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On Some Negative Integer Moments of Quasi-Negative-Binomial Distribution

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Negative integer moments of the quasi-negative-binomial distribution (QNBD) are investigated. This distribution includes recurrence relations which are helpful in the solution of many applied statistical problems, particularly in life testing and survey sampling, where ratio estimators are useful. Results study show the negative-binomial distribution when the parameter θ_2 is zero and also indicate the mean of the QNBD model when its parameters are changed.

Key words: Quasi-negative-binomial distribution, recurrence relations, Abel series expansion, negative-binomial distribution.

Introduction

The quasi-negative-binomial distribution (QNBD) was introduced in different forms by Janardan (1975), Nandi and Das (1994) and Sen and Jain (1996) but has not been studied in detail. The discrete probability function of the QNBD is given by

$$P_x(a,\theta_1,\theta_2) = \frac{(a+x-1)!}{(a-1)!\,x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{x+a}},$$
$$x = 0,1,2,..... \tag{1.1}$$

$$(\theta_1+x\theta_2)\geq 0;\ a>0, \theta_1>0\ \ if\ \ \theta_2>0$$

and

$$(\theta_1+x\theta_2)\geq 0;\ if\ \ \theta_2<0$$

where (a, θ_1, θ_2) are parameters of the distribution. When θ_2 is negative, the probabilities of the QNBD model become negative. In addition, there appears to be a natural truncation for x, for which $P_x(a,\theta_1,\theta_2)=0$; however, this has not been verified and requires a detailed error analysis, which is not included herein.

The QNBD model reduces to a negative-binomial distribution (NBD) model at $\theta_2=0$. It appears from the model that the β parameter in Greenwood and Yule's (1920) NBD model was replaced by $(\theta_1+x\theta_2)$, where x is the number of occurrences; this implies that, with successive occurrences, there is some changing tendency in the θ_1 parameter.

Hassan and Bilal (2008) explored the properties of the QNBD model (1.1) with mean and variance obtained in a hypergeometric function given as

$$\mu_1' = a\theta_1\ {}_2F_0[1,a+1,\_;\theta_2] \tag{1.2}$$

$$\begin{aligned}\mu_2' = {} & a\theta_1\ {}_2F_0[1,a+1,\_;\theta_2] \\ & +\theta_1(\theta_1+2\theta_2)a(a+1)\ {}_2F_0[2,a+2,\_;\theta_2] \\ & +\theta_1\theta_2^{\,2}a(a+1)(a+2)\ {}_2F_0[3,a+3,\_;\theta_2]\end{aligned} \tag{1.3}$$

where ${}_2F_0[1,a+1,\_;\theta_2]$ is a hypergeometric function defined by

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$${}_2F_0[1, a+1, \_; \theta_2] = \sum_{j=0}^{\infty} 1^{[j]} (a+1)^{[j]} \frac{\theta_2^{\,j}}{j!}.$$

Hassan and Bilal (2006) found applications for the QNBD model in queuing theory, theories of microorganisms and biology. They investigated the distribution of numbers of accidents as a QNBD model using Irwin's (1968) theory of proneness-liability model and then applied the model to hunting accidents, home injuries and strikes in industries; they obtained better model fits than Consul and Jain's (1973), using a generalized Poisson distribution (GPD) model.

A difficulty with the QNBD model is that its moments appear in an infinite series, which does not seem to converge to an expression that will produce moment estimators. This article investigates negative integer moments of the QNBD. This distribution includes recurrence relations which are helpful in the solution of many applied statistics problems. Results from this study show the negative-binomial distribution when the parameter θ_2 is zero and indicate the mean of the QNBD model when its parameters are changed.

Negative Integer Moments

Suppose that $\phi_s(k,a,\theta_1) = E[x+k]^{-s}$ denotes the sth negative integer moments of the QNBD model (1.1), then the following results on the negative integer moments are true for the proposed model:

$$\varphi_1\left(\frac{\theta_1}{\theta_2}, a, \theta_1\right) = E\left(x + \frac{\theta_1}{\theta_2}\right)^{-1} = \frac{(\theta_1 + \theta_2 - a\theta_1\theta_2)\theta_2}{\theta_1(\theta_1+\theta_2)} \quad (1.4)$$

$$\varphi_2\left(\frac{\theta_1}{\theta_2}, a, \theta_1\right) = E\left(x + \frac{\theta_1}{\theta_2}\right)^{-2} = \frac{\theta_2}{\theta_1}\left(\frac{\theta_2}{\theta_1} - \frac{a\theta_2^2}{(\theta_1+\theta_2)}\right) + \frac{a\theta_2^2}{(\theta_1+\theta_2)} \times \left(\frac{\theta_2}{(\theta_1+\theta_2)} - \frac{(a+1)\theta_2^2}{(\theta_1+2\theta_2)}\right) \quad (1.5)$$

$$\varphi_s\left(\frac{\theta_1}{\theta_2}, a, \theta_1\right) = \frac{\theta_2}{\theta_1}\varphi_{s-1}\left(\frac{\theta_1}{\theta_2}, a, \theta_1\right) - \frac{a\theta_2^2}{(\theta_1+\theta_2)}\varphi_{s-1} \times \left(\frac{\theta_1+\theta_2}{\theta_2}, a+1, \theta_1+\theta_2\right) \quad (1.6)$$

$$\varphi_1\left(\frac{1+\theta_1}{\theta_2}, a, \theta_1\right) = E\left(x + \frac{1+\theta_1}{\theta_2}\right)^{-1} = \frac{\theta_2(a\theta_2 - 1)}{(a\theta_2 - \theta_1 - 1)} \quad (1.7)$$

$$\varphi_2\left(\frac{1+\theta_1}{\theta_2}, a, \theta_1\right) = E\left(x + \frac{1+\theta_1}{\theta_2}\right)^{-2} = \frac{\theta_2^2(1-a\theta_2)}{(1+\theta_1-a\theta_2)^2} - \frac{a\theta_2^3(1-(a+1)\theta_2)}{(1+\theta_1-(a+1)\theta_2)(1+\theta_1-a\theta_2)} \quad (1.8)$$

$$\varphi_s\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)=\frac{\theta_2}{(1+\theta_1-a\theta_2)}\varphi_{s-1}\times\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)-\frac{a\theta_2^{\,2}}{(1+\theta_1-a\theta_2)}\varphi_{s-1}\times\left(\frac{1+\theta_1}{\theta_2},a+1,\theta_1\right) \quad (1.9)$$

$$\phi_1(a,a+1,\theta_1)=\frac{(1-a\theta_2)}{a(1+\theta_1-a\theta_2)} \quad (1.10)$$

$$\phi_1(1,a+1,\theta_1+\theta_2)=\frac{(\theta_1+\theta_2-a\theta_1\theta_2)}{a\theta_1^{\,2}} \quad (1.11)$$

$$\varphi_1(k,a,\theta_1)=\frac{(a-k-1)}{(a-1)(1+\theta_1-k\theta_2)}\varphi_1(k,a-1,\theta_1)+\frac{1-\theta_2(a-1)}{(1+\theta_1-k\theta_2)(a-1)} \quad (1.12)$$

$$\varphi_s(0,a,\theta_1)=\frac{a\theta_1^{\,2}}{(\theta_1+\theta_2)}\varphi_{s+1}(1,a+1,\theta_1+\theta_2)+\frac{a\theta_1\theta_2}{(\theta_1+\theta_2)}\varphi_s(1,a+1,\theta_1+\theta_2) \quad (1.13)$$

Proof

Taking the summation of (1.1) and differentiating it with respect to θ_1, results in the simplification

$$\frac{1}{\theta_1}+\sum_{X=0}^{\infty}\frac{(x-1)}{(\theta_1+x\theta_2)}\frac{(a+x-1)!}{(a-1)!\,x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{x+a}}=a, \quad (1.14)$$

and writing $\theta_2(x-1)=(\theta_1+x\theta_2)-(\theta_1+\theta_2)$, results in

$$\frac{(\theta_1+\theta_2)}{\theta_1\theta_2}-\frac{(\theta_1+\theta_2)}{\theta_2^{\,2}}E\left(x+\frac{\theta_1}{\theta_2}\right)^{-1}=a.$$

After rearranging the terms in the equation result (1.4) follows. Similarly, result (1.5) can be obtained by differentiating (1.14) with respect to θ_1 and simplifying the resulting equation. The results represented by (1.4) and (1.5) can also be obtained from recurrence relation (1.6), which is proven by:

$$\varphi_s\left(\frac{\theta_1}{\theta_2},a,\theta_1\right)$$

$$=E\left(x+\frac{\theta_1}{\theta_2}\right)^{-s}$$

$$=\sum_{x=0}^{\infty}\left(x+\frac{\theta_1}{\theta_2}\right)^{-s}\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}$$

$$=\left[\frac{\theta_2}{\theta_1}\sum_{x=0}^{\infty}\left(x+\frac{\theta_1}{\theta_2}\right)^{-s+1}\times\left(1-\frac{x\theta_2}{(\theta_1+x\theta_2)}\right)\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}\right]$$

$$=\left[\frac{\theta_2}{\theta_1}\sum_{x=0}^{\infty}\left(x+\frac{\theta_1}{\theta_2}\right)^{-s+1}P_x(a,\theta_1,\theta_2)-a\theta_2^{\,2}\sum_{x=1}^{\infty}\left(x+\frac{\theta_1}{\theta_2}\right)^{-s+1}\frac{(a+x-1)!}{a!\,(x-1)!}\frac{(\theta_1+x\theta_2)^{x-2}}{(1+\theta_1+x\theta_2)^{a+x}}\right]$$

where $P_x(a,\theta_1,\theta_2)$ is defined by (1.1). Replacing x with $(x+1)$ in the second component of the equation results in:

$$\varphi_s\left(\frac{\theta_1}{\theta_2},a,\theta_1\right)=$$
$$\frac{\theta_2}{\theta_1}\varphi_{s-1}\left(\frac{\theta_1}{\theta_2},a,\theta_1\right)-$$
$$\frac{a\theta_2^{\,2}}{(\theta_1+\theta_2)}\sum_{x=0}^{\infty}\left(x+\frac{(\theta_1+\theta_2)}{\theta_2}\right)P_x(a+1,\theta_1+\theta_2,\theta_2),$$

which gives recurrence relation (1.6). To prove result (1.7), take the summation of QNBD model (1.1) with parameters $(a+1,\theta_1,\theta_2)$ to yield:

$$\sum_{X=0}^{\infty}\frac{(a+x)!}{a!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+1+x}}=1.$$

Rewriting this equation as

$$\frac{1}{a}\sum_{X=0}^{\infty}\frac{(a+x)}{(1+\theta_1+x\theta_2)}P_x(a,\theta_1,\theta_2)=1,$$

and writing $\theta_2(a+x)$ as a sum of two components $(1+\theta_1+x\theta_2)$ and $(a\theta_2-\theta_1-1)$, results in

$$\frac{(a\theta_2-\theta_1-1)}{a\theta_2^{\,2}}\varphi_1\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)+\frac{1}{a\theta_2}=1.$$

Rearranging the terms in the equation result (1.7) follows. Taking the summation of the QNBD model with parameters $(a+2,\theta_1,\theta_2)$ and proceeding in the same way, result (1.6) is obtained. To prove recurrence relation (1.9):

$$\varphi_s\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)$$
$$=\sum_{x=0}^{\infty}\left(x+\frac{1+\theta_1}{\theta_2}\right)^{-s+1}\left[\begin{array}{c}\left(\dfrac{\theta_2}{(1+\theta_1+x\theta_2)}\right)\\ \times\left(\dfrac{(a+x-1)!}{(a-1)!x!}\dfrac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}\right)\end{array}\right]$$
$$=\theta_2\sum_{x=0}^{\infty}\left(x+\frac{1+\theta_1}{\theta_2}\right)^{-s+1}\left[\begin{array}{c}\left(1-\dfrac{\theta_1+x\theta_2}{(1+\theta_1+x\theta_2)}\right)\\ \times\left(\dfrac{(a+x-1)!}{(a-1)!x!}\dfrac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}\right)\end{array}\right]$$

and writing $(\theta_1+x\theta_2)=(\theta_1-a\theta_2)+(a+x)\theta_2$, results in the simplification:

$$\varphi_s\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)=$$
$$\theta_2\varphi_{s-1}\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)-$$
$$(\theta_1-a\theta_2)\varphi_s\left(\frac{1+\theta_1}{\theta_2},a,\theta_1\right)-$$
$$a\theta_2^{\,2}\varphi_{s-1}\left(\frac{1+\theta_1}{\theta_2},a+1,\theta_1\right).$$

After rearranging the terms in the equation result (1.9) follows. The results (1.10), (1.11) and (1.12) are straightforward and can be obtained in a similar way, however, for recurrence relation (1.13):

$$\varphi_s(0,a,\theta_1)=\sum_{x=0}^{\infty}x^{-s}\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}$$
$$=\sum_{x=1}^{\infty}x^{-s-1}\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}$$

and replacing x by $(x+1)$ in the equation above results in

$$\varphi_s(0,a,\theta_1) = \frac{a\theta_1}{(\theta_1+\theta_2)} \times \sum_{x=0}^{\infty}(x+1)^{-s-1}(\theta_1+\theta_2(x+1))P_x(a+1,\theta_1+\theta_2,\theta_2)$$

which gives

$$\varphi_s(0,a,\theta_1) = \frac{a\theta_1^2}{(\theta_1+\theta_2)}\varphi_{s+1}(1,a+1,\theta_1+\theta_2) + \frac{a\theta_1\theta_2}{(\theta_1+\theta_2)}\varphi_s(1,a+1,\theta_1+\theta_2)$$

Charalambidies (1990) examined an extension of the class of power series distributions and obtained a discrete class of Abel series distributions. He also explored its properties with an application to the fluctuations of sample function of stochastic process. Nandi and Das (1994) also obtained a class of Abel series distributions. Hassan and Bilal (2008) showed that the QNBD model belongs to a family of Abel series distributions by taking the Abel series expansion of $(c-r)^{-a}$ given as

$$(c-r)^{-a} = \sum_{X=0}^{\infty}\frac{(a+x-1)!}{(a-1)!\,x!}\frac{r(r+bx)^{x-1}}{(c+bx)^{a+x}}, \tag{1.15}$$

where

$$\frac{r}{(c-r)}=\theta_1, \frac{b}{(c-r)}=\theta_2$$

and

$$\frac{c}{(c-r)}=1+\frac{r}{(c-r)}=1+\theta_1 .$$

The expression in (1.15) gives the sum of the QNBD model, which is equal to unity. The following results, obtained on the basis of (1.15), are proven as:

$$\phi_1(1,a,\theta_1)=E(x+1)^{-1}=\frac{\theta_1}{(a-1)(\theta_1-\theta_2)^2}-\frac{\theta_2}{(\theta_1-\theta_2)} \tag{1.16}$$

$$\varphi_1(2,a,\theta_1) = E(x+1)^{-2} = \frac{\theta_1}{(a-1)(\theta_1-2\theta_2)^2} - \frac{\theta_1}{(a-1)(a-2)(\theta_1-2\theta_2)^3} - \frac{\theta_2}{(\theta_1-2\theta_2)} \tag{1.17}$$

$$\varphi_1[(1,2),a,\theta_1] = E[(x+1)(x+2)]^{-1} = \frac{\theta_1\theta_2(3\theta_2-2\theta_1)}{(a-1)(\theta_1-\theta_2)^2(\theta_1-2\theta_2)^2} + \frac{\theta_1}{(a-1)(a-2)(\theta_1-2\theta_2)^3} + \frac{\theta_2}{(\theta_1-\theta_2)(\theta_1-2\theta_2)} \tag{1.18}$$

$$\varphi_1(a-1,a,\theta_1) = E(x+a-1)^{-1} = \frac{1-\theta_2(a-1)}{(a-1)[1+\theta_1-\theta_2(a-1)]} \tag{1.19}$$

$$\varphi_1(a-2,a,\theta_1) = E(x+a-2)^{-1} = \frac{\begin{bmatrix}[1-\theta_2(a-1)][1+\theta_1-\theta_2(a-1)] \\ \times(a-2)+1-\theta_2^{\,2}(a-1)(a-2)\end{bmatrix}}{\begin{bmatrix}(a-1)(a-2)[(1+\theta_1-\theta_2(a-1))^2 \\ +2\theta_2(1+\theta_1-\theta_2(a-1))+\theta_2^{\,2}]\end{bmatrix}} \tag{1.20}$$

Proof

Integrating (1.15) with respect to r, results in

$$\frac{(c-r)^{-a+1}}{(a-1)} = \sum_{X=0}^{\infty} \frac{(a+x-1)!}{(a-1)!\,x!} \begin{pmatrix} \dfrac{r(r+bx)^{x}}{x} \\ -\dfrac{(r+bx)^{x+1}}{x(x+1)} \end{pmatrix} (c+bx)^{-a-x}$$

$$= \sum_{X=0}^{\infty} \frac{(a+x-1)!}{(a-1)!\,x!} \begin{pmatrix} \dfrac{(r+bx)}{x} \\ -\dfrac{(r+bx)^{2}}{x(x+1)r} \end{pmatrix} \frac{r(r+bx)^{x-1}}{(c+bx)^{a+x}} \tag{1.21}$$

Expressing the equation in terms of θ_1 and θ_2 results in

$$\frac{1}{(a-1)} = \sum_{X=0}^{\infty} \left(\frac{(\theta_1+\theta_2 x)}{x} - \frac{(\theta_1+\theta_2 x)^2}{x(x+1)\theta_1} \right) P_x(a,\theta_1,\theta_2)$$

and writing

$$(\theta_1 + x\theta_2)^2 = (\theta_1-\theta_2)^2 + 2\theta_2(\theta_1-\theta_2)(x+1) + \theta_2{}^2(x+1)^2,$$

the result (1.16) follows based on simplifications. Again, integrating (1.21) with respect to r, result (1.17) is obtained. Result (1.15) follows from (1.13) and (1.14) by using the relation

$$E[(x+1)(x+2)]^{-1} = E(x+1)^{-1} - E(x+2)^{-1}.$$

To prove result (1.19), integrating (1.15) with respect to c, results in

$$\frac{(c-r)^{-a+1}}{(a-1)} = \sum_{X=0}^{\infty} \frac{(c+bx)}{(x+a-1)} \frac{(a+x-1)!}{(a-1)!\,x!} \frac{r(r+bx)^{x-1}}{(c+bx)^{a+x}}. \tag{1.22}$$

Expressing the equation above in terms of θ_1 and θ_2 results in

$$\frac{1}{(a-1)} = \sum_{X=0}^{\infty} \frac{(1+\theta_1+x\theta_2)}{(x+a-1)} P_x(a,\theta_1,\theta_2).$$

Writing

$$(1+\theta_1+x\theta_2) = [1+\theta_1-\theta_2(a-1)] + \theta_2(x+a-1),$$

results in

$$\frac{1}{(a-1)} = \sum_{X=0}^{\infty} \frac{[1+\theta_1-\theta_2(a-1)] + \theta_2(x+a-1)}{(x+a-1)} P_x(a,\theta_1,\theta_2)$$
$$= [1+\theta_1-\theta_2(a-1)]\varphi_1(a-1,a,\theta_1) + \theta_2$$

Rearranging the terms in the equation result (1.19) follows. Integrating (1.22) with respect to c and proceeding in the same way results in (1.20).

Another useful set of recurrence relations on the negative integer moments of the QNBD model from which a number of important results can be deduced are

$$\varphi_s(k,a+1,\theta_1) = \frac{1}{(1+\theta_1-\theta_2 k)} \begin{pmatrix} \dfrac{(a-k)}{a}\varphi_s(k,a,\theta_1) \\ +\dfrac{1}{a}\varphi_{s-1}(k,a,\theta_1) \\ -\theta_2\varphi_{s-1}(k,a+1,\theta_1) \end{pmatrix} \tag{1.23}$$

$$\varphi_s(k,a,\theta_1) = \int_{k\theta_2}^{\theta_1} (\theta_1-k\theta_2)^k (1+\theta_1-k\theta_2)^{a-k} [G(a,\theta_1) + G(a+1,\theta_1)] \partial\theta_1 \tag{1.24}$$

where $G(a,\theta_1)$ and $G(a+1,\theta_1)$ are defined as

$$G(a,\theta_1) = \begin{bmatrix} \left(\dfrac{\theta_1+\theta_2}{\theta_1^2} - \dfrac{(\theta_1-k\theta_2)}{\theta_1(1+\theta_1-k\theta_2)}\right)\varphi_{s-1}(k,a,\theta_1) \\ -\dfrac{\theta_2}{\theta_1}\varphi'_{s-1}(k,a,\theta_1) \end{bmatrix} \tag{1.25}$$

$$G(a+1,\theta_1) = \frac{a\theta_2}{\theta_1(1+\theta_1-k\theta_2)}\varphi_{s-1}(k,a+1,\theta_1) \tag{1.26}$$

Proof

First, result (1.23) is proven, which is subsequently required in the derivation of (1.24). Writing

$$\varphi_s(k,a+1,\theta_1) = \sum_{X=0}^{\infty}(x+k)^{-s}\frac{(a+x)!}{a!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+1+x}}$$

simplifies to

$$\begin{aligned}\varphi_s(k,a+1,\theta_1) &= \frac{1}{a}\sum_{X=0}^{\infty}(x+k)^{-s}\frac{(a+x)}{(1+\theta_1+x\theta_2)}P_x(a,\theta_1,\theta_2) \\ &= \frac{1}{a}\sum_{X=0}^{\infty}(x+k)^{-s}(a+x)\left(1-\frac{(\theta_1+x\theta_2)}{(1+\theta_1+x\theta_2)}\right)P_x(a,\theta_1,\theta_2)\end{aligned}$$

Writing $(a+x)=(a-k)+(x+k)$ and $(\theta_1+x\theta_2)=(\theta_1-k\theta_2)+\theta_2(x+k)$ in the second component of the equation results in

$$\begin{aligned}\varphi_s(k,a+1,\theta_1) &= \frac{(a-k)}{a}\varphi_s(k,a,\theta_1)+\frac{1}{a}\varphi_{s-1}(k,a,\theta_1) \\ &\quad -(\theta_1-k\theta_2)\varphi_s(k,a+1,\theta_1)-\theta_2\varphi_{s-1}(k,a+1,\theta_1)\end{aligned}$$

Rearranging the terms, results in (1.23). To prove (1.24), suppose

$$\begin{aligned}U(x) &= \sum_{x=0}^{\infty}(x+k)^{-s}\frac{(a+x-1)!}{(a-1)!x!}\frac{(\theta_1+x\theta_2)^x}{(1+\theta_1+x\theta_2)^{a+x}} \\ &= \frac{1}{\theta_1}\sum_{x=0}^{\infty}(x+k)^{-s}(\theta_1+x\theta_2)\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}\end{aligned} \tag{1.27}$$

Writing

$$(\theta_1+x\theta_2)=(\theta_1-k\theta_2)+\theta_2(x+k)$$

results in

$$U(x)=\frac{(\theta_1-k\theta_2)}{\theta_1}\phi_s(k,a,\theta_1)+\frac{\theta_2}{\theta_1}\phi_{s-1}(k,a,\theta_1)$$

and differentiating the equation with respect to θ_1, results in

$$\begin{aligned}U'(x) &= \frac{(\theta_1-k\theta_2)}{\theta_1}\varphi'_s(k,a,\theta_1)+\frac{k\theta_2}{\theta_1^2}\varphi_s(k,a,\theta_1) \\ &\quad -\frac{\theta_2}{\theta_1^2}\varphi_{s-1}(a)+\frac{\theta_2}{\theta_1}\varphi'_{s-1}(k,a,\theta_1)\end{aligned} \tag{1.28}$$

where

$$U'(x) = \sum_{x=0}^{\infty}\frac{(x+k)^{-s}}{\theta_1}\left(x-\frac{(a+x)(\theta_1+x\theta_2)}{(1+\theta_1+x\theta_2)}\right)P_x(a,\theta_1,\theta_2)$$

Writing

$$(\theta_1+x\theta_2)=(\theta_1-k\theta_2)+\theta_2(x+k)$$

in the equation results in

$$
\begin{aligned}
U'(x) &= \frac{1}{\theta_1}\sum_{x=0}^{\infty}(x+k)^{-s}[(x+k)-k]P_x(a,\theta_1,\theta_2) \\
&\quad -\frac{a}{\theta_1}\sum_{x=0}^{\infty}(x+k)^{-s}\begin{bmatrix}(\theta_1-k\theta_2)\\ +\theta_2(x+k)\end{bmatrix}P_x(a+1,\theta_1,\theta_2) \\
&= \frac{1}{\theta_1}\varphi_{s-1}(k,a,\theta_1)-\frac{k}{\theta_1}\varphi_s(k,a,\theta_1) \\
&\quad -\frac{a(\theta_1-k\theta_2)}{\theta_1}\varphi_s(k,a+1,\theta_1). \\
&\quad -\frac{a\theta_2}{\theta_1}\varphi_{s-1}(k,a+1,\theta_1)
\end{aligned}
$$

Using this equation in (1.28) results in

$$
\begin{bmatrix}
\frac{1}{\theta_1}\varphi_{s-1}(k,a,\theta_1)\\
-\frac{k}{\theta_1}\varphi_s(k,a,\theta_1)\\
-\frac{a(\theta_1-k\theta_2)}{\theta_1}\varphi_s(k,a+1,\theta_1)\\
-\frac{a\theta_2}{\theta_1}\varphi_{s-1}(k,a+1,\theta_1)
\end{bmatrix}
=
\begin{bmatrix}
\frac{(\theta_1-k\theta_2)}{\theta_1}\varphi_s^{'}(k,a,\theta_1)\\
+\frac{k\theta_2}{\theta_1^2}\varphi_s(k,a,\theta_1)\\
-\frac{\theta_2}{\theta_1^2}\varphi_{s-1}(k,a,\theta_1)\\
+\frac{\theta_2}{\theta_1}\varphi_{s-1}^{'}(k,a,\theta_1)
\end{bmatrix}
$$

Rearranging the terms,

$$
\begin{bmatrix}
\frac{(\theta_1-k\theta_2)}{\theta_1}\varphi_s^{'}(k,a,\theta_1)\\
+\frac{k(\theta_1+\theta_2)}{\theta_1^2}\varphi_s(k,a,\theta_1)
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
\frac{(\theta_1+\theta_2)}{\theta_1^2}\varphi_{s-1}(k,a,\theta_1)\\
-\frac{\theta_2}{\theta_1}\varphi_{s-1}^{'}(k,a,\theta_1)\\
-\frac{a(\theta_1-k\theta_2)}{\theta_1}\varphi_s(k,a+1,\theta_1)\\
-\frac{a\theta_2}{\theta_1}\varphi_{s-1}(k,a+1,\theta_1)
\end{bmatrix}
$$

and using (1.23) results in

$$
\begin{bmatrix}
\frac{(\theta_1-k\theta_2)}{\theta_1}\varphi_s^{'}(k,a,\theta_1)\\
+\frac{k(\theta_1+\theta_2)}{\theta_1^2}\varphi_s(k,a,\theta_1)
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
\begin{pmatrix}
\frac{(\theta_1+\theta_2)}{\theta_1^2}\varphi_{s-1}(k,a,\theta_1)\\
-\frac{\theta_2}{\theta_1}\varphi_{s-1}^{'}(k,a,\theta_1)\\
-\frac{a(\theta_1-k\theta_2)}{\theta_1(1+\theta_1-k\theta_2)}
\end{pmatrix}\\
\times\begin{pmatrix}
\frac{(a-k)}{a}\varphi_s(k,a,\theta_1)\\
+\frac{1}{a}\varphi_{s-1}(k,a,\theta_1)\\
-\theta_2\varphi_{s-1}(k,a+1,\theta_1)
\end{pmatrix}-\frac{a\theta_2}{\theta_1}\varphi_{s-1}(k,a+1,\theta_1)
\end{bmatrix}
$$

Adding similar terms results in

$$
\begin{bmatrix}
\frac{(\theta_1-k\theta_2)}{\theta_1}\varphi_s^{'}(k,a,\theta_1)\\
+\left(\frac{k(\theta_1+\theta_2)}{\theta_1^2}+\frac{(a-k)(\theta_1-k\theta_2)}{\theta_1(1+\theta_1-k\theta_2)}\right)\varphi_s(k,a,\theta_1)
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
\varphi_s^{'}(k,a,\theta_1)+\frac{\theta_1}{(\theta_1-k\theta_2)}\\
\times\left(\frac{k(\theta_1+\theta_2)}{\theta_1^2}+\frac{(a-k)(\theta_1-k\theta_2)}{\theta_1(1+\theta_1-k\theta_2)}\right)\varphi_s(k,a,\theta_1)
\end{bmatrix}
=
$$

$$
\frac{\theta_1}{(\theta_1-k\theta_2)}\left[G(a,\theta_1)-G(a+1,\theta_1)\right]
$$

which gives the linear differential equation

$$\varphi_s'(k,a,\theta_1)+\frac{\theta_1}{(\theta_1-k\theta_2)}\left(\begin{array}{c}\frac{k(\theta_1+\theta_2)}{\theta_1^2}\\+\frac{(a-k)(\theta_1-k\theta_2)}{\theta_1(1+\theta_1-k\theta_2)}\end{array}\right)\varphi_s(k,a,\theta_1)=\frac{\theta_1}{(\theta_1-k\theta_2)}\left[G(a,\theta_1)-G(a+1,\theta_1)\right] \qquad (1.29)$$

Where $G(a,\theta_1)$ and $G(a+1,\theta_1)$ are defined in (1.25) and (1.26) respectively. The integrating factor for the differential equation is

$$I.F.=\exp\left\{\int\left(\begin{array}{c}\frac{k(\theta_1+\theta_2)}{\theta_1(\theta_1-k\theta_2)}\\+\frac{(a-k)}{(1+\theta_1-k\theta_2)}\end{array}\right)\partial\theta_1\right\}=\exp\left\{\int\left(\begin{array}{c}\frac{\theta_1(k+2)-(2\theta_1-k\theta_2)}{\theta_1(\theta_1-k\theta_2)}\\+\frac{(a-k)}{(1+\theta_1-k\theta_2)}\end{array}\right)\partial\theta_1\right\}.$$

Simplifying this equation gives the integrating factor

$$I.F.=\frac{(\theta_1-k\theta_2)^{k+1}(1+\theta_1-k\theta_2)^{a-k}}{\theta_1}.$$

Multiplying (1.29) with this integrating factor and integrating it with respect to θ_1 from $k\theta_2$ to θ_1, result (1.24) follows. Note that, taking $\theta_2=0$ in (1.24), the recurrence relation for the NBD model is obtained and is given by:

$$\varphi_s(k,a,\theta_1)=\int_0^{\theta_1}\theta_1^{k-1}(1+\theta_1)^{a-k-1}\varphi_{s-1}(k,a,\theta_1)\partial\theta_1.$$

The mean of QNBD model (1.1) results in an infinite series which renders it useful for estimating parameters by a method of moments. Next, a couple of recurrence relations between two means when their parameters are changed are proven. Suppose $\mu(a,\theta_1,\theta_2)$ represents the mean of the QNBD model with parameters (a,θ_1,θ_2), then the ratio of the mean – when the parameter θ_1 is changed to $(\theta_1+\theta_2)$ – to the mean when parameters are unchanged is independent of parameter a but is equal to the ratio $\frac{\theta_1+\theta_2}{\theta_1}$, that is,

$$\frac{\mu(a,\theta_1+\theta_2,\theta_2)}{\mu(a,\theta_1,\theta_2)}=\frac{\theta_1+\theta_2}{\theta_1} \qquad (1.30)$$

and

$$\mu(a+1,\theta_1+\theta_2,\theta_2)=\frac{(\theta_1+\theta_2)}{a\theta_1\theta_2}\mu(a,\theta_1,\theta_2)-\frac{(\theta_1+\theta_2)}{\theta_2} \qquad (1.31)$$

Proof

The mean $\mu(a,\theta_1,\theta_2)$ of the QNBD model is defined as:

$$\mu(a,\theta_1,\theta_2)=E(X)=\sum_{x=0}^{\infty}x\frac{(a+x-1)!}{(a-1)!x!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}=\sum_{x=1}^{\infty}\frac{(a+x-1)!}{(a-1)!(x-1)!}\frac{\theta_1(\theta_1+x\theta_2)^{x-1}}{(1+\theta_1+x\theta_2)^{a+x}}$$

Replacing x by $(x+1)$ in the equation above results in

$$\mu(a,\theta_1,\theta_2)=\frac{\theta_1}{(\theta_1+\theta_2)}\sum_{x=0}^{\infty}\frac{(a+x)(\theta_1+\theta_2+x\theta_2)}{(1+\theta_1+\theta_2+x\theta_2)}P_x(a,\theta_1,\theta_2) \qquad (1.32)$$

Rewriting the equation as

$$\mu(a,\theta_1,\theta_2)=\frac{\theta_1}{(\theta_1+\theta_2)}\sum_{x=0}^{\infty}(a+x)\left(\begin{array}{c}1-(1+\theta_1\\+\theta_2+x\theta_2)^{-1}\end{array}\right)P_x(a,\theta_1,\theta_2)$$

gives (1.30) after simplifying. Expressing (1.32) as

$$\mu(a,\theta_1,\theta_2) = \left[\left(\frac{a\theta_1}{(\theta_1+\theta_2)} \sum_{x=0}^{\infty} (\theta_1+\theta_2+x\theta_2) \right) \times \left(\frac{(a+x)!}{a!\,x!} \frac{(\theta_1+\theta_2)(\theta_1+\theta_2+x\theta_2)^{x-1}}{(1+\theta_1+\theta_2+x\theta_2)^{a+1+x}} \right) \right]$$

$$= a\theta_1 + \frac{a\theta_1\theta_2}{(\theta_1+\theta_2)} \mu(a+1,\theta_1+\theta_2,\theta_2)$$

and rearranging the terms results in (1.31).All results shown herein for the QNBD model are also true for the NBD model by taking $\theta_2 = 0$.

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