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*P** Index of Segregation: Distribution Under Reassignment

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Students of intergroup relations have measured segregation with a *P** index. In this article, we describe the distribution of this index under a stochastic model. We derive exact, closed-form expressions for the mean, variance, and skewness of *P** under random segregation. These yield equivalent expressions for a second segregation index: η^2. Our analytic results reveal some of the distributional properties of these indices, inform new standardizations of the indices, and enable small-sample significance testing. Two illustrative examples are presented.

Key words: Segregation index, randomization methods, quadratic assignment, Eta-squared

Introduction

Bell (1954) developed a way to measure the amount of contact between two groups. This widely-used measure has gone by several names. It has been called the exposure index (James, 1986) and the interaction index (Massey & Denton, 1986). We, however, follow Lieberson (1980) in referring to a measure of sort devised by Bell as a *P** index. It is intended to measure the probability that individuals from two different groups will have contact with one another.

The *P** index has been used in studies of residential segregation – when data are available on the number of members of a minority group (*j*) and a majority group (*k*) who live in a particular spatially-defined unit (on the same city block, for example, or in the same census tract). It requires data on the number of minority and majority residents in a number of such units. Then *P** is the probability that a randomly selected member of group *j* lives in the same unit as a member of group *k*. The index is defined as

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$$ {}_jP_k^* = \frac{1}{N_{\bullet j}} \sum_{i=1}^{u} \frac{N_{ij} N_{ik}}{N_{i\bullet}} \qquad (1) $$

where $N_{\bullet j}$ is the total number of members of group *j*, *u* is the number of units; and N_{ij} , N_{ik} , and $N_{i\bullet}$ are the number of members of group j in unit *i*, the number of members of group *k* in unit *i*, and the total number of people in unit *i*, respectively.

*P** plays a role in the study of segregation. It has been used to document school, as well as residential segregation (Coleman, Kelly, & Moore, 1975; Krivo & Kaufman, 1999). It complements alternative indices by tapping a distinct dimension of segregation (Massey, White, & Phua, 1996; Stearns & Logan, 1986). Despite recurrent criticism (Taeuber & Taeuber, 1965), the *P** index of segregation has found application in a variety of contexts for nearly 50 years.

Researchers who measure segregation with the *P** index have an obligation to interpret their results. *P** is a probability. It varies between 0 to 1. However, the probability of a member of one group being exposed to a member of another group could be misleading, depending (as it does) on relative group size. To facilitate interpretation, researchers often compare an observed value of ${}_jP_k^*$ with the value that would have been observed if there had been no segregation – that is, if the proportion of members of group *j* and group *k* within each unit equaled the overall proportion of members of those groups across all units. Then ${}_jP_k^*$ would equal $N_{\bullet k} N_{\bullet\bullet}^{-1}$, and the probability of a member of group *j* being in the same unit as a member of group *k* would be the proportion of members of group *k* across all units (Lieberson, 1980).

No doubt, students of segregation enhance understanding by providing comparison values for their measures. We wonder, however, if it is best to compare P^* to a value which assumes that there is no segregation. After all, even if the members of two groups made residential choices entirely at random, some degree of segregation could be expected by chance (cf., Winship, 1977). In some contexts it would be informative to compare P^* to the value it would attain under a random degree of segregation. Unfortunately, this comparison has not been possible to date because the value of P^* that would be produced by random segregation has not been known.

In the current article, we describe the distribution of P^* under random segregation. We develop an analytic method for determining whether the amount of intergroup contact in a particular setting differs from the amount that would be expected by chance. Exact, closed-form expressions for the expected value, variance, and skewness of P^* under random segregation are presented. These imply equivalent expressions for a second segregation index: Bell's eta-square. Our analytic results reveal some of the distributional properties of these segregation indices, inform new standardizations of the indices, and enable small-sample significance testing. For statistical characterizations of P^*, see Zoloth (1974). For distributional analyses of the widely-used *index of dissimilarity*, see the papers by Winship (1977) and Inman and Bradley (1991).

Formulation of the Problem

Our goal is to determine the distribution of the statistic in equation 1) under a stochastic model. We begin by assuming that the total number of individuals in each of *u* units is fixed – as is the total number of members of group *j* and group *k*. Our model is that each individual is randomly assigned to a unit. We seek to determine the distribution of the P^* index under all possible assignments of individuals to units – assuming that each assignment that preserves the marginal totals is as likely as every other such assignment.

If all possible assignments of individuals to units could be made, then the distribution of P^* could be constructed empirically. Ordinarily, the number of assignments will be prohibitive, however, and other methods will be required. Monte Carlo techniques could be used (cf. Taeuber & Taeuber, 1965); but these are computationally intensive and yield no exact distributional information. Here we derive the exact mean, variance, and skewness of P^* under all possible assignments of individuals to units.

Our derivation treats the distribution of P^* as a quadratic assignment problem (Hubert, 1987). We begin by representing P^* in a form that is amenable to quadratic assignment methods, so that we can draw on existing analytic results.

Denoting the total number of individuals in the analysis as $N_{\bullet\bullet}$, P^* is represented in two $N_{\bullet\bullet} \times N_{\bullet\bullet}$ matrices. Each row of each matrix will denote a particular individual, as will the corresponding column of the matrix. Hence, each entry in each matrix will denote a pair of individuals, matrix element *s,t* denoting the dyad that consists of individual *s* and individual *t*. This representation is familiar to students of social networks (Wasserman & Faust, 1994). We define the two $N_{\bullet\bullet} \times N_{\bullet\bullet}$ matrices: Q (which we call the cross-group membership matrix) and R (the unit co-occupancy matrix). Both matrices are symmetric.

The cross-group membership matrix Q identifies dyads in which the intergroup contact of interest could, in principle, occur. If the researcher wishes to measure the likelihood that a member of group *j* will have contact with a member of group *k*, the entry in the *s*th row and *t*th column of this matrix is set to $(2N_{\bullet j})^{-1}$ whenever one of the two individuals in the dyad (*s* or *t*) belongs to group *j* and the other individual belongs to group *k*. All other entries of the Q matrix are set to 0.

The unit co-occupancy matrix R identifies individuals who are in the same unit. The entry in the *s*th row and *t*th column of the R matrix is set to $N_{i\bullet}^{-1}$ if the two individuals (*s* and *t*) are in unit *i*. All elements along the diagonal of R are set to 0, as are any off-diagonal elements that denote two individuals who are in different units.

Denoting element *s,t* of matrix Q by q_{st} and the corresponding element of matrix R as r_s - algebra reveals that

$$_j P_k^* = \sum_{s=1}^{N_{\bullet\bullet}} \sum_{t=1}^{N_{\bullet\bullet}} q_{st} r_{st} \qquad (2)$$

Thus, the *P** index can be expressed as the sum of the products of corresponding elements of two matrices. Such statistics can be analyzed with quadratic assignment methods (Hubert, 1987).

Our goal is describe the distribution of P^* under all possible assignments of individuals to units. In our formulation, individuals are implicitly assigned to units by the R matrix. We could change the assignment of individuals to units by reordering the rows and corresponding columns of R.

Having expressed *P** as the sum of the products of corresponding elements of two matrices, we can draw on formulas that have been derived for the mean, variance, and skewness of such statistics under all possible re-orderings of the rows and columns of one of those matrices (Hubert, 1987) These provide the desired distributional information.

Analytic Results

Our quadratic assignment formulation yields the following results. The mean of $_jP_k^*$ under all possible assignments of individuals to units is

$$E(_j P_k^*) = [N_{\bullet k} N_{\bullet\bullet}^{-1}][(N_{\bullet\bullet} - u)(N_{\bullet\bullet} - 1)^{-1}] \quad (3)$$

all symbols having been defined above.

The variance of $_jP_k^*$ under all possible assignments of individuals to units has a more complicated mathematical expression. In fact,

$$Var(_j P_k^*) = a(b+c+d) - [E(_j P_k^*)]^2 \qquad (4)$$

where

$$a = N_{\bullet k}[N_{\bullet j} N_{\bullet\bullet}(N_{\bullet\bullet} - 1)(N_{\bullet\bullet} - 2)]^{-1}$$

$$b = (N_{\bullet\bullet} - 2)\sum_{i=1}^{u}[(N_{i\bullet} - 1)N_{i\bullet}^{-1}]$$

$$c = (N_{\bullet j} + N_{\bullet k} - 2)\sum_{i=1}^{u}[(N_{i\bullet} - 1)(N_{i\bullet} - 2)N_{i\bullet}^{-1}]$$

and

$$d = [(N_{\bullet j} - 1)(N_{\bullet k} - 1)(N_{\bullet\bullet} - 3)^{-1}]$$
$$\{(N_{\bullet\bullet} - u)^2 - 2\sum_{i=1}^{u}[(N_{i\bullet} - 1)(2N_{i\bullet} - 3)N_{i\bullet}^{-1}]\}.$$

We have derived the coefficient of skewness of $_jP_k^*$ under all possible assignments of individuals to units. Appendix A presents an analytic expression for this statistic, which we symbolize $\gamma_1(_jP_k^*)$.

Because these analytic expressions are intricate, it may be helpful to begin by noting some quantities they omit. Neither the mean, the variance, nor the skewness of $_jP_k^*$ are affected by the number of members of group *j* or *k* in any particular unit. These expressions reflect only marginal totals – the size of the two groups, and the size of the *u* units. Values that would appear as entries in a unit x group contingency table do not enter into the equations because these are moments of a distribution of the possible values of $_jP_k^*$ over all possible entries that would preserve the marginal totals.

Equation (3) yields insight into the impact of random segregation on *P**. In the absence of any segregation, the probability of a member of group *j* being in the same unit with a member of group *k* equals $N_{\bullet k} N_{\bullet\bullet}^{-1}$, as earlier researchers noted. This probability is lower under random segregation. Relative to the probability of intergroup exposure under no segregation, the random expectation for $_jP_k^*$ is lower by a factor of $(N_{\bullet\bullet} - u)(N_{\bullet\bullet} - 1)^{-1}$, as equation 3) indicates. This difference might be negligible if the units under analysis were sufficiently large; it could be appreciable if the units were sufficiently small.

Other $P*$ Indices

The probability of a member of one group interacting with a member of a second group will not, in general, equal the probability of a member of the second group interacting with a member of the first (Lieberson, 1980). However, these probabilities have a simple relationship to one another.

$$ {}_kP_j^* = N_{\bullet j} N_{\bullet k}^{-1} \; {}_jP_k^* \tag{5}$$

This implies that

$$\begin{aligned} E({}_kP_j^*) &= N_{\bullet j} N_{\bullet k}^{-1} \; E({}_jP_k^*) \qquad (6)\\ Var({}_kP_j^*) &= N_{\bullet j}^{2} N_{\bullet k}^{-2} \; Var({}_jP_k^*)\\ \text{and} \quad \alpha_1({}_jP_k^*) &= \alpha_1({}_kP_j^*) \end{aligned}$$

These equations permit a comparison of the distributions of complementary exposure indices. In skewness, the distributions of ${}_jP_k^*$ and ${}_kP_j^*$ are identical. In expectation and variance, these two distributions are identical if groups j and k are the same size. If group j is smaller than group k, then ${}_jP_k^*$ will have a higher expectation and greater variability than ${}_kP_j^*$. If group j is larger than group k, then ${}_jP_k^*$ will have a lower expectation and less variability than ${}_kP_j^*$. The greater the difference in the size of two groups, the greater will be the difference in expectation and variability of the two exposure indices involving those groups.

Often students of segregation wish to measure the likelihood that a member of a group will be in the same unit as other members of that group. They have done so with a isolation index (Bell, 1954).

$$ {}_jP_j^* = \frac{1}{N_{\bullet j}} \sum_{i=1}^{u} \frac{N_{ij} N_{ij}}{N_{i\bullet}} \tag{7}$$

The methods above must be adapted to describe the distribution of ${}_jP_j^*$. One begins by applying the formulas above to an index for the exposure of individuals who are members of group j to individuals who are not members of group j. Having obtained results for the exposure index jPnot$_{-j}^*$, results for the corresponding isolation index follow when one recognizes that

$$ {}_jP_j^* = 1 - {}_jP_{not\text{-}j}^* \tag{8}$$

Then it should be apparent that

$$\begin{aligned} E({}_jP_j^*) &= 1 - E({}_jP_{not\text{-}j}^*) \qquad (9)\\ Var({}_jP_j^*) &= Var({}_jP_{not\text{-}j}^*)\\ \text{And} \quad \gamma_1({}_jP_j^*) &= -\gamma_1({}_jP_{not\text{-}j}^*) \end{aligned}$$

Eta-square

Bell (1954) also proposed a revised index of isolation

$$ I = \left({}_jP_j^* - \frac{N_{\bullet j}}{N_{\bullet\bullet}} \right) \left(1 - \frac{N_{\bullet j}}{N_{\bullet\bullet}} \right)^{-1} \tag{10}$$

noting that the value of this statistic would invariably lie between 0 and 1.

As Duncan and Duncan (1955) observed, Bell's revised index of isolation is identical to η^2 for predicting a dicotomous group membership variable (1= member of group j, 0= not a member of group j) from unit. η^2, the correlation ratio, equals the percentage of variance in group membership accounted for by unit, a familiar metric for describing strength of association.

Having derived the mean, variance, and skewness of the distribution of ${}_jP_j^*$ under random segregation, we can use equation 10) to obtain equivalent expressions for Bell's η^2 measure

$$\begin{aligned} E(\eta^2) &= (u-1)(N_{\bullet\bullet}-1)^{-1} \qquad (11)\\ Var(\eta^2) &= Var({}_jP_{not-j}^*) N_{\bullet\bullet}^2 (N_{\bullet\bullet} - N_{\bullet j})^{-2}\\ \gamma_1(\eta^2) &= -\gamma_1\left({}_jP_{not-j}^*\right) \end{aligned}$$

where $Var({}_jP_{not-j}^*)$ can be obtained from equation 4) above and $\gamma_1\left({}_jP_{not-j}^*\right)$ can be obtained from Appendix A.

Standardization and Significance Testing

Often, researchers want to compare the levels of intergroup contact in different locales. Locales may differ from one another in a number of ways – in-group composition, for example, and in the size of units. If some researchers want their comparisons of intergroup contact to reflect differences in-group composition across locales (Massey, White, & Phua, 1996), others would prefer to make these comparisons in a standardized metric.

Many scholars have treated Bell's η^2 index as a standardized measure of intergroup contact. In this role, η^2 has some limitations. Neither the expectation nor the variance of η^2 are fixed under the assumption of random segregation, as the equations in 11) reveal. Given completely random segregation in two locales, the expected value of η^2 in the two locales would not in general be equal. Ordinarily, one of the locales would have larger units than the other, hence a lower expected η^2.

For standardized comparisons of intergroup contact, we propose the following measure

$$Z = [{}_jP_k^* - E({}_jP_k^*)]\,[\mathrm{Var}({}_jP_k^*)]^{-.5} \qquad (12)$$

or the analogous Z-statistic for η^2. Under random segregation, these statistics would have an expected value of <u>0</u> and a variance of <u>1</u> in <u>any</u> locale – regardless of group composition or unit size.

Researchers may wish to determine whether an observed level of intergroup contact differs to a statistically significant degree from the level that would be produced by random segregation. Although in principle an exact test might be constructed with the multiple hypergeometric distribution (Agresti, 1990), we propose a less cumbersome alternative. We suggest that segregation researchers refer the Z-statistic of equation 12) to some reference value. Liberal reference values could be taken from the standard normal distribution, conservative reference values from Chebyschev's inequality. These would imply that intergroup contact departs from the level expected under random segregation at an alpha-level of .05 if the absolute value of Z exceeds 1.96 (by the normal criterion) or 4.47 (by the Chebyschev criterion). Intermediate reference values could be obtained by incorporating the skewness of the segregation measure into a Type III Fisher's distribution. See Hubert (1987) for details.

For samples of the size analyzed in many studies of residential segregation, significance testing may be moot. In such large samples, every departure from expectation may be highly significant (Taeuber & Taeuber, 1965), and associations between group membership and unit occupancy may be amenable to traditional chi-square tests. Our standardization methods would nonetheless be of value.

The inferential test we are proposing should be more useful for small data sets, where the statistical significance of intergroup contact is not a foregone conclusion, and chi-square approximations would be suspect. Such data sets may be uniquely suited to a *P** analysis – the members of a unit being most likely to have contact with one another when the units are small.

Examples

For illustrative purposes, we will analyze intergroup contact at a mid-sized American University. We will consider two examples – an example of contact between minority and non-minority faculty members, and an example of contact between minority and non-minority students.

Table 1 presents data on the number of minority and non-minority faculty members serving in eight different units of this University, as published by the University's Office of Institutional Research. These units are housed in different buildings. Faculty tend to interact within these units of the University, not across units.

To assess intergroup contact in this setting, we begin by computing the probability of a minority faculty member serving in the same unit of the University as a non-minority faculty member. Results show that ${}_mP_{non\text{-}m}^* = .8724$, a sizeable probability. Of course, one needs to consider that the overall proportion of non-minority faculty members is .8868. It is noteworthy that the observed probability of a minority faculty member serving in the same unit as a non-minority is slightly lower than the proportion of non-minorities as a whole. Does this imply that minority faculty members tend to be isolated from non-minorities? Is this tendency greater than would be expected if these faculty members were distributed across the eight units at random?

Table 1. Faculty Members at an American University, By Educational Unit and Minority Status

Educational Unit:	Minority	Non-Minority
Humanities	11	48
Social Science	5	47
Natural Science	7	76
Fine Arts	6	42
Nursing	1	21
Business	7	42
Education	3	21
Divinity	2	12

To answer these questions, we used the present analytic methods. Application of equation 3) above shows that the observed level of minority exposure to non-minorities (${}_{m}P_{non\text{-}m}{}^{*}$ = .8724) is slightly greater than what would be produced by random segregation: E(${}_{m}P_{non\text{-}m}{}^{*}$) = .8700. There is little dispersion in the values of ${}_{m}P_{non\text{-}m}{}^{*}$ across all possible assignments of these faculty members to the 8 units; the square root of equation 4) yields S.D. (${}_{m}P_{non\text{-}m}{}^{*}$) = .0089. Applying the equations in the Appendix, we find that the distribution of ${}_{m}P_{non\text{-}m}{}^{*}$ is negatively skewed: γ_1(${}_{m}P_{non\text{-}m}{}^{*}$) = -1.25. Plugging into equation 12), a standardized measure of minority faculty exposure to non-minorities is Z = +.27. By any significance testing criterion, this level of the intergroup contact could have been produced by chance.

Although the isolation of minority faculty members could be expressed in terms of a complementary $P*$ index (${}_{m}P_{m}{}^{*}$ = .1132 with Z = -.27), we will express it in terms of Bell's η^2. The observed value of η^2 = .0162 – a value that is close to what would be expected under random segregation: E(η^2) = .0189. For an analogue to the Z-statistic in equation 12), we could divide the difference between observed and expected values of η^2 by .01004 (the standard deviation of η^2) and find that in this standardized metric Z = -.27 – the same value that was found for the P^* isolation index. These values will always be the same.

Even if minority faculty are integrated at this institution, students may be segregated. We checked for segregation among some undergraduates who were enrolled in a Psychology course. Weekly, students choose to attend any one of the six laboratory sessions that are taught in conjunction with the course. Table 2 depicts the number of minority and non-minority students who attended different laboratory sessions one week during the Spring semester of 2000. Each student's minority status was reported by a laboratory supervisor who was unaware of the purpose of the report.

Table 2. Students Enrolled in a Psychology Class, By Laboratory and Minority Status

Laboratory Attended:	Minority	Non-Minority
Monday 1:00 PM	1	7
Monday 3:00 PM	0	6
Wednesday 1:00 PM	0	2
Wednesday 3:00 PM	2	4
Friday 1:00 PM	2	4
Friday 3:00 PM	2	0

Conclusion

Do students avoid intergroup contact by choosing to attend laboratories with peers of their own ethnicity? To address this question, we computed the probability of a minority student attending the same laboratory session as another minority student. Computations showed that the isolation index ${}_mP_m^* = .494$ – far greater than total proportion of minority students in this sample (.233), and somewhat greater than the isolation index that random laboratory choices would have produced: $E({}_mP_m^*) = .366$.

In this sample, random laboratory choices produce sufficient variability in values of the isolation index $[S.D.({}_mP_m^*) = .073]$ that the observed degree of minority isolation would not (by a two-tailed test) differ significantly from its expected value ($Z = +1.75$). Bell's η^2 index (.340) also exceeds its expected value (.172) by an amount that yields the same value of Z (+1.75, with S.D. = .095). Of course, these small-scale examples are only illustrative. Larger data sets might yield different conclusions.

We hope that these analytic techniques will be useful to students of segregation. They require no assumption about the sampling of observations, or the form of any population distribution. They reflect randomizations of the data at hand (Edington, 1995).

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Appendix

The coefficient of skewness of P^* is defined as $\gamma_1({}_jP_k{}^*) = E[{}_jP_k^* - E({}_jP_k^*)]^3\,[Var({}_jP_k^*)]^{-3/2}$. Below we present an expression for $E[({}_jP_k{}^*)^3]$. Skewness can be computed from

$$\gamma_1({}_jP_k{}^*) = \{E[({}_jP_k^*)^3] - [E({}_jP_k^*)]^3 - 3E({}_jP_k^*)Var({}_jP_k^*)\}\,[Var({}_jP_k^*)]^{-3/2}$$

$$E[({}_jP_k{}^*)^3] = N_{\bullet k}[N_{\bullet\bullet}(N_{\bullet\bullet}-1)]^{-1}N_{\bullet j}^{-2}\,[A+B+C+D+F+G],$$ where

$$A=\sum_{i=1}^{u}[(N_{i\bullet}-1)N_{i\bullet}^{-2}]\ ,\quad B=3(N_{\bullet j}+N_{\bullet k}-2)(N_{\bullet\bullet}-2)^{-1}\{u+\sum_{i=1}^{u}[(2-3N_{i\bullet})N_{i\bullet}^{-2}]\},$$

$$C=3(N_{\bullet j}-1)(N_{\bullet k}-1)[(N_{\bullet\bullet}-2)(N_{\bullet\bullet}-3)]^{-1}[u(u-N_{\bullet\bullet})\sum_{i=1}^{u}N_{i\bullet}^{-1}+2(N_{\bullet\bullet}-8u+16\sum_{i=1}^{u}N_{i\bullet}^{-1}-9\sum_{i=1}^{u}N_{i\bullet}^{-2})],$$

$$D=[(N_{\bullet j}-1)(N_{\bullet j}-2)+(N_{\bullet k}-1)(N_{\bullet k}-2)][(N_{\bullet\bullet}-2)(N_{\bullet\bullet}-3)]^{-1}$$
$$(N_{\bullet\bullet}-6u+11\sum_{i=1}^{u}N_{i\bullet}^{-1}-6\sum_{i=1}^{u}N_{i\bullet}^{-2}),$$

$$F=3(N_{\bullet j}-1)(N_{\bullet k}-1)(N_{\bullet j}+N_{\bullet k}-4)[(N_{\bullet\bullet}-2)(N_{\bullet\bullet}-3)(N_{\bullet\bullet}-4)]^{-1}\sum_{i=1}^{u}(N_{i\bullet}-1)(N_{i\bullet}-2)[N_{i\bullet}(N_{\bullet\bullet}-u)-6(N_{i\bullet}-2)]N_{i\bullet}^{-2}$$

$$G=\left(N_{\bullet j}-1\right)\left(N_{\bullet j}-2\right)\left(N_{\bullet k}-1\right)\left(N_{\bullet k}-2\right)\left[\left(N_{\bullet\bullet}-2\right)\left(N_{\bullet\bullet}-3\right)\left(N_{\bullet\bullet}-4\right)\left(N_{\bullet\bullet-5}\right)\right]^{-1}$$
$$\left[\left(N_{\bullet\bullet}-u\right)^3-6\left(N_{\bullet\bullet}-u\right)\left(2N_{\bullet\bullet}-5u+3\sum_{i=1}^{u}N_{i\bullet}^{-1}\right)+8\left(5N_{\bullet\bullet}-22u+32\sum_{i=1}^{u}N_{i\bullet}^{-1}-15\sum_{i=1}^{u}N_{i\bullet}^{-2}\right)\right].$$