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Gap Labeling

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Benameur, Moulay-Tahar (F-LYON-GD); **Oyono-Oyono, Hervé** (F-CLEF2-LPM)
Gap-labelling for quasi-crystals (proving a conjecture by J. Bellissard). (English summary)
Operator algebras and mathematical physics (*Constanţa*, 2001), 11–22, Theta, Bucharest, 2003.

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Kaminker, Jerome (1-INPI); **Putnam, Ian** (3-VCTR)
A proof of the gap labeling conjecture.
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FEATURED REVIEW.

This joint review covers the two papers in the heading as well as an article by J. Bellissard, R. Benedetti and J.-M. Gambaudo ["Spaces of tilings, finite telescopic approximations, and gap-labeling", Comm. Math. Phys., to appear] (hereafter BO, KP and BBG, respectively).

The gap labeling theorem was originally conjectured by Bellissard [in *From number theory to physics (Les Houches, 1989)*, 538–630, Springer, Berlin, 1992; MR1221111 (94e:46120)]. The problem arises in a mathematical version of solid state physics in the context of aperiodic tilings. Its three proofs, discovered independently by the authors above, all lie in K-theory. Here is the core result of these papers:

Let Σ be a Cantor set and let

$$\Sigma \times \mathbb{Z}^d \to \Sigma$$

be a free and minimal action of $\mathbb{Z}^d$ on Σ with invariant probability measure μ. Let

$$\mu\colon C(\Sigma) \to \mathbb{C}$$

and

$$\tau_\mu\colon C(\Sigma) \rtimes \mathbb{Z}^d \to \mathbb{C}$$

be the traces induced by μ and denote likewise their induced maps in K-theory. Then

$$\mu\left(K_0(C(\Sigma))\right) = \tau_\mu\left(K_0(C(\Sigma) \times \mathbb{Z}^d)\right)$$

as subsets of $\mathbb{R}$.

We shall try to explain why this core result has anything to do with something called gap labeling.

This review is organized as follows:

(1) The origin of the problem and its formulation in mathematical terms.
(2) Foliated spaces as a setting for the common formalism.
(3) The BBG proof.
(4) Common features of the BO and KP proofs.
(5) The BO proof.
(6) The KP proof.

(7) Earlier partial results.

1. The origin of the problem. We model the motion of a particle in a solid via the tight binding approximation as follows. The solid is modeled by a tiling, where the tiles represent the locations of the atoms, and the particle hops from tile to tile. The (simplified!) quantum mechanical model of this motion is a certain self-adjoint Schrödinger operator on the space of square summable functions on the set of tiles. So the position of the particle is represented by a tile and momentum corresponds to translation. We are interested in the spectrum of this operator. In the crystal context, Bloch theory shows that the periodic structure of the atoms leads to a spectrum consisting of bands—i.e., a union of closed intervals, and hence there are gaps in the spectrum. The challenge in the present problem is to determine the gaps in the spectrum of the Schrödinger operator in a solid which is not periodic but is almost periodic.

More formally, a tiling T of $\mathbb{R}^d$ is a collection of subsets $\{t_1, t_2, \ldots\}$ called tiles, such that their union is $\mathbb{R}^d$ and their interiors are pairwise disjoint. We assume that each tile is homeomorphic to a closed ball. Any translate $T + x$ of T by some $x \in \mathbb{R}^d$ is again a tiling. Take the set $T + \mathbb{R}^d$ of all translates and endow it with a metric: for $0 < \varepsilon < 1$, say, the distance between T_1 and T_2 in $T + \mathbb{R}^d$ is less than ε if there are vectors x_1, x_2 of length less than ε such that $T_1 + x_1$ and $T_2 + x_2$ coincide on the open ball $B(0, 1/\varepsilon)$. Let Ω denote the closure of $T + \mathbb{R}^d$ in this metric. Then $\mathbb{R}^d$ acts on Ω; the action is denoted ω. The space Ω is the continuous hull of the tiling. (This is the quick and dirty definition of the metric and Ω: there are much better and more natural definitions—cf. BBG.)

We assume that for any $R > 0$ there are only finitely many subsets of T whose union has diameter less than R (the so-called finite pattern condition), which ensures that Ω is compact. The orbit of T is obviously dense. We assume that every orbit is dense: in other words, that the $\mathbb{R}^d$ action on Ω is minimal. This is the case if and only if for every finite patch P in T there is some $R > 0$ such that for each $x \in \mathbb{R}^d$ there is a translate of P contained in $T \cap B(x, R)$. This is called the repetitivity condition.

There is an equivalent version of this construction using the notion of repetitive Delone sets of finite type due to Lagarias. Bellissard, D. J. L. Herrmann and M. Zarrouati [in *Directions in mathematical quasicrystals*, 207–258, Amer. Math. Soc., Providence, RI, 2000; MR1798994 (2002a:82101)] replaced a discrete point set by the sum of mass one Dirac measures at each site. The compactness of the hull is then a trivial consequence of well-known theorems in measure theory. This point of view is more natural from the point of view of the hull topology.

A tiling T is aperiodic if $T + x \neq T$ for all $x \in \mathbb{R}^d \setminus \{0\}$, and a tiling is strongly aperiodic if Ω contains no periodic tilings. Assume henceforth that T is strongly aperiodic and satisfies the finite pattern and repetitivity conditions; thus Ω is compact with a free and minimal $\mathbb{R}^d$-action ω. There is a natural C^*-algebra to model the situation, namely $C(\Omega) \rtimes \mathbb{R}^d$, referred to by Bellissard as the noncommutative Brillouin zone. Bellissard's deep insight was to regard the dynamical system via this C^*-algebra as a noncommutative space, in the sense introduced by Alain Connes, and to show that the resolvents of the Schrödinger operator lie in it. Gaps in the spectrum will yield projections in $C(\Omega) \rtimes \mathbb{R}^d$ and the classes of those projections lie in $K_0(C(\Omega) \rtimes \mathbb{R}^d)$. Any trace on the C^*-algebra yields a homomorphism $K_0(C(\Omega) \rtimes \mathbb{R}^d) \to \mathbb{R}$ whose image is a countable subgroup of $\mathbb{R}$. For natural choices of the trace, these numbers have physical and mathematical meaning. They are related to the integrated density of states and also can be obtained experimentally. Thus it is worthwhile to try to determine this subgroup of $\mathbb{R}$.

L. A. Sadun and R. F. Williams [Ergodic Theory Dynam. Systems **23** (2003), no. 1, 307–316; MR1971208 (2004a:37023)] showed that the hull Ω contains a Cantor set Σ with a minimal $\mathbb{Z}^d$-action such that there is a homeomorphism

$$\Sigma \times_{\mathbb{Z}^d} \mathbb{R}^d \cong \Omega.$$

(The set Σ can be constructed as a canonical transversal. To do so, each prototile is associated with a point in its interior; then take Σ to be the union of the tilings having one tile with point at the origin. Then Σ is defined modulo the choice of a point in each prototile. In the description via a Delone set there is no choice, since the position of atoms is already fixed and therefore the transversal becomes "canonical".) This homeomorphism does not conjugate the $\mathbb{R}^d$-actions. However, KP show that there is a strong Morita equivalence of associated C^*-algebras

$$C(\Sigma) \rtimes \mathbb{Z}^d \approx C(\Omega) \rtimes \mathbb{R}^d$$

and so these two C^*-algebras have isomorphic K-theory groups. We regard Σ with its $\mathbb{Z}^d$-action as a discrete model for the foliated space Ω.

2. Foliated spaces. Every point in Ω has an open neighborhood of the form $U \times N$, where U is open in $\mathbb{R}^d$ and N is a Borel subset of Ω. If N were an open subset of Euclidean space then this would be the local picture of a foliated manifold. This is not the case generally. Instead, this is the local picture of a foliated space.

A side note on terminology. In ancient times a lamination was a space obtained by deleting some leaves of a foliated manifold. C. C. Moore and the reviewer [*Global analysis on foliated spaces*, Springer, New York, 1988; MR0918974 (89h:58184); second edition, Cambridge Univ. Press, to appear] introduced foliated space to describe a space whose local picture is $U \times N$ as above. This includes laminations as well as other situations such as the continuous hull. This usage is found, e.g., in [A. Candel and L. Conlon, *Foliations. I*, Amer. Math. Soc., Providence, RI, 2000; MR1732868 (2002f:57058)]. More recently, É. Ghys [in *Dynamique et géométrie complexes (Lyon, 1997)*, ix, xi, 49–95, Soc. Math. France, Paris, 1999; MR1760843 (2001g:37068)] and others have taken to using lamination for this more general concept. We will stick with the foliated space terminology.

Suppose given a $\mathbb{Z}^d$-invariant probability measure $\mu\colon C(\Sigma) \to \mathbb{C}$. This gives rise to an invariant transverse measure on Ω with corresponding Ruelle-Sullivan current C_μ and associated homology class

$$[C_\mu] \in H_d^\tau(\Omega).$$

Here H_*^τ denotes tangential homology [cf. C. C. Moore and C. L. Schochet, op. cit. (Chapter III)]. This gives rise to traces

$$\mu\colon C(\Sigma) \to \mathbb{C} \quad \text{and} \quad \tau_\mu\colon C(\Sigma) \rtimes \mathbb{Z}^d \to \mathbb{C}$$

and associated homomorphisms

$$\mu\colon K_0(C(\Sigma)) \to \mathbb{R} \quad \text{and} \quad \tau_\mu\colon K_0(C(\Sigma) \rtimes \mathbb{Z}^d) \to \mathbb{R}.$$

The group $K_0(C(\Sigma))$ is isomorphic to $C(\Sigma, \mathbb{Z})$, the continuous, integer-valued functions on Σ, and we may describe its image under the trace $\mu(K_0(C(\Sigma)))$ as the subgroup of $\mathbb{R}$ generated by the measures of the clopen sets of Σ. It is not very hard to prove that

$$\mu\Big(K_0(C(\Sigma))\Big) \subseteq \tau_\mu\Big(K_0(C(\Sigma) \rtimes \mathbb{Z}^d)\Big).$$

The deepest part of the gap labeling theorem is to demonstrate that this inclusion is actually an equality of sets.

Note that each gap in the spectrum of the self-adjoint operator associated to the motion of the particle in the initial tiling corresponds to a projection in the C^*-algebra $C(\Omega) \rtimes \mathbb{R}^d$ of the foliated space Ω and hence to a class in

$$K_0(C(\Omega) \rtimes \mathbb{R}^d) \cong K_0(C(\Sigma) \rtimes \mathbb{Z}^d).$$

Bellissard, Herrmann and Zarrouati [op. cit.] proved that the integrated density of the states of the operator depends only upon the noncommutative space Ω itself, and not upon the operator. Thus the possible values of the gap labeling are independent of the choice of operator; they depend only upon the noncommutative topology of Ω.

All three proofs of the gap labeling theorem proceed by translating the gap labeling problem to tangential cohomology via some version of the Chern character and then by a combination of direct computation and deep general results.

3. The BBG proof. BBG consider a somewhat more general situation than described above. As this review focuses upon the K-theory result, we must omit details. We urge the reader to study the paper, as it has interesting applications beyond the immediate K-theoretic concern of the gap labeling theorem. BBG provide a geometric analysis of the foliated space itself. They represent Ω as topologically conjugate to the inverse limit of expanding flattening sequences of branched oriented flat manifolds of dimension d (BOF d-manifolds) with $\mathbb{R}^d$ action by parallel transport under constant vector fields. The cohomology of the BOF manifolds is analyzed combinatorially via cellular cohomology and a spectral sequence is used to calculate the K-theory of the associated C^*-algebras. Taking direct limits then yields a very concrete description of $K_*(C(\Omega \rtimes \mathbb{R}^d))$. The associated Ruelle-Sullivan maps are then explicitly calculated. BBG use a partial Chern character map c and then must deal with the diagram

$$\begin{array}{ccc} K_0(C(\Omega) \rtimes \mathbb{R}^d) & \xrightarrow{c} & H^d_\tau(C(\Omega) \rtimes \mathbb{R}^d) \\ \downarrow{\scriptstyle \tau_\mu} & & \downarrow{\scriptstyle (-)\cap C_\mu} \\ \mathbb{R} & \xrightarrow{\cong} & \mathbb{R} \end{array}$$

where $\cap C_\mu$ is the cap product by the class $[C_\mu] \in H_d^\tau(\Omega)$ of the Ruelle-Sullivan current induced by the trace. Their proof requires the use of Connes' Thom isomorphism theorem [A. Connes, Adv. in Math. **39** (1981), no. 1, 31–55; MR0605351 (82j:46084)] as well as cyclic cohomology [A. Connes, in *Geometric methods in operator algebras (Kyoto, 1983)*, 52–144, Longman Sci. Tech., Harlow, 1986; MR0866491 (88k:58149)].

4. Common features of the BO and the KP proofs. Consider the diagram

$$\begin{array}{ccc}
K_0(C(\Sigma)) & \xrightarrow{\mu} & \mathbb{R} \\
\big\downarrow i_* & & \big\downarrow \cong \\
K_0(C(\Sigma \rtimes \mathbb{Z}^d)) & \xrightarrow{\tau_\mu} & \mathbb{R} \\
\big\downarrow \text{m.e.} & & \big\downarrow \equiv \\
K_0(C(\Omega \rtimes \mathbb{R}^d)) & \xrightarrow{\overline{\tau}_\mu} & \mathbb{R} \\
\big\uparrow \cong \varphi_c & & \big\uparrow \equiv \\
K_d(C(\Omega)) & & \mathbb{R} \\
\big\downarrow \text{ch}^d & & \big\downarrow \equiv \\
H_\tau^d(\Omega; \mathbb{R}) & \xrightarrow{\cap C_\mu} & \mathbb{R}
\end{array}$$

where Σ is the given Cantor set with the given $\mathbb{Z}^d$ action,

$$\Omega = \Sigma \times_{\mathbb{Z}^d} \mathbb{R}^d$$

is the suspension of the action, φ_c is Connes' Thom isomorphism, i_* is the map induced by the inclusion of C^*-algebras, m.e. is the isomorphism induced by the Morita equivalence of the C^*-algebras, and (this is a result of T. Fack and G. Skandalis [Invent. Math. **64** (1981), no. 1, 7–14; MR0621767 (82g:46113)])

$$\varphi_c([E]) = \text{ind}_a[D_E] \in K_0(C(\Omega \rtimes \mathbb{R}^d))$$

where $\text{ind}_a[D_E]$ is the analytic index of the Dirac D operator twisted by the bundle E. The top square commutes by the definition of the traces. The middle square is shown to commute by looking carefully at properties of the Morita equivalence.

The bottom rectangle commutes by the index theorem for foliated spaces. KP prove this as KP 2.4, and BO prove it as BO 4.2. (Since Ω is a foliated space but not a foliated manifold, one needs the version of the Index Theorem established by Moore and the reviewer [op. cit.].) The analogous result of BBG is Theorem 6.1, which they prove by reduction to a result in cyclic cohomology [A. Connes, op. cit.; MR0866491 (88k:58149)].

5. The BO proof. BO filter the leaves of Ω and obtain a pair of spectral sequences

$$E^2 = H_*(\mathbb{Z}^d; C(\Omega, \mathbb{Z})) \Longrightarrow K_*(C^*(C(\Omega) \rtimes \mathbb{R}^d))$$

and

$$\tilde{E}^2 = H_*(\mathbb{Z}^d; C(\Omega, \mathbb{R})) \Longrightarrow H_\tau^*(C(\Omega) \rtimes \mathbb{R}^d).$$

The Chern character induces a natural transformation

$$\text{ch} : E^r \to \tilde{E}^r.$$

Both spectral sequences collapse at the E^2 level, essentially because $\text{ch} \otimes \mathbb{Q}$ is an isomorphism. This makes it possible to explicitly identify the image of

$$\text{ch} : K_*(C^*(C(\Omega) \rtimes \mathbb{R}^d)) \longrightarrow H_\tau^*(C(\Omega) \rtimes \mathbb{R}^d)$$

as $H_*(\mathbb{Z}^d; C(\Omega, \mathbb{Z}))$. This integrality result leads to an identification of the top component of the

Chern character and implies the gap labeling theorem.

6. The KP proof. KP rely on a less commutative approach. Let $\pi : \Sigma \times \mathbb{R}^d \to \Omega$ be the quotient map, let L denote the union of the hyperplanes parallel to the coordinate axis and going through the points of $\mathbb{Z}^d$, let $Y = \pi(\Sigma \times L)$ and $j\colon \Omega - Y \to \Omega$ be the inclusion, and let

$$\alpha : K_0(C(\Sigma)) \to K_0(C(\Omega) \rtimes \mathbb{R}^d)$$

be the map described by Connes in [*Noncommutative geometry*, Academic Press, San Diego, CA, 1994; MR1303779 (95j:46063) (p. 120)] (modified for foliated spaces) that associates—to a clopen set in a transversal to a foliation—a projection in its foliation algebra. They show (KP 3.2) that the natural diagram

$$\begin{array}{ccc} K_0(C(\Sigma)) & \xrightarrow{\alpha} & K_0(C(\Omega) \rtimes \mathbb{R}^d) \\ \downarrow \beta & & \uparrow \varphi_c \\ K_d(C_0(\Omega - Y)) & \xrightarrow{j_*} & K_d(C(\Omega)) \end{array}$$

commutes, where β is Bott periodicity. Then an explicit study of the partial Chern character ch_n implies the gap labeling theorem.

7. Earlier results. To complete this review, we note that there were previous partial results on the problem. The conjecture was first established in the case $d = 1$ by Bellissard [op. cit.] using the Pimsner-Voiculescu long exact sequence, and the case $d = 2$ was done by A. van Elst [Rev. Math. Phys. **6** (1994), no. 2, 319–342; MR1269302 (95f:46122)] using a similar technique. The $d = 2$ result was reestablished using the Kasparov spectral sequence in [J. Bellissard, E. Contensou and A. Legrand, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 2, 197–200; MR1646928 (99h:46131)]. In the case where the hull is given by an action of $\mathbb{Z}^d$ on a Cantor set Σ, A. H. Forrest and J. R. Hunton [Ergodic Theory Dynam. Systems **19** (1999), no. 3, 611–625; MR1695911 (2000e:19006)] used spectral sequence techniques to prove that the K-theory group is isomorphic to the group cohomology $H^*(\mathbb{Z}^d; C(\Sigma, \mathbb{Z}))$, which made calculation possible in many practical situations that occur in physics, as well as the case $d = 3$.

{Editorial remark: The paper by Bellissard, Benedetti and Gambaudo (BBG) is expected to appear, and will be linked to this review once it is indexed in the MR database.}

Reviewed by *Claude Schochet*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.