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Auxiliary information is used along with ranking information to derive several classes of estimators to estimate the population mean of a variable of interest based on RSS (ranked set sample). The properties of these newly suggested estimators were examined. Comparisons between special cases of these estimators and other known estimators are made using a real data set. Some of the new estimators are superior to the old ones in terms of bias and mean square error.

Keywords: Auxiliary variables, efficiency, ranking, ranked set sample.

## Introduction

Many authors have discussed the use of supplementary information of auxiliary variables in survey sampling to improve the existing estimators (for example, Cochran, 1977). The ratio estimator is among the most commonly adopted to estimate: (1) population means, or (2) the total of some variable of interest from a finite population with the help of an auxiliary variable when the correlation coefficient between the two variables is positive. When the correlation coefficient between the two variables is negative, the product estimator is used. These estimators are more efficient, i.e. have smaller variances than the usual estimators of the population mean based on the sample mean of a simple random sample (SRS).

Ranked set sampling (RSS) can be used when the measurement of sample units drawn from a population of interest is very laborious or costly, but several elements can be easily arranged (ranked) in the order of magnitude. Takahasi and Wakimoto (1968) established the theory of RSS. They showed that the mean of the RSS is an unbiased estimator for the population mean and is more efficient than the mean of SRS. Dell and Clutter (1972) studied the effect of ranking error on the efficiency of RSS. The RSS has many statistical applications in biology and environmental studies (Barabesi & El-Sharaawi, 2001), for example, McIntyre (1952) first suggested using RSS to estimate the yield of pasture. In addition, RSS has been investigated by many researchers (Stokes, 1977; Stokes & Sager, 1988; Lam, et al., 1994, 1980; Mode, et al., 1999; Al-Saleh & Al-Shrafat, 2001; Al-Saleh & Zheng, 2000; Al-Saleh & Al-Omary, 2000), for more details about RSS, see Kaur, et al., 1995.

The RSS method can be summarized as follows: Select m random samples of size m units each and rank the units within each sample with respect to the variable of interest by a visual inspection or some other simple method.

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Next, select for actual measurement the $i^{th}$ smallest unit from the $i^{th}$ sample for $i$ =1, 2,..., m. In this way, a total of m measured units are obtained, one from each sample. The cycle may be repeated r times to get a sample of size $n = rm$. These $n = rm$ units form the RSS data. Note that in RSS, $rm^2$ elements are identified, but only $rm$ of them are quantified. Thus, comparing this sample with a simple random sample (SRS) of size $rm$ is reasonable.

Some Notions and Preliminaries

Let $Y$ denote the variable of interest whose population mean and variance are $\mu_y$ and $\sigma_y^2$ respectively. Estimate $\mu_y$ using the information provided by one or two auxiliary variables $X_1$ and $X_2$ based on SRS and RSS will be considered. Let $\mu_{x_i}$ and $\sigma_{x_i}^2$ be the population mean and variance for $X_i$, $i = 1, 2$. Let $Y_j$, $X_{1j}$ and $X_{2j}$ denote the values of the variables $Y, X_1$ and $X_2$ respectively, on the $j^{th}$ unit of the population. The population means $\mu_{x_1}$ and $\mu_{x_2}$ of the auxiliary variables are assumed to be known.

Let $Y_{(i)j}$, $X_{1(i)j}$ and $X_{2(i)j}$ represent the $i^{th}$ order statistics of a sample of size m in the $j^{th}$ cycle of the variables $Y, X_1$ and $X_2$ respectively based on a RSS of size $n = rm$ drawn from the population. The sample mean for each variable using RSS data are defined as follows:

$$\bar{Y}_{(n)} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i)j},$$

$$\bar{X}_{1(n)} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{1(i)j},$$

and

$$\bar{X}_{2(n)} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{2(i)j}.$$

Consider the following notations:

$$T_{y(i)} = \left(\mu_{y_{(i)}} - \mu_y\right), \; T_{x_{1(i)}} = \left(\mu_{x_{1(i)}} - \mu_{x_1}\right),$$

$$T_{x_{2(i)}} = \left(\mu_{x_{2(i)}} - \mu_{x_2}\right),$$

$$T_{yx_{1(i)}} = \left(\mu_{y_{(i)}} - \mu_y\right)\left(\mu_{x_{1(i)}} - \mu_{x_1}\right),$$

$$T_{yx_{2(i)}} = \left(\mu_{y_{(i)}} - \mu_y\right)\left(\mu_{x_{2(i)}} - \mu_{x_2}\right),$$

$$T_{x_1x_{2(i)}} = \left(\mu_{x_{1(i)}} - \mu_{x_1}\right)\left(\mu_{x_{2(i)}} - \mu_{x_2}\right),$$

$$\sigma_{yx_{1(i)}} = E\left(Y_{(i)} - \mu_{y(i)}\right)\left(X_{1(i)} - \mu_{x_1(i)}\right),$$

$$\sigma_{yx_{2(i)}} = E\left(Y_{(i)} - \mu_{y(i)}\right)\left(X_{2(i)} - \mu_{x_2(i)}\right),$$

$$\sigma_{x_1x_{2(i)}} = E\left(X_{1(i)} - \mu_{x_1(i)}\right)\left(X_{2(i)} - \mu_{x_2(i)}\right).$$

then:

$$\sum_{i=1}^{n} T_{y(i)} = 0, \quad \sum_{i=1}^{n} T_{x_1(i)} = 0, \quad \sum_{i=1}^{n} T_{x_2(i)} = 0,$$

$$\sum_{i=1}^{n} \sigma_{y(i)}^2 = n\sigma_y^2 - \sum_{i=1}^{n} T_{y(i)}^2,$$

$$\sum_{i=1}^{n} \sigma_{x_1(i)}^2 = n\sigma_{x_1}^2 - \sum_{i=1}^{n} T_{x_1(i)}^2,$$

$$\sum_{i=1}^{n} \sigma_{yx_{1(i)}} = n\sigma_{yx_2} - \sum_{i=1}^{n} T_{yx_{1(i)}},$$

$$\sum_{i=1}^{n} \sigma_{yx_{2(i)}} = n\sigma_{yx_2} - \sum_{i=1}^{n} T_{yx_{2(i)}}$$

$$\sum_{i=1}^{n} \sigma_{x_1x_{2(i)}} = n\sigma_{x_1x_2} - \sum_{i=1}^{n} T_{x_1x_{2(i)}}$$

and

$$\sum_{i=1}^{n} \sigma_{x_2(i)}^2 = n\sigma_{x_2}^2 - \sum_{i=1}^{n} T_{x_2(i)}^2.$$

The following classes of estimators of the mean of the variable $Y$ based on RSS are:

$$\tilde{Y}_{a_1,a_2} = \bar{Y}_{(n)}\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1}\left(\frac{\bar{X}_{2(n)}}{\mu_{x_2}}\right)^{a_2} \qquad (2.1)$$

and

$$\tilde{Y}_{w_1, w_2} = \bar{Y}_{(n)}\left[w_1\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1} + w_2\left(\frac{\bar{X}_{2(n)}}{\mu_{x_2}}\right)^{a_2}\right] \quad (2.2)$$

where $a_1, a_2, w_1$, $w_2$ are constants and $w_1 + w_2 = 1$.

Estimators Based on RSS and One or Two Auxiliary Variables

It is not possible to rank two or more dimensional data, therefore, ranking one of the variables and taking the corresponding values of other variables is an option. Assuming that the variable can be ranked perfectly - there are no errors in ranking the units, there will be errors in ranking the other variables.

Ranking on Study Variable *Y*

Assume that the ranking on variable $Y$ is perfect while the ranking on variables $X_1$ and $X_2$ will have errors; the estimators (2.1) and (2.2) are respectively given by:

$$\tilde{Y}_a = \bar{Y}_{(n)}\left(\frac{\bar{X}_{1[n]}}{\mu_{x_1}}\right)^{a_1}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2}, \quad (3.1)$$

and

$$\tilde{Y}_w = w_1\bar{Y}_{(n)}\left(\frac{\bar{X}_{1[n]}}{\mu_{x_1}}\right)^{a_1} + w_2\bar{Y}_{(n)}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2}. \quad (3.2)$$

where

$$\bar{X}_{1[n]} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{1[i]j}$$

and

$$\bar{X}_{2[n]} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} X_{2[i]j}$$

are the sample means of the RSS for $X_1$ and $X_2$ respectively and $X_{1[i]j}$ and $X_{2[i]j}$ are the $i^{th}$ judgment order statistic of the $i^{th}$ sample of the $j^{th}$ cycle, of the variables $X_1$ and $X_2$ respectively.

Let

$$e_0 = \frac{\bar{Y}_{(n)} - \mu_y}{\mu_y}, \quad e_1 = \frac{\bar{X}_{1[n]} - \mu_{x_1}}{\mu_{x_1}}$$

and

$$e_2 = \frac{\bar{X}_{2[n]} - \mu_{x_2}}{\mu_{x_2}}.$$

Obtain the bias and the *MSE* of the estimators $\tilde{Y}_a$ and $\tilde{Y}_w$ respectively up to the order of $n^{-1}$ as follows:

$$B\left(\tilde{Y}_a\right) = \frac{a_1}{rm^2\mu_{x_1}}(m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}}) + \frac{a_2}{rm^2\mu_{x_2}}(m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}}) + \frac{\mu_y a_1(a_1 - 1)}{2rm^2\mu_{x_2}^2}(m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2) + \frac{\mu_y a_2(a_2 - 1)}{2rm^2\mu_{x_2}^2}(m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2) + \frac{\mu_y a_1 a_2}{rm^2\mu_{x_1}\mu_{x_2}}(m\sigma_{x_1x_2} - \sum_{i=1}^{m} T_{x_1x_{2[i]}}) \quad (3.3)$$

The *MSE* of $\tilde{Y}_a$ when ranking on variable *Y* is:

$$MSE\left(\tilde{Y}_a\right) = \frac{m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2}{rm^2} + \frac{\mu_y^2 a_1^2(m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2)}{rm^2\mu_{x_1}^2} + \frac{\mu_y^2 a_2^2(m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2)}{rm^2\mu_{x_2}^2} + \frac{2\mu_y^2 a_1(m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}})}{rm^2\mu_y\mu_{x_1}} + \frac{2\mu_y^2 a_2(m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}})}{rm^2\mu_y\mu_{x_2}} + \frac{2\mu_y^2 a_1 a_2(m\sigma_{x_1x_2} - \sum_{i=1}^{m} T_{x_1x_{2[i]}})}{rm^2\mu_{x_1}\mu_{x_2}} \quad (3.4)$$

up to the order of $n^{-1}$. The optimum values of $a_1$ and $a_2$, which minimize the *MSE* of $\widetilde{Y}_a$, are obtained by the derivation of (3.4) with respect to $a_1$ and $a_2$ respectively

$$a_1^* = -\frac{\mu_{x_1}(m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}})(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2)-(m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}})(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})}{\mu_y(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2)(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2)-(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})} \quad (3.5)$$

$$a_2^* = -\frac{\mu_{x_2}(m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}})(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2)-(m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}})(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})}{\mu_y(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2)(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2)-(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})} \quad (3.6)$$

The minimum *MSE* up to terms of $n^{-1}$ for the class $\widetilde{Y}_{a^*}$ is:

$$MSE_{\min}\left(\tilde{Y}_{a^*}\right) = \frac{1}{rm^2}[(m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}})^2(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2}^2) +(m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}})^2(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1}^2)-(m\sigma_{y}^2-\sum_{i=1}^{m}T_{y(i)}^2) - \frac{2(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1}^2)(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2}^2)(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})}{(m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1}^2)(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2}^2)-(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}})^2}] \quad (3.7)$$

If $a_1$ and $a_2$ take the values in (3.5) and (3.6) respectively, the bias of $\widetilde{Y}_a$ from (3.3) is given by:

$$B_{\min}\left(\tilde{Y}_{a^*}\right) = \frac{g_1}{rm^2\mu_y^2\{[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2][m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2]-[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}]^2\}} \quad (3.8)$$

where $g_1$ is given in the Appendix.

The bias and the *MSE* of the estimators of (3.2) are given by:

$$B\left(\tilde{Y}_w\right) = \mu_y\{\frac{a_1w_1}{rm^2\mu_y\mu_{x_1}}[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}] +\frac{a_2w_2}{rm^2\mu_y\mu_{x_2}}[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}] +\frac{w_1a_1(a_1-1)}{2rm^2\mu_{x_2}^2}[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2] +\frac{w_2a_2(a_2-1)}{2rm^2\mu_{x_2}^2}[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2]\}, \quad (3.9)$$

up to the order of $n^{-1}$. The *MSE* of the estimator $\tilde{Y}_w$ if ranking on variable $Y$ is:

$$MSE\left(\tilde{Y}_w\right)=\mu_y^2\{\frac{m\sigma_y^2-\sum_{i=1}^{m}T_{y_{(i)}}^2}{rm^2\mu_y^2}+\frac{w_1^2a_1^2[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2]}{rm^2\mu_{x_1}^2}+$$

$$\frac{w_2^2a_2^2[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2]}{rm^2\mu_{x_2}^2}+\frac{2w_1a_1[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}]}{rm^2\mu_y\mu_{x_1}}+$$

$$\frac{2w_2a_2[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}]}{rm^2\mu_y\mu_{x_2}}+\frac{2w_1w_2a_1a_2[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}]}{rm^2\mu_{x_1}\mu_{x_2}}\}$$

(3.10)

if $a_1$ and $a_2$ are both known and take the values in (3.5) and (3.6) respectively, up to order $n^{-1}$.

The optimum values of $w_1$ and $w_2$, which minimize the *MSE* of $\tilde{Y}_w$, obtained by the derivation of equation (3.10) with respect to $w_1$ under the restriction $w_1+w_2=1$, are given by:

$$w_1^*=$$

$$\frac{a_2^2\left(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right)-a_1\left(m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right)+a_2\left(m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right)-a_1a_2\left(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right)}{\left(a_1^2m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right)+a_2^2\left(m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right)-2a_1a_2\left(m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right)}$$

and the *MSE* of

$$\tilde{Y}_{w^*}=w_1^*\bar{Y}_{(n)}\left(\frac{\bar{X}_{1[n]}}{\mu_{x_1}}\right)^{a_1}+w_2^*\bar{Y}_{(n)}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2}$$

(3.12)

is the minimum *MSE* up to terms of $n^{-1}$. As for the class $\tilde{Y}_w$:

$$MSE_{\min}\left(\tilde{Y}_{w^*}\right)=$$

$$\frac{1}{rm^2}\left\{\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y_{(i)}}^2\right]+w_1^{*2}\left(a_1^2\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\right.\right.$$

$$\left.+a_2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]-2a_1a_2\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\right)$$

$$-2w_1^*\left(a_2^2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]-a_1\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]\right.$$

$$\left.+a_2\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]-a_1a_2\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\right)$$

$$\left.+a_2^2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]+2a_2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\right\}$$

(3.13)

If $w$ takes the value in (3.11), then bias of $\tilde{Y}_w$ from (3.9) is given by:

$$B\left(\tilde{Y}_{w^*}\right)=$$

$$\mu_y\left\{w_1^*\left(a_1\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]-a_2\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\right.\right.$$

$$\left.+\frac{a_1(a_1-1)}{2}+\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]-\frac{a_2(a_2-1)}{2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\right)$$

$$\left.+\frac{a_2(a_2-1)}{2}+\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]+a_2\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\right\}$$

(3.14)

Ranking on One Auxiliary Variable

If the ranking of $X_1$ is perfect, then the two estimators (2.1) and (2.2) are given by:

$$\tilde{Y}_a=\bar{Y}_{[n]}\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2} \quad (3.15)$$

and

$$\tilde{Y}_w = w_1\bar{Y}_{[n]}\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1} + w_2\bar{Y}_{(n)}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2} \quad (3.16)$$

The formulas for the bias and *MSE* of estimators (3.15) and (3.16) respectively will be the same as in 3.1 except for the current estimators replace $[\ ]$ by $(\ )$ in $X_1$, and $(\ )$ by $[\ ]$ in $Y$.

Similarly if the ranking on $X_2$ is perfect, then the estimators:

$$\tilde{Y}_a = \bar{Y}_{[n]}\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2} \quad (3.17)$$

$$\tilde{Y}_w = w_1\bar{Y}_{[n]}\left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}}\right)^{a_1} + w_2\bar{Y}_{(n)}\left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right)^{a_2} \quad (3.18)$$

result.

The formulas for the bias and the *MSE* of estimators (3.17) and (3.18) respectively, will be the same as in 3.1 except for the case of replacing $[\ ]$ by $(\ )$ in $X_2$, and $(\ )$ by $[\ ]$ in $Y$ ($a_1 = 0$ or $a_2 = 0$ in (2.1) and $w_1 = 0$ or $w_2 = 0$ correspond to the case of one auxiliary variable).

Comparisons of Estimators

Consider the following known estimators. The RSS sample mean of the data:

$$\bar{Y}_{(n)} = \frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} Y_{(i)j}$$

is an unbiased estimator for the population mean and its variance is given by:

$$Var\left(\bar{Y}_{(n)}\right) = \frac{1}{rm^2}[m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2]$$

(Takahasi & Wakimoto, 1968).

The Ratio estimator using RSS data is defined as:

$$\bar{Y}_R = \bar{Y}_{(n)}\frac{\mu_x}{\bar{X}_{[n]}}.$$

This estimator is a special case of the estimator in equation (1) where $a_1 = -1$ *and* $a_1 = 0$. The bias and the $MSE$ of this estimator are respectively given by:

$$B\left(\bar{Y}_R\right) = \frac{\mu_y}{rm^2\mu_x}\left[\left(m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}}\right) - \frac{1}{\mu_x}\left(m\sigma_x^2 - \sum_{i=1}^{m} T_{x_{[i]}}^2\right)\right]$$

$$MSE\left(\bar{Y}_R\right) = \frac{\mu_y^2}{rm^2}\left[\frac{1}{\mu_y^2}\left(m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2\right) + \frac{1}{\mu_x^2}\left(m\sigma_x^2 - \sum_{i=1}^{m} T_{x_{[i]}}^2\right) - \frac{2}{\mu_y\mu_x}\left(m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}}\right)\right]$$

(Samawi & Muttlak, 1996).

The product estimator using RSS data is defined as:

$$\bar{Y}_P = \bar{Y}_{(n)}\frac{\bar{X}_{[n]}}{\mu_x}$$

This estimator is a special case for the estimator in equation (1) where $a_1 = 1$ *and* $a_1 = 0$. The new estimator is called the product estimator and its bias and $MSE$ respectively are given by

$$B\left(\bar{Y}_P\right) = \frac{\mu_y}{rm^2}[m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}}]$$

$$MSE\ \left(\bar{Y}_P\right)=\frac{\mu_y^2}{rm^2}\{\frac{1}{\mu_y^2}[m\sigma_y^2-\sum_{i=1}^{m}T_{y(i)}^2]+$$
$$\frac{1}{\mu_x^2}[m\sigma_x^2-\sum_{i=1}^{m}T_{x[i]}^2]+\frac{2}{\mu_y\mu_x}[m\sigma_{yx}-\sum_{i=1}^{m}T_{yx[i]}]\}$$

If $a_1 = a_2 = -1$ is set in the estimator of equation (2) the following new estimator results:

$$\widehat{\boldsymbol{Y}}_{a} = \bar{\boldsymbol{Y}}_{(n)} \frac{\mu_{x_1}}{\bar{\boldsymbol{X}}_{1[n]}} \frac{\mu_{x_2}}{\bar{\boldsymbol{X}}_{2[n]}}.$$

The bias and the $MSE$ are respectively given by

$$B\left(\widehat{Y}_a\right)=\frac{\mu_y}{rm^2}\left\{\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]\right.$$
$$+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]+\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]$$
$$\left.-\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]-\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\right\}$$

and

$$MSE\left(\widehat{Y}_a\right)=\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y(i)}^2\right]\right.$$
$$+\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]$$
$$-\frac{2}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]-\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]$$
$$\left.+\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right\}$$

Setting $a_1 = a_2 = 1$ in the estimator of equation (2) results in a new estimator defined as:

$$\ddot{Y}_a = \bar{Y}_{(n)} \frac{\bar{X}_{1[n]}}{\mu_{x_1}} \frac{\bar{X}_{2[n]}}{\mu_{x_2}}.$$

The bias and the $MSE$ are respectfully given by

$$B\ \left(\ddot{Y}_a\right)=$$
$$\frac{\mu_y}{rm^2}\left\{\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]+\right.$$
$$\left.\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]+\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\right\}$$

and

$$MSE\ \left(\ddot{Y}_a\right)=$$
$$\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y(i)}^2\right]+\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]\right.$$
$$+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]+\frac{2}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]$$
$$\left.+\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]+\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right\}$$

If $a_1 = a_2 = 1$ in the estimator of equation 3 is set, a new estimator called the Multivariate ratio estimator using RSS can be defined as

$$\breve{Y}_w = \bar{Y}_{(n)}\left\{w_1 \frac{\bar{X}_{1[n]}}{\mu_{x_1}} + w_2 \frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right\}.$$

The bias and the $MSE$ are respectively given by

$$B\left(\breve{Y}_w\right)=\mu_y\left\{\frac{w_1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]+\right.$$
$$\left.\frac{w_2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\right\}$$

and

$$MSE\left(\breve{Y}_w\right)=\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y(i)}^2\right]\right.$$

$$+w_1^2\left(\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]\right.$$

$$\left.-\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right)-2w_1\left(\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]\right.$$

$$-\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]+\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]$$

$$\left.-\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right)$$

$$\left.\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]+\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\right\}.$$

The comparison between the estimators proposed is illustrated by using a real data set. The data for the illustration was taken from Ahmed (1995); the population consists of 332 villages. Consider the variables, $Y$, $X_1$ and $X_2$ where $Y$ is number of cultivators, $X_1$ is the area of the village and $X_2$ is the number of household in the village.

The following steps summarize the simulation procedure to find the bias and $MSE$ of an estimator for the population mean using perfect ranking on the variable of interest Y.

Step 1:

Simulate $rm^2$ observations from the 332 real data values with replacement and perform the RSS procedure with *m* =5 and *r* = 16 to get sample of size $n=rm=80$.

Step 2:

Use the data in Step 1 to calculate

$$\hat{Y}_{(n)}=\frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m}\hat{Y}_{(i:m)j}=\frac{1}{80}\sum_{j=1}^{16}\sum_{i=1}^{5}\hat{Y}_{(i:m)j}$$

where $\hat{Y}_{(i:m)j}$ is the $i^{th}$ smallest in the sample of size *m* =5 in the $j^{th}$ cycle.

Step 3:

Repeat steps 1 and 2 (30,000) times, using these 30,000 values to obtain

$$\bar{\hat{Y}}_{(n)}=\frac{1}{30000}\sum_{i=1}^{30000}\hat{Y}_{(n)i}$$

Step 4:

Find the approximate bias and MSE for $\hat{Y}_{(n)}$. The bias is obtained by

$$B\left(\hat{Y}_{(n)}\right)=\frac{1}{30000}\sum_{i=1}^{30000}\hat{Y}_{(n)i}-\mu_y,$$

and the $MSE$ of $\hat{Y}_{(n)}$ is obtained as

$$MSE\left(\hat{Y}_{(n)}\right)=\frac{1}{30000}\sum_{i-1}^{30000}\left(\hat{Y}_{(n)i}-\bar{\hat{Y}}_{(n)}\right)^2.$$

The above simulation was preformed for all other estimators suggested ranking on one of the variables Y, $X_1$ or $X_2$. Calculate the efficiency of these estimators with respect to the $MSE\left(\hat{Y}_{(n)}\right)=Var(\bar{Y}_{(n)})$ estimator using

$$e\left(\hat{Y}\right)=\frac{MSE\left(\hat{Y}_{(n)}\right)}{MSE\left(\hat{Y}\right)},$$

where $\hat{Y}$ represents any of the estimators given.

In Tables 1-3, $MSE$, bias, and efficiency have been calculated for each of the suggested estimators. In Table 1, ranking on the variable Y is shown (i.e., the ranking of variable Y will be perfect while the ranking of the other variables will be with errors in ranking). Tables 2 and 3 show the ranking on the variables $X_1$ and $X_2$ respectively.

Considering the results of Tables 1-3 it is observed that $\tilde{Y}_{a^*}$ dominates all other estimators and achieved the highest efficiency. Its efficiency is more than 22 times higher than the

Table 1: The Bias, $MSE$ and the Efficiency for all Estimators Based on Ranking of the Variable $Y$

| Estimator | Auxiliary variable | MSE | Efficiency | Bias |
|---|---|---|---|---|
| $\overline{Y}_{(n)}$ | None | 8374.579 | 1 | 0 |
| $\widetilde{Y}^{*}$ | $x_1$ | 820.252 | 10.2041 | 0.843 |
| $\widetilde{Y}^{*}$ | $x_2$ | 909.625 | 9.17431 | 0.752 |
| $\widetilde{Y}_{a^*}$ | $x_1, x_2$ | 379.579 | 22.2222 | 0.253 |
| $\widetilde{Y}_{w^*}$ | $x_1, x_2$ | 582.065 | 14.4928 | -0.521 |
| $\overline{Y}_R$ | $x_1$ | 4403.464 | 1.8939 | 0.422 |
| $\overline{Y}_R$ | $x_2$ | 2217.261 | 3.7594 | 0.998 |
| $\overline{Y}_P$ | $x_1$ | 33092.8 | 0.25163 | 12.522 |
| $\overline{Y}_P$ | $x_2$ | 30979.3 | 0.2688 | 7.121 |
| $\widehat{Y}_a$ | $x_1, x_2$ | 8479.4 | 0.98231 | 14.153 |
| $\ddot{Y}_a$ | $x_1, x_2$ | 69893.8 | 0.119147 | 15.332 |
| $\breve{Y}_w$ | $x_1, x_2$ | 598.243 | 14.0845 | 7.151 |

Table 2: The Bias, $MSE$ and the Efficiency for all Estimators Based on Ranking of the Variable $X_1$

| Estimator | Auxiliary variable | MSE | Efficiency | Bias |
|---|---|---|---|---|
| $\overline{Y}_{[n]}$ | None | 8311.06 | 1 | 0 |
| $\widetilde{Y}^*$ | $x_1$ | 899.012 | 9.2592 | 0.899 |
| $\widetilde{Y}^*$ | $x_2$ | 1250.112 | 6.6666 | 0.822 |
| $\widetilde{Y}_{a^*}$ | $x_1, x_2$ | 378.865 | 22.2222 | 0.299 |
| $\widetilde{Y}_{w^*}$ | $x_1, x_2$ | 581.231 | 14.4928 | -0.675 |
| $\overline{Y}_R$ | $x_1$ | 4403.511 | 1.8903 | 0.533 |
| $\overline{Y}_R$ | $x_2$ | 2213.96 | 3.7594 | 1.228 |
| $\overline{Y}_P$ | $x_1$ | 33077.6 | 0.2513 | 14.532 |
| $\overline{Y}_P$ | $x_2$ | 30895.8 | 0.26903 | 9.217 |
| $\widehat{Y}_a$ | $x_1, x_2$ | 8711.43 | 0.98231 | 15.533 |
| $\ddot{Y}_a$ | $x_1, x_2$ | 69790.4 | 0.11909 | 17.222 |
| $\breve{Y}_w$ | $x_1, x_2$ | 612.103 | 13.6986 | 10.511 |

Table 3: The Bias, $MSE$ and the Efficiency for all Estimators Based on Ranking of the Variable $X_1$

| Estimator | Auxiliary Variable | MSE | Efficiency | Bias |
|---|---|---|---|---|
| $\overline{Y}_{[n]}$ | None | 8311.06 | 1 | 0 |
| $\widetilde{Y}^*$ | $x_1$ | 899.012 | 9.2592 | 0.899 |
| $\widetilde{Y}^*$ | $x_2$ | 1250.112 | 6.6666 | 0.822 |
| $\widetilde{Y}_{a^*}$ | $x_1, x_2$ | 378.865 | 22.2222 | 0.299 |
| $\widetilde{Y}_{w^*}$ | $x_1, x_2$ | 581.231 | 14.4928 | -0.675 |
| $\overline{Y}_R$ | $x_1$ | 4403.511 | 1.8903 | 0.533 |
| $\overline{Y}_R$ | $x_2$ | 2213.96 | 3.7594 | 1.228 |
| $\overline{Y}_P$ | $x_1$ | 33077.6 | 0.2513 | 14.532 |
| $\overline{Y}_P$ | $x_2$ | 30895.8 | 0.26903 | 9.217 |
| $\widehat{Y}_a$ | $x_1, x_2$ | 8711.43 | 0.98231 | 15.533 |
| $\ddot{Y}_a$ | $x_1, x_2$ | 69790.4 | 0.11909 | 17.222 |
| $\breve{Y}_w$ | $x_1, x_2$ | 612.103 | 13.6986 | 10.511 |

RSS estimator. Some other estimators achieved higher efficiency than $\widehat{Y}_{(n)}$, theses estimators are: $\widetilde{Y}_{w^*}, \breve{Y}_w, \widetilde{Y}^*$, and $\overline{Y}_R$.

The estimators achieved about the same efficiency no matter which variable was ranked on. This provides greater flexibility in choosing the variable to rank on, since some of the variables are more difficult to rank than others.

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## Appendix

$$g_{1=}\Bigg\{\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\Bigg(2\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]$$

$$-2\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]^2-\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]$$

$$\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]+\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]^2\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]+$$

$$\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]^2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]+\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]$$

$$\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]^2-\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]+$$

$$\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]^2\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]+\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]^2\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]$$

$$\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]+\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]^2\Bigg)-\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]$$

$$\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\Bigg(\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]^2+\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]^2$$

$$+\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]+\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]$$

$$\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\Bigg)\Bigg\}$$