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Makarand V. Ratnaparkhi
Wright State University, makarand.ratnaparkhi@wright.edu

Vasant B. Waikar
Miami University, waikarvb@muohio.edu

Fredrick J. Schuurmann
Miami University, schuurf@muohio.edu

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The Bootstrap Method for the Selection of a Shrinkage Factor in Two-stage Estimation of the Reliability Function of an Exponential Distribution

Makarand V. Ratnaparkhi Vasant B. Waikar Fredrick J. Schuurmann
Wright State University Miami University

An application of a bootstrap method for selecting a suitable shrinkage factor for the two-stage shrinkage estimator of a reliability function for the exponential distribution is discussed. The estimator obtained here has higher efficiency as compared to the one where the shrinkage factor is not subjected to bootstrapping.

Key words: Reliability function, two-stage estimation, shrinkage, bootstrap, efficiency.

Introduction

The estimation of the reliability function for an exponential distribution is introduced. The estimation procedure is similar to the two-stage shrinkage estimation procedure discussed by Adke, et al. (1987) for the estimation of the mean of an exponential distribution. A brief background of their procedure is provided; such background is applicable in the case described herein.

In certain studies the experimenter may have some knowledge about the value of a parameter (or has a hypothesis about the value of a parameter). The use of such information in estimation is considered in certain shrinkage estimation methods. For example, Thompson (1968) defined the shrinkage estimator of the normal mean μ in terms of the value μ_0 of μ . H also has a commented about the use of μ_0 instead of the Bayesian estimation for the normal mean. More aggressive use of μ_0 after testing the related hypothesis about μ can be found in two-stage shrinkage estimation method for μ developed by Waikar, et al. (1984). Further, the shrinkage factors considered in Thompson (1968), Waikar, et al. (1984) and Adke, et al. (1987) are based either on the corresponding test statistics or their respective functions.

The shrinkage estimation of reliability for exponentially distributed life times has also been discussed in Tse and Tso (1996). However, their method is different from the one that is discussed in this article. In particular, Tse and Tso (1996) do not use the two-stage estimation.

Makarand V. Ratnaparkhi is a Professor of Statistics at Wright State University in Dayton, Ohio, USA. Email him at: makarand.ratnaparkhi@wright.edu. Vasant B. Waikar is a retired Professor of Statistics, Miami University, Oxford, Ohio, USA. Email: waikarvb@muohio,edu. Fredrick J. Schuurmann is an Emiratis Professor of Mathematics, Miami University, Oxford, Ohio, USA. Email: schuurf@muohio.edu.

Methodology

Two-Stage Shrinkage Estimator of Reliability

Let T, a r.v. representing the failure time, have an exponential distribution with pdf

$$f(t) = (1/\beta)\exp(-t/\beta)\ ,\ t > 0, \beta > 0. \quad (1)$$

then the corresponding reliability function is:

$$R(t) = P\ (T \geq t) = \exp(-t/\beta). \quad (2)$$

The First Stage Estimation

The two- stage shrinkage estimation procedure for $R(t) = \exp(-t/\beta)$ is as follows:

1)

Select a sample of size n_1 on T.

Let T_{1i} , $i = 1, 2, \ldots\ldots, n_1$ be the first sample.

Let $T_1 = \sum T_{1i}$.

Then, $\overline{T}_1 = T_1 / n_1$ is the mean of the first stage sample.

2)

Test the prior knowledge about , $\beta = \beta_0$, *i.e.*

test $H_0 : \beta = \beta_0$ *versus* $H_a : \beta \neq \beta_0$ at level α.

3)

The rejection region is given by $T_1 \leq a_1$ or $T_1 \geq a_2$,

where a_1 and a_2 are given by:

$$\Gamma_1(n_1, a_2/\beta_0) - \Gamma_1(n_1, a_1/\beta_0) = 1 - \alpha, \text{ and}$$

$$\Gamma_1(n_1+1, a_2/\beta_0) - \Gamma_1(n_1+1, a_1/\beta_0) = 1 - \alpha$$

where $\Gamma_1(.)$ denotes the incomplete gamma function. [See Bain (1991) for details.]

4)

If $H_0 : \beta = \beta_0$ is not rejected, then the shrikage estimator of reliability is:

$$\hat{R}(t) = k \exp(-t/\overline{T}_1) + (1-k)\exp(-t/\beta_0) \quad (3)$$

where the shrikage factor k, $0 < k < 1$, is given by

$$k = |\overline{T}_1 - \beta_0| / [(a_2 - a_1)/n_1]. \quad (4)$$

The Second Stage Estimation

If H_0 is rejected then select the second sample of size n_2, T_{2i} , $i = 1, 2, \cdots\cdots, n_2$.

Let $\overline{T}_2$ be the sample mean of the second sample .

Calculate:

$$\overline{T} = \frac{n_1\overline{T}_1 + n_2\overline{T}_2}{n_1 + n_2}, \quad (5)$$

and define the estimator of the reliability as

$$\hat{R}(t) = \exp(-t/\overline{T}).$$

Thus, the two-stage shrinkage estimator of the reliability function denoted by is given by $\hat{R}_s(t)$:

$$\hat{R}_s(t) = k \exp(-t/\overline{T}_1) + (1-k)\exp(-t/\beta_0)$$

if H_0 is not rejected,

and

$$\hat{R}_s(t) = \exp(-t/\overline{T})$$

if H_0 is rejected. (6)

Bootstrapping the Shrinkage Factor k and Related Two-stage Estimator of Reliability

The shrinkage estimators and the choice of shrinkage factor have been studied for over the last five decades for various applications. In what follows, the use of bootstrap technique for selecting a shrinkage factor k in the above estimator (6) is investigated.

First, note that the efficiency of (6) is a function of as k defined above in (4). Further, for given α, the factor k is a function of $\overline{T}_1$, the mean of the first-stage sample, and hence is a random variable. Therefore, the bootstrapping for the first-stage sample T_{1i} , $i = 1, 2, \ldots\ldots, n_1$ and the corresponding k is considered as follows.

Generating a Set of k's Using Bootstrap Method

First, proceed as in the steps (1)-(4) described above in the methods section. If *Ho* is not rejected, then the bootstrap method is used as follows for the observed data on T.

1. Generate a bootstrap sample $T^*_{1\,i}$, $i = 1, 2, \ldots\ldots, n_1$, from the first stage sample T_{1i} , $i = 1, 2, \ldots\ldots, n_1$. (The * denotes the bootstrapping sample operation). Let $\overline{T}_1{*}$ denote the bootstrap mean and let k* denote the corresponding shrinkage factor. Thus, $k{*} = |\overline{T}^*_1 - \beta_0| / [(a_2 - a_1)/n_1]$ with the property $0 < k{*} < 1$.

2. Repeat the bootstrap procedure and calculate k^* until a set of predetermined B values of k^* (where $0 < k^* < 1$) is generated.

3. Several ways of using this sequence of k^* values are available for defining the shrinkage factor. Here, the mean of B values of k^* is selected. Let $\bar{k}^*$ denote this mean. Now, the two-stage bootstrap shrinkage estimator of the reliability function, denoted by $\hat{R}_b(t)$, is defined as,

$$\hat{R}_b(t) = \bar{k}^* \exp(-t/\bar{T}_1) + (1-\bar{k}^*)\exp(-t/\beta_0)$$
if H_0 is not rejected, and
$$\hat{R}_b(t) = \exp(-t/\bar{T})$$
if H_0 is rejected. (7)

Since, the derivations for the mean and the mean squared errors for $\hat{R}_s(t)$ and $\hat{R}_b(t)$ are not straightforward the values of ($\hat{R}_s(t)$, $\hat{R}_b(t)$) were simulated for the comparison of bias and the MSE's of these estimators.

Results

Fifteen thousand repetitions were carried out for different combinations of the parameter $\beta = 1$ and specified values of $(R(t), \beta_0, \alpha, n_1, n_2)$. For each repetition, B = 100 bootstrap samples were selected from the first stage sample. The simulation results are shown in Table 1.

Conclusion

The simulation results show that the estimator of the reliability function based on the mean of the values of the shrinkage factor obtained using the bootstrap procedure is more efficient as compared to the one without such bootstrapping. The same conclusion for the other values of the parameters and the sample sizes n_1, n_2 is applicable. For brevity, the other simulations results are not included here.

This article is demonstrates the use of a bootstrap method for generating a set of shrinkage factors. Using this, a final shrinkage factor can be defined based on these bootstrapped shrinkage factors as appropriate for a given problem. In the above discussion the mean of the set of shrinkage factor values is used, however, other possible selections are the median, the maximum or the minimum of the set of k^*'s. In fact, the bootstrapping of the shrinkage factor can be used in many other shrinkage estimator settings where such factor is a function of a sample statistic.

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Table 1: The Bias and Mean Squared Error for Estimators $\hat{R}_s(t)$ and $\widehat{R}_b(t)$
(β=1.00, β_0 = 1.00, α= 0.05)

$R(t) = 0.9, n_1 = 10, \quad n_2 = 10$

	$\widehat{R}_s(t)$	$\widehat{R}_b(t)$	$\widehat{R}_s(t) \ / \widehat{R}_b(t)$
Bias	-0.00229	-0.00213	-
MSE	0.00019	0.00016	1.19

$R(t) = 0.8, n_1 = 10, \quad n_2 = 10$

	$\widehat{R}_s(t)$	$\widehat{R}_b(t)$	$\widehat{R}_s(t) \ / \widehat{R}_b(t)$
Bias	-0.00387	-0.00374	-
MSE	0.00065	0.00055	1.20

$R(t) = 0.9, n_1 = 10, \quad n_2 = 15$

	$\widehat{R}_s(t)$	$\widehat{R}_b(t)$	$\widehat{R}_s(t) \ / \widehat{R}_b(t)$
Bias	-0.00189	-0.00182	-
MSE	0.00015	0.00012	1.20