 100 .

For each combination, each sample was divided into $(K+1)$ groups where $K = 2, 4$ and 8 . The censoring proportion (cp) is taken as $0.2, 0.4$ and 0.6 . The empirical bootstrap distribution was constructed using $B = 2000$ replications. Then, the following quantities are simulated for each interval: Lower error rates (L): The fraction of intervals that fall entirely above the true parameter; Upper error rates (U): The fraction of intervals that fall entirely below the true parameter and Total error rates (T): The fraction of intervals that did not contain the true parameter value. The results are given in Tables 1-3.

Conclusion

We have compared the performance of several versions of bootstrap confidence intervals together and with the approximated (CM) confidence interval. Bootstrap confidence intervals outperform the CM interval in terms of

total error rates and symmetry in many cases with large sample sizes and appear to be better for small sample sizes.

It can be noted that for $k = 2$, small sample size ($n = 20, 30$) and with censoring proportion ($cp = 0.4$), the CM intervals tend to be anti-conservative. This is also true for JAC, BTS and BCa intervals. On the other hand, the BC, BSD and PRC intervals tend to attain the nominal sizes. As the censoring proportion is light to moderate with $cp = 0.8$, the JAC and the CM intervals tend to be equivalent and grossly anti-conservative while the BC and BCa intervals tend to be grossly conservative. For larger sample sizes ($n = 50, 100$) all intervals attain their nominal sizes except for the BC and BCa intervals where they remain anticonservative. In situations where $k = 2$ and small sample size, all intervals are asymmetric. As k increases, the intervals tend generally to be more symmetric.

The performance of the PRC, BC and BCa intervals improves considerably for larger values of k . Also their performance improves for higher values of r , that is, the more symmetric the parent BurrX distribution, the more symmetric the PRC, BC and BCa intervals tend to be. In conclusion, it appears that the intervals proposed by Chen and Mi (2001) have a good performance except for situations of small sample size and heavy censoring. In this case the BTS, JAC and especially BSD intervals provide better alternatives.

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Table.1 95% confidence interval for θ based on $k = 2$.

R	CP	n	20			30			50			100		
			Method	L	U	T	L	U	T	L	U	T	L	U
0.4	0.4	CM	0.000	0.023	0.023	0.020	0.019	0.039	0.012	0.027	0.039	0.015	0.027	0.042
		BTS	0.000	0.008	0.008	0.020	0.011	0.031	0.012	0.010	0.022	0.015	0.040	0.055
		PRC	0.000	0.026	0.026	0.000	0.045	0.045	0.000	0.096	0.096	0.002	0.087	0.088
		BC	0.000	0.070	0.070	0.000	0.048	0.048	0.012	0.066	0.078	0.052	0.035	0.087
		BCA	0.000	0.085	0.085	0.000	0.068	0.068	0.008	0.089	0.097	0.040	0.047	0.087
		JAC	0.000	0.023	0.023	0.020	0.019	0.039	0.012	0.027	0.039	0.042	0.027	0.069
		BSD	0.006	0.046	0.052	0.013	0.024	0.037	0.025	0.025	0.050	0.020	0.030	0.050
	0.6	CM	0.013	0.024	0.036	0.007	0.026	0.033	0.017	0.029	0.045	0.023	0.022	0.045
		BTS	0.013	0.009	0.022	0.007	0.020	0.027	0.017	0.005	0.022	0.022	0.020	0.041
		PRC	0.000	0.009	0.009	0.002	0.023	0.025	0.005	0.027	0.031	0.006	0.037	0.043
		BC	0.013	0.082	0.095	0.008	0.059	0.067	0.041	0.043	0.083	0.029	0.036	0.064
		BCA	0.000	0.083	0.083	0.007	0.064	0.071	0.041	0.046	0.087	0.023	0.042	0.065
		JAC	0.013	0.024	0.036	0.007	0.026	0.033	0.017	0.029	0.045	0.023	0.022	0.045
		BSD	0.018	0.037	0.054	0.016	0.033	0.049	0.022	0.021	0.043	0.021	0.027	0.048
	0.8	CM	0.029	0.049	0.078	0.018	0.024	0.042	0.015	0.027	0.042	0.024	0.034	0.058
		BTS	0.028	0.017	0.044	0.018	0.007	0.025	0.017	0.011	0.027	0.024	0.016	0.039
		PRC	0.006	0.017	0.022	0.005	0.023	0.028	0.006	0.015	0.021	0.019	0.016	0.034
		BC	0.006	0.091	0.097	0.008	0.101	0.109	0.013	0.091	0.104	0.037	0.056	0.093
		BCA	0.003	0.091	0.094	0.008	0.104	0.112	0.008	0.093	0.101	0.031	0.057	0.088
		JAC	0.029	0.049	0.078	0.018	0.024	0.042	0.015	0.027	0.042	0.024	0.034	0.058
		BSD	0.022	0.043	0.064	0.018	0.043	0.061	0.017	0.029	0.046	0.024	0.033	0.057
0.8	0.4	CM	0.000	0.020	0.020	0.024	0.021	0.045	0.014	0.027	0.041	0.019	0.023	0.042
		BTS	0.000	0.005	0.005	0.024	0.015	0.039	0.014	0.009	0.023	0.019	0.038	0.057
		PRC	0.000	0.022	0.022	0.000	0.050	0.050	0.000	0.090	0.090	0.001	0.086	0.087
		BC	0.000	0.060	0.060	0.001	0.052	0.053	0.014	0.068	0.082	0.059	0.032	0.091
		BCA	0.000	0.078	0.078	0.000	0.076	0.076	0.013	0.086	0.098	0.048	0.044	0.092
		JAC	0.000	0.020	0.020	0.024	0.021	0.045	0.014	0.027	0.041	0.041	0.023	0.064
		BSD	0.007	0.037	0.043	0.016	0.033	0.049	0.021	0.028	0.049	0.026	0.031	0.056
	0.6	CM	0.012	0.027	0.039	0.014	0.023	0.037	0.018	0.040	0.058	0.022	0.028	0.050
		BTS	0.012	0.011	0.022	0.014	0.015	0.029	0.018	0.011	0.029	0.022	0.025	0.046
		PRC	0.000	0.011	0.011	0.001	0.019	0.020	0.008	0.036	0.044	0.007	0.044	0.051
		BC	0.012	0.076	0.088	0.014	0.046	0.059	0.047	0.057	0.104	0.027	0.042	0.068
		BCA	0.000	0.077	0.077	0.014	0.050	0.063	0.047	0.062	0.108	0.022	0.050	0.072
		JAC	0.012	0.027	0.039	0.014	0.023	0.037	0.018	0.040	0.058	0.022	0.028	0.050
		BSD	0.017	0.035	0.052	0.023	0.037	0.060	0.024	0.041	0.065	0.025	0.035	0.060
	0.8	CM	0.029	0.072	0.100	0.018	0.032	0.050	0.020	0.035	0.055	0.023	0.030	0.053
		BTS	0.028	0.022	0.050	0.018	0.013	0.030	0.023	0.010	0.032	0.023	0.019	0.042
		PRC	0.007	0.022	0.029	0.003	0.031	0.034	0.011	0.018	0.028	0.015	0.019	0.033
		BC	0.008	0.107	0.115	0.005	0.097	0.102	0.018	0.099	0.117	0.035	0.050	0.085
		BCA	0.005	0.108	0.112	0.005	0.099	0.104	0.012	0.100	0.112	0.028	0.051	0.078
		JAC	0.029	0.072	0.100	0.018	0.032	0.050	0.020	0.035	0.055	0.023	0.030	0.053
		BSD	0.025	0.054	0.079	0.020	0.049	0.069	0.026	0.041	0.066	0.023	0.030	0.053

Table 1 Continued:

2	0.4	CM	0.000	0.024	0.024	0.024	0.016	0.040	0.013	0.032	0.044	0.019	0.024	0.043
		BTS	0.000	0.006	0.006	0.024	0.011	0.034	0.013	0.012	0.025	0.019	0.038	0.057
		PRC	0.000	0.028	0.028	0.000	0.049	0.049	0.000	0.085	0.085	0.000	0.090	0.090
		BC	0.000	0.074	0.074	0.005	0.055	0.060	0.013	0.047	0.060	0.032	0.036	0.067
		BCA	0.000	0.084	0.084	0.002	0.100	0.102	0.009	0.066	0.075	0.021	0.045	0.066
		JAC	0.000	0.024	0.024	0.024	0.016	0.040	0.013	0.032	0.044	0.043	0.024	0.067
		BSD	0.007	0.048	0.054	0.016	0.032	0.048	0.026	0.030	0.056	0.022	0.025	0.046
	0.6	CM	0.015	0.024	0.039	0.011	0.015	0.026	0.020	0.033	0.053	0.024	0.020	0.044
		BTS	0.015	0.011	0.026	0.011	0.011	0.022	0.020	0.011	0.030	0.024	0.019	0.042
		PRC	0.000	0.011	0.011	0.001	0.013	0.013	0.007	0.031	0.038	0.007	0.034	0.041
		BC	0.015	0.078	0.093	0.011	0.046	0.057	0.068	0.058	0.126	0.039	0.037	0.076
		BCA	0.000	0.079	0.079	0.011	0.052	0.063	0.056	0.062	0.118	0.037	0.039	0.076
		JAC	0.015	0.024	0.039	0.011	0.015	0.026	0.020	0.033	0.053	0.024	0.020	0.044
		BSD	0.018	0.034	0.051	0.018	0.027	0.045	0.022	0.032	0.054	0.022	0.028	0.050
	0.8	CM	0.023	0.067	0.090	0.024	0.032	0.056	0.015	0.037	0.052	0.020	0.034	0.054
		BTS	0.022	0.020	0.042	0.024	0.007	0.030	0.016	0.012	0.028	0.020	0.019	0.038
		PRC	0.005	0.020	0.025	0.005	0.032	0.037	0.007	0.022	0.029	0.017	0.019	0.035
		BC	0.008	0.127	0.135	0.005	0.133	0.138	0.017	0.094	0.111	0.023	0.049	0.072
		BCA	0.005	0.128	0.132	0.005	0.134	0.139	0.011	0.094	0.105	0.022	0.050	0.072
		JAC	0.023	0.067	0.090	0.024	0.032	0.056	0.015	0.037	0.052	0.020	0.034	0.054
		BSD	0.018	0.053	0.070	0.027	0.049	0.075	0.020	0.041	0.061	0.018	0.031	0.048

Table 2: 95% confidence interval for θ based on $k = 4$.

R	CP	n	20			30			50			100		
			L	U	T	L	U	T	L	U	T	L	U	T
0.4	0.4	CM	0.000	0.036	0.036	0.005	0.026	0.031	0.009	0.025	0.034	0.014	0.023	0.037
		BTS	0.000	0.016	0.016	0.001	0.024	0.025	0.003	0.053	0.056	0.003	0.072	0.075
		PRC	0.000	0.067	0.067	0.000	0.078	0.078	0.000	0.097	0.097	0.001	0.104	0.104
		BC	0.014	0.035	0.049	0.032	0.019	0.051	0.048	0.013	0.061	0.069	0.010	0.078
		BCA	0.007	0.045	0.051	0.023	0.028	0.051	0.043	0.020	0.063	0.062	0.011	0.073
		JAC	0.000	0.039	0.039	0.005	0.029	0.033	0.011	0.028	0.039	0.015	0.024	0.039
		BSD	0.002	0.037	0.039	0.007	0.030	0.037	0.010	0.026	0.036	0.016	0.024	0.040
	0.6	CM	0.007	0.046	0.053	0.008	0.044	0.052	0.013	0.037	0.050	0.024	0.024	0.048
		BTS	0.002	0.021	0.022	0.004	0.030	0.033	0.005	0.038	0.043	0.010	0.042	0.052
		PRC	0.000	0.046	0.046	0.000	0.069	0.069	0.001	0.057	0.058	0.002	0.057	0.058
		BC	0.022	0.052	0.074	0.023	0.056	0.079	0.035	0.033	0.068	0.051	0.017	0.068
		BCA	0.012	0.055	0.067	0.017	0.063	0.079	0.029	0.038	0.067	0.046	0.019	0.065
		JAC	0.008	0.047	0.055	0.009	0.045	0.053	0.013	0.041	0.054	0.024	0.024	0.048
		BSD	0.006	0.047	0.053	0.009	0.045	0.053	0.013	0.041	0.054	0.022	0.028	0.050
	0.8	CM	0.008	0.057	0.065	0.012	0.042	0.054	0.014	0.039	0.053	0.019	0.031	0.050
		BTS	0.005	0.022	0.026	0.007	0.021	0.027	0.008	0.025	0.033	0.009	0.025	0.033
		PRC	0.001	0.027	0.028	0.002	0.030	0.031	0.003	0.028	0.031	0.006	0.028	0.034
		BC	0.014	0.092	0.106	0.014	0.072	0.086	0.021	0.050	0.071	0.026	0.040	0.065
		BCA	0.010	0.095	0.105	0.012	0.073	0.085	0.019	0.051	0.069	0.025	0.042	0.066
		JAC	0.011	0.057	0.068	0.012	0.043	0.054	0.015	0.040	0.055	0.020	0.031	0.051
		BSD	0.011	0.059	0.070	0.012	0.046	0.057	0.014	0.040	0.054	0.018	0.031	0.049

Table 2: Continued

1.6	0.4	CM	0.000	0.033	0.033	0.009	0.029	0.038	0.011	0.029	0.039	0.018	0.025	0.043
		BTS	0.000	0.014	0.014	0.003	0.032	0.035	0.003	0.049	0.052	0.002	0.099	0.101
		PRC	0.000	0.060	0.060	0.000	0.082	0.082	0.000	0.088	0.088	0.000	0.135	0.135
		BC	0.011	0.033	0.044	0.040	0.021	0.061	0.054	0.020	0.074	0.080	0.011	0.090
		BCA	0.006	0.040	0.046	0.027	0.030	0.057	0.044	0.024	0.068	0.075	0.014	0.089
		JAC	0.000	0.034	0.034	0.009	0.033	0.042	0.012	0.031	0.043	0.019	0.025	0.044
		BSD	0.004	0.034	0.038	0.010	0.035	0.045	0.009	0.035	0.043	0.016	0.027	0.043
	0.6	CM	0.007	0.045	0.051	0.007	0.032	0.039	0.014	0.032	0.046	0.019	0.028	0.047
		BTS	0.003	0.019	0.022	0.003	0.022	0.025	0.003	0.030	0.033	0.005	0.046	0.051
		PRC	0.000	0.047	0.047	0.000	0.072	0.072	0.000	0.053	0.053	0.001	0.065	0.065
		BC	0.020	0.051	0.071	0.019	0.054	0.073	0.039	0.027	0.066	0.046	0.018	0.064
		BCA	0.011	0.059	0.070	0.014	0.059	0.073	0.033	0.032	0.065	0.044	0.023	0.067
		JAC	0.010	0.047	0.057	0.007	0.038	0.045	0.014	0.034	0.048	0.019	0.028	0.047
		BSD	0.010	0.046	0.056	0.006	0.037	0.042	0.014	0.030	0.044	0.015	0.032	0.047
	0.8	CM	0.008	0.056	0.063	0.011	0.045	0.056	0.014	0.034	0.048	0.024	0.032	0.056
		BTS	0.005	0.020	0.025	0.005	0.022	0.026	0.006	0.017	0.023	0.015	0.021	0.036
		PRC	0.001	0.026	0.027	0.002	0.031	0.032	0.002	0.018	0.020	0.010	0.025	0.034
		BC	0.010	0.093	0.103	0.014	0.077	0.090	0.021	0.045	0.065	0.032	0.038	0.070
		BCA	0.009	0.094	0.102	0.009	0.077	0.086	0.019	0.047	0.066	0.030	0.039	0.069
		JAC	0.012	0.056	0.068	0.012	0.046	0.057	0.016	0.034	0.050	0.024	0.033	0.057
		BSD	0.009	0.058	0.067	0.009	0.047	0.056	0.012	0.038	0.049	0.021	0.030	0.050
2	0.4	CM	0.000	0.031	0.031	0.004	0.026	0.030	0.011	0.021	0.032	0.018	0.023	0.041
		BTS	0.000	0.015	0.015	0.002	0.030	0.031	0.003	0.051	0.054	0.004	0.090	0.094
		PRC	0.000	0.063	0.063	0.000	0.075	0.075	0.000	0.094	0.094	0.000	0.115	0.115
		BC	0.005	0.031	0.036	0.033	0.023	0.056	0.055	0.011	0.066	0.079	0.010	0.089
		BCA	0.000	0.041	0.041	0.021	0.029	0.050	0.050	0.015	0.065	0.073	0.014	0.087
		JAC	0.000	0.035	0.035	0.004	0.031	0.035	0.014	0.023	0.037	0.020	0.024	0.044
		BSD	0.004	0.038	0.042	0.009	0.029	0.038	0.012	0.026	0.038	0.019	0.026	0.045
	0.6	CM	0.005	0.042	0.047	0.007	0.041	0.047	0.016	0.038	0.053	0.017	0.032	0.049
		BTS	0.002	0.014	0.016	0.003	0.028	0.031	0.003	0.037	0.040	0.007	0.053	0.060
		PRC	0.000	0.044	0.044	0.000	0.074	0.074	0.000	0.058	0.058	0.001	0.071	0.072
		BC	0.012	0.052	0.064	0.018	0.055	0.073	0.034	0.032	0.065	0.048	0.024	0.072
		BCA	0.006	0.057	0.063	0.012	0.068	0.079	0.025	0.038	0.063	0.043	0.027	0.069
		JAC	0.008	0.043	0.050	0.007	0.043	0.050	0.016	0.041	0.057	0.017	0.033	0.050
		BSD	0.007	0.046	0.053	0.009	0.043	0.052	0.013	0.036	0.049	0.016	0.035	0.051
	0.8	CM	0.008	0.053	0.061	0.013	0.043	0.056	0.010	0.038	0.047	0.020	0.030	0.050
		BTS	0.006	0.020	0.026	0.006	0.022	0.027	0.004	0.022	0.026	0.013	0.022	0.035
		PRC	0.001	0.025	0.025	0.001	0.030	0.030	0.001	0.024	0.025	0.009	0.025	0.034
		BC	0.012	0.088	0.099	0.015	0.064	0.079	0.016	0.053	0.069	0.027	0.044	0.070
		BCA	0.008	0.088	0.095	0.012	0.065	0.077	0.013	0.055	0.068	0.025	0.045	0.070
		JAC	0.010	0.053	0.063	0.014	0.043	0.056	0.010	0.039	0.049	0.020	0.031	0.051
		BSD	0.009	0.053	0.062	0.011	0.042	0.053	0.010	0.041	0.051	0.020	0.032	0.052

Table 3: 95% confidence interval for θ based on $k = 8$.

R	CP	n	Method	20			30			50			100		
				L	U	T	L	U	T	L	U	T	L	U	T
0.4	0.4	CM	0.000	0.033	0.033	0.003	0.027	0.030	0.011	0.029	0.040	0.013	0.028	0.041	
		BTS	0.000	0.033	0.033	0.000	0.055	0.055	0.001	0.102	0.103	0.001	0.156	0.157	
		PRC	0.000	0.153	0.153	0.000	0.143	0.143	0.000	0.164	0.164	0.000	0.190	0.190	
		BC	0.012	0.030	0.042	0.040	0.015	0.054	0.073	0.013	0.086	0.092	0.011	0.102	
		BCA	0.005	0.041	0.045	0.033	0.021	0.054	0.069	0.018	0.087	0.088	0.014	0.102	
		JAC	0.000	0.040	0.040	0.004	0.030	0.034	0.011	0.030	0.041	0.013	0.029	0.042	
		BSD	0.000	0.038	0.038	0.005	0.029	0.034	0.009	0.033	0.042	0.012	0.029	0.041	
	0.6	CM	0.004	0.046	0.050	0.010	0.034	0.044	0.010	0.026	0.036	0.020	0.030	0.050	
		BTS	0.001	0.035	0.036	0.004	0.037	0.041	0.004	0.046	0.050	0.006	0.073	0.078	
		PRC	0.000	0.080	0.080	0.000	0.066	0.066	0.000	0.074	0.074	0.001	0.092	0.092	
		BC	0.019	0.041	0.060	0.034	0.025	0.059	0.042	0.018	0.060	0.067	0.016	0.083	
		BCA	0.013	0.046	0.059	0.027	0.028	0.054	0.035	0.020	0.055	0.063	0.019	0.082	
		JAC	0.004	0.054	0.057	0.012	0.038	0.050	0.010	0.028	0.038	0.020	0.030	0.050	
		BSD	0.003	0.049	0.052	0.011	0.036	0.047	0.010	0.029	0.038	0.021	0.029	0.050	
	0.8	CM	0.005	0.060	0.065	0.005	0.057	0.062	0.013	0.044	0.057	0.016	0.035	0.050	
		BTS	0.001	0.035	0.035	0.003	0.040	0.043	0.005	0.039	0.044	0.005	0.039	0.044	
		PRC	0.000	0.043	0.043	0.000	0.051	0.051	0.003	0.047	0.050	0.002	0.043	0.045	
		BC	0.012	0.076	0.088	0.015	0.065	0.080	0.028	0.047	0.075	0.036	0.030	0.065	
		BCA	0.011	0.077	0.087	0.011	0.067	0.078	0.027	0.048	0.075	0.033	0.030	0.063	
		JAC	0.006	0.068	0.074	0.005	0.061	0.066	0.013	0.047	0.060	0.016	0.035	0.051	
		BSD	0.005	0.065	0.069	0.005	0.059	0.064	0.014	0.047	0.061	0.016	0.036	0.052	
0.8	0.4	CM	0.000	0.035	0.035	0.003	0.031	0.034	0.007	0.024	0.031	0.013	0.024	0.037	
		BTS	0.000	0.033	0.033	0.001	0.061	0.062	0.002	0.100	0.102	0.001	0.148	0.149	
		PRC	0.000	0.146	0.146	0.000	0.144	0.144	0.000	0.162	0.162	0.000	0.191	0.191	
		BC	0.014	0.028	0.042	0.038	0.018	0.056	0.070	0.008	0.078	0.108	0.009	0.116	
		BCA	0.009	0.041	0.049	0.031	0.025	0.056	0.064	0.014	0.078	0.104	0.010	0.113	
		JAC	0.000	0.037	0.037	0.005	0.034	0.038	0.007	0.027	0.034	0.013	0.025	0.038	
		BSD	0.001	0.038	0.039	0.005	0.031	0.036	0.008	0.029	0.037	0.014	0.026	0.040	
	0.6	CM	0.004	0.050	0.054	0.005	0.039	0.044	0.012	0.039	0.051	0.015	0.032	0.046	
		BTS	0.001	0.033	0.034	0.001	0.044	0.045	0.001	0.065	0.066	0.002	0.073	0.075	
		PRC	0.000	0.076	0.076	0.000	0.081	0.081	0.000	0.093	0.093	0.001	0.086	0.087	
		BC	0.018	0.042	0.060	0.024	0.030	0.054	0.043	0.026	0.069	0.067	0.015	0.082	
		BCA	0.013	0.048	0.061	0.021	0.035	0.056	0.040	0.029	0.069	0.064	0.017	0.081	
		JAC	0.004	0.056	0.060	0.006	0.042	0.048	0.013	0.041	0.054	0.015	0.032	0.047	
		BSD	0.004	0.054	0.058	0.005	0.040	0.045	0.013	0.041	0.054	0.016	0.033	0.048	
	0.8	CM	0.003	0.066	0.069	0.009	0.056	0.065	0.013	0.043	0.056	0.019	0.032	0.051	
		BTS	0.002	0.042	0.044	0.002	0.036	0.038	0.005	0.038	0.043	0.008	0.038	0.046	
		PRC	0.001	0.050	0.051	0.000	0.048	0.048	0.002	0.043	0.044	0.005	0.042	0.046	
		BC	0.011	0.083	0.093	0.019	0.066	0.085	0.026	0.043	0.069	0.039	0.029	0.067	
		BCA	0.007	0.081	0.088	0.016	0.067	0.083	0.025	0.043	0.068	0.037	0.030	0.067	
		JAC	0.004	0.072	0.076	0.011	0.059	0.070	0.013	0.044	0.057	0.019	0.033	0.052	
		BSD	0.003	0.068	0.070	0.008	0.060	0.067	0.013	0.043	0.056	0.020	0.035	0.055	

Table 3: Continued

1.2	0.4	CM	0.000	0.036	0.036	0.004	0.038	0.041	0.011	0.026	0.037	0.014	0.027	0.041
		BTS	0.000	0.036	0.036	0.001	0.059	0.059	0.001	0.098	0.099	0.001	0.147	0.148
		PRC	0.000	0.148	0.148	0.000	0.144	0.144	0.000	0.156	0.156	0.000	0.184	0.184
		BC	0.013	0.034	0.047	0.040	0.025	0.065	0.059	0.010	0.069	0.099	0.011	0.110
		BCA	0.005	0.042	0.047	0.033	0.030	0.063	0.057	0.015	0.071	0.096	0.016	0.111
		JAC	0.000	0.040	0.040	0.004	0.041	0.045	0.011	0.029	0.040	0.014	0.031	0.045
		BSD	0.001	0.042	0.043	0.005	0.039	0.044	0.011	0.029	0.039	0.015	0.028	0.043
	0.6	CM	0.007	0.052	0.059	0.008	0.050	0.058	0.012	0.038	0.050	0.016	0.031	0.046
		BTS	0.001	0.040	0.041	0.001	0.055	0.056	0.002	0.061	0.063	0.001	0.081	0.081
		PRC	0.000	0.070	0.070	0.000	0.086	0.086	0.000	0.084	0.084	0.000	0.099	0.099
		BC	0.019	0.047	0.066	0.032	0.038	0.069	0.045	0.026	0.071	0.069	0.016	0.084
		BCA	0.014	0.051	0.065	0.025	0.042	0.066	0.040	0.030	0.069	0.067	0.017	0.084
		JAC	0.007	0.056	0.063	0.008	0.054	0.061	0.012	0.040	0.052	0.016	0.032	0.047
		BSD	0.007	0.054	0.061	0.007	0.051	0.057	0.012	0.041	0.053	0.015	0.035	0.049
	0.8	CM	0.008	0.055	0.063	0.010	0.051	0.061	0.016	0.044	0.060	0.017	0.038	0.055
		BTS	0.004	0.032	0.036	0.002	0.038	0.040	0.008	0.038	0.045	0.006	0.041	0.047
		PRC	0.001	0.042	0.043	0.000	0.048	0.048	0.002	0.043	0.045	0.004	0.047	0.051
		BC	0.014	0.074	0.088	0.020	0.065	0.085	0.031	0.044	0.075	0.031	0.032	0.063
		BCA	0.011	0.073	0.084	0.016	0.064	0.080	0.029	0.044	0.073	0.029	0.033	0.062
		JAC	0.010	0.059	0.068	0.011	0.057	0.068	0.017	0.045	0.062	0.018	0.038	0.055
		BSD	0.009	0.058	0.066	0.010	0.054	0.064	0.016	0.046	0.062	0.019	0.037	0.055
1.6	0.4	CM	0.000	0.033	0.033	0.003	0.030	0.033	0.012	0.027	0.038	0.017	0.026	0.043
		BTS	0.000	0.030	0.030	0.000	0.055	0.055	0.002	0.093	0.095	0.002	0.160	0.162
		PRC	0.000	0.141	0.141	0.000	0.140	0.140	0.000	0.155	0.155	0.000	0.192	0.192
		BC	0.005	0.031	0.036	0.036	0.018	0.053	0.075	0.014	0.089	0.112	0.010	0.121
		BCA	0.002	0.042	0.044	0.030	0.024	0.054	0.071	0.017	0.087	0.109	0.012	0.121
		JAC	0.000	0.043	0.043	0.004	0.032	0.035	0.012	0.028	0.039	0.017	0.026	0.043
		BSD	0.000	0.036	0.036	0.005	0.033	0.037	0.012	0.028	0.040	0.017	0.026	0.043
	0.6	CM	0.004	0.045	0.049	0.008	0.038	0.045	0.010	0.038	0.048	0.019	0.030	0.049
		BTS	0.000	0.032	0.032	0.003	0.043	0.046	0.002	0.063	0.065	0.004	0.074	0.078
		PRC	0.000	0.067	0.067	0.000	0.076	0.076	0.000	0.081	0.081	0.000	0.091	0.091
		BC	0.014	0.038	0.052	0.030	0.029	0.059	0.039	0.027	0.066	0.066	0.017	0.082
		BCA	0.009	0.044	0.052	0.022	0.031	0.053	0.037	0.030	0.066	0.063	0.019	0.081
		JAC	0.004	0.048	0.052	0.009	0.042	0.051	0.010	0.040	0.050	0.020	0.031	0.050
		BSD	0.003	0.046	0.049	0.007	0.036	0.043	0.009	0.040	0.049	0.018	0.032	0.050
	0.8	CM	0.007	0.055	0.062	0.013	0.048	0.060	0.013	0.036	0.049	0.019	0.032	0.051
		BTS	0.002	0.026	0.028	0.005	0.026	0.031	0.004	0.033	0.037	0.006	0.039	0.045
		PRC	0.000	0.036	0.036	0.001	0.042	0.042	0.001	0.036	0.037	0.003	0.045	0.048
		BC	0.014	0.069	0.082	0.023	0.056	0.079	0.026	0.038	0.063	0.038	0.028	0.066
		BCA	0.011	0.068	0.079	0.021	0.055	0.076	0.023	0.037	0.060	0.037	0.029	0.066
		JAC	0.008	0.060	0.068	0.013	0.050	0.063	0.013	0.038	0.051	0.020	0.033	0.053
		BSD	0.007	0.056	0.063	0.012	0.048	0.060	0.013	0.038	0.051	0.017	0.033	0.050

Table 3: Continued

2	0.4	CM	0.000	0.031	0.031	0.007	0.031	0.038	0.006	0.029	0.034	0.017	0.034	0.051
		BTS	0.000	0.032	0.032	0.002	0.060	0.062	0.001	0.102	0.103	0.000	0.147	0.147
		PRC	0.000	0.133	0.133	0.000	0.143	0.143	0.000	0.158	0.158	0.000	0.178	0.178
		BC	0.006	0.031	0.037	0.042	0.021	0.063	0.064	0.013	0.077	0.113	0.011	0.123
		BCA	0.004	0.038	0.041	0.033	0.027	0.060	0.059	0.017	0.076	0.107	0.015	0.122
		JAC	0.000	0.039	0.039	0.007	0.035	0.041	0.006	0.029	0.035	0.017	0.035	0.052
		BSD	0.000	0.038	0.038	0.008	0.031	0.039	0.007	0.028	0.035	0.016	0.034	0.050
		0.6	CM	0.003	0.045	0.048	0.009	0.045	0.054	0.009	0.035	0.044	0.015	0.032
BTS	0.001	0.031	0.032	0.003	0.049	0.052	0.002	0.058	0.060	0.002	0.092	0.093		
PRC	0.000	0.067	0.067	0.000	0.080	0.080	0.000	0.083	0.083	0.000	0.108	0.108		
BC	0.015	0.039	0.054	0.032	0.034	0.066	0.040	0.023	0.062	0.059	0.015	0.074		
BCA	0.009	0.046	0.055	0.029	0.040	0.068	0.035	0.025	0.060	0.055	0.016	0.071		
JAC	0.003	0.049	0.052	0.010	0.047	0.057	0.010	0.038	0.048	0.016	0.033	0.049		
BSD	0.002	0.046	0.048	0.009	0.046	0.055	0.009	0.039	0.048	0.016	0.033	0.049		
0.8	CM	0.004	0.057	0.061	0.009	0.046	0.055	0.014	0.040	0.054	0.018	0.031	0.049	
	BTS	0.001	0.033	0.034	0.004	0.027	0.031	0.005	0.039	0.043	0.005	0.033	0.038	
	PRC	0.000	0.043	0.043	0.001	0.042	0.042	0.001	0.043	0.044	0.003	0.039	0.041	
	BC	0.011	0.071	0.081	0.017	0.059	0.075	0.034	0.042	0.076	0.035	0.026	0.061	
	BCA	0.006	0.070	0.076	0.016	0.059	0.074	0.030	0.043	0.072	0.033	0.026	0.059	
	JAC	0.004	0.062	0.066	0.010	0.050	0.060	0.015	0.042	0.057	0.019	0.032	0.050	
	BSD	0.004	0.061	0.065	0.008	0.048	0.056	0.015	0.042	0.057	0.017	0.032	0.048	