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# Estimation of Process Variances in Robust Parameter Designs

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The modeling of variation through interactions is appealing in crossed array design as it leads to greater robustness to certain type of model misspecification. As an alternative to signal-to-noise analysis, a new, systematic method based on Taguchi type crossed array design is given. It is shown in this article that when fractional factorial design is used for the outer array, the crossed array design is not robust to the presence of noise-noise interactions and a method of rectifying the problem is suggested.

Keywords: Inner and outer arrays, interactions, off-line quality control, orthogonal polynomial, PerMIA, Taguchi experiment.

## Introduction

Robust design has been widely used in industry to improve productivity and achieve higher quality at a lower cost. The main idea in robust design is to develop product and process designs that can deliver at a minimal cost units of target performance which are usable or functional with maintained quality under all intended operating conditions.

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Thus, one major approach in robust design is to reduce variation in the quality characteristic without actually eliminating the causes of variation (the noise factors). Instead of replacing some components with more expensive ones to achieve smaller variation from target, robust design methodology seeks combinations of levels of factors affecting the quality characteristics that are least sensitive to environmental changes in production or operating conditions. This adjustment to the optimal levels are usually less expensive and are achieved through parameter design.

In parameter design, techniques of design of experiments are widely used to obtain data for a number of experimental runs corresponding to different combinations of the factors. An analysis of the resulting data is performed to approximate the optimal combination yielding the smallest variation from the target. In these regards, Taguchi-type experiments consisting of crossed arrays are sometimes performed, and the experimental data are analyzed using signal to noise ratio as a performance measure. A factor affecting response or product characteristic can be classified as a control factor or a noise factor (internal or external). Control factors are factors the levels or values of which are controllable during production. In contrast, the levels of the noise factors are expensive to control in production or uncontrollable during use in the lifetime of the product. However, for the purpose of assessing their effects on the quality characteristics, the levels of the noise factors may also be controlled in the experimental runs in parameter design. In crossed array designs, each treatment combination of the control factors considered appears with every member in a set of treatment combinations of noise factors.

Taguchi's crossed array design and the signal-to-noise ratio analysis were criticized in the literature (Box, 1988). Some major difficulties in Taguchi's approach are summarized in Barreau et al. (1999). Crossed array design generally calls for a larger number of experimental runs which may be deemed unnecessary when some of the interactions may be safely assumed to be zero (Shoemaker et al., 1991). Furthermore, the use of signal-to-noise ratio may not always be appropriate as a performance measure to be minimized (Box, 1988), and modeling directly the signal to noise ration as the response in ANOVA is generally not intuitive and problematic. As an alternative design, the use of combined arrays has been suggested in the literature (Welch et al., 1990; Shoemaker et al., 1991).

In combined array design, both the control and noise factors are integrated into the same array, resulting in less number of experimental runs. The resulting data are then analyzed differently, with the control factors affecting variance through their interactions with the noise factors (O'Donnell and Vining, 1997; Myers, 1997). Engel and Huele (1996) used a generalized linear modeling approach to analyze combined array designs.

It is interesting to note that similar approach of modeling through interactions between the control and the noise factors is in fact more appropriate for crossed array designs (Barreau, et al., 1999). Despite some of its major drawbacks, Taguchi's approach is still embraced by many practitioners, largely because of its conceptual simplicity and easier implementation that requires less sophisticated analytical tools. Furthermore, the combined array methodology, though more economical, is less robust than the crossed array design to model misspecification especially when certain significant interactions among control factors are accidentally omitted in the design and analysis.

The number of experimental runs required in a crossed array design can be substantially reduced by employing fractional factorial designs for the inner (involving control factors) and outer array (involving noise factors). Barreau, et al. (1999) examined the role of interactions between control and noise factors in a Taguchi type experiment. These approaches of design and analysis have the advantages of being more economical, and yet are capable of retaining the benefits of having crossed inner and outer arrays.

The use of interaction analysis also throws light on how the noise variables affect the response, and provides a more natural analysis than a direct modeling of the signal-to-noise ratio as a response variable. Design of resolution III can be used for the inner array without any adverse effects on the study of variation or performance measure even if some interactions exist between control factors. However, complication arises when two factor interactions exist between noise factors. Such interactions do not appear in the true unknown objective function to be minimized for finding optimal levels, but it is shown in this paper that they can seriously bias the estimation of this objective function.

It is suggested that this potential bias be corrected based on a small confirmatory experiment. It is also proposed to use orthogonal polynomials in the analysis to facilitate the identification of adjustment variables, variables that only affect variation through the mean function. It is well known that the use of adjustment variables greatly simplifies the process of minimizing variation while having the mean on target. Furthermore, the use of orthogonal polynomials when some variables are quantitative allows one to better relate the analysis to response surface methodology and to obtain interpolated values for improved results in variance minimization.

## Methodology

In this section, an outline of a systematic approach for analyzing data from a crossed array design is given. The details are best explained by a practical example, which will be left to the next section. Let $y$ be the response variable representing a certain product characteristic. Suppose there are $c$ control variables each with $k_c$ levels, and $n$ noise variables each has $k_n$ levels. For the ease of discussion, all the control and noise variables are assumed to be quantitative, but the necessary modifications when there are both quantitative and qualitative variables will be demonstrated with a real example in the next section.

Suppose that there are $N_c$ treatment combinations in the inner array, which is an orthogonal resolution III main effect plan. Similarly, there are $N_n$ treatment combinations in the outer array, which is an orthogonal resolution III main effect plan. Assume all interactions involving three or more factors (both control and noise factors) are non-significant. For the $i^{th}$ control factor $x_i$, there are $k_c$ levels corresponding to $k_c$ numeric coded values. Denote the set of the $k_c$ numeric coded values by $W$. Let $u_1(x),...,u_{k_c-1}(x)$ be orthogonal polynomials where $u_j(x)$ is a polynomial of degree $j$ such that $\sum_{x_i \in W} u_j(x_i) = 0$, $\sum_{x_i \in W} u_j(x_i) u_{j'}(x_i) = 0$, for all $j$ and $j \neq j'$.

The $n$ noise factors $z_1,...,z_n$ are random variables assumed to be independent and, without loss of generality, to have mean 0 and standard deviation 1. Thus if all the two factor control-control and noise-noise interactions are suppressed, a linear model for the response $y$ conditional also on $z_1,...,z_n$ can be formulated as:

$$
\begin{aligned}
y &= f(x_1,...,x_c, z_1,...,z_c) + e \\
&= \mu + \sum_{i=1}^{c} \alpha_i^T u(x_i) + \sum_{i'=1}^{n} \gamma_{i'} z_{i'} \\
&\quad + \sum_{i'=1}^{n} \sum_{i=1}^{c} \beta_{i'i}^T u(x_i) z_{i'} + e,
\end{aligned}
$$

where $\alpha_i$ is a $k_c \times 1$ vector, $\gamma_{i'}$ is a scalar, $\beta_{i'i}$ is a $k_c \times 1$ vector of unknown coefficients, and $u(x) = (u_1(x),...,u_{k_c-1}(x))^T$. Here the error term $e$ has mean 0 and constant variance $\sigma_e^2$. Thus for given $x_1,...,x_c$, treating $z_1,...,z_n$ as random, the variance of $y$ is therefore

$$
\sigma^2(x_1,...,x_c) = \sum_{i'=1}^{n} V_{i'}^2 + \sigma_e^2 \tag{1}
$$

where

$$
V_{i'} = (\gamma_{i'} + \sum_{i=1}^{c} \beta_{ii'}^T u(x_i)).
$$

Thus to estimate the unknown $\alpha_i$, $\gamma_{i'}$ and $\beta_{ii'}$ can be estimated by the least squares estimators $\hat{\alpha}_i$, $\hat{\gamma}_{i'}$ and $\hat{\beta}_{ii'}$ using data collected from a crossed array design where the outer array is an orthogonal Resolution III main effect plan with each noise factors set at two levels -1 and +1 (corresponding to $\pm 1$ standard deviation). The optimal solution for achieving smallest variation is obtained by minimizing the objective function (1). To obtain an approximate solution for smallest variation, one can minimize with respect to $x_1,...,x_c$, the estimated objective function:

$$
\begin{aligned}
\hat{h}(x_1,...,x_c) &= \sum_{i'=1}^{n} \hat{V}_{i'}^2 \\
&= (\hat{\gamma}_{i'} + \sum_{i=1}^{c} \hat{\beta}_{ii'}^T u(x_i)).
\end{aligned}
$$

How is this variance minimization procedure affected if some or all of the two factor noise-noise interactions are in fact non-negligible? It is not difficult to see that in such cases, for given $x_1,...,x_c$ the variance of $y$ differs from (2.1) by a positive term that does not involve $x_1,...,x_c$. Thus one might want to minimize the same function $\hat{h}(x_1,...,x_c)$. However, because the main effects in the outer array are aliased with certain two factor noise-noise interactions, the estimator $\hat{\gamma}_{i'}$ no longer estimates $\gamma_{i'}$ alone, but the sum of $\gamma_{i'}$ and the effects of the two factor noise-noise interactions in the same alias set. Thus it is not appropriate to minimize directly $\hat{h}(x_1,...,x_c)$ without adjustment. It is proposed here that a follow up $2^n$ factorial (or a faction of $2^n$) experiment of the *n* noise factors be performed to estimate all the two factor noise-noise interactions independently. The estimates obtained are used to correct for bias of the estimated coefficients in the function $\hat{h}(x_1,...,x_c)$. This procedure will be illustrated with the example in next Section.

If for a control factor $x_i$, the vector $\beta_{ii'} = 0$ for all $i' = 1,...,n$, then $x_i$ does not appear in the objective function and the optimal solution does not depend on $x_i$. This kind of control factor is called adjustment factor. Their existence greatly simplifies the procedure of minimizing variance while the mean is made on target, as the variation can first be minimized using the non-adjustment control variables, and then the values of the adjustment variable is set to give the targeted mean value. The identification of adjustment variables can be done by examining the magnitudes of the two factor control-noise interactions using graphical technique such as the half normal probability plot (Box, 1988).

With the present formulation through orthogonal polynomials, one can also examine the sum of squares of the orthogonal contrasts corresponding to these interactions. It is also suggested that the effects of the interactions of each control variable with the noise variables on the results of variance minimization be studied for this purpose.

These approaches will also be illustrated with an example in the next section. If the constant variance in the assumed model is violated, one might have to transform the response variable to attain approximate homogeneity of variances. As explained in Box (1988), the minimization of variance in the transformed metric can be seen as approximately minimizing a performance measure independent of the mean (PerMIA).

## Results

The new methods are outlined to re-analyze the data from a crossed array design, studied by Vandenbrande (2000), using signal-to-noise ratio. The data involve a car body paint spray process in which it is required to spray paint on a plate evenly to a desirable width. Although the surface has to be adequately covered, overspray would result in unnecessarily higher cost in paint as well as causing quality problems on other part of the car body. The response measurement *y* is the width of the paint pattern.

There are four control variables: type of gun $x_1$ (a qualitative variable with values 1, 2 and 3 representing three different guns), paint flow $x_2$, paint airflow $x_3$ and atomizing airflow $x_4$. The last three variables are quantitative and each is set at 3 levels (low, medium and high) which we take to be equally spaced and coded as -1, 0, +1. There are three noise factors: color $z_1$, input air pressure $z_2$, and paint viscosity $z_3$. Each of the three noise factors has two levels: -1 and +1. A Taguchi type of crossed array experiment is performed using the $L_9$ and $L_4$ orthogonal arrays for, respectively, the inner and outer arrays, as displayed in Table 1.

There are therefore 36 experimental runs, determined by crossing the 4 treatment combinations in the outer array with each of the 9 treatment combinations in the inner array. The observed data are given in (Vandenbrande, 1998, 1999).

The first step in the analysis involves defining indicator variables for any qualitative control variables and finding orthogonal polynomials for the quantitative control variables. Here, only type of gun is qualitative and we define $x_{11}$ to be equal to 1 for type 1 and 0 otherwise, $x_{12}$ equal to 1 for type 2 and 0 otherwise. The linear and quadratic orthogonal polynomials used for $x_2$, $x_3$ and $x_4$ are $u_1(x)=x$, $u_2(x)=2-3x^2$.

The coefficients of the linear contrast corresponding to x =-1, 0, +1, are $u_1(x)$=-1, 0, +1, and that of the quadratic contrast corresponding to x =-1, 0 ,+1, are $u_2(x)$=-1,2,-1.

Table 1. Inner and outer array layout

| Inner Array | | | |
|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |
| 1 | -1 | -1 | -1 |
| 2 | -1 | 0 | 1 |
| 2 | 0 | 1 | -1 |
| 2 | 1 | -1 | 0 |
| 3 | -1 | 1 | 0 |
| 3 | 0 | -1 | 1 |
| 3 | 1 | 0 | -1 |
| Outer array | | | |
| $z_1$ | -1 | 1 | 1 |
| $z_2$ | -1 | 1 | -1 |
| $z_3$ | -1 | -1 | 1 |

Our model, suppressing two factor control-control, noise-noise as well as higher order interactions is therefore:

$$y=\mu+(\alpha_{11}x_{11}+\alpha_{12}x_{12}) +\sum_{i=2}^{4}(\alpha_{i1}u_1(x_i)+\alpha_{i2}u_2(x_i))$$

$$+\sum_{i'=1}^{3}\gamma_{i'}z_{i'}+\sum_{i'=1}^{3}(\beta_{1i'1}x_{11}z_{i'}+\beta_{1i'2}x_{12}z_{i'})$$

$$+\sum_{i=2}^{4}\sum_{i'=1}^{3}(\beta_{ii'1}u_1(x_i)z_{i'}+\beta_{ii'2}u_2(x_i)z_{i'})+e \qquad (2)$$

The least squares estimates of $\alpha_{ij}$, $\gamma_{i'}$ and $\beta_{ii'j}$, $i=1,\ldots 4$, $i'=1,2,3$, $j=1,2$, and the broken down sum of squares for each degree of freedom are given in Table 2.

In the second step, one may proceed if desirable to identify adjustment variables which do not interact with any of the noise variables. Specifically, we look for quantitative adjustment variables as these variables can be used to make continuous adjustment of the mean to the target value. By looking at the sum of squares (SS) corresponding to the orthogonal contrasts $u(x_i)z_{i'}$, it is seen that the control factor paint flow $x_2$ has small SS of interactions with all three noise factors. This suggests that using $x_2$ as an adjustment variable and drop it from the variance function (1). The effect of excluding $x_2$ from the study of variance will be examined later.

In step 3, minimize the estimated objective function $\hat{h}$ defined in Section 2, or equivalently, the estimated variance function of *y* given $x_1, x_3$ and $x_4$. In principle, the mean and variance (treating $z_1, z_2, z_3$ as random along with *e*) of *y* given $x_1, x_2, x_3$ and $x_4$ can be estimated based on the analytical expression for the mean and variance derived from (3.1). However, an equivalent but more intuitive and easily programmable procedure is to calculate the mean and variance based on generated pseudo observations.

To generate these pseudo observations, we first set a new variable $z_4$ to two levels at -1 and +1 as other noise factors. Also let $\hat{\gamma}_4=\sqrt{MSE}$. The pseudo observations are generated using (3.1) with the least square estimates replacing the unknown coefficients and also the error *e* by $\hat{\gamma}_4 z_4$. Here, the $z_i$, i=1,…, 4 can be -1 or +1, yielding a total of $2^4$ pseudo observations. The conditional mean and variance of *y* given $x_1, x_2, x_3$ and $x_4$ can then be estimated by the usual mean and variance of the pseudo observations (with $2^4$ as the divisor in calculating variance). This procedure is justified as it is equivalent to using Gaussian Quadrature to evaluate the first two moments, and the two point Gaussian Quadrature is known to yield exact integral for polynomial of degree 3.

The added advantage of using the approach of pseudo observations is that it can be readily applied to evaluate any expected loss function *L(y),* not just the quadratic loss function, by calculating the mean loss at the values of the pseudo observations. This can be particularly helpful if an analytical expression for the expected loss is difficult to obtain.

Table 2. Estimates and sum of squares:

$\hat{y}$ = 39.6 + 1.02 $x_{11}$ - 2.57 $x_{12}$ + 3.84 $u_1(x_2)$+0.604 $u_2(x_2)$ + 3.64 $u_1(x_3)$-1.69 $u_2(x_3)$
-2.99 $u_1(x_4)$ +1.37 $u_2(x_4)$ -3.63 $z_1$+ 0.308 $z_2$ - 0.0417 $z_3$ + 3.48 $x_{11}$ $z_1$ + 2.58 $x_{12}$ $z_1$
+ 0.550 $x_{11}$ $z_2$ - 0.0500 $x_{12}$ $z_2$ - 1.15 $x_{11}$ $z_3$ + 0.233 $x_{12}$ $z_3$ - 0.0125 $u_1(x_2)$ $z_1$
+ 0.0931 $u_2(x_2)$ $z_1$+0.438 $u_1(x_2)z_2$ +0.121 $u_2(x_2)$ $z_2$-0.221 $u_1(x_2)z_3$+0.290 $u_2(x_2)$ $z_3$
-1.46 $u_1(x_3)$ $z_1$-0.253 $u_2(x_3)$ $z_1$-0.550 $u_1(x_3)$ $z_2$+0.717 $u_2(x_3)$ $z_2$ 0.783 $u_1(x_3)$ $z_3$
- 0.889 $u_2(x_3)$ $z_3$+1.73 $u_1(x_4)$ $z_1$-0.519 $u_2(x_4)$ $z_1$-1.08 $u_1(x_4)$ $z_2$-0.717 $u_2(x_4)$ $z_2$
+ 0.850 $u_1(x_4)$ $z_3$ + 0.369 $u_2(x_4)$ $z_3$.

| Control factor $x_2$ | | Control factor $x_3$ | | Control factor $x_4$ | |
|---|---|---|---|---|---|
| Effects | Sum of squares | Effects | Sum of squares | Effects | Sum of squares |
| $u_1(x_2)$ $z_1$ | 0.004 | $u_1(x_3)$ $z_1$ | 51.042 | $u_1(x_4)$ $z_1$ | 72.107 |
| $u_2(x_2)$ $z_1$ | 0 .623 | $u_2(x_3)$ $z_1$ | 4.601 | $u_2(x_4)$ $z_1$ | 19.427 |
| $u_1(x_2)z_2$ | 4.594 | $u_1(x_3)$ $z_2$ | 7.260 | $u_1(x_4)$ $z_2$ | 27.735 |
| $u_2(x_2)$ $z_2$ | 1.051 | $u_2(x_3)$ $z_2$ | 36.980 | $u_2(x_4)$ $z_2$ | 36.980 |
| $u_1(x_2)z_3$ | 1.170 | $u_1(x_3)$ $z_3$ | 14.727 | $u_1(x_4)$ $z_3$ | 17.340 |
| $u_2(x_2)$ $z_3$ | 6.067 | $u_2(x_3)$ $z_3$ | 56.889 | $u_2(x_4)$ $z_3$ | 9.827 |

Table 3 gives the estimated standard deviation (column (1)) for all 27 treatment combinations of $x_1, x_3$, and $x_4$. The combination $x_1 = 3$, $x_3 = -1$, $x_4 = 1$, yields the smallest value of standard deviation of 1.6. However, because of practical consideration, high atomizing air must be combined with somewhat higher fan air.

One might consider the next best combination at $x_1 = 1$, $x_3 = -1$, $x_4 = 0$, with an estimated standard deviation of 1.8. The use of orthogonal polynomials allows interpolation to obtain improved results at $x_1 = 1$, $x_3 = -1.1$, $x_4 = -0.4$, yielding a smaller standard deviation of 1.6. The last few columns of Table 3 give the mean and standard deviation for each of $x_2$ = -1, 0, +1 when $x_2$ is also included in the variance analysis. The difference in standard deviations from column (1) is minimal.

Furthermore, if a target mean of 45 is desired, then $x_2$ should be set around $x_2$ = 1. As pointed out in the last section, the procedure of minimizing variance can be adversely affected if some of the two factor noise-noise interactions are non-zero. Thus we suggest, as a safeguard against this potential problem by assessing these interactions with small number of additional experimental runs. In the present example, each

Table 3. Means and standard deviations

| | | | | | $x_2$ = -1 | | $x_2$ = 0 | | $x_2$ = +1 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | $x_3$ | $x_4$ | (1) | (2) | mean | SD | mean | SD | mean | SD |
| 1 | -1 | -1 | 3.7 | 3.5 | 35.8 | 3.1 | 41.5 | 3.3 | 43.5 | 4.0 |
| 1 | -1 | 0 | 1.8 | 3.0 | 36.9 | 1.4 | 42.6 | 0.9 | 44.6 | 1.0 |
| 1 | -1 | 1 | 4.2 | 4.7 | 29.8 | 3.8 | 35.5 | 4.0 | 37.5 | 3.9 |
| 1 | 0 | -1 | 6.3 | 5.6 | 34.4 | 5.9 | 40.0 | 5.9 | 42.1 | 6.7 |
| 1 | 0 | 0 | 3.3 | 3.1 | 35.5 | 2.9 | 41.2 | 2.5 | 43.2 | 3.5 |
| 1 | 0 | 1 | 3.8 | 3.5 | 28.4 | 3.3 | 34.1 | 3.4 | 36.1 | 4.0 |
| 1 | 1 | -1 | 3.4 | 3.9 | 43.1 | 2.9 | 48.8 | 2.9 | 50.8 | 3.4 |
| 1 | 1 | 0 | 3.6 | 4.9 | 44.2 | 3.6 | 49.9 | 3.3 | 51.9 | 3.0 |
| 1 | 1 | 1 | 2.1 | 3.9 | 37.1 | 1.8 | 42.8 | 2.0 | 44.8 | 1.0 |
| 2 | -1 | -1 | 2.6 | 3.2 | 32.2 | 1.8 | 37.9 | 2.2 | 39.9 | 2.7 |
| 2 | -1 | 0 | 2.3 | 4.0 | 33.4 | 2.2 | 39.0 | 2.0 | 41.0 | 1.2 |
| 2 | -1 | 1 | 3.4 | 4.6 | 26.3 | 3.0 | 31.9 | 3.4 | 33.9 | 2.8 |
| 2 | 0 | -1 | 5.5 | 5.1 | 30.8 | 5.0 | 36.5 | 5.0 | 38.5 | 5.8 |
| 2 | 0 | 0 | 3.1 | 3.7 | 31.9 | 2.8 | 37.6 | 2.5 | 39.6 | 3.1 |
| 2 | 0 | 1 | 2.4 | 2.9 | 24.8 | 1.5 | 30.5 | 1.9 | 32.5 | 2.4 |
| 2 | 1 | -1 | 3.9 | 5.0 | 39.5 | 3.6 | 45.2 | 3.6 | 47.2 | 3.7 |
| 2 | 1 | 0 | 5.1 | 6.5 | 40.6 | 5.2 | 46.3 | 5.0 | 48.3 | 4.5 |
| 2 | 1 | 1 | 3.1 | 5.0 | 33.5 | 3.0 | 39.2 | 3.1 | 41.2 | 2.1 |
| 3 | -1 | -1 | 4.1 | 4.4 | 34.8 | 3.7 | 40.5 | 3.7 | 42.5 | 4.3 |
| 3 | -1 | 0 | 3.6 | 4.8 | 35.9 | 3.6 | 41.6 | 3.3 | 43.6 | 3.2 |
| 3 | -1 | 1 | 1.6 | 3.4 | 28.8 | 0.9 | 34.5 | 1.2 | 36.5 | 0.3 |
| 3 | 0 | -1 | 7.2 | 6.9 | 33.4 | 6.9 | 39.0 | 6.8 | 41.1 | 7.6 |
| 3 | 0 | 0 | 5.5 | 5.8 | 34.5 | 5.4 | 40.1 | 5.0 | 42.2 | 5.5 |
| 3 | 0 | 1 | 3.1 | 3.4 | 27.4 | 2.5 | 33.0 | 2.5 | 35.1 | 3.2 |
| 3 | 1 | -1 | 6.3 | 6.9 | 42.1 | 6.1 | 47.7 | 6.0 | 49.8 | 6.3 |
| 3 | 1 | 0 | 6.9 | 8.0 | 43.2 | 7.0 | 48.9 | 6.8 | 50.9 | 6.6 |
| 3 | 1 | 1 | 3.9 | 5.5 | 36.1 | 3.8 | 41.8 | 3.8 | 43.8 | 3.3 |

main effect in the outer array is aliased with the interaction between the remaining two noise factors. For instance, the coefficient $\hat{\gamma}_3$ of the noise factor "viscosity" is small, but since $z_3$ is aliased with $z_1z_2$, it actually estimates the sum of $\gamma_3 + \gamma_{12}$, where $\gamma_{12}$ is the coefficient of $z_1z_2$.

In the last step, we propose to have a $2^2$ factorial (or a factional factorial so that the interactions suspected to be significant are estimable) of the noise factors conducted at the solution obtained in step 3, i.e. $x_1 = 1$, $x_3 = -1.1$, $x_4 = -0.4$. To estimate $\gamma_{12}$, first subject the fitted value based on (3.1) from each of the $y$ values from the new experiment and estimate $\gamma_{12}$ by the slope of the regression of the adjusted $y$ on $z_1z_2 - z_3$.

As an illustrative example, suppose an estimate $\hat{\gamma}_{12} = -1.855$ is obtained. Then the coefficient $\gamma_3$ can be re-estimated as -0.042-(-1.855) = 1.813. Column (2) of Table 3 now gives the standard deviations based on the new model (model (2) together with the additional term $\gamma_{12}z_1z_2$). The results are markedly different from column (1), and the smallest value no longer occurs at $x_1 = 3$, $x_3 = -1$, $x_4 = 1$, suggesting that such adjustment might be necessary.

## Conclusion

We have suggested in this article a systematic approach in analyzing crossed array designs, where fractional factorial design may be employed in the outer array. This kind of designs is still popular because of its simplicity and its greater robustness than combined array designs to certain type of model mis-specification. It is however demonstrated that non-ignorable noise-noise interactions may still create problems with the crossed array design. A method of rectifying these difficulties is proposed, but the problem of finding cost effective follow up design to complement the original design is worth studying.

Our approach also assumes the constant variance assumption conditional on values of both the control and noise factors. If this assumption is violated, the response variable may have to be transformed to attain constant variances before the suggested analysis can be carried out.

Alternatively, the use of generalized linear model (Nelder and Lee, 1991) or the approach of Engel (1982) may also be appropriate. The choice of an appropriate transformation may be facilitated using the graphical plot of Box (1988), or the analysis of Chan and Mak (1997). However, even if the quadratic loss function is used in the original metric, the induced loss function in the transformed scale is no longer quadratic. In this case, the expected loss can be approximated using the idea of pseudo observations. This approach is equivalent to using Gaussian Quadrature to carry out the integration in computing the expected loss. As is well known the approximation can be improved by using more data points for the noise factors in generating the pseudo observations. Details will not be given here.

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