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JMASM Algorithms and Code
JMASM20: Exact Permutation Critical Values For The Kruskal-Wallis One-Way ANOVA

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The exhaustive enumeration of all the permutations of the observations in an experiment is the only possible way of truly constructing exact tests of significance. The permutation paradigm requires no distributional assumptions and works well with values that are normal, almost normal and non-normally distributed. The Kruskal-Wallis test does not require the assumptions that the samples are from normal populations and that the samples have the same standard deviation. In this article, the exact permutation distribution of the Kruskal-Wallis test statistic is generated empirically by actually obtaining all the distinct permutations of an experiment. The tables of exact critical values for the Kruskal-Wallis one-way ANOVA are produced.

Keywords: Permutation test, Kruskal-Wallis test, p-value, permutation algorithm, one-way ANOVA.

Introduction

Variation is inherent in nature and errors are made occasionally when inferences are drawn from experiments. The risk in decision making cannot be totally eliminated but it can be controlled if correct statistical procedures are employed. The unconditional permutation approach is a statistical procedure that ensures that the probability of a type I error is exactly α and ensures that the resulting distribution of the test statistic is exact (Agresti, 1992; Good, 2000; Pesarin, 2001).

Scheffe (1943) demonstrated that for a general class of problems, the permutation approach is the only possible method of constructing exact tests of significance. It is asymptotically as powerful as the best parametric test (Hoeffding, 1952). In this article, consideration is given to the exhaustive permutation of the ranks of the observations in a single factor multi-sample experiment to arrive at the exact distribution of the Kruskal-Wallis (K-W) test statistic.

The method of obtaining an exact test of significance originated with Fisher (1935). The essential feature is that all the distinct arrangements of the observations are considered, with the proviso that all permutations are equally likely under the null hypothesis. An exact test on the level of significance α is constructed by choosing a proportion, α, of the permutation as the critical region.

Statisticians have considered for some decades the possibility of generating exact critical values for the common test statistics that are in use today. This has resulted in the development of several ways such as the exact conditional permutation approach (Fisher, 1935; Agresti, 1992), the Monte Carlo approaches such as the Bootstrap (Efron, 1979; Efron and Tibshirani, 1993), the Bayesian approach (Casella & Robert, 2004), and the likelihood approach (Owen, 1988; Barndorff-Nielsen & Hall, 1988).

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The works of Siegel and Castellan (1989), Conover (1999), Headrick (2003), Bagui & Bagui (2004) are contributions to the quest for exact critical values but the distributions are obtained from either simulation or asymptotic approximations of the distribution of the K-W test statistic. For small samples, $n_i \leq 5$, i = 1(1)p in a p-sample experiment, the null distribution of K-W statistic is not known and a chi-square approximation will not be a good approximation, (see Bagui & Bagui (2004)). The consideration given in this article produces the exact distribution of the K-W test statistic for small samples.

Distribution-free analysis of variance

The single-factor ANOVA model for comparing p populations or treatment means assumes that for i = 1, 2, …, p, a random sample of size n is drawn from a normal population with mean μ_i and variance σ^2. The normality assumption is required for the validity of the F test while the validity of the Kruskal-Wallis test for testing equality of the μ_i's (Kruskal & Wallis, 1952) depends only on the amount by which observed values deviate from their means μ_i's (random error) having the same continuous distribution.

Given a multisample experiment with

$$X_i = \left(X_{i1}, X_{i2}, \ldots, X_{in_i}\right)^T, \text{ i = 1(1)p}$$

and

$$\mathbf{X_N} = \left(X_1, X_2, \ldots, X_p\right),$$

where $N = \sum_{i=1}^{p} n_i$, the total number of observations in the data set. Suppose that one ranks all the N observations from 1 (smallest X_{ij}) to N (largest X_{ij}), the permutation test procedure presented in this article, computes an empirical estimate of the cumulative distribution of the test statistic T under the null hypothesis. Let the layout of the ranks of the observations X_{ij} be as follows:

$$R_i = \left(r_{i1}, r_{i2}, \ldots, r_{in_i}\right)^T, \text{ i = 1(1)p.}$$

and

$$\mathbf{R_N} = \left(R_1, R_2, \ldots, R_p\right), \text{ N} = \sum_{i=1}^{p} n_i .$$

Under the null hypothesis, $\mathbf{R_N}$ is composed of N independent and identically distributed random variables and hence conditioned on the observed data set. An exhaustive permutation of the ranks yields

$$M = \frac{N!}{\prod_{i=1}^{P}\left[(n_i)!\right]}$$

permutations of the N ranks of the variates of p subsets of size n_i, i = 1(1)p which are equally likely, each having the conditional probability M^{-1}.

When $H_0 : \mu_1 = \mu_2 = \ldots = \mu_p$ is true, the N observations are assumed to have come from the same distribution, in which case all possible assignments of the rank 1, 2, …, N to the p samples are equally likely and the ranks will be intermingled in these samples. Let R_{ij} denote the rank of the jth observation in the ith treatment X_{ij}. Let $R_{i\cdot}$ and $\overline{R}_{i\cdot}$ denote respectively the total and mean of the ranks in the ith treatment. The K-W test statistic is a measure of the extent to which the $\overline{R}_{i\cdot}$'s deviate from their common expected value $\frac{N+1}{2}$, and H_0 is rejected if the computed value of the statistic indicates too great a discrepancy between observed and expected rank averages. The K-W test statistic is

$$H = \frac{12}{N(N+1)}\sum_{i=1}^{p}\frac{R_{i\bullet}^2}{n_i} - 3(N+1).$$

If H_0 is rejected when $H \geq c$, then c should be chosen so that the test has level α. That is, c should be the upper-tail critical value of the distribution of H when H_0 is true. Under H_0, each possible assignment can be enumerated, the value of H determined for each one, and the null distribution obtained by counting the number of times each value of H occurs. When H_0 is true, the large-sample approximation is applied if p = 3, $n_i \geq 6$, i = 1(1)3 or $p > 3$, $n_i \geq 5$, i = 1(1)p (Devore, 1982; Rohatgi, 1984). H has approximately a chi-squared distribution with p – 1 degrees of freedom. An approximate level α test is given by: Reject H_0 if $H \geq \chi^2_{\alpha,\ p-1}$.

Methodology

The process of obtaining the permutations starts by choosing the test statistic T and the acceptable significance level α. Let π_1, π_2, …, π_n be a set of all distinct permutations of the ranks of the data set in the experiment. The permutation test procedure is as follows:

1. Rank the observations of the experiment as required by the K-W test.

2. Compute the observed value of the K-W test statistic ($H_1 = t_0$).

3. Obtain a distinct permutation π_i, of the ranks in Step 1.

4. Compute the K-W test statistic H_i for permutation π_i in Step 3, that is, H_i = H(π_i).

5. Repeat Steps 3 and 4 for i = 2, 3, …, M.

6. Construct an empirical cumulative distribution for H

$$p_0 = \mathrm{p}(\mathrm{H} \leq \mathrm{H_i}) = \frac{1}{\mathrm{M}}\sum_{\mathrm{i=1}}^{\mathrm{M}} \psi(\mathrm{t_0} - \mathrm{H_i}),$$

where

$$\psi(\cdot) = \begin{cases} 1, & \text{if } \mathrm{t_0} \geq H_{\mathrm{i}} \\ 0, & \text{if } \mathrm{t_0} < H_{\mathrm{i}} \end{cases}.$$

7.Under the empirical distribution, if $p_0 \leq \alpha$, reject the null hypothesis.

The complexity in permutation test lies in obtaining all the distinct permutations of the observations in a given experiment. For example, a four-sample experiment with six variates in each sample requires 2,308,743,493,056 permutations. The frequency distribution is constructed for all the distinct occurrences of the test statistic from which the probability distribution of the test statistic is computed.

The number of permutations of the ranks of a two-sample experiment is

$$\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}, \text{ n = min } (n_1, n_2),$$

see Odiase & Ogbonmwan (2005) for details.

After obtaining the permutations of the ranks of a two sample experiment, the number of ways to permute the ranks of any n_3 of the combined ranks ($n_1 + n_2 + n_3$) of the variates of the three-sample experiment yields

$$\binom{n_1+n_2+n_3}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} = \binom{\sum_{k=1}^{3} n_k}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}$$

A complete enumeration of the distinct permutations of the ranks of a four-sample experiment yields

$$\binom{\sum_{k=1}^{4} n_k}{n_4}\binom{\sum_{k=1}^{3} n_k}{n_3}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} = \prod_{j=3}^{4}\binom{\sum_{k=1}^{j} n_k}{n_j}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i}$$

Continuing in this manner, for p ≥ 3 treatments, the distinct permutations of the ranks of the variates are enumerated through

$$\prod_{j=3}^{p}\binom{\sum_{k=1}^{j} n_k}{n_j}\sum_{i=0}^{n}\binom{n_1}{i}\binom{n_2}{i} = \prod_{j=1}^{p}\binom{\sum_{k=1}^{j} n_k}{n_j}.$$

For the balanced case, $n_1 = n_2 = \ldots = n_p = n$, the number of distinct permutations of the ranks of the variates is $\prod_{j=1}^{p} \binom{jn}{n}$. As an illustration, let

$$R_i = \left(r_{i1}, r_{i2}, \ldots, r_{in_i}\right)^T, \ i = 1(1)p$$

and

$$\mathbf{R_N} = \left(R_1, R_2, \ldots, R_p\right).$$

Consider a three-sample experiment with observations x_{ij}, $n_1 = 3$, $n_2 = n_3 = 2$, that is, $\begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & & \end{pmatrix}$. Assuming there are no ties, the configuration of the ranks of the experiment can be taken as $\begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & & \end{pmatrix}$. An exhaustive permutation of this experiment yields 210 distinct permutations of the ranks.

First obtain the 6 permutations of the ranks of the 4 variates of the last two treatments, that is,

$$\begin{pmatrix} r_{21} & r_{31} \\ r_{22} & r_{32} \end{pmatrix}, \begin{pmatrix} r_{31} & r_{21} \\ r_{22} & r_{32} \end{pmatrix}, \begin{pmatrix} r_{32} & r_{31} \\ r_{22} & r_{21} \end{pmatrix},$$
$$\begin{pmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{pmatrix}, \begin{pmatrix} r_{21} & r_{31} \\ r_{32} & r_{22} \end{pmatrix}, \begin{pmatrix} r_{31} & r_{21} \\ r_{32} & r_{22} \end{pmatrix}.$$

There are 35 ways to permute any 3 ranks of the combined 7 ranks of the variates of the experiment.

$$\begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{12} \\ r_{21} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{21} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{12} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{12} \\ r_{22} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{22} \\ r_{13} \end{pmatrix},$$
$$\begin{pmatrix} r_{22} \\ r_{12} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{12} \\ r_{31} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{31} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{31} \\ r_{12} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{12} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{32} \\ r_{13} \end{pmatrix},$$

$$\begin{pmatrix} r_{32} \\ r_{12} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{21} \\ r_{22} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{12} \\ r_{22} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{22} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{21} \\ r_{31} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{12} \\ r_{31} \end{pmatrix},$$
$$\begin{pmatrix} r_{21} \\ r_{31} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{21} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{12} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{32} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{22} \\ r_{31} \end{pmatrix}, \begin{pmatrix} r_{22} \\ r_{12} \\ r_{31} \end{pmatrix},$$
$$\begin{pmatrix} r_{22} \\ r_{31} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{22} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{22} \\ r_{12} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{22} \\ r_{32} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{11} \\ r_{31} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{31} \\ r_{12} \\ r_{32} \end{pmatrix},$$
$$\begin{pmatrix} r_{31} \\ r_{32} \\ r_{13} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{22} \\ r_{31} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{22} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{21} \\ r_{31} \\ r_{32} \end{pmatrix}, \begin{pmatrix} r_{22} \\ r_{31} \\ r_{32} \end{pmatrix}.$$

Each of the 35 ways will combine with the 6 permutations of the remaining 4 ranks of the variates making up the last two treatments in any configuration of the experiment, that is,

$$\binom{7}{3} \sum_{i=0}^{2} \binom{2}{i} \binom{2}{i}.$$

Consider the set of all these 210 permutations, for each one of them, compute the test statistic of interest and hence calculate the probability of the different values of the test statistic based on the number of times each is occurring. When ties occur in the data set, the tied observations are usually assigned the mean of the ranks they would have been assigned if they were distinct. Ties do not pose any problem to the permutation test presented in this article. Assuming no ties, the experiment just presented will have ranks {1, 2, 3, 4, 5, 6, 7} represented as $\begin{pmatrix} 1 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & & \end{pmatrix}$ and the distinct permutations of these ranks lead to the remaining 209 permutations.

Permutation algorithms

Considering the associated complexity in a complete enumeration of the distinct permutations necessary for the compilation of the distribution of the K-W test statistic, computer algorithms for an exhaustive enumeration are now presented.

The first step in developing permutation algorithm is to formulate an initial configuration of the ranks of the variates of an experiment by taking the trivial configuration given below as:

$$\begin{pmatrix} 1 & n_1+1 & n_1+n_2+1 & \cdots & \sum_{i=1}^{p-1} n_i+1 \\ 2 & : & : & : & \\ 3 & : & : & : & \\ 4 & : & : & : & \\ : & : & : & : & \\ : & & & & \\ n_1 & n_1+n_2 & n_1+n_2+n_3 & \cdots & \sum_{i=1}^{p} n_i \end{pmatrix}$$

Algorithm (PERMUTATION) of Odiase & Ogbonmwan (2005) can handle the permutation of the ranks of the variates in a two-sample experiment. Algorithm 1 in this article generates the distinct permutations of the ranks of the variates of a three-sample experiment and relies on the permutation of the ranks of the variates in a two-sample experiment.

Algorithm 2 calls Algorithm 1 and then generates the distinct permutations of the ranks of the variates of a four-sample experiment. Algorithms 1 and 2 can be extended to take care of the sample sizes under consideration.

Results

Critical values for the K-W test statistic

The algorithms were implemented in Intel Visual Fortran. Figures 1 – 10 show the small sample distribution of the K-W test statistic for different sample sizes for 3 and 4 samples. The resulting tables of exact critical values as obtained from the exact permutation distribution of the K-W test statistic are presented in Tables 1 and 2.

Conclusion

Figures 1 and 2 reveal the fact that the chi squared distribution, which is the large sample approximation of the K-W test statistic, will poorly approximate the exact distribution of the K-W test statistic for very small sample sizes. As sample sizes increase, the shape of the chi squared distribution begins to emerge as seen in Figures 3 – 10.

The critical values for a test statistic are usually determined by cutting off the most extreme 100α% of the theoretical frequency distribution of the test statistic, where α is the level of significance, see Siegel and Castellan (1989). The critical values of the K-W test statistic contained in Tables 1 and 2 are obtained from the enumeration of all the distinct permutations of the ranks of the variates in an experiment. These critical values are exact and therefore ensures that the probability of a type I error in decisions arising from the use of the K-W test is exactly α.

Algorithm 1 (3 samples)

```
for II10 ← 1, P do
for JJ10 ← 1, K(II10) do
Y(JJ10, II10) ← Z1(JJ10, II10)
Y1(JJ10, II10) ← Z1(JJ10, II10)
end for
end for
Obtain a distinct permutation of ranks in the last two samples
        Exchange one rank
for JJ1 ← 1, K(2) do
TEMPA ← Y1(JJ1, P - 2)
for II1 ← P-1, P do
for 110 JJ2 ← 1, K(II1) do
Y1(JJ1, P - 2) ← Y1(JJ2, II1)
Y1(JJ2, II1) ← TEMPA
Obtain a distinct permutation of ranks in the last two samples
end for
end for
end for
        Exchange two ranks
for II ← 1, K(2) – 1 do
TEMPA1 ← Y1(II, P – 2)
for JJ ← II + 1, K(2) do
TEMPA2 ← Y1(JJ, P – 2)
for LL ← P – 1, P do
for II1 ← 1, K(LL) do
for LL1 ← LL, P do
if LL ← LL1 then
TT ← II1 + 1
else
TT ← 1
end if
for JJ1 ← TT, K(LL1) do
Y1(II, P - 2) ← Y1(II1, LL)
Y1(II1, LL) ← TEMPA1
Y1(JJ, P - 2) ← Y1(JJ1, LL1)
Y1(JJ1, LL1) ← TEMPA2
Obtain a distinct permutation of ranks in the last two samples
end for
end for
end for
end for
end for
end for
…
        Restore original ranks
for II0 ← 1, P do
for JJ0 ← 1, K(II0) do
Z1(JJ0, II0) ← Z(JJ0, II0)
end for
end for
```

Algorithm 2 (4 samples)

Generate ranks

1: KK ← 0
2: for I ← 1, P do
3: KK ← KK + K(I-1)
4: for J ← 1, K(I) do
5: Z(J, I) ← KK + J
6: Z1(J, I) ← Z(J, I)
7: Y(J, I) ← Z(J, I)
8: Y1(J, I) ← Y(J, I)
9: X(J, I) ← Z(J, I)
10: X1(J, I) ← X(J, I)
11: end for
12: end for
13: call Algorithm 1
14: for R2 ← 1, P do
15: for R3 ← 1, K do
16: Y(R3,R2) ← Z1(R3,R2)
17: Y1(R3,R2) ← Z1(R3,R2)
18: end for
19: end for

Adjust Algorithm 1 as follows and insert here:

20: Change all the loop variables
21: Change the variable names TEMPA, TEMPA1, TEMPA2, TT, TT1, …
22: Replace Steps 10, 22, … with [Variable name ← P – 2, P]
23: Replace all [P – 2] with [P – 3]
24: Replace [Y1] with [Z1]
25: Replace [Obtain a distinct permutation of ranks in the last two samples] with [Call Algorithm 1]
26: Construct the empirical distribution of H
27: Sort values of H in ascending order of magnitude
28: Construct the CDF for H

Figures 1 – 10: Distribution of Kruskal-Wallis test statistic for different sample sizes

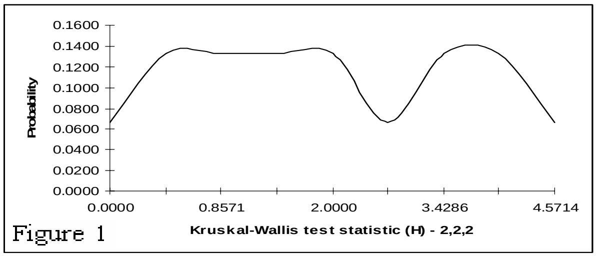

Figure 1

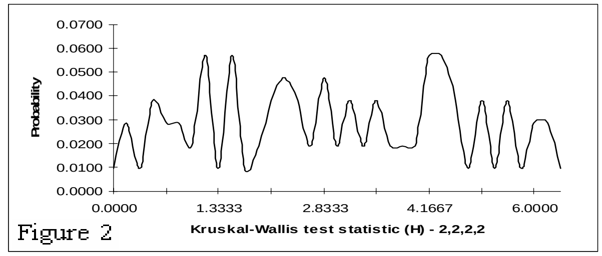

Figure 2

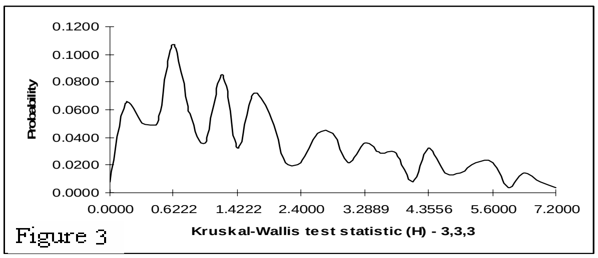

Figure 3

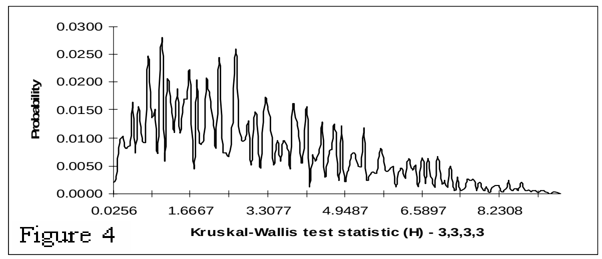

Figure 4

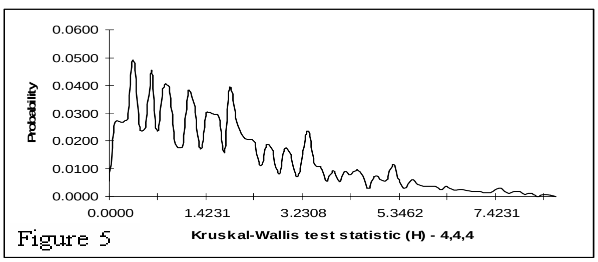

Figure 5

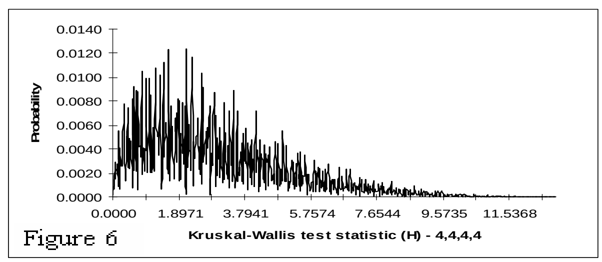

Figure 6

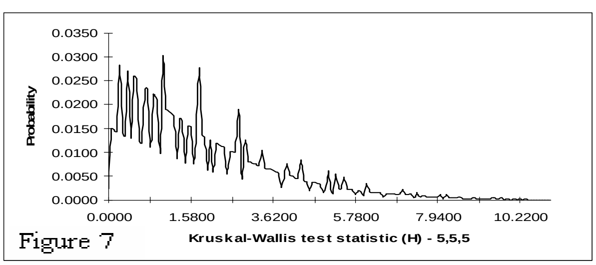

Figure 7

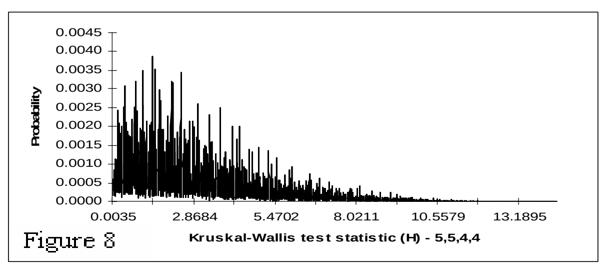

Figure 8

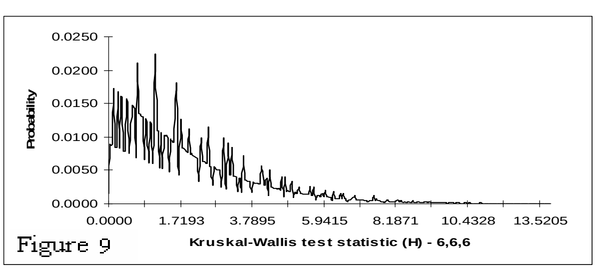

Figure 9

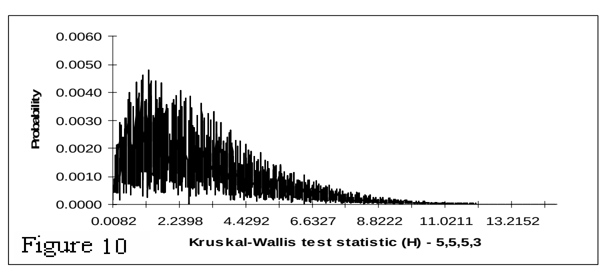

Figure 10

Table 1: Critical values for Kruskal-Wallis test statistic (3 samples)

Sample Size	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
2,2,1	3.6000						
2,2,2	3.7143	4.5714					
3,2,1	4.2857						
3,2,2	4.4643	4.5000	5.3571				
3,3,1	4.5714						
3,3,2	4.5556	5.1389	5.5556	6.2500			
3,3,3	4.6222	5.6000	5.9556	6.4889			
4,2,1	4.0179	4.8214					
4,2,2	4.4583	5.1250	5.3333	6.0000			
4,3,1	3.8889	5.4000	5.3889				
4,3,2	4.4444	5.4000	5.8000	6.3000	6.4444	7.0000	
4,3,3	4.7000	5.7273	6.0182	6.7455	7.0000	7.3182	8.0182
4,4,1	4.0667	4.8667	6.0000	6.1667			
4,4,2	4.4455	5.2364	6.0818	6.8727	7.0364	7.2818	7.8545
4,4,3	4.4773	5.5758	6.3864	7.1364	7.4773	7.8485	8.3258
4,4,4	4.5000	5.6538	6.5769	7.5385	7.7308	8.1154	8.7692
5,1,1	3.8571						
5,2,1	4.0500	4.4500	5.2500				
5,2,2	4.2933	5.0400	5.6933	6.1333			
5,3,1	3.8400	4.8711	5.7600	6.4000			
5,3,2	4.4945	5.1055	5.9491	6.8218	6.9491	7.1818	7.6364
5,3,3	4.4121	5.5152	6.3030	6.9818	7.5152	7.8788	8.2424
5,4,1	3.9600	4.8600	5.7764	6.8400	6.9545		
5,4,2	4.5182	5.2682	6.0409	7.1182	7.5682	7.8136	8.1136
5,4,3	4.5231	5.6308	6.3949	7.3949	7.9064	8.2564	8.6256
5,4,4	4.6187	5.6176	6.5967	7.7440	8.1560	8.7033	9.1286
5,5,1	4.0364	4.9091	5.7818	6.8364	7.7455	7.7455	8.1818
5,5,2	4.5077	5.2462	6.2308	7.2692	8.0769	8.2923	8.6846
5,5,3	4.5363	5.6264	6.4879	7.5429	8.2637	8.7912	9.2835
5,5,4	4.5200	5.6429	6.6714	7.7914	8.4629	9.0257	9.5057
5,5,5	4.5000	5.6600	6.7200	7.9800	8.7200	9.3800	9.9200
6,1,1	4.0833						
6,2,1	3.8222	4.6222	5.4000				
6,2,2	4.4364	5.0182	5.5273	6.5455	6.6545		
6,3,1	3.8182	4.8545	5.8545	6.5818			
6,3,2	4.5455	5.2273	6.0606	6.7273	7.5000	7.5758	8.1818
6,3,3	4.5385	5.5513	6.3846	7.1923	7.6154	8.3205	8.6282
6,4,1	3.8636	4.9242	5.6970	7.0833	7.5000	7.9545	
6,4,2	4.4359	5.2628	6.1090	7.2115	7.8205	8.3077	8.6667
6,4,3	4.5989	5.6044	6.5000	7.4670	8.0275	8.6538	9.1703
6,4,4	4.5238	5.6667	6.5952	7.7238	8.3238	8.8810	9.6286
6,5,1	3.9205	4.8359	5.8615	6.9974	8.0667	8.4359	8.8846
6,5,2	4.4747	5.3187	6.1890	7.2989	8.1868	8.7473	9.1890
6,5,3	4.4971	5.6000	6.6210	7.5600	8.2971	9.0286	9.6686
6,5,4	4.5000	5.6558	6.7358	7.8958	8.6400	9.2933	9.9600
6,5,5	4.5294	5.6985	6.7809	8.0118	8.8353	9.5809	10.2706
6,6,1	3.9780	4.8571	5.9121	7.0659	7.9341	8.9231	9.3077
6,6,2	4.4190	5.3524	6.1714	7.4095	8.1524	8.9333	9.6762
6,6,3	4.5250	5.6000	6.6833	7.6833	8.4167	9.2250	10.1250
6,6,4	4.5184	5.7206	6.7831	7.9890	8.7206	9.4118	10.3419
6,6,5	4.5412	5.7516	6.8379	8.1190	8.9817	9.7242	10.5242
6,6,6	4.5380	5.7193	6.8772	8.1871	9.0877	9.8713	10.8421

Table 2: Critical values for Kruskal-Wallis test statistic (4 samples)

Sample Size	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
2,2,1,1	4.7143						
2,2,2,1	5.0357	5.3571	5.6786				
2,2,2,2	5.5000	6.0000	6.1667				
3,2,1,1	4.8929	5.4643					
3,2,2,1	5.3889	5.8056	6.0556	6.5000			
3,2,2,2	5.6444	6.2444	6.6444	7.0000	7.1333	7.5333	
3,3,1,1	5.2222	5.8889					
3,3,2,1	5.6222	6.1556	6.5111	7.0444	7.2000	7.4000	
3,3,2,2	5.7273	6.4727	7.0000	7.6364	7.7273	8.0000	8.1273
3,3,3,1	5.5818	6.5273	6.8909	7.3273	7.7636	8.0545	8.3455
3,3,3,2	5.8182	6.6818	7.4697	7.9545	8.3182	8.5606	8.9242
3,3,3,3	5.9744	6.8974	7.6154	8.4359	8.7436	9.1538	9.4615
4,2,1,1	5.2083	5.4583	6.0833				
4,2,2,1	5.5000	6.0000	6.5000	6.8000			
4,2,2,2	5.6727	6.4364	6.9818	7.3091	7.8545	7.9636	8.2909
4,3,1,1	4.9778	6.0444	6.5667	6.7111			
4,3,2,1	5.5727	6.3000	6.9091	7.3636	7.7273	7.8909	8.1818
4,3,2,2	5.7121	6.6136	7.3182	7.8485	8.2500	8.5909	8.8939
4,3,3,1	5.6667	6.5379	7.2727	7.7500	8.1212	8.3561	8.8409
4,3,3,2	5.8590	6.7821	7.5577	8.3205	8.7179	9.0577	9.4038
4,3,3,3	6.0000	6.9670	7.7582	8.6538	9.2308	9.5769	10.0000
4,4,1,1	5.1273	5.8636	6.9273	7.5000			
4,4,2,1	5.5455	6.3636	7.1364	7.8864	8.2273	8.5682	8.7045
4,4,2,2	5.7692	6.6923	7.5192	8.3077	8.6731	9.0577	9.4423
4,4,3,1	5.6603	6.6154	7.4808	8.2179	8.5769	8.8654	9.2949
4,4,3,2	5.8901	6.8626	7.7363	8.6099	9.1538	9.4835	9.9121
4,4,3,3	6.0048	7.0333	7.9238	8.8667	9.4905	9.9667	10.4619
4,4,4,1	5.6374	6.7088	7.6319	8.5714	8.9505	9.2473	9.7253
4,4,4,2	5.9000	6.9429	7.8857	8.8571	9.4714	9.9143	10.4000
4,4,4,3	6.0292	7.1292	8.0542	9.0667	9.7167	10.3417	10.9000
4,4,4,4	6.0662	7.2132	8.2059	9.2647	9.9485	10.5662	11.3382
5,2,1,1	5.1067	5.7600	6.0667	6.6000			
5,2,2,1	5.5309	6.0327	6.5782	7.2000	7.4727	7.8000	
5,2,2,2	5.6182	6.5273	7.1545	7.6636	8.0182	8.3818	8.6818
5,3,1,1	5.1309	6.0036	6.8764	7.1673	7.4000		
5,3,2,1	5.5030	6.3303	7.0939	7.7455	8.1818	8.2909	8.7273
5,3,2,2	5.7538	6.6564	7.4641	8.1949	8.6256	8.9333	9.4231
5,3,3,1	5.6564	6.6000	7.4205	8.1179	8.5282	8.8974	9.2564
5,3,3,2	5.8571	6.8220	7.6505	8.5912	9.0571	9.4176	9.8549
5,3,3,3	5.9981	7.0114	7.8267	8.8400	9.4571	9.9067	10.4095
5,4,1,1	5.2000	6.0182	6.8000	7.8591	8.2000	8.2955	8.6364
5,4,2,1	5.5615	6.4077	7.2115	8.1692	8.5731	8.9423	9.3231
5,4,2,2	5.7725	6.7220	7.5989	8.4692	9.0495	9.4451	9.8604
5,4,3,1	5.6396	6.6813	7.5253	8.3989	8.9802	9.3484	9.7934
5,4,3,2	5.8933	6.9171	7.7933	8.8000	9.3933	9.8733	10.3543
5,4,3,3	6.0292	7.0892	7.9892	9.0292	9.6958	10.2892	10.8558
5,4,4,1	5.6686	6.7429	7.6743	8.7171	9.3029	9.6971	10.2114
5,4,4,2	5.9400	6.9850	7.9475	9.0000	9.6625	10.2525	10.7875
5,4,4,3	6.0346	7.1669	8.1346	9.2118	9.9397	10.5574	11.2963
5,4,4,4	6.0608	7.2569	8.2725	9.3902	10.1373	10.8020	11.5882
5,5,1,1	5.0923	6.0154	6.8769	8.0769	8.6000	8.9077	9.0923

Table 2: Continued

Sample Size	$H_{0.9000}$	$H_{0.9500}$	$H_{0.9750}$	$H_{0.9900}$	$H_{0.9950}$	$H_{0.9975}$	$H_{0.9990}$
5,5,2,1	5.5648	6.5341	7.2725	8.3077	9.0198	9.4352	9.7582
5,5,2,2	5.7943	6.7714	7.6457	8.6286	9.2914	9.8800	10.3429
5,5,3,1	5.6476	6.7371	7.6286	8.5962	9.2743	9.7619	10.2191
5,5,3,2	5.9150	6.9417	7.8750	8.9467	9.6350	10.1667	10.8200
5,5,3,3	6.0118	7.1176	8.0588	9.1882	9.9176	10.5529	11.2353
5,5,4,1	5.6625	6.7800	7.7625	8.8625	9.5500	10.1025	10.5900
5,5,4,2	5.9338	7.0279	8.0162	9.1500	9.8868	10.5154	11.1904
5,5,4,3	6.0523	7.2157	8.2092	9.3562	10.1307	10.7895	11.5739
5,5,4,4	6.0684	7.2895	8.3421	9.5351	10.3281	11.0228	11.8439
5,5,5,1	5.6824	6.8294	7.8000	9.0176	9.7588	10.3941	10.9588
5,5,5,2	5.9451	7.0745	8.0941	9.2863	10.0980	10.7451	11.5137
5,5,5,3	6.0433	7.2456	8.2889	9.4959	10.3193	10.9930	11.8257

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