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Two Dimension Marginal Distributions of Crossing Time and Renewal Numbers Related to Two-Stage Erlang Processes

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The two dimensional marginal transform, probability density and cumulative probability distribution functions for the random variables T_{ξ_N} (time taken by servers during vacations), ξ_N (number of vacations taken by servers) and η_N (number of customers or units arriving in the system) are derived by taking combinations of these random variables. One random variable is controlled at one time to determine the effect of the other two random variables simultaneously.

Key Words: Two dimensional marginal distribution, Erlang processes, Markov processes, renewal processes.

Introduction

Biggins and Cannings (1987) found that a Markov renewal process $\{(X_n, T_n) : n \geq 0\}$ might have two constituents, and that $\{X_n : n \geq 0\}$ is a homogenous Markov chain where $(T_{n+1} - T_n)$ is the sojourn time in $X_n (T_0 = 0)$. Thus, X_n could be the state entered at T_n and left at T_{n+1} , assuming that $\{X_n : n \geq 0\}$ and $\{T_{n+1} - T_n \geq 0\}$ are independent, and the distribution of $(T_{n+1} - T_n)$ is dependent on $\{X_n : n \geq 0\}$ through X_n and X_{n+1} only (otherwise not dependent on n). It is assumed that sojourn time is always strictly positive. When the initial state is i , which is $X_0 = i$, the return to state i is an ordinary renewal

process, and the visit to $j \neq i$ forms a delayed renewal process (the delay being the elapsed time until the first visit to j). Thus, Cinlar (1969) proved the theory of Markov renewal processes which generalizes those of renewal processes and the Markov chain and is a blend of the two. Biggins and Cannings (1987) applied the Markov Chain $\{X_n : n \geq 0\}$ to a finite state space assuming it to be the case in what followed, so that all introduced matrices are finite. In addition, the time T_n is integer-valued for transforms used for generating functions (with argument z) and Laplace transforms. They showed that the Markov renewal process theory provided a useful framework for the Markov chain model with wider applicability to the occurrence of sequences in the Markov chains, specifically on type one counters. Results are applied to problems regarding the reliability of the consecutive k-out-of-n:F system (Koutras & Papastavirdis, 1993; Godbole, 1993; Fu & Koutras, 1994). The geometric distribution of order k was one of the simplest waiting time distributions. Several waiting time problems have been studied in more general situations (Ebnesahrshoob & Sobel, 1990; Kreos, 1992; Aki, 1992; Aki & Hirano, 1989, 1993, 1999; Mohanty, 1994). A class of waiting time problems was proposed by Ebnesahrshoob and Sobel (1990), who obtained the probability

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generating functions (PGF) of waiting time distributions for a run of 0 of length r and a run of 1 of length k . Ling (1990) studied the distribution of waiting time for first time occurrence among E 's when X 's are independently and identically distributed (IID) and finite valued random variables, and all k 's had the same value. Aki and Hirano (1993) obtained the PGF's of the distributions of the sooner or later waiting time for the same event as Ebneshrshoob and Sobel. Talpur and Shi (1994) found the two dimension marginal distributions of crossing time and renewal numbers related with two Poisson processes using probability arguments, and constructing an absorbing Markov process. In this article, the same technique is extended for the case of the two stage Erlang process.

Methodology

An extensive literature review has shown that renewal processes are widely used in reliability theory and in models of queuing theory. Both theories are based on counting processes, and situations where the differences between two or more counting processes examined are common. Stochastic processes can be helpful in analyzing such situations. Kroese (1992) showed the difference process of the two counting processes as

$$D(t) = N_1(t) - N_2(t),$$

where $N_1(t)$ and $N_2(t)$ are two counting processes associated with the corresponding renewal

sequences of $\{X_i\}$ and $\{Y_j\}$. The problem considered here is extended from the work of Kroese (1992) and Talpur & Shi (1994) and is based upon the renewal sequence of two variables $\{X_i\}$ and $\{Y_j\}$ as shown in Figure 1.

Let

$$\xi_N = \min_{n \rightarrow \infty} \{n / T_n \geq S_N\},$$

where ξ_N is a random variable and N is a constant.

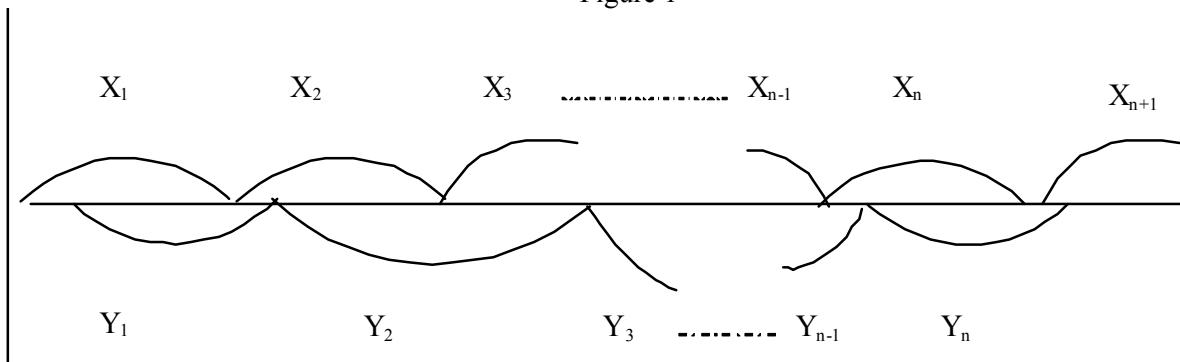
$$S_0 = 0, S_N = X_1 + X_2 + \dots + X_n$$

$$T_0 = 0, T_n = Y_1 + Y_2 + \dots + Y_n$$

$$T_{\xi_N} = \sum_{j=1}^{\xi_N} Y_j,$$

X represents the inter arrival, and Y is the number of vacations performed by the server. Both variables are discrete and have renewal processes at each occurrence. The level of absorption is achieved at the n^{th} arrival of X_n ; after the n^{th} arrival, the n^{th} vacation Y_n of the server occurs. The difference between the times at which the n^{th} vacation occurred and the n^{th} customer arrived is the crossing time of the server. The probability generating function, probability density function, and cumulative probability distribution function for the two dimensional marginal distribution for the three random variables T_{ξ_N} (time taken by servers during vacations), ξ_N (number of vacations taken by servers), and η_N (number of customers or units arriving in the system) are thus obtained.

Figure 1



Assumptions

Let N be a constant and X_i and Y_j be two sequences of random variables. Assume that X_i ($i = 1, 2, 3, \dots$) is independently and identically distributed (IID) with a finite mean, λ^{-1} , and that Y_j ($j = 1, 2, 3, \dots$) is IID with a finite mean μ^{-1} . Assume also that $N_1(t)$ is the Erlang process associated with X_i , in which the distribution of X_i is the 2-stage Erlang distribution, and $N_2(t)$ is the Erlang process associated with Y_j , in which the distribution of Y_j is the 2-stage Erlang distribution. In addition, assume that X_i and Y_j are mutually independent.

Absorbing Markov process and absorbing time distribution

Consider a Markov process $\{X(t), t \geq 0\}$ on the state space $E = (0, 1, 2, \dots)$. If E_0 and E_1 are two non-null subsets of E and they satisfy the cases

$$E_0 \cup E_1 = E$$

and

$$E_0 \cap E_1 = \emptyset,$$

then, E_0 and E_1 are called a partition of E . If E_0 is the absorbing state set and E_1 is the transient state set, and α_E is the initial condition, the absorbing Markov process (AMP) is constructed to analyze the problem. Consider the AMP $\{N_1(t), N_2(t), I(t), J(t)\}$, in which $N_1(t)$ and $N_2(t)$ are the counting processes associated with X_i

and Y_j respectively. $I(t)$ and $J(t)$ represent the phases of X_i and Y_j at time t respectively, and its state space is

$$E = \{(i, k, j, l), (i', j') / i, j = 0, 1, \dots; k, l = 1, 2; i' = N', N' + 1', \dots; j' = 1', 2', \dots\}$$

where (i', j') are absorbing states. The transition of states is illustrated in Figure 2. Let

$$P_{ij}(k, l, t) = p\{N_1(t) = i, N_2(t) = j, I(t) = k, J(t) = l\}$$

and

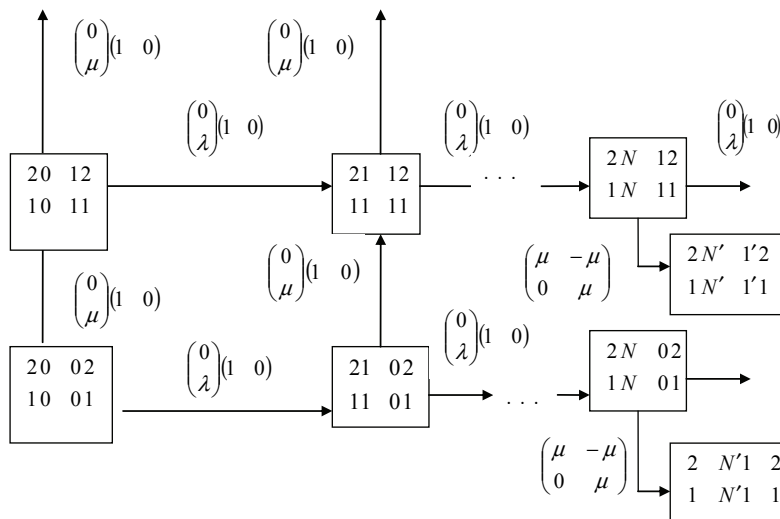
$$P_{ij}(t) = [p_{ij}(1, 1, t), \dots, p_{ij}(1, n, t), \dots, p_{ij}(m, 1, t), \dots, p_{ij}(m, n, t)].$$

From the transition rate diagram, the systems differential equations are as follows

$$P'_{ij}(t) = p_{ij}(t) \left\{ - \left[\begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & -\mu \\ 0 & \mu \end{pmatrix} \right] \right\} + p_{i-1, j}(t) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + p_{ij-1}(t) \begin{pmatrix} 0 \\ \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

where $i = 0, 1, \dots, N-1; j = 0, 1, 2, \dots; \dots$ (2.1)

Figure 2: Transition Rate Diagram



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and

$$P'_{ij}(t) = p_{ij}(t) \left\{ - \left\{ \begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & -\mu \\ 0 & \mu \end{pmatrix} \right\} \right\} + p_{i-1,j}(t) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix},$$

where $i = N, N+1, \dots; j = 0, 1, \dots$ (2.2)

Using these differential equations, Talpur and Iffat (2007) obtained the joint distribution for three random variables. The two dimension marginal distributions for the same problem were also obtained in this study.

Two dimensional marginal probability distribution function for T_{ξ_N}, ξ_N

The number of arriving customers is fixed in order to observe the effect of time taken by server vacations and the number of vacations taken. The two dimensional marginal probability generating function (probability transform function), two dimensional marginal probability density function, and the two dimensional marginal cumulative probability distribution function for random variables T_{ξ_N}, ξ_N are computed under the following theorems.

Theorem 3.1

The two dimensional marginal probability generating function of the two random variables T_{ξ_N}, ξ_N is given by:

$$f^*(s, u) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}.$$

Proof 3.1

The two dimensional marginal probability generating function of two random variables T_{ξ_N}, ξ_N is calculated from the joint

probability generating function of three random variables T_{ξ_N}, ξ_N and η_N (Talpur & Iffat, 2007):

$$f^*(s, u, z) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

If z is close to 1, then the two dimensional marginal probability generating function is:

$$f^*(s, u) = u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}. \tag{3.1}$$

Theorem 3.2

The two dimensional marginal probability density function of two random variables T_{ξ_N}, ξ_N is given by:

$$p\{T_{\xi_N} \leq t, \xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left[\lambda^i \mu^j (\lambda + \mu)^{j+i-1} \frac{t^{2j+2i-2}}{(2j+2i-2)!} e^{-(\lambda+\mu)t} + \lambda^i \mu^j (\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} e^{-(\lambda+\mu)t} \right] dt$$

Proof 3.2

As used by Talpur and Shi (1994), the following equation can be obtained by definition of the z and L transform

$$f^*(s, u) = \sum_{j=1}^{\infty} \int_0^{\infty} \exp(-st) dp \{T_{\xi_N} \leq t, \xi_N = j\} u^j$$

Inserting the value from equation (3.1) results in:

$$f^*(s, u) = u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N$$

$$\begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Let $a = s + \lambda + \mu$ and applying the rule of power series as in Pipes and Harwil (1970) results in:

$$f^*(s, u) = \frac{u}{a} \left\{ \frac{\lambda}{a} \sum_{k=0}^{\infty} \left(\frac{\lambda + \mu}{a} \right)^{k+1} \left(\frac{u\mu}{a} \right)^k \right\}^N$$

$$\mu \sum_{l=0}^{\infty} \left(\frac{\lambda + \mu}{a} \right)^l \left(\frac{\lambda}{a} \right)^l \left\{ 1 + \frac{\lambda + \mu}{a} \right\}$$

Applying the negative binomial distribution and simplifying the series (Hogg & Craig, 2006), and setting $j = k + 1$ and $i = N + l$, results in:

$$f^*(s, u) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-1}{j-1} \left(\frac{u}{s+\lambda+\mu} \right)^j \left(\frac{\lambda}{s+\lambda+\mu} \right)^i$$

$$\left\{ \left(\frac{\lambda + \mu}{s + \lambda + \mu} \right)^{j+i-1} + \left(\frac{\lambda + \mu}{s + \lambda + \mu} \right)^{j+i} \right\} u^{k+1}$$

After comparing the coefficient of u^j and taking the inverse of the Laplace transform, the two dimensional probability density function for the two random variables time taken by vacation of servers with respect to number of vacations is obtained as follows:

$$p\{T_{\xi_N} \leq t, \xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j$$

$$\left[\begin{aligned} & (\lambda + \mu)^{j+i-1} \frac{t^{2j+2i-2}}{(2j+2i-2)!} \\ & + (\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} \end{aligned} \right] e^{-(\lambda+\mu)t} \tag{3.2}$$

Theorem 3.3

The two dimensional marginal cumulative probability distribution function of two random variables T_{ξ_N}, ξ_N is given by:

$$p\{T_{\xi_N} \leq t, \xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda + \mu} \right)^i \left(\frac{\mu}{\lambda + \mu} \right)^j$$

$$\left[\sum_{r=0}^{2i+2j-1} \frac{[(\lambda + \mu)t]^r}{r!} + \sum_{r=1}^{2i+2j} \frac{[(\lambda + \mu)t]^r}{r!} \right] e^{-(\lambda+\mu)t}$$

Proof 3.3

The two dimensional marginal cumulative probability distribution function T_{ξ_N}, ξ_N , is obtained by integrating the probability density function (Medhi, 1982).

$$p\{T_{\xi_N} \leq t, \xi_N = j\} = \int_0^{\infty} dp \{T_{\xi_N} \leq t, \xi_N = j\} dt$$

$$= \int_0^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j$$

$$\left[\begin{aligned} & (\lambda + \mu)^{j+i-1} \frac{t^{2j+2i-2}}{(2j+2i-2)!} \\ & + (\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} \end{aligned} \right] e^{-(\lambda+\mu)t} dt$$

Integration by parts is applied to obtain the cumulative probability distribution function for the length of vacations taken by servers with the number of vacations taken by servers:

$$\begin{aligned}
 P\{T_{\xi_N} \leq t, \xi_N = j\} &= F(t, u) \\
 &= \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^j \\
 &\quad \left[\sum_{r=0}^{2i+2j-1} \frac{[(\lambda+\mu)t]^r}{r!} + \sum_{r=1}^{2i+2j} \frac{[(\lambda+\mu)t]^r}{r!} \right] e^{-(\lambda+\mu)t}
 \end{aligned} \tag{3.3}$$

Two dimensional marginal probability distribution functions for T_{ξ_N}, η_N

The effect of the number of vacations taken by servers combined with the numbers of customers arriving is studied by controlling the number of vacations taken by the servers. The two dimensional marginal probability transform function (probability generating function), two dimensional probability density function, and two dimensional marginal cumulative probability distribution function are obtained. The time taken by number of vacations by servers with the number of arriving units are represented by random variables T_{ξ_N}, η_N .

Theorem 4.1

The two dimensional marginal probability generating function (probability transform function) for random variables T_{ξ_N}, η_N is:

$$\begin{aligned}
 f^*(s, z) &= \\
 (1 \ 0) &\left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\
 &\quad \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}
 \end{aligned}$$

Proof 4.1

The two dimension marginal probability generating function for the two random variables, T_{ξ_N}, η_N , is calculated from the joint

probability generating function (joint probability transform function) for the three random variables T_{ξ_N}, ξ_N and η_N as given by Talpur and Iffat (2007):

$$\begin{aligned}
 f^*(s, u, z) &= \\
 u(1 \ 0) &\left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\
 &\quad \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}
 \end{aligned}$$

Let u be close to 1 for controlling the effect of the random variable ξ_N . Then we get the two dimensional marginal probability generating function of the two random variables T_{ξ_N}, η_N as

$$\begin{aligned}
 f^*(s, z) &= \\
 (1 \ 0) &\left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\
 &\quad \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}
 \end{aligned} \tag{4.1}$$

Theorem 4.2

The two dimensional marginal probability density function for random variables T_{ξ_N}, η_N is:

$$\begin{aligned}
 f(t, z) &= \\
 &\sum_{j=0}^{\infty} \binom{N+j-1}{j-1} \lambda^i \mu^j \\
 &\quad \left[(\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} + (\lambda + \mu)^{j+i-1} \frac{t^{2j+2i}}{(2j+2i)!} \right] e^{-(\lambda+\mu)t} dt
 \end{aligned}$$

Proof 4.2

The following equation as given by Talpur and Shi (1994) can be expressed by the definition of z and L transform as:

$$\begin{aligned}
 f^*(s, z) &= \sum_{i=N}^{\infty} \int_0^{\infty} \exp(-st) dp \{T_{\xi_N} \leq t, \eta_N = i\} z^i \\
 &= (1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\
 &\quad \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}.
 \end{aligned}$$

Let $a = s + \lambda + \mu$, the following results after algebraic manipulation:

$$f^*(s, z) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\mu}{a}\right)^j \left(\frac{\lambda}{a}\right)^i \left\{ \left(\frac{\lambda + \mu}{a}\right)^{j+i} + \left(\frac{\lambda + \mu}{a}\right)^{j+i+1} \right\} z^i.$$

After substituting the value of a , comparing the coefficient of z^i , and taking the inverse of the Laplace transform, the two dimensional marginal probability density function for the variables time taken by servers vacations with respect to the number of customers arriving, T_{ξ_N}, η_N , is established by:

$$\begin{aligned}
 f(t, z) &= \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j \left[\begin{aligned} &(\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} \\ &+ (\lambda + \mu)^{j+i+1} \frac{t^{2j+2i}}{(2j+2i)!} \end{aligned} \right] e^{-(\lambda + \mu)t} \tag{4.2}
 \end{aligned}$$

Theorem 4.3

The two dimensional marginal probability distribution function of random variables T_{ξ_N}, η_N is:

$$\begin{aligned}
 p\{T_{\xi_N} \leq t, \eta_N = i\} &= \\
 F(t, z) &= \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda + \mu}\right)^i \left(\frac{\mu}{\lambda + \mu}\right)^j
 \end{aligned}$$

$$\left[\sum_{r=0}^{2j+2i-1} \frac{[(\lambda + \mu)t]^r}{r!} + \sum_{r=1}^{2j+2i} \frac{[(\lambda + \mu)t]^r}{r!} \right] e^{-(\lambda + \mu)t} \tag{4.3}$$

Proof 4.3

The two dimensional cumulative probability distribution function for two random variables, T_{ξ_N}, η_N , is obtained by integrating the two dimension marginal probability density function for the same random variables.

$$\begin{aligned}
 p\{T_{\xi_N} \leq t, \eta_N = i\} &= \sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j \int_0^{\infty} \left[\begin{aligned} &(\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} \\ &+ (\lambda + \mu)^{j+i+1} \frac{t^{2j+2i}}{(2j+2i)!} \end{aligned} \right] e^{-(\lambda + \mu)t} dt
 \end{aligned}$$

After algebraic manipulation and applying the integration by parts the proof is obvious.

Two dimensional marginal distribution functions for ξ_N, η_N

The effect of the number of vacations taken by servers on the number of customer arrivals was studied by controlling the time taken with the number of vacations made by servers. The two dimensional marginal probability transform function (probability generating function), two dimensional probability density function and two dimensional marginal cumulative probability distribution function for the number of server vacations with number of arriving customers, as represented by random variables ξ_N, η_N , are now calculated.

Theorem 5.1

The two dimensional marginal probability generating function (probability transform function) for the two random variables ξ_N, η_N is:

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$$f'(u, z) = u(1 \ 0)$$

$$\left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}.$$

The two dimensional probability density function for random variables ξ_N, η_N is:

$$f(u, z) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} 2 \binom{N+j-2}{j-1} \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i.$$

The two dimensional marginal cumulative probability distribution function for the random variables ξ_N, η_N is:

$$p\{\xi_N = j, \eta_N = i\} = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} 2 \binom{N+j-2}{j-1} \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i.$$

Proof 5.1

The two dimensional marginal probability generating function (probability transform function) for the two random variables ξ_N, η_N is obtained from the joint probability transform function of the three random variables as:

$$f^*(s, u, z) = u(1 \ 0) \\ \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Let s be close to 0^+ to control the effect of time by the number of vacations taken by the number of servers. The two dimensional marginal probability generating function for random variables ξ_N, η_N is obtained using:

$$f'(u, z) = u(1 \ 0)$$

$$\left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad (5.1a)$$

The definition of the z transform is expressed by the following equation (Talpur & Shi, 1994):

$$f(u, z) = \sum_{i=N}^{\infty} p\{\xi_N = j, \eta_N = i\} u^j z^i = u(1 \ 0) \\ \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \\ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -\lambda z & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Using the same process as in theorem 2.2 results in:

$$f(u, z) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i u^j z^i,$$

and comparing the coefficients u and z , the following proof is obtained:

$$f(u, z) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i u^j z^i \quad (5.1b)$$

Two dimensional marginal cumulative probability distribution functions for two random variables, ξ_N and η_N , was obtained by summing their density function and the number of vacations made by servers with the number of arriving units.

Results

As shown in Table 1, the two dimensional marginal probability distributions of random variables involving the crossing time spent for the number of vacations taken by servers (T_{ξ_N}) followed by the number of vacations by the service channels (ξ_N) shows a two stage Erlang distribution for the probability density function for achieving the absorption state. The cumulative probability distribution function is found to be a Gamma distribution.

The two dimensional marginal probability distribution for random variables involving the crossing time spent for the number of vacations taken by servers (T_{ξ_N}) with a

reasonable number of arriving customers (η_N) for achieving the absorption state is shown in Table 2. The two variables show a two stage Erlang distribution for the probability density function. The Gamma distribution is satisfied for the cumulative probability distribution function.

The probability density function for two random variables is expressed as a negative binomial distribution. The cumulative probability distribution function also satisfied the negative binomial distribution. As Medhi (1982) expressed, if the parameter λ (intensity function) of a Poisson process is a random variable with Gamma distribution, then the mixed Poisson distribution is Negative binomial.

Table 1: Two Dimensional marginal probability distributions of random variables T_{ξ_N}, ξ_N

Transform function $f^*(s, u)$	$u(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -u\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N$ $\begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$
Probability density function $f(t, u) =$ $\{T_{\xi_N} \leq t, \xi_N = j\}$	$\sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j$ $\left[(\lambda + \mu)^{j+i-1} \frac{t^{2j+2i-2}}{(2j+2i-2)!} + (\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t}$
Probability distribution function $p\{T_{\xi_N} \leq t, \xi_N = j\}$	$\sum_{i=N}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda + \mu} \right)^i \left(\frac{\mu}{\lambda + \mu} \right)^j \left[\sum_{r=0}^{2j+2i-1} \frac{[(\lambda + \mu)t]^r}{r!} e^{-(\lambda+\mu)t} + \sum_{r=0}^{2j+2i} \frac{[(\lambda + \mu)t]^r}{r!} e^{-(\lambda+\mu)t} \right]$

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Table 2: Two Dimensional marginal probability distributions of random variables T_{ξ_N}, η_N

Transform function $f^*(s, z)$	$(1 \ 0) \left\{ \begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -\mu & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N$ $\begin{pmatrix} s + \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & s + \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$
Probability density function $f(t, z) =$ $\{T_{\xi_N} \leq t, \eta_N = i\}$	$\sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \lambda^i \mu^j$ $\left[(\lambda + \mu)^{j+i} \frac{t^{2j+2i-1}}{(2j+2i-1)!} + (\lambda + \mu)^{j+i+1} \frac{t^{2j+2i}}{(2j+2i)!} \right] e^{-(\lambda+\mu)t}$
Probability distribution function $p\{T_{\xi_N} \leq t, \eta_N = i\}$	$\sum_{j=1}^{\infty} \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda + \mu} \right)^i \left(\frac{\mu}{\lambda + \mu} \right)^j$ $\left[\sum_{r=0}^{2j+2i-1} \frac{[(\lambda + \mu)t]^r}{r!} e^{-(\lambda+\mu)t} \right.$ $\left. + \sum_{r=0}^{2j+2i} \frac{[(\lambda + \mu)t]^r}{r!} e^{-(\lambda+\mu)t} \right]$

Table 3: Two dimensional marginal probability distribution functions of random variables ξ_N, η_N

Transform function $f^*(u, z)$	$(1 \ 0) \left\{ \begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -u\mu & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N$ $\begin{pmatrix} \lambda + \mu & -(\lambda + \mu) \\ -z\lambda & \lambda + \mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$
Probability density function $f(u, z)$	$\binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda + \mu} \right)^j \left(\frac{\lambda}{\lambda + \mu} \right)^i$
Probability distribution function $F(u, z) =$ $p\{T_{\xi_N} \leq t, \eta_N = i\}$	$\sum_{j=1}^{\infty} \sum_{i=N}^{\infty} 2 \binom{N+j-2}{j-1} \left(\frac{\lambda}{\lambda + \mu} \right)^i \left(\frac{\mu}{\lambda + \mu} \right)^j$

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