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The progressive Type-II censoring scheme has become quite popular. A drawback of a progressive censoring scheme is that the length of the experiment can be very large if the items are highly reliable. Recently, Kundu and Joarder (2006) introduced the Type-II progressively hybrid censored scheme and analyzed the data assuming that the lifetimes of the items are exponentially distributed. This article presents the analysis of Type-II progressively hybrid censored data when the lifetime distributions of the items follow Weibull distributions. Maximum likelihood estimators and approximate maximum likelihood estimators are developed for estimating the unknown parameters. Asymptotic confidence intervals based on maximum likelihood estimators and approximate maximum likelihood estimators are proposed. Different methods are compared using Monte Carlo simulations and one real data set is analyzed.

Key words: Maximum likelihood estimators; approximate maximum likelihood estimators; Type-I censoring; Type-II censoring; Monte Carlo simulation.

Introduction

The Type-II progressive censoring scheme has become very popular. It can be described as follows: consider n units in a study and suppose $m < n$ is fixed before the experiment, in addition, m other integers, $R_1,..,R_m$ are also fixed so that $R_1 + ... + R_m + m = n$. At the time of the first failure, for example, $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, for example, $Y_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the $m-th$ failure, $Y_{m:m:n}$, the remaining R_m units are removed. Extensive work has been conducted on this particular scheme during the last ten years; see Balakrishnan and Aggarwala (2000) and Balakrishnan (2007).

Unfortunately the major problem with the Type-II progressive censoring scheme is that the time length of the experiment can be very large. Due to this problem, Kundu and Joarder (2006) introduced a new censoring scheme named Type-II Progressively Hybrid Censoring, which ensures that the length of the experiment cannot exceed a pre-specified time point T. The detailed description and advantages of the Type-II progressively hybrid censoring is presented in Kundu and Joarder (2006) (see also Childs, Chandrasekar & Balakrishnan, 2007); in both publications the authors assumed the lifetime distributions of the items to be exponential.

Because the exponential distribution has limitations, this article considers the Type-II progressively hybrid censored lifetime data, when the lifetime follows a two-parameter Weibull distribution. Maximum likelihood estimators (MLEs) of the unknown parameters are provided and it was observed that the MLEs cannot be obtained in explicit forms. MLEs can

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be obtained by solving a non-linear equation and a simple iterative scheme is proposed to solve the non-linear equation. Approximate maximum likelihood estimators (AMLEs), which have explicit expressions are also suggested. It is not possible to compute the exact distributions of the MLEs, so the asymptotic distribution is used to construct confidence intervals. Monte Carlo simulations are used to compare different methods and one data analysis is performed for illustrative purposes.

Type-II Progressively Hybrid Censoring Scheme Models

If it is assumed that the lifetime random variable Y has a Weibull distribution with shape and scale parameters α and λ respectively, then the probability density function (PDF) of Y is

$$f_Y(y;\alpha,\gamma)=\frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{y}{\lambda}\right)^{\alpha}};\quad y>0, \quad (1)$$

where $\alpha>0,\ \lambda>0$ are the natural parameter space. If the random variable Y has the density function (1), then $X=\ln Y$ has the extreme value distribution with the PDF

$$f_X(x;\mu,\sigma)=\frac{1}{\sigma}e^{\left(\frac{x-\mu}{\sigma}-e^{\frac{x-\mu}{\sigma}}\right)};\quad -\infty<x<\infty, \quad (2)$$

where $\mu=\ln\lambda,\ \sigma=1/\alpha$. The density function as described by (2) is known as the density function of an extreme value distribution with location and scale parameters μ and σ respectively. Models (1) and (2) are equivalent models in the sense that the procedure developed under one model can be easily used for the other model. Although, they are equivalent models, (2) can be the easier with which to work compared to model (1), because in model (2) the two parameters μ and σ appear as location and scale parameters. For $\mu=0$ and $\sigma=1$, model (2) is known as the standard extreme value distribution and has the following PDF

$$f_Z(z;0,1)=e^{\left(z-e^{z}\right)};\quad -\infty<z<\infty. \quad (3)$$

Type-II Progressively Hybrid Censoring Scheme Data

Under the Type-II progressively hybrid censoring scheme, it is assumed that n identical items are put on a test and the lifetime distributions of the n items are denoted by $Y_1,..,Y_n$. The integer $m<n$ is pre-fixed, $R_1,..,R_m$ are m pre-fixed integers satisfying $R_1+.....+R_m+m=n$, and T is a pre-fixed time point. At the time of the first failure $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $Y_{2:m:n}$, R_2 of the remaining units are removed and so on. If the $m-th$ failure $Y_{m:m:n}$ occurs before time T, the experiment stops at time point $Y_{m:m:n}$. If, however, the m-th failure does not occur before time point T and only J failures occur before T (where $0\le J<m$), then at time T all remaining R_J^* units are removed and the experiment terminates. Note that $R_J^*=n-(R_1+...+R_J)-J$. The two cases are denoted as Case I and Case II respectively and this is called the censoring scheme as the Type-II progressively hybrid censoring scheme (Kundu and Joarder, 2006).

In the presence of the Type-II progressively hybrid censoring scheme, one of the following is observed

Case I:

$$\{Y_{1:m:n},...,Y_{m:m:n}\};\ \text{if } Y_{m:m:n}<T, \quad (4)$$

or

Case II:

$$\{Y_{1:m:n},...,Y_{J:m:n}\};\ \text{if } Y_{J:m:n}<T<Y_{J+1:m:n}. \quad (5)$$

For Case II, although $Y_{J+1:m:n}$ is not observed, but $Y_{J:m:n}<T<Y_{J+1:m:n}$ means that the $J-th$ failure took place before T and no failure took place between $Y_{J:m:n}$ and T (i.e., $Y_{J+1:m:n},..,Y_{m:m:n}$) are not observed.

The conventional Type-I progressive censoring scheme needs the pre-specification of $R_1,..,R_m$ and also $T_1,...,T_m$ (see Cohen 1963, 1966 for details). The choices of $T_1,..,T_m$ are not trivial. For the conventional Type-II progressive censoring scheme the experimental time is unbounded. In the proposed censoring scheme the choice of T depends on how much maximum experimental time the experimenter can afford to continue and also the experimental time is bounded.

Maximum Likelihood Estimators (MLEs)

Based on the observed data, the likelihood function for Case I is

$$l(\alpha,\lambda) = K_1\left(\frac{\alpha}{\lambda}\right)^m \prod_{i=1}^{m}\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha-1} e^{-\left[\sum_{i=1}^{m}(1+R_i)\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha}\right]}, \tag{6}$$

and for Case II, the MLE is

$$l(\alpha,\lambda) = K_2\left(\frac{\alpha}{\lambda}\right)^J \prod_{i=1}^{J}\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha-1} e^{-\left[\sum_{i=1}^{J}(1+R_i)\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha}+R_J^*\left(\frac{T}{\lambda}\right)^{\alpha}\right]} \quad \text{if } J>0, \tag{7}$$

$$= e^{-n\left(\frac{T}{\lambda}\right)^{\alpha}}, \quad \text{if } J=0 \tag{8}$$

where

$$K_1 = \prod_{i=1}^{m}\left[n-\sum_{k=1}^{i-1}(1+R_k)\right]$$

and

$$K_2 = \prod_{i=1}^{J}\left[n-\sum_{k=1}^{i-1}(1+R_k)\right],$$

both are constant.

The logarithm of (6) and (7), can be written without the constant terms as

$$L(\alpha,\lambda) = d(\ln\alpha-\ln\lambda)+(\alpha-1)\left[\sum_{i=1}^{d}\ln y_{i:m:n}-d\ln\lambda\right]-\frac{1}{\lambda^{\alpha}}W(\alpha). \tag{9}$$

Here $d=m,\ W(\alpha)=\sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}$ and $d=J,\ W(\alpha)=\sum_{i=1}^{J}(1+R_i)y_{i:m:n}^{\alpha}+R_J^*T^{\alpha}$ for Case-I and Case-II respectively. It is assumed that $d>0$, otherwise the MLEs do not exist.

Taking derivatives with respect to α and λ of (9) and equating them to zero results in

$$\frac{\partial L(\alpha,\lambda)}{\partial\lambda} = -\frac{d\alpha}{\lambda}+\frac{\alpha}{\lambda^{\alpha+1}}W(\alpha)=0, \tag{10}$$

$$\frac{\partial L(\alpha,\lambda)}{\partial\alpha} = \frac{d}{\alpha}+\sum_{i=1}^{d}\ln y_{i:m:n}-d\ln\lambda-\frac{1}{\lambda^{\alpha}}V(\alpha)+\frac{1}{\lambda^{\alpha}}W(\alpha)\ln\lambda=0. \tag{11}$$

Here, $V(\alpha)=\sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}$ and $V(\alpha)=\sum_{i=1}^{J}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}+R_J^*T^{\alpha}\ln T$, for Case-I and Case-II respectively. Note that

$$\lambda^{\alpha} = \frac{W(\alpha)}{d} = u(\alpha) \quad (say) \tag{12}$$

and the MLE of α can be obtained by solving

$$\alpha = h(\alpha), \tag{13}$$

where

$$h(\alpha) = \frac{d}{-\sum_{i=1}^{d}\ln y_{i:m:n}+\frac{1}{u(\alpha)}W(\alpha)}.$$

A simple iterative scheme is proposed to obtain the MLE of α from (13). Starting with an initial guess of α, for example, $\alpha^{(0)}$, obtain $\alpha^{(1)}=h(\alpha^{(0)})$ and proceed in this way to obtain $\alpha^{(n+1)}=h(\alpha^{(n)})$. The iterative procedures stops when $\left|\alpha^{(n+1)}-\alpha^{(n)}\right|<\in$, which is some pre-assigned tolerance limit. Once the MLE of α is obtained the MLE of λ can be obtained from (12). Since the MLE's, when they exist, are not in compact forms, the following approximate MLE's and its' explicit expressions are proposed.

Approximate Maximum Likelihood Estimators (AMLEs)

Using the following notations $x_{i:m:n}=\ln y_{i:m:n}$ and $S=\ln T$, the likelihood equation of the observed data $x_{i:m:n}$ for Case-I is

$$l(\mu,\sigma)=\frac{1}{\sigma^m}\prod_{i=1}^{m}\left[n-\sum_{k=1}^{i-1}(1+R_K)\right]g(z_{i:m:n})\left(\bar{G}(z_{i:m:n})\right)^{R_i}, \tag{14}$$

and for Case II is

$$l(\mu,\sigma)=\frac{1}{\sigma^J}\prod_{i=1}^{J}\left[n-\sum_{k=1}^{i-1}(1+R_K)\right]g(z_{i:m:n})\left(\bar{G}(z_{i:m:n})\right)^{R_i}\left(\bar{G}(V)\right)^{R_J^*} \tag{15}$$

where ,

$z_{i:m:n}=(x_{i:m:n}-\mu)/\sigma$, $\quad V=(S-\mu)/\sigma$, $g(x)=e^{x-e^x}$, $\bar{G}(x)=e^{-e^x}$, $\mu=\ln\lambda$ and $\sigma=1/\alpha$.

Ignoring the constant term, the following log-likelihood results from (15) is

$$L(\mu,\sigma)=\ln\left[l(\mu,\sigma)\right]=-m\ln\sigma+\sum_{i=1}^{m}\ln\left(g(z_{i:m:n})\right)+\sum_{i=1}^{m}R_i\ln\left(\bar{G}(z_{i:m:n})\right). \tag{16}$$

From (16) the following approximate MLE's of μ and σ are obtained (see Appendix 1),

$$\tilde{\mu}=\frac{(c_1-c_2-m)\hat{\sigma}+d_1}{c_1},\ \tilde{\sigma}=\frac{-B+\sqrt{B^2-4AC}}{2A} \tag{17}$$

where

$c_1=\sum_{i=1}^{m}D_ie^{\mu_i}$, $c_2=\sum_{i=1}^{m}D_i\mu_ie^{\mu_i}$, $d_1=\sum_{i=1}^{m}D_iX_{i:m:n}e^{\mu_i}$,

$d_2=\sum_{i=1}^{m}D_iX_{i:m:n}^2e^{\mu_i}$, $d_3=\sum_{i=1}^{m}D_i\mu_iX_{i:m:n}e^{\mu_i}$, $A=mc_1$,

$B=c_1(d_3+m\bar{X})-d_1(c_2+m)$, $\quad C=d_1^2-c_1d_2$,

$\mu_i=G^{-1}(p_i)=\ln(-\ln q_i)$, $\quad p_i=i/(n+1)$,

$q_i=1-p_i$ and $D_i=1+R_i$, for $i=1,\cdots,m$.

For Case-II, ignoring the constant term, the log-likelihood is obtained as

$$L(\mu,\sigma)=\ln\left[l(\mu,\sigma)\right]=-J\ln\sigma+\sum_{i=1}^{J}\ln\left(g(z_{i:m:n})\right)+\sum_{i=1}^{J}R_i\ln\left(\bar{G}(z_{i:m:n})\right)+R_J^*\ln\bar{G}(V). \tag{18}$$

In this case the approximate MLE's are (see Appendix 2)

$$\tilde{\mu}=\frac{(c_1'-c_2'-J)\tilde{\sigma}+d_1'}{c_1'},\ \ \tilde{\sigma}=\frac{-B'+\sqrt{B'^2-4A'C'}}{2A'} \tag{19}$$

where,

$c_1'=\sum_{i=1}^{J}D_ie^{\mu_i}+R_J^*e^{\mu_J^*}$,

$c_2'=\sum_{i=1}^{J}D_i\mu_ie^{\mu_i}+R_J^*\mu_J^*e^{\mu_J^*}$,

$d_1'=\sum_{i=1}^{J}D_iX_{i:m:n}e^{\mu_i}+R_J^*Se^{\mu_J^*}$,

$d_2'=\sum_{i=1}^{J}D_iX_{i:m:n}^2e^{\mu_i}+R_J^*S^2e^{\mu_J^*}$,

$d_3'=\sum_{i=1}^{J}D_i\mu_iX_{i:m:n}e^{\mu_i}+R_J^*\mu_J^*Se^{\mu_J^*}$,

$A'=Jc_1'$, $\quad B'=c_1'\left(d_3'+J\bar{X}\right)-d_1'\left(c_2'+J\right)$,

$C'=d_1^2-c_1'd_2'$.

Here μ_i and D_i are the same as above for $i=1,..,J$, $\mu_J^*=G^{-1}(p_J^*)=\ln(-\ln q_J^*)$, $p_J^*=(p_J+p_{J+1})/2$, and $q_J^*=1-p_J^*$.

Results

Because the performance of the different methods cannot be compared theoretically, Monte Carlo simulations are used to compare the performances of the different methods proposed for different parameter values and for different sampling schemes. The term different sampling schemes mean different sets of R_i's and different T values. The performances of the MLEs and AMLEs estimators of the unknown parameters are compared in terms of their biases and mean squared errors (MSEs) for different censoring schemes. The average lengths of the asymptotic confidence intervals and their coverage percentages are also compared. All computations were performed using a Pentium IV processor and a FORTRAN-77 program. In all cases the random deviate generator RAN2 was used as proposed in Press, et al. (1991).

Because λ is the scale parameter, all cases $\lambda=1$ have been taken in without loss of generality. For simulation purposes, the results are presented when T is of the form $T^{1/\alpha}$. The reason for choosing T in that form is as follows: if $\hat{\alpha}$ represents the MLE or AMLE of α, then the distribution of $\hat{\alpha}/\alpha$ becomes independent of α in the case for $\lambda=1$. For that purpose the result is reported only for $\alpha=1$ without loss of generality, however, these results can be used for any other α also.

Type-II progressively hybrid censored data is generated for a given set $n, m, R_1, ..., R_m$ and T by using the following transformation for exponential distribution as suggested in Balakrishnan and Aggarwala (2000).

$$
\begin{aligned}
Z_1 &= nE_{1:m:n} \\
Z_2 &= (n - R_1 - 1)(E_{2:m:n} - E_{1:m:n}) \\
&\vdots \\
Z_m &= (n - R_1 - ... - R_{m-1} - m + 1)(E_{m:m:n} - E_{m-1:m:n})
\end{aligned}
\quad (20)
$$

It is known that if E_i's are i.i.d. standard exponential, then the spacing Z_i's are also i.i.d. standard exponential random variables. From (20) it follows that

$$
\begin{aligned}
E_{1:m:n} &= \frac{1}{n} Z_1 \\
E_{2:m:n} &= \frac{1}{n - R_1 - 1} Z_2 + \frac{1}{n} Z_1 \\
&\vdots \\
E_{m:m:n} &= \frac{1}{n - R_1 - ... - R_{m-1} - m + 1} Z_m + ... + \frac{1}{n} Z_1.
\end{aligned}
\quad (21)
$$

Using (21) and parameters α and λ, Type-II progressively hybrid censored data for the Weibull distribution can be generated for a given $n, m, R_1, .., R_m$, $Y_{1:m:n}, ..., Y_{m:m:n}$. If $Y_{m:m:n} < T$, then Case I results and the corresponding sample is $\{(Y_{1:m:n}, R_1), ..., (Y_{m:m:n}, R_m)\}$. If $Y_{m:m:n} > T$, then Case II results and J is found such that $Y_{J:m:n} < T < Y_{J+1:m:n}$. The corresponding Type-II hybrid censored sample is $\{(Y_{1:m:n}, R_1), ..., (Y_{J:m:n}, R_J)\}$ and R_J^*, where R_J^* is same as defined before.

Consider different n, m and T. Two different sampling schemes have been used, namely,

Scheme 1:

$R_1 = ... = R_{m-1} = 0$ and $R_m = n - m.$

Scheme 2:

$R_1 = ... = R_{m-1} = 1$ and $R_m = n - 2m + 1.$

Note that Scheme 1 is the conventional Type-II censoring scheme and Scheme 2, is a typical progressive censoring scheme. In each case the MLEs and AMLEs are computed as estimates of the unknown parameters. The 95% asymptotic confidence intervals are calculated based on MLEs by replacing the MLEs by AMLEs. The process was replicated 1,000 times. Average estimates, MSEs and average confidence lengths with coverage percentages were reported in Tables 1-8.

Based on Tables 1-4 (for MLEs) and Tables 5-8 (for AMLEs), the following observations are made: As expected, for fixed n, as m increases the biases and the MSEs decrease for both α and λ, however, for fixed m as n increases this may not be true. This shows that the effective sample size (m) plays an important role when considering the actual sample size (n). It is also observed that the MLEs for schemes 1 and 2 behave quite similarly in terms of biases and MSEs, unless both n and m are small. The performances in terms of biases and MSEs improve as T increases. Similar results are also observed for AMLEs.

Comparing different confidence intervals in terms of average lengths and coverage probabilities, it is generally observed that both the methods work well even for small n and m. For both methods, it is observed that the average confidence lengths decrease as n increases for fixed m, or vice versa. For both the MLE and AMLE methods, scheme 1 and scheme 2 behave very similarly although the confidence intervals for scheme 1 tend to be slightly shorter than scheme 2.

Data Analysis

Kundu and Joarder (2006) analyzed the following two data sets obtained from Lawless (1982) using exponential distributions.

Data Set 1

In this case $n = 36$ and, if $m = 10$, $T = 2600$, $R_1 = R_2 = ... = R_9 = 2, R_{10} = 8$, then the Type II progressively hybrid censored sample is: 11, 35, 49, 170, 329, 958, 1,925, 2,223, 2,400, 2,568. From the above sample data, $D = m = 10$ is obtained, which yields α and λ of based on MLEs and AMLEs are $(\hat{\alpha}=6.29773\times10^{-1}, \hat{\lambda}=8113.80176)$, $(\tilde{\alpha}=6.33116\times10^{-1}, \tilde{\lambda}=6511.83036)$ respectively. Using the above estimates the 95% asymptotic confidence interval for α and λ is obtained based on MLEs and AMLEs which are

$$(6.29773\times10^{-1}, 6.29882\times10^{-1}),$$
$$(8113.40869, 8114.19482),$$

and

$$(6.33116\times10^{-1}, 6.33176\times10^{-1}),$$
$$(6511.4344, 6512.2264)$$

respectively.

Data Set 2

Consider $m = 10, T = 2000,$ and R_i's are same as Data Set 1. In this case the progressively hybrid censored sample obtained as: 11, 35, 49, 170, 329, 958, 1,925 and $D = J = 7$. The MLE and AMLEs of α and λ are $(\hat{\alpha} = 4.77441\times10^{-1}, \hat{\lambda} = 25148.8613)$ and $(\tilde{\alpha} = 4.77589\times10^{-1}, \tilde{\lambda} = 23092.3759)$ respectively. From the above estimates the 95% asymptotic confidence intervals are obtained for α and λ based on MLEs and AMLEs, which are

$$(4.77383\times10^{-1}, 4.77499\times10^{-1}),$$
$$(25148.5078, 25149.2148)$$

and

$$(4.77529\times10^{-1}, 4.77649\times10^{-1}),$$
$$(23092.0219, 23092,7299)$$

respectively.

In both cases it is clear that if the tested hypothesis is $H_0 : \alpha = 1$, it will be rejected, this implies that in this case the Weibull distribution should be used rather than exponential.

Conclusion

This article discussed the Type-II progressively hybrid censored data for the two parameters Weibull distribution. It was observed that the maximum likelihood estimator of the shape parameter could be obtained by using an iterative procedure. The proposed approximate maximum likelihood estimators of the shape and scale parameters could be obtained in explicit forms. Although the exact confidence intervals could not be constructed, it was observed that the asymptotic confidence intervals work reasonably well for MLEs. Although the frequentest approach was used, Bayes estimates and credible intervals can also be obtained under suitable priors along the same line as Kundu (2007).

References

Arnold, B. C., & Balakrishnan, N. (1989). *Relations bounds and approximations for order statistics, lecture notes in statistics (53)*. New York: Springer Verlag.

Balakrishnan, N. (2007). Progressive censoring: An apprisal (with discussions). In press. To appear in *TEST*.

Balakrishnan, N., & Aggrwala, R. (2000). *Progressive censoring: Theory, methods and applications*. Boston, MA: Birkhauser.

Balakrishnan, N., & Varadan, J. (1991). Approximate MLEs for the location and scale parameters of the extreme value distribution with censoring. *IEEE Transactions on Reliability*, *40*, 146-151.

Childs, A., Chandrasekhar, B., & Balakrishnan, N. (2007). *Exact likelihood inference for an exponential parameter under progressively hybrid schemes: Statistical models and methods for biomedical and technical systems*. F. Vonta, M. Nikulin, N. Limnios, & C. Huber-Carod (Eds.). Boston, MA: Birkhauser.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, *5*, 327-329.

Cohen, A. C. (1966). Life-testing and early failure. *Technometrics*, *8*, 539-549.

David, H. A. (1981). *Order Statistics* (*2nd Ed.*). New York: John Wiley and Sons.

Kundu, D., & Joarder, A. (2006). Analysis of type II progressively hybrid censored data. *Computational Statistics and Data Analysis*, *50*, 2509-2528.

Kundu, D. (2007). On hybrid censored Weibull distribution. *Journal of Statistical Planning and Inference*, *137*, 2127-2142.

Table 1: MLE Estimate for T = 0.75

N.M.		Scheme 1	Scheme 2
30, 15	α	1.0968(0.0862), 1.2913(94.5)	1.0751(0.0838), 1.1898(93.5)
	λ	1.0358(0.1611),1.6019(89.1)	1.0760(0.2937), 1.6015(88.8)
40, 20	α	1.0898(0.0623), 1.0099(96.6)	1.0750(0.0626), 1.0167(94.9)
	λ	1.0111(0.0934), 1.2453(92.3)	1.413(0.1321), 1.3662(90.7)
60, 20	α	1.1046(0.0644), 1.1701(92.3)	1.0916(0.0554), 1.0255(94.7)
	λ	0.9777(0.0962), 1.6693(88.5)	0.9842(0.0902), 1.4432(91.2)
60, 30	α	1.0473(0.0342), 0.7386(96.5)	1.0385(0.0364), 0.7681(95.1)
	λ	1.0109(0.0653), 0.9055(92.7)	1.0350(0.0962), 1.0315(90.9)
80, 30	α	1.0566(0.0344), 0.7918(95.6)	1.0435(0.0302), 0.7074(96.1)
	λ	0.9913(0.0633), 1.0782(92.5)	1.0081(0.0731), 0.9630(92.6)
80, 40	α	1.0401(0.0252), 0.6275(97.3)	1.0301(0.0269), 0.6501(95.6)
	λ	1.0060(0.0449), 0.7670(93.2)	1.0261(0.0614), 0.8732(91.7)
100, 40	α	1.0471(0.0256), 0.6620(97.4)	1.0323(0.0219), 0.5932(96.4)
	λ	0.9904(0.0406), 0.878(93.4)	1.0096(0.0465), 07985(93.8)
100, 50	α	1.0369(0.0209), 05544(96.2)	1,0281(0.0232), 0.5811(95.6)
	λ	0.9996(0.0292), 0.6760(93.6)	1.0185(0.0418), 0.7800(93.0)

Table 2: MLE Estimate for T = 1.00

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1102(0.0841), 1.2367(96.0)	1.0719(0.0730), 1.0287(95.6)
	λ	0.9982(0.1171), 1.5080(92.1)	1.0383(0.1397), 1.3333(91.2)
40, 20	α	1.0983(0.0600), 0.9891(97.7)	1.0704(0.0518), 0.8833(96.4)
	λ	0.9864(0.0629), 1.2035(93.7)	1.0179(0.0817), 1.1445(92.1)
60, 20	α	1.1046(0.0644), 1.1701(92.3)	1.0933(0.0550), 1.0249(95.2)
	λ	0.9781(0.0982), 1.6692(88,5)	0.9776(0.0793), 1.4394(91.6)
60, 30	α	1.0539(0.0329), 0.7320(97.0)	1.0358(0.0291), 0.6855(95.9)
	λ	0.9945(0.0510), 0.8876(94.2)	1.0157(0.0616), 0.8892(92.3)
80, 30	α	1.0567(0.0344), 0.7918(95.7)	1.0487(0.0291), 0.7049(96.9)
	λ	0.9906(0.0605), 1.0781(92.5)	0.9926(0.0553), 0.9508(93.9)
80, 40	α	1.0456(0.0246), 06214(97.8)	0.0313(0.0225), 0.5879(97.0)
	λ	0.9927(0.0331), 0.7531(94.1)	1.0110(0.0429), 0.7624(92.2)
100, 40	α	1.0473(0.0255), 0.6621(97.4)	1.0396(0.0211), 0.5788(97.4)
	λ	0.9895(0.0385), 0.8781(93.4)	0.9936(0.0364), 0.7655(94.0)
100, 50	α	1.0397(0.0205), 0.5493(96.9)	1.0252(0.0190), 0.5216(94.7)
	λ	0.9927(0.0243), 0.6653(94.0)	1.0120(0.0301), 0.6773(93.5)

Table 3: MLE Estimate for T = 1.50

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1130(0.0833),1.2367(96.3)	1.0727(0.0630), 0.9343(95.8)
	λ	0.9857(0.0820), 1.5075(92.7)	1.0196(0.1079), 1.2004(92.7)
40, 20	α	1.0992(0.0599), 0.9886(97.8)	1.0682(0.0430), 0.7962(97.5)
	λ	0.9841(0.0600), 1.2025(93.6)	1.0025(0.0593), 1.0237(94.4)
60, 20	α	1.1046(0.0644), 1.1701(92.3)	1.0932(0.0550), 1.0248(95.2)
	λ	0.9781(0.0982), 1.6692(88.5)	0.9779(0.0807), 1.4394(91.6)
60, 30	α	1.0544(0.0327), 0.7320(97.2)	1.0366(0.0259), 0.6251(94.9)
	λ	0.9920(0.0451), 0.8875(94.2)	1.0054(0.0498), 0.8042(93.0)
80, 30	α	1.0567(0.0344), 0.7918(95.7)	1.0492(0.0288), 0.7051(97.0)
	λ	0.9906(0.0605), 1.0781(92.5)	0.9900(0.0503), 09508(93.8)
80, 40	α	1.0458(0.0245), 0.6215(97.8)	1.0308(0.0192), 0.5357(96.8)
	λ	0.9919(0.0312),0.7531) (94.1)	1.0031(0.0319), 0.6896(93.6)
100, 40	α	1.0473(0.0255),0.6621(97.4)	1.0407(0.0209), 0.5785(97.7)
	λ	0.9895(0.0385), 08781(93.4)	0.9901(0.0322), 0.7645(94.0)
100, 50	α	1.0397(0.0205), 0.5492(96.9)	1.0277(0.0156), 0.4768(94.7)
	λ	0.9928(0.0243), 0.6652(94.0)	1.0008(0.0231), 0.6138(94.7)

Table 4: MLE Estimate for T = 2.00

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1130(0.0833),1.2367(96.3)	1.0754(0.062), 0.9106(96.1)
	λ	0.9857(0.0820), 1.5075(92.7)	1.0045(0.0882), 1.1750(92.6)
40, 20	α	1.0992(0.0599), 0.9886(97.8)	1.0695(0.0423),0.7720(95.8)
	λ	0.9841(0.0600), 1.2025(93.6)	0.9966(0.0538), 0.9983(94.6)
60, 20	α	1.1046(0.0644), 1.1701(92.3)	1.0932(0.0550),1.0248(95.2)
	λ	0.9781(0.0982), 1.6692(88.5)	0.9779(0.0807), 1.4394(91.6)
60, 30	α	1.0544(0.0327), 0.7320(97.2)	1.0379(0.0248), 0.6054(95.6)
	λ	0.9920(0.0451), 0.8875(94.2)	1.0004(0.0433), 0.7836(94.2)
80, 30	α	1.0567(0.0344), 0.7918(95.7)	1.0492(0.0288), 0.7051(97.0)
	λ	0.9906(0.0605), 1.0781(92.5)	0.9900(0.0503), 0.9508(93.8)
80, 40	α	1.0458(0.0245), 0.6215(97.8)	1.0321(0.0176),0.5179(96.6)
	λ	0.9919(0.0312),0.7531) (94.1)	0.9986(0.0283), 0.6709(94.1)
100, 40	α	1.0473(0.0255),0.6621(97.4)	1.0407(0.0209), 0.5785(97.7)
	λ	0.9895(0.0385), 08781(93.4)	0.9901(0.0322), 0.7645(94.0)
100, 50	α	1.0397(0.0205), 0.5492(96.9)	1.0286(0.0149), 0.4608(94.6)
	λ	0.9928(0.0243), 0.6652(94.0)	0.9986(0.0219), 0.5969(94.1)

Table 5: Approximate MLE Estimate for T = 0.75

N.M.		Scheme 1	Scheme 2
30, 15	α	1.0873(0.0847), 1.2073(94.2)	1.0814(0.0889), 1.2070(93.3)
	λ	1.0354(0.1640), 1.4941(89.1)	1.0104(0.3033), 1.5970(88.6)
40, 20	α	1.0832(0.0615), 0.9924(96.2)	1.0837(0.06559), 1.0651(95.4)
	λ	1.01103(0.0941), 1.2224(92.2)	0.9752(0.1333), 1.4107(91.3)
60, 20	α	1.0998(0.0638), 1.1226(92.1)	1.0915(0.0554), 1.0236(94.5)
	λ	0.9792(0.0968), 1.6005(88.4)	0.9435(0.0823), 1.4270(92.8)
60, 30	α	1.0432(0.0340), 0.7349(96.5)	1.0486(0.0386), 0.7959(95.5)
	λ	1.0102(0.0655), 0.900(92.7)	0.9679(1.0962), 1.0530(92.1)
80, 30	α	1.0533(0.0342), 07870(95.5)	1.0492(0.0308), 0.7288(96.4)
	λ	0.9920(0.0635), 1.0714(92.5)	0.9520(0.0688), 0.9797(94.7)
80, 40	α	1.0372(0.0251), 0.6253(97.1)	1.0409(0.0284), 0.6735(96.3)
	λ	1,0054(0.0450), 0.7640(93.2)	0.9588(0.0620), 0.8914(92.2)
100, 40	α	1,0447(0,0255), 0,6593(97.2)	1.0417(0.0227), 0.6140(97.4)
	λ	0.9906(0.0407), 08743(93.4)	0.9463(0.0453), 08151(94.8)
100, 50	α	1.0346(0.0208), 0.5529(96.2)	1.0392(0.0243), 0.6029(96.4)
	λ	0.9991(0.0292), 0.6739(93.6)	0.9512(0.0421), 0.7976(93.9)

Table 6: Approximate MLE Estimate for T = 1.00

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1003(0.827),1.1683(95.3)	1.0921(0.0811), 1.0824(96.1)
	λ	0.9968(0.1177), 1.4208(92.0)	0.9378(0.1395), 1.3736(92.3)
40, 20	α	1.0916(0.0592), 0.9731(97.4)	1.0936(0.0582), 0.9316(97.0)
	λ	0.9851(0.0628), 1.1827(93.8)	0.9175(0.0809), 1.1.822(94.3)
60, 20	α	1.0998(0.0638), 1.1226(92.1)	1.0933(0.0550),1.0232(94.9)
	λ	0.9797(0.0989), 1.6004(88.4)	0.9359(0.0703), 1.4234(93.4)
60, 30	α	1.0497(0.0327), 0.7283(97.0)	1.0586(0.0326), 0.7217(96.7)
	λ	0.9936(0.0510), 0.8827(94.2)	0.9150(0.0608), 0.9169(92.7)
80, 30	α	1.0534(0.0342), 0.7870(95.6)	1.0555(0.0296), 0.7275(97.0)
	λ	0.9912(0.0607), 1.0712(92.5)	0.9309(0.0476), 0.9685(96.6)
80, 40	α	1.0426(0.0244), 0.6193(97.7)	1.0546(0.0251), 0.6189(97.2)
	λ	0.9921(0.0330), 0.7502(94.2)	0.9102(0.0429), 0.7863(92.3)
100, 40	α	1.0448(0.0254), 0.6594(97.2)	1.0518(0.0218), 0.6009(98.0)
	λ	0.9897(0.0385), 0.8743(93.4)	0.9183(0.0311), 0.7825(95.7)
100, 50	α	1.0374(0.0204), 0.5478(96.8)	1.0484(0.0211), 0.5489(95.6)
	λ	0.9922(0.0243), 0.6633(94.0)	0.9112(0.0299), 0.6984(93.7)

Table 7: Approximate MLE Estimate for T = 1.50

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1030(0.0819), 1.1681(95.7)	1.1153(0.0735), 1.0048(96.1)
	λ	0.9841(0.0820), 1.4202(92.7)	0.8709(0.0978), 1.2553(93.6)
40, 20	α	1.0925(0.0592), 0.9726(97.6)	1.1158(0.0519), 0.8603(96.9)
	λ	0.9827(0.0598), 1.1818(93.7)	0.8541(0.0524), 1.0754(95.5)
60, 20	α	1.0998(0.0638), 1.1226(92.1)	1.0932(0.0550), 1.0231(94.9)
	λ	0.9797(0.0989), 1.6004(88.4)	0.9361(0.0712), 1.4233(93.4)
60, 30	α	1.0502(0.0325), 0.7284(97.1)	1.0832(0.0313), 0.6753(95.1)
	λ	0.9910(0.0450), 0.8826(94.2)	0.8563(0.0443), 0.8445(92.8)
80, 30	α	1.0534(0.0342), 0.7870(95.6)	1.0560(0.0293), 0.7277(97.2)
	λ	0.9912(0.0607), 1.0712(92.5)	0.9280(0.0424), 0.9684(96.6)
80, 40	α	1.0428(0.0244), 0.6193(97.7)	1.0778(0.0232), 0.5787(95.8)
	λ	0.9912(0.0311), 0.7502(94.2)	0.8540(0.0287), 0.7242(92.0)
100, 40	α	1.0448(0.0254), 0.6594(97.2)	1.0532(0.0215), 0.6009(98.3)
	λ	0.9897(0.0385), 0.8743(93.4)	0.9136(0.0262), 0.7816(96.8)
100, 50	α	1.0374(0.0204), 0.5477(96.8)	1.0748(0.0188), 0.5152(94.4)
	λ	0.9922(0.0243), 0.6632(94.0)	0.8514(0.0205), 0.6447(91.6)

Table 8: Approximate MLE Estimate for T = 2.00

N.M.		Scheme 1	Scheme 2
30, 15	α	1.1030(0.0819), 1.1681(95.7)	1.1337(0.0710), 0.9924(96.1)
	λ	0.9841(0.0820), 1.4202(92.7)	0.8327(0.0690), 1.2439(95.0)
40, 20	α	1.0925(0.0592), 0.9726(97.6)	1.1326(0.0512), 0.8454(96.0)
	λ	0.9827(0.0598), 1.1818(93.7)	0.8245(0.0414), 1.0610(96.1)
60, 20	α	1.0998(0.0638), 1.1226(92.1)	1.0932(0.0550), 1.0231(94.9)
	λ	0.9797(0.0989), 1.6004(88.4)	0.9361(0.0712), 1.4233(93.4)
60, 30	α	1.0502(0.0325), 0.7284(97.1)	1.0990(0.0302), 0.6630(94.8)
	λ	0.9910(0.0450), 0.8826(94.2)	0.8269(0.0336), 0.8319(92.5)
80, 30	α	1.0534(0.0342), 0.7870(95.6)	1.0560(0.0293), 0.7277(97.2)
	λ	0.9912(0.0607), 1.0712(92.5)	0.9280(0.0424), 0.9684(96.6)
80, 40	α	1.0428(0.0244), 0.6193(97.7)	1.0946(0.0217), 0.5678(94.6)
	λ	0.9912(0.0311), 0.7502(94.2)	0.8247(0.0222), 0.7129(90.8)
100, 40	α	1.0448(0.0254), 0.6594(97.2)	1.0532(0.0215), 0.6009(98.3)
	λ	0.9897(0.0385), 0.8743(93.4)	0.9136(0.0262), 0.7816(96.8)
100, 50	α	1.0374(0.0204), 0.5477(96.8)	1.0916(0.0185), 0.5053(92.4)
	λ	0.9922(0.0243), 0.6632(94.0)	0.8245(0.0170), 0.6346(89.4)

Lawless, J. F. (1982). *Statistical models and methods for lifetime data*. New York: Wiley.

Mann, N. R. (1971). Best linear invariant estimation for Weibull parameters under progressive censoring. *Technometrics*, *13*, 521-533.

Press, W. H., Flannery, B. P., Teukolsky, S. A., & Vetterling, W. T. (1991). *Numerical recipes: The art of scientific computing*. Cambridge, U.K.: Cambridge University Press.

Thomas, D. R., & Wilson, W. M. (1972). Linear order statistics estimation for the two-parameter Weibull and extreme value distribution from Type-II progressively censored samples. *Technometrics*, *14*, 679-691.

Appendix 1

For case-I, taking derivatives with respect to μ and σ of $L(\mu,\sigma)$ as defined in (16), results in

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{m} R_i \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{m} \frac{g'(z_{i:m:n})}{g(z_{i:m:n})}\right] = 0 \tag{22}$$

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{m} R_i z_{i:m:n} \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{m} z_{i:m:n} \frac{g'(z_{i:m:n})}{g(z_{i:m:n})} - m\right] = 0. \tag{23}$$

Clearly, (22) and (23) do not have explicit analytical solutions. Consider a first-order Taylor approximation to $g'(z_{i:m:n})/g(z_{i:m:n})$ and $g(z_{i:m:n})/\bar{G}(z_{i:m:n})$ by expanding around the actual mean μ_i of the standardized order statistic $Z_{i:m:n}$, where $\mu_i = G^{-1}(p_i) = \ln(-\ln q_i)$, and $p_i = i/(n+1)$, $q_i = 1 - p_i$ for $i = 1,\ldots,m$, similar to Balakrishnan and Varadan (1991), David (1981) or Arnold and Balakrishnan (1989). Otherwise, the necessary procedures for obtaining μ_i, $i = 1,..,m$, were made available by Mann (1971) and Thomas and Wilson (1972). Note that for $i = 1,\ldots,m$

$$g'(z_{i:m:n})/g(z_{i:m:n}) \approx \alpha_i - \beta_i z_{i:m:n} \tag{24}$$

$$g(z_{i:m:n})/\bar{G}(z_{i:m:n}) \approx 1 - \alpha_i + \beta_i z_{i:m:n} \tag{25}$$

where,

$$\alpha_i = \frac{g'(\mu_i)}{g(\mu_i)} - \mu_i\left[\frac{g''(\mu_i)}{g'(\mu_i)} - \left(\frac{g'(\mu_i)}{g'(\mu_i)}\right)^2\right]$$
$$= 1 + \ln q_i(1 - \ln(-\ln q_i)),$$

$$\beta_i = \left[-\frac{g''(\mu_i)}{g'(\mu_i)} + \left(\frac{g'(\mu_i)}{g(\mu_i)}\right)^2\right] = -\ln q_i$$

Using the approximation (24) and (25) in (22) and (23), results get

$$\left[\sum_{i=1}^{m} D_i e^{\mu_i} - \sum_{i=1}^{m} D_i \mu_i e^{\mu_i} - m\right]\sigma + \sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i} - \mu \sum_{i=1}^{m} D_i e^{\mu_i} = 0 \tag{26}$$

and

$$\left[m\sum_{i=1}^{m} D_i e^{\mu_i}\right]\sigma^2 +$$
$$\left[\sum_{i=1}^{m} D_i e^{\mu_i}\left(\sum_{i=1}^{m} D_i \mu_i X_{i:m:n} e^{\mu_i} + m\bar{X})\right)\right]\sigma +$$
$$\left[\sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}\right]^2$$
$$-\left[\sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}\left(\sum_{i=1}^{m} D_i \mu_i e^{\mu_i} + m\right)\right]\sigma$$
$$-\left[\sum_{i=1}^{m} D_i e^{\mu_i}\right]\left[\sum_{i=1}^{m} D_i X_{i:m:n}^2 e^{\mu_i}\right] = 0 \tag{27}$$

The above two equations (26) and (27) can be written as

$$(c_1 - c_2 - m)\sigma + d_1 - \mu c_1 = 0 \tag{28}$$

$$A\sigma^2 + B\sigma + C = 0 \tag{29}$$

where

$$c_1 = \sum_{i=1}^{m} D_i e^{\mu_i},\ c_2 = \sum_{i=1}^{m} D_i \mu_i e^{\mu_i},\ d_1 = \sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i},$$
$$d_2 = \sum_{i=1}^{m} D_i X_{i:m:n}^2 e^{\mu_i},$$
$$d_3 = \sum_{i=1}^{m} D_i \mu_i X_{i:m:n} e^{\mu_i},\ A = mc_1,\ B = c_1(d_3 + m\bar{X}) - d_1(c_2 + m),$$
$$C = d_1^2 - c_1 d_2$$

and $D_i = 1 + R_i$ for $i = 1,..,m$. The solution to the preceding equations yields the approximate MLE's are

$$\tilde{\mu} = \frac{(c_1 - c_2 - m)\tilde{\sigma} + d_1}{c_1} \tag{30}$$

$$\tilde{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{31}$$

Consider only positive root of σ; these approximate estimators are equivalent but not unbiased. Unfortunately, it is not possible to compute the exact bias of $\tilde{\mu}$ and $\tilde{\sigma}$ theoretically because of intractability encountered in finding the expectation of $\sqrt{B^2 - 4AC}$.

Appendix 2

For case-II, taking derivatives with respect to μ and σ of $L(\mu,\sigma)$ as defined in (18), gives (similar to Case-I)

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{J} R_i \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{J} \frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + R_J^* \frac{g(V)}{\bar{G}(V)}\right] = 0 \tag{32}$$

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{J} R_i z_{i:m:n} \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{J} z_{i:m:n} \frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + R_J^* V \frac{g(V)}{\bar{G}(V)} - J\right] = 0. \tag{33}$$

Here again consider the first-order Taylor approximation to $g'(z_{i:m:n})/g(z_{i:m:n})$ and $g(z_{i:m:n})/\bar{G}(z_{i:m:n})$ by expanding around the actual mean μ_i of the standardized order statistic $Z_{i:m:n}$, where μ_i's are defined in Appendix 1. Here $g(V)/\bar{G}(V)$ is also exploded in the Taylor series around the point μ_J^*, where

$\mu_J^* = G^{-1}\left(p_J^*\right) = \ln\left(-\ln q_J^*\right)$, $p_J^* = \left(p_J + p_{J+1}\right)/2$ and $q_J^* = 1 - p_J^*$.

Note that

$$\frac{g'(V)}{g(V)} \approx \alpha_J^* - \beta_J^* V \tag{34}$$

$$\frac{g(V)}{\bar{G}(V)} \approx 1 - \alpha_J^* + \beta_J^* V \tag{35}$$

where

$$\alpha_J^* = \frac{g'\left(\mu_J^*\right)}{g\left(\mu_J^*\right)} - \mu_J^*\left[\frac{g''\left(\mu_J^*\right)}{g'\left(\mu_J^*\right)} - \left(\frac{g'\left(\mu_J^*\right)}{g\left(\mu_J^*\right)}\right)^2\right] = 1 + \ln q_J^*\left(1 - \ln\left(-\ln q_J^*\right)\right),$$

$$\beta_J^* = \left[-\frac{g''\left(\mu_J^*\right)}{g'\left(\mu_J^*\right)} + \left(\frac{g'\left(\mu_J^*\right)}{g\left(\mu_J^*\right)}\right)^2\right] = -\ln q_J^*$$

Using the approximation (24), (25), (34) and (35) in (32) and (33) gives

$$\left[\left(\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}\right) - \left(\sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*}\right) - J\right]\sigma$$
$$\left[\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}\right] - \mu\left[\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}\right] = 0 \tag{36}$$

and

$$\left[J\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}\right]\sigma^2 + \left[\left(\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}\right)\left(\sum_{i=1}^{J} D_i \mu_i X_{i:m:n} e^{\mu_i} + R_J^* \mu_J^* S e^{\mu_J^*} + J\bar{X}\right)\right]\sigma$$
$$-\left[\left(\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}\right)\left(\sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*} + J\right)\right]\sigma + \left(\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}\right)^2$$
$$-\left[\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}\right]\left[\sum_{i=1}^{J} D_i X_{i:m:n}^2 e^{\mu_i} + R_J^* S^2 e^{\mu_J^*}\right] = 0. \tag{37}$$

The above two equations (36) and (37) can be written as

$$(c_1' - c_2' - J)\sigma + d_1' - \mu c_1' = 0 \tag{38}$$

$$A'\sigma^2 + B'\sigma + C' = 0 \tag{39}$$

where

$c_1' = \sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}$,

$c_2' = \sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*}$,

$d_1' = \sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}$,

$d_2' = \sum_{i=1}^{J} D_i X_{i:m:n}^2 e^{\mu_i} + R_J^* S^2 e^{\mu_J^*}$,

$d_3' = \sum_{i=1}^{J} D_i \mu_i X_{i:m:n} e^{\mu_i} + R_J^* \mu_J^* S e^{\mu_J^*}$,

$A' = Jc_1'$, $B' = c_1'\left(d_3' + J\bar{X}\right) - d_1'\left(c_2' + J\right)$,

$C' = d_1^{'2} - c_1' d_2'$ and $D_i = 1 + R_i$, for $i = 1, \cdots, J$.

The solution to the preceding equations yields the approximate MLE's are

$$\tilde{\mu} = \frac{(c_1^{'} - c_2^{'} - J)\tilde{\sigma} + d_1^{'}}{c_1^{'}} \tag{40}$$

$$\tilde{\sigma} = \frac{-B' + \sqrt{B^{'2} - 4A'C'}}{2A'} \tag{41}$$

Consider only positive root of σ; these approximate estimators are equivalent but not unbiased. Unfortunately, it is not possible to compute the exact bias of $\tilde{\mu}$ and $\tilde{\sigma}$ theoretically because of intractability encountered in finding the expectation of $\sqrt{B^{'2} - 4A'C'}$.