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# Bayesian Threshold Moving Average Models

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A Bayesian approach in threshold moving average model for time series with two regimes is provided. The posterior distribution of the delay and threshold parameters are used to examine and investigate the intrinsic characteristics of this nonlinear time series model. The proposed approach is applied to both simulated data and a real data set obtained from a chemical system.

Key words: Threshold time series, moving average model, Bayesian estimation, simulation, chemical data.

## Introduction

One class of nonlinear time series models is the threshold time series models which are extensively reported in literature. Among these, Tong and Lim (1980) introduced threshold autoregressive (TAR) models with statistical inference and applications. Bayesian inference for threshold autoregressive models have been investigated by different authors.

Geweke and Terui (1991) derived an exact posterior distribution of the delay and threshold parameters. Cathy, et al. (1995) used Monte Carlo Markov chain (MCMC) methods to implement a Bayesian inference on TAR models, and Broemeling and Cook (1992) performed a Bayesian analysis on TAR models. However, most of the literature emphasizes the threshold autoregressive models. Wang, et al. (1984) introduced the threshold autoregressive moving average (TARMA) models and considered the estimation of the model parameters. De Gooijer (1998) studied various problems associated with the identification, estimation and testing of threshold moving average models. Ling and Tong (2005) considered a quasi-likelihood ratio test for the threshold in moving average models. Amendola, et al. (2009) discussed the stochastic structure of the self-exiting TARMA model; they specified sufficient conditions for weak stationarity and showed that the self-exiting TARMA model belongs to the class of the random coefficients autoregressive models. Smadi (1997) used the Bayesian approach for exploration of the joint posterior distribution for TARMA models using MCMC methods: he assumed noninformative priors, fixing the delay parameter $d$. In addition, he used a modified Gibbs sampling scheme, which is a hybrid strategy of Gibbs sampler, random walk Metropolis, and importance sampling. Safadi and Morettin (2000) considered a Bayesian analysis for threshold autoregressive moving average models and a hierarchical prior to perform Bayesian analysis using a rearranged procedure with MCMC methods.

The objective of this study is to provide a Bayesian approach in a threshold moving average model for time series with two regimes. The posterior distribution of the delay and the threshold parameters are used to examine and investigate the characteristics which are intrinsic to this nonlinear time series model. The

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proposed approach is applied to both simulated data and a real data set obtained from a chemical system.

## Methodology

### The Threshold Models

Let $\{Y_t, t \geq 1\}$ be a time series, the threshold autoregressive moving average models with two regimes. Wang, et al. (1984) symbolized *TARMA(2,(p₁, q₁),(p₂, q₂.))*, given by:

$$Y_t = \begin{cases} \sum_{i=1}^{p_1} \phi_i^{(1)} Y_{t-i} + \sum_{i=1}^{q_1} \theta_i^{(1)} Z_{t-i}^{(1)} + Z_t^{(1)}, & \mathrm{Y_{t\text{-}d}} \leq r, \\ \sum_{i=1}^{p_2} \phi_i^{(2)} Y_{t-i} + \sum_{i=1}^{q_2} \theta_i^{(2)} Z_{t-i}^{(2)} + Z_t^{(2)}, & \mathrm{Y_{t\text{-}d}} > r, \end{cases} \quad (1)$$

where $\varphi_1^{(j)},...,\varphi_{p_j}^{(j)}, \theta_1^{(j)},...,\theta_{q_j}^{(j)}, (j=1,2)$ are the model coefficients of the two ARMA subsystems, *r* is called the threshold parameter and *d* is the delay parameter; assume the innovations $Z_t^{(j)} : N(0,\sigma_j^2), j = 1, 2$.

A special case of equation (1), is the threshold autoregressive model *TAR(2;p₁,p₂)*:

$$Y_t = \begin{cases} \sum_{i=1}^{p_1} \varphi_i^{(1)} Y_{t-i} + \mathrm{Z_t^{(1)}} & \mathrm{Y_{t\text{-}d}} \leq r, \\ \sum_{i=1}^{p_2} \varphi_i^{(2)} Y_{t-i} + Z_t^{(2)} & \mathrm{Y_{t\text{-}d}} > r. \end{cases} \quad (2)$$

Another special case of equation (1) is the threshold Moving Average model *TMA(2;q₁,q₂)*:

$$Y_t = \begin{cases} \sum_{i=1}^{q_1} \theta_i^{(1)} Z_{t-i}^{(1)} + \mathrm{Z_t^{(1)}} & \mathrm{Y_{t\text{-}d}} \leq r, \\ \sum_{i=1}^{q_2} \theta_i^{(2)} Z_{t-i}^{(2)} + Z_t^{(2)} & \mathrm{Y_{t\text{-}d}} > r. \end{cases} \quad (3)$$

### Posteriors Distribution

The approximate posterior distribution of the delay and threshold parameters (*d, r*) for threshold moving average models (3) is based on using estimated residuals instead of the true innovations. Broemeling and Shaarawy (1986) implemented the estimated innovations for Bayesian analysis of ARMA models. Smadi (1997) and Safadi and Morettin (2000) have used this estimated innovation approach to explore the posterior distributions of the threshold autoregressive moving average models.

Defining $\phi = (d, r)$, and the set $B_j = (\theta_1^{(j)}, \theta_2^{(j)}, ..., \theta_{q_j}^{(j)})$, and under the normality assumption, that is, $Z_t^{(j)}$ $i.i.d$ $N(0, \sigma_j^2)$, a prior $\pi(\phi)$ could be any form as long as

$$\sum_{d=1}^{R} \int_{-\infty}^{\infty} \pi(d, r) dr = 1.$$

Conditional on $\phi$, independent priors on $B_j$ and $\sigma_j$ of standard Jeffreys prior can be expressed as:

$$\Pi\{B_j, \sigma_j (j=1,2), \varphi\} = \pi_c(B_j, \sigma_j (j=1,2), \varphi)\pi(\varphi) \quad (4)$$

where $\pi_c(B_j, \sigma_j, j = 1,2 \mid \phi) = (\sigma_1 \sigma_2)^{-1}$. It is also assumed that $Z_0^{(1)} = Z_{-1}^{(1)} = ... = Z_{1-q_1}^{(1)} = 0$ and $Z_0^{(2)} = Z_{-1}^{(2)} = ... = Z_{1-q_2}^{(2)} = 0$. Conditioning on $\phi = (d, r)$, estimates of the innovations $Z_t^{(1)}$ and $Z_t^{(2)}$ can be obtained using least squares estimates. In this case, the following model is obtained:

$$Y_t = \begin{cases} \sum_{i=1}^{q_1} \theta_i^{(1)} \hat{Z}_{t-i}^{(1)} + \mathrm{Z_t^{(1)}} & \mathrm{Y_{t\text{-}d}} \leq r, \\ \sum_{i=1}^{q_2} \theta_i^{(2)} \hat{Z}_{t-i}^{(2)} + Z_t^{(2)} & \mathrm{Y_{t\text{-}d}} > r. \end{cases} \quad (5)$$

Derivation of the approximate posterior of the delay and the threshold parameters of the threshold moving average model is similar to the threshold autoregressive model (2) reported by Geweke and Terui (1993). After estimating the innovations and using the model (3), conditional on $\phi = (d, r)$, let $W_1$ be a vector consisting of $N_1$ ordered observations on $\{Y_t\}$ such that $Y_{t-d} \le r$, and let $W_2$ be a vector consisting of $N_2$ ordered observations on $\{Y_t\}$ such that $Y_{t-d} > r$. Let $X_j$ be a $N_j \times q_j$ matrix of lagged variables on the estimated innovations $\{\hat{Z}_t\}$ corresponding to $W_j\ (j = 1,2)$. Then, the approximate posterior density of $(B_1, \sigma_1, B_2, \sigma_2)$ conditional on *d* and *r* is the product of two posterior densities, that is:

$$P(B_j, \sigma_j, j = 1,2 \mid \varphi, Y\} = \prod_{j=1}^{2} \sigma_j^{-(N_j+1)} (2\pi)^{-(N_j/2)} \exp\left\{ -\frac{(W_j - X_j B_j)'(W_j - X_j B_j)}{2\sigma_j^2} \right\} \quad (6)$$

The posterior distribution of $\phi$ can be derived by integrating this expression with respect to $(B_1, \sigma_1, B_2, \sigma_2)$. The problem is to integrate the following expression with respect to σ:

$$\sigma^{-(T+1)} \exp\left\{ -\frac{(W - XB)'(W - XB)}{2\sigma^2} \right\} = \sigma^{-(T+1)} \exp\left\{ -\frac{\upsilon s^2 + (B - \hat{B})' X' X (B - \hat{B})}{2\sigma^2} \right\} \quad (7)$$

where

$$B = (X'X)^{-1} X'W,\ s^2 = (W - X\hat{B})'(W - X\hat{B}) / \nu,\quad \nu = N - K - 1.$$

Integrating over $\sigma_1$ and $\sigma_2$ on the right hand side of equation (7), it is possible to obtain:

$$P\{B_j, j = 1,2 \mid \varphi, Y\} = \prod_{j=1}^{2} (2\pi)^{-(N_j/2)} \int_0^\infty \sigma_j^{-(N_j+1)} \exp\left\{ -\frac{\nu_j s_j^2 + (B_j - \hat{B}_j)' W_j' W_j (B_j - \hat{B}_j)}{2\sigma_j^2} \right\} d\sigma_j$$

$$P\{B_j, j = 1,2 \mid \varphi, Y\} = \prod_{j=1}^{2} (2\pi)^{-(N_j/2)} 2^{(N_j/2-1)} \Gamma\left( \frac{N_j}{2} \right) \{\nu_j s_j^2 + (B_j - \hat{B}_j)' W_j' W (B_j - \hat{B}_j)\}^{-(N_j/2)} \quad (8)$$

multiplying equation (8) by $\pi(\phi)$ and integrating $B_j$, for $j = 1, 2$ out results in the posterior distribution of $\phi$ (Geweke & Terui, 1993):

$$P(\varphi \mid Y) \propto \pi(\varphi) \prod_{j=1}^{2} (2\pi)^{-(N_j/2)} 2^{(N_j/2-1)} \Gamma\left( \frac{N_j}{2} \right) (s_j^2)^{-(N_j/2)} \times \int_{R^{(k_j+1)}} \left\{ \nu_j + (B_j - \hat{B}_j)' \frac{W_j' W_j}{s_j^2} (B_j - \hat{B}_j) \right\}^{-(N_j/2)} dB_j$$

$$= \prod_{j=1}^{2} 2^{-(\nu_j/2+1)} (\pi)^{-(\nu_j/2)} \Gamma\left( \frac{\nu_j}{2} \right) \left( \frac{\nu_j s_j^2}{2} \right)^{-(\nu_j/2)} |W_j' W_j|^{-1/2} \quad (9)$$

where

$$s_j^2 = (W_j - X_j \hat{B}_j)'(W_j - X_j \hat{B}_j) / \nu_j,\ \hat{B}_j = (X_j' X_j)^{-1} X_j' W_j,\quad \nu_j = N_j - K_j - 1.$$

## Results

The characteristics of the posterior distribution of $\phi = (d, r)$ were investigated for simulated data and a real data set obtained from a chemical system. As a set of possible values of the threshold parameter *r*, the order statistics $[Y^{(i)}, i = 1, ..., q(\le N)]$ of observations was used.

### Simulation Examples

Simulation results were based on both one realization and 100 realizations. The TMA(2;1,1) model was considered, where $\theta^{(1)} = -0.4, \theta^{(2)} = 0.4, \sigma_1^2 = 1, \sigma_2^2 = 1$, $d$ = *1*,and $r$ = *0*. A one realization was generated with series length of 50. As a set of possible values of the threshold parameter *r*, $[r_0, r_L]$ was chosen as large as possible and the delay parameter *d* was selected as *d = 1, 2, 3, 4* and *5*.

Simulation results demonstrated that the posterior mass was concentrated at *d=1, 2* and *3*. Summary results of the joint posterior distribution of $\phi = (d, r)$ are presented in Table 1. The marginal posterior distribution of *d = 1, 2, 3* have probabilities of 0.5566, 0.2500 and 0.1800 respectively. The posterior probability concentrates predominantly on few points, namely (*d, r*) = (1, 0.0036) and (1, 0.0350) with respective probabilities of 0.05582 and 0.06987.

Simulation results based on 100 realizations with series length of 100 were analyzed. For each realization, $\phi = (d, r)$ was estimated based on modal value of the posterior distribution. The results yield relative frequencies of 87%, 7%, and 6% for *d = 1, 2,* and *3* respectively. The marginal posterior of *r* is shown in Figure 1; as expected, the model value is concentrated around the true threshold value *r = 0*.

Real Data Example

Series A, which consists of 197 observations and represents the concentration of a chemical process, was considered (Box & Jenkins, 1976). The differenced time series was considered. Fitting MA(1) model yields

$$X_t = -0.69 Z_{t-1} + Z_t \qquad (15)$$

Smadi (1997) used the MCMC technique for exploration of the posterior distribution of the threshold parameter *r*. The methodology proposed herein is applied to the differenced series in order to examine the posterior distribution of the threshold and the delay parameter $\phi = (d, r)$. The number of threshold points is reduced from 196 to 22 points because some differences have the same values. Values of [–0.4, 0.4] were assigned for [$r_0$, $r_L$]. For the delay parameter *d*, the set *d = 1, 2, 3, 4* and *5* were selected. It was found that the posterior mass was concentrated at *d = 1, 2* and *3*.

Summary results of the joint posterior distribution of $\phi = (d, r)$ are presented in Table 2. It can be seen that the marginal posterior distribution of *d = 1, 2, 3* have probabilities of 0.8344, 0.13163 and 0.03398, respectively. Also, the posterior probability concentrates on (*d, r*) = (1, 0.0) with probability of 0.17; this corresponds to the largest mode of the posterior density. Conditioning on (*d, r*) = (1, 0.0), the fitted TMA(2;1,1) model is

$$X_t = \begin{cases} -0.72 Z_{t-1}^{(1)} + Z_t^{(1)} & \text{if } X_{t-1} \le 0, \\ -0.66 Z_{t-1}^{(2)} + Z_t^{(2)} & \text{if } X_{t-1} > 0. \end{cases}$$

where $\hat{\sigma}_1^2 = 0.15$ and $\hat{\sigma}_2^2 = 0.12$.

Conclusion

From the proposed methodology and numerical results it can be concluded that the threshold moving average models are tractable from a Bayesian point of view. The nonlinearity threshold-type for moving average models can be detected by examining the marginal posterior distribution of the threshold parameter.

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Table 1: Summary of Joint Posterior Densities of (*r, d*)

| (*r, d*) | Integrated Density | (*r, d*) | Integrated Density | (*r, d*) | Integrated Density |
|---|---|---|---|---|---|
| (-1.1832, 1) | 0.02234 | (-1.1832, 2) | 0.00345 | (-1.1832, 3) | 0.01233 |
| (-1.0101, 1) | 0.02278 | (-1.0101, 2) | 0.00263 | (-1.0101, 3) | 0.00771 |
| (-0.8562, 1) | 0.02282 | (-0.8562, 2) | 0.00223 | (-0.8562, 3) | 0.00513 |
| (-0.7346, 1) | 0.01532 | (-0.7346, 2) | 0.00192 | (-0.7346, 3) | 0.00510 |
| (-0.6573, 1) | 0.01058 | (-0.6573, 2) | 0.00181 | (-0.6573, 3) | 0.00321 |
| (-0.6326, 1) | 0.00839 | (-0.6326, 2) | 0.00181 | (-0.6326, 3) | 0.00321 |
| (-0.5574, 1) | 0.00777 | (-0.5574, 2) | 0.00163 | (-0.5574, 3) | 0.00462 |
| (-0.5478, 1) | 0.00752 | (-0.5478, 2) | 0.00171 | (-0.5478, 3) | 0.00303 |
| (-0.5237, 1) | 0.00287 | (-0.5237, 2) | 0.00172 | (-0.5237, 3) | 0.00281 |
| (-0.4982, 1) | 0.00255 | (-0.4982, 2) | 0.00148 | (-0.4982, 3) | 0.00196 |
| (-0.4631, 1) | 0.00195 | (-0.4631, 2) | 0.00166 | (-0.4631, 3) | 0.00400 |
| (-0.4336, 1) | 0.00170 | (-0.4336, 2) | 0.00187 | (-0.4336, 3) | 0.00272 |
| (-0.3709, 1) | 0.00164 | (-0.3709, 2) | 0.00197 | (-0.3709, 3) | 0.00193 |
| (-0.2798, 1) | 0.00154 | (-0.2798, 2) | 0.00218 | (-0.2798, 3) | 0.00259 |
| (-0.1939, 1) | 0.00159 | (-0.1939, 2) | 0.00298 | (-0.1939, 3) | 0.00199 |
| (-0.1918, 1) | 0.00143 | (-0.1918, 2) | 0.00326 | (-0.1918, 3) | 0.00289 |
| (-0.1837, 1) | 0.00205 | (-0.1837, 2) | 0.00556 | (-0.1837, 3) | 0.00211 |
| (-0.1450, 1) | 0.00173 | (-0.1450, 2) | 0.00760 | (-0.1450, 3) | 0.00170 |
| (-0.1430, 1) | 0.00161 | (-0.1430, 2) | 0.00568 | (-0.1430, 3) | 0.00128 |
| (-0.1151, 1) | 0.00423 | (-0.1151, 2) | 0.00220 | (-0.1151, 3) | 0.01389 |
| (-0.0884, 1) | 0.00351 | (-0.0884, 2) | 0.00175 | (-0.0884, 3) | 0.02066 |
| (-0.0560, 1) | 0.00514 | (-0.0560, 2) | 0.00176 | (-0.0560, 3) | 0.01154 |
| (-0.0124, 1) | 0.00380 | (-0.0124, 2) | 0.00214 | (-0.0124, 3) | 0.01050 |
| (0.0000, 1) | 0.00296 | (0.0000, 2) | 0.00284 | (0.0000, 3) | 0.00881 |
| (0.0036, 1) | 0.05584 | (0.0036, 2) | 0.00492 | (0.0036, 3) | 0.00569 |
| (0.0350, 1) | 0.06987 | (0.0350, 2) | 0.00677 | (0.0350, 3) | 0.00355 |
| (0.0879, 1) | 0.03947 | (0.0879, 2) | 0.00821 | (0.0879, 3) | 0.00258 |
| (0.1222, 1) | 0.03186 | (0.1222, 2) | 0.01139 | (0.1222, 3) | 0.00189 |
| (0.2192, 1) | 0.02311 | (0.2192, 2) | 0.01434 | (0.2192, 3) | 0.00215 |
| (0.2223, 1) | 0.01459 | (0.2223, 2) | 0.00523 | (0.2223, 3) | 0.00155 |
| (0.2748, 1) | 0.01459 | (0.2748, 2) | 0.00523 | (0.2748, 3) | 0.00155 |
| (0.3154, 1) | 0.01221 | (0.3154, 2) | 0.00334 | (0.3154, 3) | 0.00129 |
| (0.4112, 1) | 0.01151 | (0.4112, 2) | 0.00458 | (0.4112, 3) | 0.00129 |
| (0.4434, 1) | 0.01184 | (0.4434, 2) | 0.00738 | (0.4434, 3) | 0.00115 |
| (0.4484, 1) | 0.00883 | (0.4484, 2) | 0.01028 | (0.4484, 3) | 0.00105 |
| (0.4677, 1) | 0.00626 | (0.4677, 2) | 0.01178 | (0.4677, 3) | 0.00102 |
| (0.5505, 1) | 0.01574 | (0.5505, 2) | 0.01142 | (0.5505, 3) | 0.00159 |
| (0.5662, 1) | 0.02498 | (0.5662, 2) | 0.02024 | (0.5662, 3) | 0.00177 |
| (0.7154, 1) | 0.01499 | (0.7154, 2) | 0.03736 | (0.7154, 3) | 0.00173 |
| (0.8502, 1) | 0.02341 | (0.8502, 2) | 0.0229 | (0.8502, 3) | 0.00286 |
| (1.1930, 1) | 0.01985 | (1.1930, 2) | 0.00789 | (1.1930, 3) | 0.01769 |

Figure 1: Marginal Posterior Density of *r*

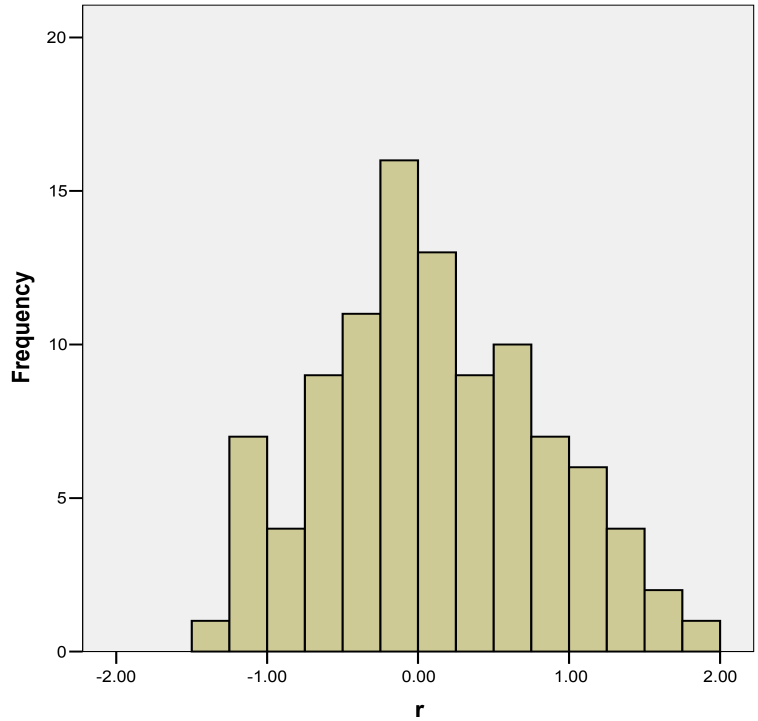


Table 2: Summary of Joint Posterior Densities of (*d, r*) for Chemical Data

| *(d, r)* | Integrated Density | *(d, r)* | Integrated Density | *(d, r)* | Integrated Density |
|---|---|---|---|---|---|
| (1, -0.4) | 0.03665 | (2, -0.4) | 0.01669 | (2, -0.4) | 0.00088 |
| (1, -0.3) | 0.15757 | (2, -0.3) | 0.01319 | (2, -0.3) | 0.00061 |
| (1, -0.2) | 0.15757 | (2, -0.2) | 0.01319 | (2, -0.2) | 0.00061 |
| (1, -0.1) | 0.09583 | (2, -0.1) | 0.00858 | (2, -0.1) | 0.00031 |
| (1, 0.0) | 0.17483 | (2, 0.0) | 0.01402 | (2, 0.0) | 0.00030 |
| (1, 0.1) | 0.04738 | (2, 0.1) | 0.02424 | (2, 0.1) | 0.00026 |
| (1, 0.2) | 0.03386 | (2, 0.2) | 0.02145 | (2, 0.2) | 0.00064 |
| (1, 0.3) | 0.03344 | (2, 0.3) | 0.01352 | (2, 0.3) | 0.00304 |
| (1, 0.4) | 0.09726 | (2, 0.4) | 0.00675 | (2, 0.4) | 0.02733 |

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