Journal of Modern Applied Statistical Methods

Volume 11 | Issue 1 Article 15

5-1-2012

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Cover Page Footnote
The authors are thankful to the referees and editor for their valuable comments. This work was partially supported by the Tehran Payame Noor University (PNU) through a grant.

Journal of Modern Applied Statistical Methods
May 2012, Vol. 11, No. 1, 179-189

1538 – 9472/12/$95.00

Statistical Inferences for Lomax Distribution Based on Record Values (Bayesian and Classical)

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A maximum likelihood estimation (MLE) based on records is obtained and a proper prior distribution to attain a Bayes estimation (both informative and non-informative) based on records for quadratic loss and squared error loss functions is also calculated. The study considers the shortest confidence interval and Highest Posterior Distribution confidence interval based on records, and using Mean Square Error MSE criteria for point estimation and length criteria for interval estimation, their appropriateness to each other is examined.

Key words: Lomax distribution; record values, maximum likelihood estimation, method of moment, Bayesian estimation, shortest interval, highest posterior density (HPD) interval, quadratic loss function, squared error loss function, prior density, posterior density, simulation, MSE.

Introduction

Let $X_1, X_2, X_3,$ be a sequence of independent and identically (iid) random variable with cumulative distribution (cdf) function $F(x)$ and probability density (pdf) $f(x)$ For $n \geq 1$ define

$$T(1) = 1, T(n+1) = \min\left\{ j : X_j \geq X_{T_{(n)}} \right\}.$$

The sequence $\{X_{T(n)}\}_{n=1}^{\infty}$ is known as an upper record value statistic and the sequence $\{T(n)\}_{n=1}^{\infty}$ is known as a record time sequence (Arnold, Balakrishnan & Nagaraja, 1998). Chandler (1952) was one of the first to study record theory and he defined a mathematical model for record values. Record values arise naturally in many applications involving data relating to weather, sports, economics and life testing studies. Many authors have studied records and their associated statistics as well as inference-based testing on records. Some of the best examples may be found in the works of Balakrishnan, Arnold, Nagaraja (1998), Ahsanullah (1998) and Nevzoroz (1946).

Sevgi, et al. (2005) examined the relationship between order statistics and records. Mohammad (2002) and Balakrishnan (1994) examined the recurrent relations between the moments for the generalized exponential and Lomax distributions. Ahsanullah (1974) studied record values received from Lomax distribution, and Ahsanullah and Holland (1994) discussed both scale and location estimation of the distribution of generalized extreme values based on records. Asgharzadeh (2009) discussed both MLE and Bayesian estimation based on record values and Chan (1998) presents interval estimation according to records for groups of scales and locations. Soliman and Abd Ellah (2006) compared Bayesian and Non-Bayesian estimation based on records.

The Lomax distribution plays an important role in reliability. Consider the one-parameter Lomax distribution with pdf

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$$f(x;\theta)=\frac{1}{\theta}(1+x)^{-\left(\frac{1}{\theta}+1\right)} \quad x\ge 0,\theta>0, \tag{1}$$

and cdf

$$F(x;\theta)=1-(1+x)^{-\frac{1}{\theta}} \quad x\ge 0,\theta>0.$$

An application of the Lomax distribution in receiver operating characteristic (ROC) was presented by Campbell and Ratnaparkhi (1993). Distributional properties and recurrence relation moments of record values was studied by Balakrishnan (1994) and Ahsanullah (1991). Much work has been done with respect to estimating the parameters using both classical and Bayesian techniques, and parametric and nonparametric inference based on record values have also been studied extensively (for example, see Ahmadia, et al., 2009; Soliman & Al-Abound, 2008; Baklizi, 2008).

This study has several components: It considers Lomax parameter estimation based on record values. It estimates the parameter θ using maximum likelihood and method of moment (MME) based on record values. It uses an appropriate selection of density function for a prior distribution to derive a Bayesian estimation based on record values. For the latter, by applying an appropriate selection for the prior density, the society parameter is controlled; this means that the Mean Square Error MSE) of the Bayesian estimation is controlled by controlling the parameters of this distribution. Finally, it derives the shortest interval estimation and Highest Posterior Density (HPD) interval estimation based on record values. Examples are used to illustrate the various components.

Point Estimation of θ Parameter: The Method of Maximum Likelihood Estimation

If $X_{T(1)},X_{T(2)},\ldots,X_{T(n)}$ represents the first n upper record values from the Lomax distribution in (1), then the joint distribution of $X_{T(1)},X_{T(2)},\ldots,X_{T(n)}$ is

$$f\left(x_{T(1)},x_{T(2)},\ldots,x_{T(n)}\right)=f\left(x_{T(n)};\theta\right)\prod_{i=1}^{n-1}h\left(x_{T(i)};\theta\right), \tag{2}$$

where

$$h\left(x_{T(i)};\theta\right)=\frac{f(x_{T(i)})}{1-F(x_{T(i)})}.$$

Thus, for the Lomax distribution,

$$f\left(x_{T(1)},x_{T(2)},\ldots,x_{T(n)}\right)=\theta^{-n}\frac{(1+x_{T(n)})^{-\frac{1}{\theta}}}{\prod_{i=1}^{n}(1+x_{T(i)})}, \tag{3}$$

and the log likelihood function is

$$L=-nln\theta-\left(\frac{1}{\theta}\right)\ln\left(1+x_{T(n)}\right)-\sum_{i=1}^{n}\ln\left(1+x_{T(i)}\right). \tag{4}$$

The maximum likelihood estimation (MLE) based on records can be obtained from (4) as

$$-\frac{n}{\theta}+\frac{\ln\left(1+x_{T(n)}\right)}{\theta^2}=0$$

and

$$\hat{\theta}=\frac{\ln(1+x_{T(n)})}{n}. \tag{5}$$

Using (2), the marginal pdf of $X_{T(n)}$ can be derived as

$$f\left(x_{T(n)}\right)=\frac{1}{\theta^n(n-1)!}\left(1+x_{T(n)}\right)^{-\left(\frac{1}{\theta}+1\right)}(\ln(1+x_{T(n)}))^{n-1}, \tag{6}$$

therefore,

$$\left(\hat{\theta}_{MLE}\right)=\theta, Var\left(\hat{\theta}_{mle}\right)=\frac{\theta^2}{n}.$$

conversely, if (3) is rewritten as

$$f\left(x_{T(1)},x_{T(2)},\ldots,x_{T(n)}\right)=$$
$$\exp(-nln\theta-\frac{1}{\theta}\ln\left(1+x_{T(n)}\right)-\sum_{i=1}^{n}\ln(1+x_{T(i)})),$$

then $\ln(1+x_n)$ is a complete sufficient statistic for parameter θ. Therefore $\hat{\theta}_{MLE}$ based on record is the equal to the Uniformly Minimum-Variance Unbiased Estimator (UMVUE) for parameter θ.

Point Estimation of θ Parameter: The Method of Moment Estimation

The MME, first introduced by Pearson (1894), was one of the first methods used to estimate the society parameter θ (for additional details and an example see Pearson, 1894). The Lomax parameter θ is estimated by the MME based on record values by using the density function (6), which results in

$$E\left(X_{T(n)}\right)=\frac{1}{(1-\theta)^{n}}-1.$$

Next, solving the equation

$$E\left(X_{T(n)}\right)=\bar{X}$$

yields a MME based on record values, where $\bar{X}$ is average of the n first records $(X_{T(1)},X_{T(2)},\ldots,X_{T(n)})$. Thus,

$$\hat{\theta}_{MME}=1-\frac{1}{(1+\bar{X})^{\frac{1}{n}}}.$$

Bayesian Estimation of θ Parameter

The Bayesian estimator of θ is obtained based on record values under the two following loss functions:

$$L\left(\hat{\theta},\theta\right)=\left(\frac{\hat{\theta}}{\theta}-1\right)^{2}, \tag{7}$$

and

$$L\left(\hat{\theta},\theta\right)=(\hat{\theta}-\theta)^{2}, \tag{8}$$

where $\hat{\theta}$ is an estimator of θ. Assuming an inverse Weibull distribution $\mathrm{IWD}(\gamma,\beta,c=1)$, the prior for θ is conjugated as

$$\pi\left(\theta\right)=\frac{1}{\Gamma(\gamma)\beta}(\frac{\beta}{\theta})^{\gamma+1}\exp\left(-\frac{\beta}{\theta}\right), \tag{9}$$

such that

$$E\left(\theta\right)=\frac{\beta}{\gamma-1},$$
$$Var\left(\theta\right)=\frac{\beta^{2}}{\left(\gamma-1\right)^{2}(\gamma-2)}$$

where $\gamma>0,\ \beta>0$. Note that $\frac{1}{\theta}\sim gamma(\gamma,\beta)$. This prior density has an advantage over other priors because it is easy to use and the parameter (γ,β) can be chosen such that prior precision for the true value of θ is fulfilled because Bayesian estimations are functions of (γ,β), therefore, the precision of the Bayesian estimations cannot be controlled by altering the prior distribution parameters. Combining likelihood function (3) with prior density (9), the posterior density of θ is obtained as

$$\pi\left(\theta|x\right)=$$
$$\left\{\begin{array}{l}\dfrac{1}{\Gamma\left(n+\gamma\right)\left[\ln\left(1+x_{T(n)}\right)+\beta\right]}\left(\dfrac{\ln\left(1+x_{T(n)}\right)+\beta}{\theta}\right)^{n+\gamma1}\\ \times\exp\left(-\dfrac{\ln\left(1+x_{T(n)}\right)+\beta}{\theta}\right)\end{array}\right\}$$

where $x=x_{T(1)},\ldots,x_{T(n)},\ \theta>0$. Note that

$$\left(\frac{1}{\theta}|x_{T(1)},\ldots,x_{T(n)}\right) \sim gamma\left(n+\gamma, \beta+\text{Ln}\left(1+x_{T(n)}\right)\right).$$

Bayesian Estimator of θ Under Quadratic Loss Function

The posterior distribution of θ is $\text{IWD}(n+\gamma, \beta+\ln\left(1+x_n\right), c=1)$, where IWD is Inverse Weibull Distribution (in other words $IWD = \frac{1}{W}$ where W is a Weibull variable) and the Bayes estimator of θ is based on record values under a quadratic loss function (7), for example $\hat{\theta}_{b,1}$, as given by Berger (1985) is

$$\begin{aligned}\hat{\theta}_{b,1} &= \frac{E\left(\omega(\theta)\mu(\theta)|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right)}{E\left(\omega(\theta)|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right)} \\ &= \frac{E\left(\frac{1}{\theta}|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right)}{E\left(\frac{1}{\theta^2}|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right)} \\ &= \frac{\beta+Ln\left(1+x_{T(n)}\right)}{n+\gamma+1}.\end{aligned} \tag{10}$$

Bayesian Estimator of θ Under Squared Error Loss Function

Considering the posterior distribution of θ and loss function (8), the Bayes estimator based on record values, for example, $\hat{\theta}_{b,2}$, is given as (Berger, 1985):

$$\begin{aligned}\hat{\theta}_{b,2} &= E\left(\mu(\theta)|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right) \\ &= E\left(\theta|X_{T(1)}, X_{T(2)},\ldots,X_{T(n)}\right) \\ &= \frac{Ln\left(1+x_{T(n)}\right)+\beta}{n+\gamma-1}.\end{aligned}$$

As a result, the Bayesian estimation is formed as a differentia combination of both prior distribution and sample distribution as:

$$\hat{\theta}_{b,2} = \frac{n}{n+\gamma-1}\frac{\ln\left(1+x_{T(n)}\right)}{n} + \frac{\gamma-1}{n+\gamma-1}\frac{\beta}{\gamma-1}. \tag{11}$$

Interval Estimation of θ Based on Record Values: The Shortest Interval Estimation

To obtain the shortest $(1-\alpha)\%$ confidence interval estimation based on record values, a pivot quantity is chosen as a function of a minimal sufficient statistic for parameter θ ($\hat{\theta}_{MLE}$) such that

$$Q = \frac{2\ln\left(1+X_{T(n)}\right)}{\theta}.$$

From (6) it is clear that the distribution of Q is χ_{2n} for any constants a and b, hence,

$$\begin{aligned}P\left(a<Q<b\right) &= \int_a^b f_Q\left(t\right)dt \\ &= 1-\alpha.\end{aligned} \tag{12}$$

Algebraic manipulation results in the confidence interval

$$\frac{2\ln\left(1+x_{T(n)}\right)}{b} < \theta < \frac{2\ln\left(1+x_{T(n)}\right)}{a},$$

thus, the length of interval is obtained as

$$L = 2\ln\left(1+x_{T(n)}\right)\left[\frac{1}{a}-\frac{1}{b}\right]. \tag{13}$$

To minimize (13) and satisfy (12), a and b are selected using the Lagrange multipliers method

$$\psi\left(a,b,\lambda\right) = 2\ln\left(1+x_{T(n)}\right)\left(\frac{1}{a}-\frac{1}{b}\right)+\lambda\left(\int_a^b f_Q\left(t\right)dt-\left(1-\alpha\right)\right).$$

After derivation by λ, a, and b, the following results:

$$\begin{cases} \int_a^b f_Q(t)\,dt = 1-\alpha \\ \dfrac{-2\ln\left(1+x_{T(n)}\right)}{a^2} - \lambda f(a) = 0 \\ \dfrac{2\ln\left(1+x_{T(n)}\right)}{b^2} + \lambda f(b) = 0 \end{cases} \Rightarrow \begin{cases} \int_a^b f_Q(t)\,dt = 1-\alpha \\ a^2 f_Q(a) = b^2 f_Q(b) \end{cases}.$$

Accordingly, a and b must satisfy (14) to yield the shortest interval estimation for θ:

$$P(a<Q<b) = \int_a^b f_Q(t)\,dt = 1-\alpha, \quad a^2 f_Q(a) = b^2 f_Q(b) \qquad (14)$$

Interval Estimation of θ Based on Record Values: Highest Posterior Density (HPD) θ Estimation

After obtaining the posterior distribution $\pi\left(\theta | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right)$, the problem of the likelihood that the parameter θ lies within the interval $[c_L, c_U]$ arises. Bayesians call the interval based on the posterior distribution a credible interval; the interval $[c_L, c_U]$ is said to be a $(1-\alpha)\%$ credible interval for θ if

$$\int_{c_L}^{c_U} \pi\left(\theta | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right) d\theta = 1-\alpha. \qquad (15)$$

The Highest Posterior Density (HPD) region is given by $\left\{A : \pi\left(\theta | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right) \geq c\right\}$ where c is chosen so that

$$\int_{c_L}^{c_U} \pi\left(\theta | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right) d\theta = 1-\alpha$$

$$\pi\left(c_L | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right) = \pi\left(c_U | X_{T(1)}, X_{T(2)}, \ldots, X_{T(n)}\right) \qquad (16)$$

The HPD interval estimation is optimal in the sense that it results in the shortest interval. Let $\lambda = \frac{1}{\theta}$, by this assumption the posterior distribution of λ is Gamma $(n+\gamma, \ln(1+x_n)+\beta)$. After algebraic manipulation, an HPD estimation $(1-\alpha)\%$ for parameter θ based on records is given by

$$\frac{\Gamma^*(n+\gamma, Ac_L, Ac_U)}{\Gamma(n+\gamma)} = 1-\alpha \quad, \left(\frac{c_L}{c_U}\right)^{n+\gamma-1} = \exp\left(c_L A - c_U A\right), \qquad (17)$$

where $A = \beta + \ln\left(1+x_{T(n)}\right)$ and Γ^* is the generalized incomplete Gamma function. Therefore HPD interval estimation based on record values can be obtained as:

$$\theta \in \left[\frac{1}{c_U}, \frac{1}{c_L}\right]. \qquad (18)$$

Simulation and Examples

MSE and Bias

To illustrate the estimation techniques developed, consider the following simulated data from the Lomax distribution:

3.286379	2.652416	1.325698
1.895476	16.420820	10.123657
1.254875	14.852147	12.985314
11.684235	15.365742	1085.950045
50.254198	850.569874	32.154875
950.548796	2423.065086	1.989562
84.254187	1240.325487	7372.085167
2.658474	352.325469	6524.123548
15.987455	33.659874	5487.214587
1.235478	3658.125489	9083.239327
48.236584	6985.125489	6.325698
448.125634	8754.215487	47739.689056
125.258643	25.365987	12543.2158746
25413.125487	256.326598	1254.365241
1.36548	16845.362545	25.326874
6985.125469	7.365214	121942.356923

This data was obtained by using the transformation $x_i = \frac{1}{(1-u_i)^{\theta}} - 1$, where u_i is a uniformly distributed random variable. If only the upper record values have been observed, these are:

3.286379	7372.085167
16.420820	9083.239327
1085.950045	47739.689056
2423.065086	121942.356923

for a non-informative prior distribution with $\gamma = 0, \beta = 1$, and $\gamma = 8, \beta = 7.56$ for an informative prior distribution. Results from equations (5), (10) and (11) for the parameter θ computed for n=4, 5, 6, 7, 8 are presented in Table1.

Interval Estimation

Results from using equations (14) and (15) for the parameter, $\lambda = \frac{1}{\theta}$ computed for n=4, 5, 6, 7, 8 are presented in Tables 2 and 3.

Table 1: Estimation, Bias and MSE

Number of Records (n=)		Estimate	Bias	MSE
4 5 6 7 8	$\hat{\theta}_{MLE}$	1.948300 1.781118 1.519049 1.539077 1.463914	0 0 0 0 0	0.9489683 0.6344765 0.3845852 0.3383940 0.2678805
4 5 6 7 8	$\hat{\theta}_{b,1}$ Non-Informative	1.758640 1.650932 1.444899 1.471692 1.412368	-0.18966006 -0.13018638 -0.07414991 -0.06738463 -0.05154600	0.6433108 0.4575572 0.2880506 0.2636236 0.2143157
4 5 6 7 8	$\hat{\theta}_{b,2}$ Non-Informative	2.931067 2.476398 2.022859 1.962257 1.815902	0.9827668 0.6952796 0.5308099 0.4231795 0.3519877	2.6528857 1.4747832 0.8076270 0.6396728 0.4737801
4 5 6 7 8	$\hat{\theta}_{b,1}$ Informative	1.181015 1.176114 1.111620 1.145846 1.133607	-0.7672848 -0.6050046 -0.4074296 -0.3932308 -0.3303074	0.6785692 0.4469587 0.2275325 0.2194012 0.1684260
4 5 6 7 8	$\hat{\theta}_{b,2}$ Informative	1.395746 1.372133 1.282638 1.309539 1.284754	-0.5525547 -0.4089857 -0.2364112 -0.2295385 -0.1791599	0.4308002 0.2774215 0.1378137 0.1372864 0.1082954

Table2: Shortest $(1-\alpha)\%$ Confidence Interval Estimation Based On Record Values

Number of Records (n=)	Interval		Length	$(1-\alpha)\%$ Confidence
4	0.7071710	4.549181	3.842010	90%
5	0.7314714	3.743103	3.011632	
6	0.6822589	2.965349	2.283090	
7	0.7401722	2.843073	2.102900	
8	0.7432804	2.589680	1.846400	
4	0.6255906	5.766975	5.141384	95%
5	0.6532331	4.580130	3.930780	
6	0.6139227	3.545798	2.931875	
7	0.6765692	3.340115	2.669904	
8	0.6765692	3.001246	2.324677	
4	0.5012662	9.505643	9.003677	99%
5	0.5321775	7.002903	6.470725	
6	0.5064018	5.142492	4.636090	
7	0.5586008	4.663650	4.104764	
8	0.5689107	4.069324	3.500414	

Table3: Highest Posterior Distribution (HPD) $(1-\alpha)\%$ Interval Estimation Based On Record Statistics

Number of Records (n=)	$\lambda \in (\lambda_L, \lambda_U)$		$\theta \in [\frac{1}{\lambda_U}, \frac{1}{\lambda_L}]$		Length	$(1-\alpha)\%$ Confidence
4	0.414500	1.137700	0.878966	2. 412545	1.533579	90%
5	0.430000	1.140900	0.876501	2.325581	1.449080	
6	0.472900	1.196200	0.835980	2.114612	1.278631	
7	0.467300	1.164000	0.859100	2.139900	1.280846	
8	0.483600	1.173200	0.852369	2.067825	1.215455	
4	0.3668000.	1.237600	0.808015	2.726281	1.918266	95%
5	383300	1.236500	0.808734	2.608923	1.800188	
6	0.423500	1.293800	0.772917	2.361275	1.588358	
7	0.421100	1.254100	0.797384	2.374733	1.577384	
8	0.437700	1.260800	0.793147	2.284670	1.491523	
4	0.295500	1.416700	0.705865	3.384095	2.678229	99%
5	0.312600	1.407700	0.710378	3.198976	2.488598	
6	0.348500	1.468000	0.681198	2.869400	2.188242	
7	0.350100	1.415400	0.706514	2.856327	2.149813	
8	0.366900	1.417600	0.705417	2.725538	2.020121	

Figure 1: MSE's of the Estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{b,1}$, and $\hat{\theta}_{b,2}$ Informative and Non-Informative

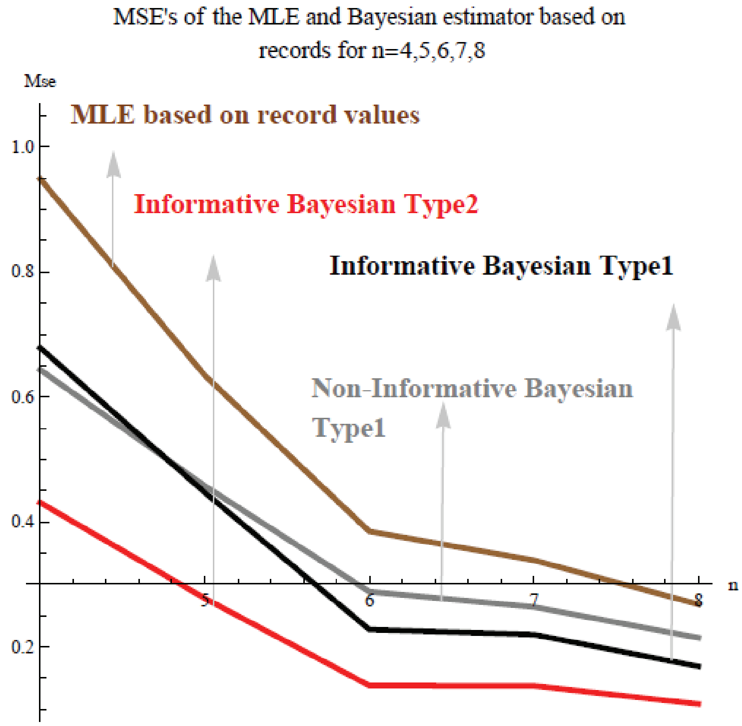


Figure 2: Lengths of the Shortest Interval and Highest Posterior Distribution (HPD) Estimations Based on Record Statistics for 90% Confidence

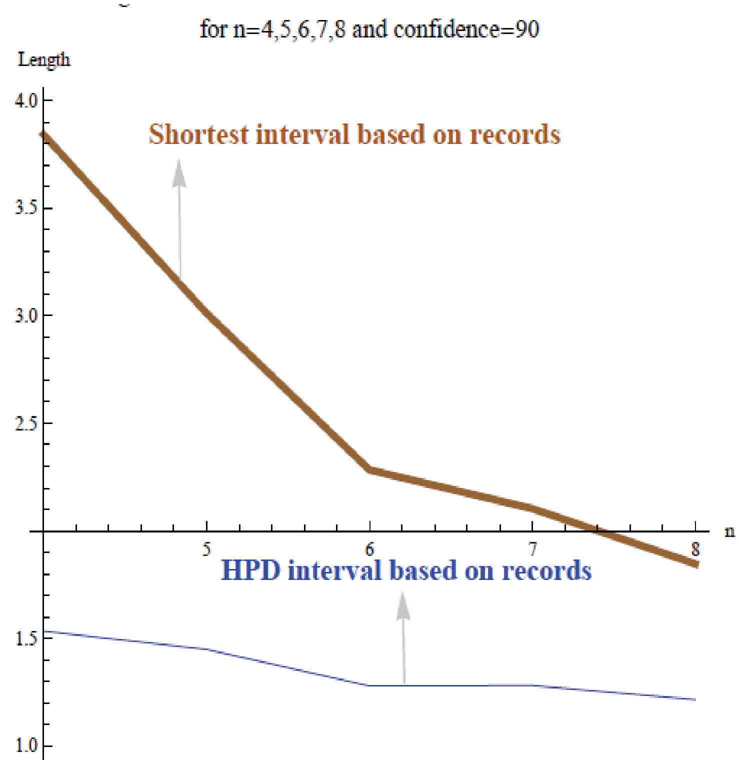


Figure 3: Lengths of the Shortest Interval and Highest Posterior Distribution (HPD) Estimations Based on Record Statistics for 95% Confidence

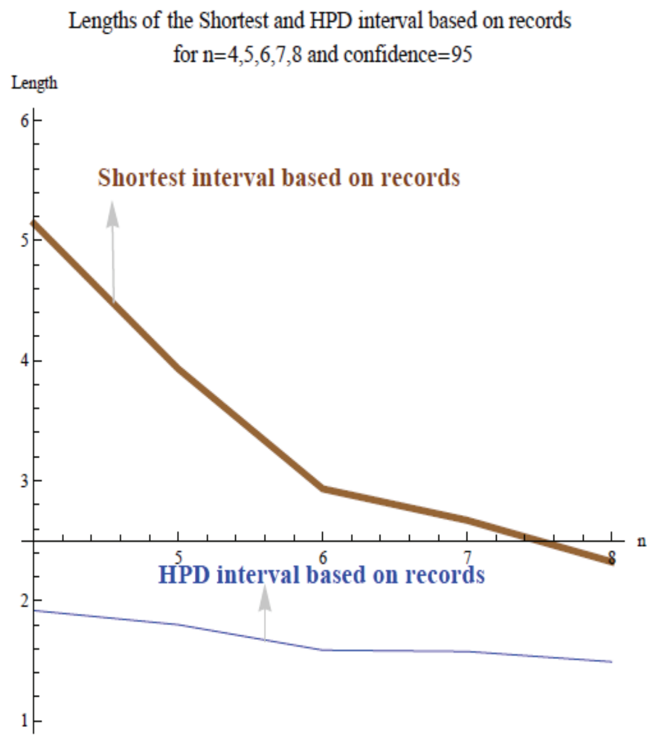


Figure 4: Lengths of the Shortest Interval and Highest Posterior Distribution (HPD) Estimations Based on Record Statistics for 99% Confidence

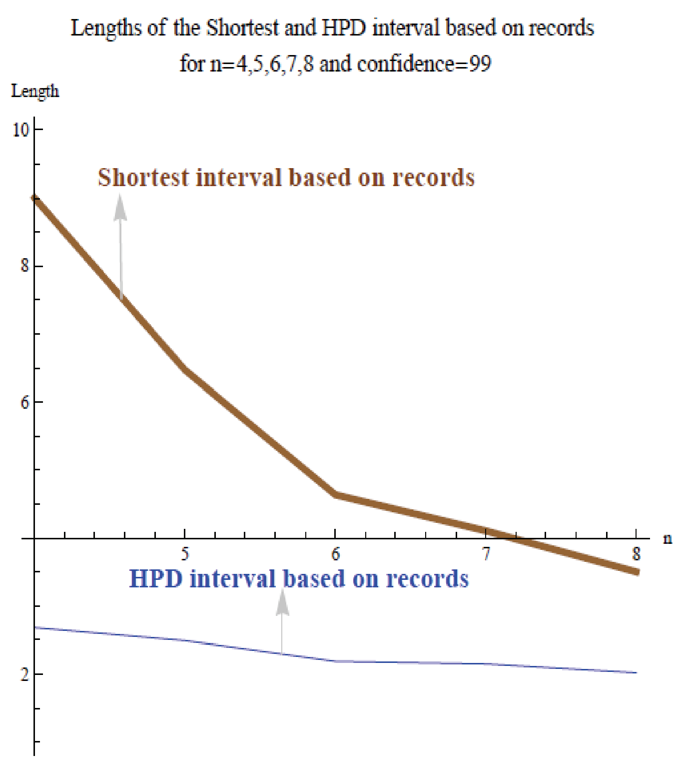
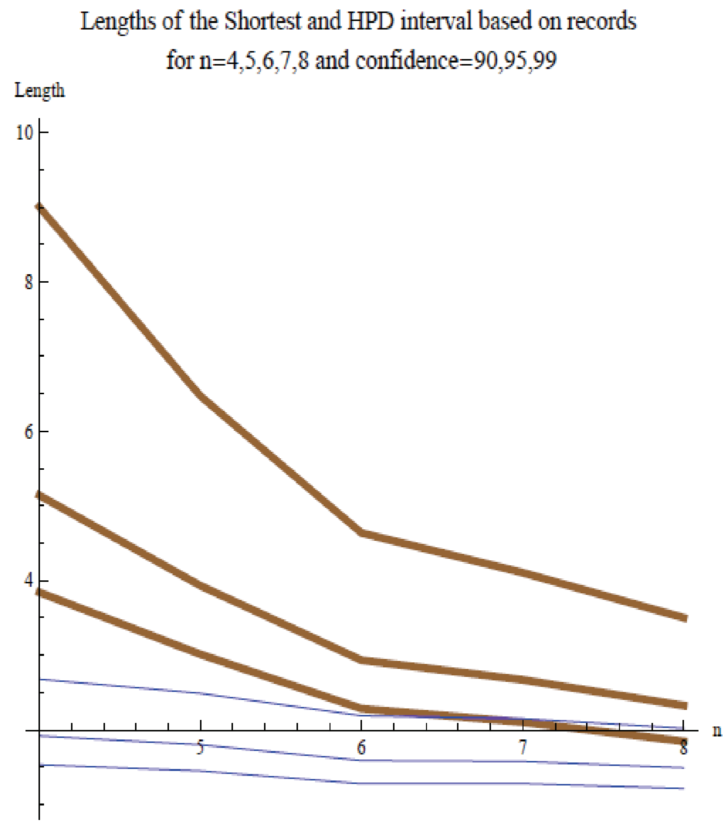


Figure 5: Lengths of the Shortest Interval and Highest Posterior Distribution (HPD) Estimations Based on Record Statistics for 90%, 95% and 99% Confidence



Conclusion

MLE and Bayesian estimations based on record values were obtained. For the Bayes estimations, in order to control the passive parameter of society, the prior distribution was assumed to be Gamma. In addition, Bayes estimations were obtained for two types of loss functions and, with a view of prior estimation, using an informative posterior density function, HPD estimations were obtained in a theoretic way (see Table 3). Conversely, the shortest confidence interval was obtained using a MLE based on records and equation (14) (Tate & Klett, 1959); see Table 2 for results.

Theoretical results of the study are explained numerically by simulation in the following ways: Table 1 shows that an informative Bayesian estimation based on records under squared error loss function has the lowest MSE compared to the informative Bayesian estimation, which is based on records under a quadratic loss function with a non-informative Bayesian estimation under a squared error loss function.. This is also compared to a MLE based on records; comparisons are shown in Figure 1. Confidence intervals and their lengths for record numbers 4, 5, 6, 7, 8 and confidence levels 90%, 95% and 99% were obtained. The longer the *n*, the shorter the interval distance (see Table 3). Comparing Tables 2 and 3, it the point at which HPD estimations have a shorter length than the confidence interval with optimal length is observed. This comparison is illustrated in Figures 2, 3 and 4 for various confidence levels; Figure 5 shows the comparison for all levels.