On kernel engineering via Paley–Wiener

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ON KERNEL ENGINEERING VIA PALEY–WIENER

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ABSTRACT. A radial basis function approximation takes the form

$$s(x) = \sum_{k=1}^{n} a_k \phi(x - b_k), \qquad x \in \mathbb{R}^d,$$

where the coefficients $a_1, \ldots, a_n$ are real numbers, the *centres* $b_1, \ldots, b_n$ are distinct points in $\mathbb{R}^d$, and the function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radially symmetric. Such functions are highly useful in practice and enjoy many beautiful theoretical properties. In particular, much work has been devoted to the *polyharmonic* radial basis functions, for which ϕ is the fundamental solution of some iterate of the Laplacian. In this note, we consider the construction of a rotation-invariant signed (Borel) measure μ for which the convolution $\psi = \mu * \phi$ is a function of compact support, and when ϕ is polyharmonic. The novelty of this construction is its use of the Paley–Wiener theorem to identify compact support via analysis of the Fourier transform of the new kernel ψ, so providing a new form of kernel engineering.

1. INTRODUCTION

A radial basis function (RBF) approximation takes the form

$$s(x) = \sum_{k=1}^{n} a_k \phi(x - b_k), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where the coefficients $a_1, \ldots, a_n$ are real numbers, the *centres* $b_1, \ldots, b_n$ are distinct points in $\mathbb{R}^d$, and the function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radially symmetric. Such functions are extremely useful in practice and enjoy many beautiful theoretical properties. Wendland's textbook [14] is an excellent source for the theory of radial basis functions in general and, in particular, for Wendland's seminal work on the construction of compactly supported, radially symmetric, positive definite functions, whilst Buhmann's textbook [5] is also highly useful.

In particular, much work has been devoted to the *polyharmonic* radial basis functions, for which ϕ is the fundamental solution of some iterate of the Laplacian. In this note, we consider the construction of a rotation-invariant signed (Borel) measure μ for which the convolution $\mu * \phi$ is a function of compact support. The novelty of our construction is its use of the Paley–Wiener theorem, and this paper extends earlier work of the author [3], in which it is shown that spherical averages of certain polyharmonic radial basis functions can be compactly supported. We shall briefly discuss this spherical averaging technique to set the stage for the present work, but the reader is referred to [3] for details.

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During his fundamental work on the convergence properties of RBFs during the late 1980s, Ian Jackson [7], then a doctoral student of M. J. D. Powell, discovered the following remarkable fact: if $\phi(x) = \|x\|$, the Euclidean norm, then the *spherical average* of (1.1) is compactly supported when the dimension d is odd and the coefficients and centres satisfy the relations

$$\sum_{k=1}^{n} a_k \|b_k\|^{2\ell} = 0, \qquad \ell = 0, 1, \ldots, (d-1)/2, \tag{1.2}$$

the spherical average As being defined by

$$As(x) = \int_{S^{d-1}} s(\|x\|\theta)\, d\mu_d(\theta), \qquad x \in \mathbb{R}^d, \tag{1.3}$$

where μ_d denotes normalized $(d-1)$-dimensional Lebesgue measure on the unit sphere S^{d-1} in $\mathbb{R}^d$. Jackson's method was to expand the integrand of (1.3) for large $\|x\|$ and, in a *tour de force* of classical analysis, to identify the result as a certain hypergeometric function, from which he was then able to deduce the compact support of As when relations (1.2) hold[1]. In contrast, the new technique presented in [3] generalizes to *any* function $\phi : \mathbb{R}^d \to \mathbb{R}$ whose (distributional) Fourier transform is of the form $\widehat{\phi}(\xi) = C\|\xi\|^{-2m}$, $\xi \neq 0$, for some positive integer m, and identifies compact support via properties of the Fourier transform. One of the crucial observations of [3] is that spherical averaging can be equivalently described via averaging with respect to Haar measure on the orthogonal group, which we briefly describe for the convenience of the reader.

Definition 1.1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be any continuous function. The *spherical average* $Af : \mathbb{R}^d \to \mathbb{R}$ can also be defined by the equation

$$Af(x) = \int_{O(d)} f(Ux)\, d\sigma_d(U), \qquad x \in \mathbb{R}^d, \tag{1.4}$$

where $O(d)$ denotes the orthogonal group, that is,

$$O(d) = \{V \in \mathbb{R}^{d\times d} : V^T V = I\},$$

and σ_d denotes the normalized Haar probability measure on $O(d)$.

Further properties of Haar measure, together with its construction, can be found in the early chapters of Milman and Schechtman [11]. For our purposes, however, the key observation is that the Fourier transform commutes with spherical averaging. Specifically, we have the following result.

Theorem 1.2. *The spherical averaging operator A and the Fourier transform operator F commute when applied to tempered distributions, that is, we obtain the commutative diagram*

$$\begin{array}{ccc} S'(\mathbb{R}^d) & \xrightarrow{A} & S'_R(\mathbb{R}^d) \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ S'(\mathbb{R}^d) & \xrightarrow{A} & S'_R(\mathbb{R}^d) \end{array}$$

where $S'_R(\mathbb{R}^d)$ denotes the subspace of rotation-invariant tempered distributions.

[1] Jackson [7] was motivated by the construction of approximate identities using compactly supported spherical averages of radial basis functions.

Proof. This is Theorem 2.2 of [3]. □

Theorem 1.3. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a radially symmetric continuous function of polynomial growth and define*

$$s(x) = \sum_{k=1}^{n} a_k \phi(x - b_k), \qquad x \in \mathbb{R}^d.$$

Then

$$\widehat{As}(\xi) = \widehat{\phi}(\xi) \sum_{k=1}^{n} a_k \Omega_d(\|b_k\|\|\xi\|), \qquad \xi \in \mathbb{R}^d, \tag{1.5}$$

where $\Omega_d : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Omega_d(t) = \widehat{\mu_d}(tu), \qquad t \in \mathbb{R}, \tag{1.6}$$

for any fixed unit vector $u \in \mathbb{R}^d$. Thus Ω_d is essentially the Fourier transform of the Haar probability measure on the sphere.

Proof. This is Theorem 3.1 of [3]. □

In [3], (1.5) is then studied via the Paley–Wiener theorem, which enables us to identify compactly supported functions via their Fourier transform. In this paper, the key observation is that (1.5) can be regarded as a form of *kernel engineering*, in the sense that it provides a new kernel of the form

$$\widehat{\psi}(\xi) = \widehat{\phi}(\xi)\widehat{\mu}(\xi), \tag{1.7}$$

where

$$\widehat{\mu}(\xi) = \sum_{k=1}^{n} a_k \Omega_d(\|b_k\|\|\xi\|), \qquad \xi \in \mathbb{R}^d. \tag{1.8}$$

In this particular case, the rotation-invariant measure μ has arisen via spherical averaging, but we shall now study kernels of the form (1.7) in their own right.

2. Compactly supported RBFs via Paley–Wiener

The results of this paper apply to all polyharmonic radial basis functions, that is, to kernels whose Fourier transform takes the form $\widehat{\phi}(\xi) = C\|\xi\|^{-2m}$, $\xi \in \mathbb{R}^d \backslash \{0\}$, for some constant $C \in \mathbb{R}$ and positive integer m. However, for clarity of exposition, we shall specialize to the special case of the Euclidean norm. Specifically, we investigate the convolution $\mu * \phi$ when μ is a compactly supported rotation-invariant measure on $\mathbb{R}^d$ and $\phi(x) = \|x\|$. We find necessary and sufficient conditions ensuring $\mu * \phi$ is a continuous function of compact support. Our technique rests on the Paley–Wiener characterization of Fourier transforms of compactly supported measures, and we address this material first.

Definition 2.1. A continuous function $f : \mathbb{C}^d \to \mathbb{C}$ is an *entire function* (of several complex variables) if the maps $\{z \mapsto f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_d) : z \in \mathbb{C}\}$ are entire functions (of one complex variable) for $j = 1, \ldots, d$. Furthermore, an entire function $f : \mathbb{C}^d \to \mathbb{C}$ has exponential type T if

$$|f(z)| \le C \exp(T|\operatorname{Im} z|), \qquad z \in \mathbb{C}^d, \tag{2.1}$$

where $\operatorname{Im} z := ((\operatorname{Im} z_1)^2 + \cdots + (\operatorname{Im} z_d)^2)^{1/2}$ for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$.

Let us write $B(\delta) := \{x \in \mathbb{R}^d : \|x\| \le \delta\}$.

Theorem 2.2 (Paley–Wiener). *Let ν be a Borel measure on $\mathbb{R}^d$ supported by the ball $B(\delta)$. Then its Fourier transform*

$$\widehat{\nu}(z) = \int_{\mathbb{R}^d} \exp(-iz^T x)\, d\nu(x), \qquad z \in \mathbb{C}^d, \tag{2.2}$$

is an entire function of exponential type δ. Further, every entire function of exponential type δ is the Fourier transform of a distribution supported by the ball $B(\delta)$.

Proof. This is a special case of the Paley–Wiener theorem proved in Theorem 7.23 of [12]. □

We shall be taking Fourier transforms of rotation-invariant measures on $\mathbb{R}^d$. Therefore spherical polar coordinates can be used to calculate the transform via a univariate integral, and we shall again need the function initially defined in (1.6), that is,

$$\Omega_d(t) = \int_{S^{d-1}} \exp(itu^T\theta)\, d\mu_d(\theta), \qquad t \in \mathbb{R},$$

where μ_d denotes the rotation-invariant probability measure on the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ and u can be any unit vector in $\mathbb{R}^d$. Thus Ω_d is essentially the Fourier transform of μ_d; specifically, we have $\widehat{\mu}(\xi) = \Omega_d(\|\xi\|u)$, for any $\xi \in \mathbb{R}^d$ and any unit vector $u \in \mathbb{R}^d$; see Part III of [6] or Schoenberg's original study [13] for further details. However, we only need the fact that Ω_d is an entire function of exponential type one, which follows from Theorem 2.2. Therefore this even function has the Taylor series

$$\Omega_d(z) = \sum_{\ell=0}^{\infty} c_\ell z^{2\ell}, \qquad z \in \mathbb{C}. \tag{2.3}$$

Now Theorem 1.2 tells us that the Fourier transform and the spherical average commute for tempered distributions. Applying Theorem 1.2 to any rotation-invariant Borel measure μ supported on the ball $B(\delta)$, we therefore deduce that

$$\begin{aligned}\widehat{\mu}(\xi) &= A\Big(\mathbb{R}^d \ni \xi \mapsto \int_{\mathbb{R}^d} \exp(-ix^T\xi)\, d\mu(x)\Big)\\ &= \int_{\mathbb{R}^d} \Omega_d(\|x\|\|\xi\|)d\mu(x),\end{aligned} \tag{2.4}$$

since μ is already rotation-invariant, and unaffected by the spherical averaging operator. Now the function and the measure of (2.4) are rotation-invariant, which implies that the integral can be rewritten in univariate form, essentially using spherical polar coordinates in a slightly less familiar form than usual. Specifically, there is a bijection between rotation-invariant measures on $B(\delta)$ and even univariate measures on $(-\delta, \delta)$: the $\tilde{\mu}$-measure of the pair of intervals $(-b, -a) \cup (a, b)$, for $0 < a < b$, is defined to be the μ-measure of the annulus with inner radius a and outer annulus b. This bijection leaves the total variation norm invariant, that is, $\|\mu\| = \|\tilde{\mu}\|$, since there is also an obvious bijection between even functions in $C[-\delta, \delta]$ and radially symmetric functions on $B(\delta)$, and this also preserves the uniform norm of the function. We now state this formally.

Lemma 2.3. *Let μ be a rotation-invariant Borel measure on $\mathbb{R}^d$ supported by the ball $B(\delta)$. Then there is a signed measure $\tilde{\mu}$ on $[-\delta, \delta]$ such that $\widehat{\mu}$ is given by the formula*

$$\widehat{\mu}(\xi) = \int_{-\delta}^{\delta} \Omega_d(t\|\xi\|)\, d\tilde{\mu}(t), \qquad \xi \in \mathbb{R}^d. \tag{2.5}$$

Proof. We use (2.4), recalling that $\tilde{\mu}$ is the even measure defined by

$$\tilde{\mu}(S) := \frac{1}{2}\mu\{x \in \mathbb{R}^d : \|x\| \in S\},$$

for every Borel subset S of $[0, \infty)$. □

Since $\phi(x) = \|x\|$ is a continuous function of algebraic growth, we can regard ϕ as a tempered distribution in the sense of Schwartz (1966). Its Fourier transform is given by

$$\widehat{\phi}(\xi) = -\lambda_d \|\xi\|^{-(d+1)}, \qquad \xi \in \mathbb{R}^d \setminus \{0\},$$

where $\lambda_d = 2^d \pi^{(d-1)/2}\Gamma((d+1)/2)$. This result can be found in Theorem 8.16 of [14].

Theorem 2.4. *Let the dimension d be an odd integer and let μ be a rotation-invariant measure supported by the ball $B(\delta)$. Then the continuous function $\mu * \phi$ is supported by $B(\delta)$ if and only if*

$$\int_{-\delta}^{\delta} p(t)\, d\tilde{\mu}(t) = 0, \qquad p \in \mathbb{P}_d, \tag{2.6}$$

where $\mathbb{P}_d$ denotes the vector space of all polynomials on $\mathbb{R}^d$.

Proof. We have the tempered distribution

$$\widehat{\mu * \phi}(\xi) = \frac{-\lambda_d}{\|\xi\|^{d+1}} \sum_{\ell=0}^{\infty} c_\ell \|\xi\|^{2\ell} \int_{-\delta}^{\delta} t^{2\ell}\, d\tilde{\mu}(t), \qquad \text{for } \xi \neq 0. \tag{2.7}$$

Hence the right hand side of (2.7), $\widehat{\psi}(\xi)$ say, is an entire function of exponential type δ if and only if

$$\int_{-\delta}^{\delta} t^{2\ell}\, d\tilde{\mu}(t) = 0, \qquad \ell = 0, 1, \ldots, (d-1)/2. \tag{2.8}$$

Therefore, since $|\widehat{\psi}(\xi)| = O(\|\xi\|^{-d-1})$, for large $\|\xi\|$, we deduce that its inverse Fourier transform ψ is a continuous function supported by the ball $B(\delta)$, by the Paley–Wiener theorem. However, because (2.7) holds only for nonzero $\xi \in \mathbb{R}^d$, we can only conclude that $\mu * \phi(x) = \psi(x) + p(x)$, where p is a (radially symmetric) polynomial; we must now prove that p is the zero polynomial.[2] However, since ψ is supported by the ball $B(\delta)$, we have

$$\mu * \phi(x) = \int_{B(\delta)} \|x - z\|\, d\mu(z) = p(x), \qquad \text{for } \|x\| > \delta.$$

[2] This important logical point was omitted from the original proof. I am grateful to an anonymous referee, who kindly pointed out this lacuna.

Further, for large $\|x\|$, we have the expansion

$$\begin{aligned}\|x-z\| &= \left(\|x\|^2 - 2x^Tz + \|z\|^2\right)^{1/2} \\ &= \|x\|\left(1 - \frac{2x^Tz}{\|x\|^2} + \frac{\|z\|^2}{\|x\|^2}\right)^{1/2} \\ &= \|x\| - \frac{x^Tz}{\|x\|} + \frac{\|x\|^2\|z\|^2 - (x^Tz)^2}{2\|x\|^3} + \cdots, \end{aligned} \tag{2.9}$$

where the final line uses the Taylor series

$$(1-w)^{1/2} = \sum_{k=0}^{\infty} \frac{(-1/2)_k w^k}{k!}, \qquad |w| < 1,$$

and the Pochhammer symbol is defined by

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

Now

$$\int_{B(\delta)} 1 \; d\mu(z) = \int_{-\delta}^{\delta} 1 \; d\tilde{\mu}(s) = 0,$$

by the assumed polynomial annihilation property of $\tilde{\mu}$, whilst the rotational invariance of the signed measure μ implies that

$$\int_{B(\delta)} z_j d\mu(z) = 0, \qquad \text{for } 1 \le j \le d.$$

Hence, substituting the expansion (2.9) into the definition of $\mu * \phi$, for large $\|x\|$, we obtain

$$\begin{aligned} p(x) &= \mu * \phi(x) \\ &= \int_{B(\delta)} \left(\frac{\|x\|^2\|z\|^2 - (x^Tz)^2}{2\|x\|^3} + \cdots\right) d\mu(z) \\ &= O(1/\|x\|), \end{aligned}$$

which implies that the polynomial satisfies $\lim_{\|x\|\to\infty} p(x) = 0$. Therefore p is, in fact, the zero polynomial, and we have established that $\mu * \phi(x) = \psi(x)$, for all $x \in \mathbb{R}^d$. □

Thus $\mu * \phi$ is compactly supported if and only if $\tilde{\mu}$ is an even measure on $[-\delta, \delta]$ annihilating polynomials of degree d.

Such compactly supported convolutions can be highly useful analytical tools in their own right. Our first application addresses the so-called local stability of the translates of ϕ: we relate the size of the coefficient a_j in the linear combination

$$s(x) = \sum_{k=1}^{n} a_k \phi(x - b_k), \qquad x \in \mathbb{R}^d, \tag{2.10}$$

to the magnitude of s near the centre b_k. The stability of translates was also addressed in [4], [8], [9] and [10], but our approach is rather different.

Theorem 2.5. *Let the dimension d be an odd integer and let s be given by* (2.10). *If $\|b_j - b_k\| \ge 1$ for $j \ne k$, then*

$$|a_j| \le \frac{\max\{|s(x)| : \|x - b_j\| \le 1\}}{\Delta_d},$$

where Δ_d denotes the distance in the uniform norm of the absolute value function from polynomials of degree d in the Banach space of continuous functions on $[-1,1]$, endowed with the uniform norm.

Proof. Let μ be a rotation-invariant measure satisfying the conditions of Theorem 2.4. We have

$$\mu * s(x) = \sum_{k=1}^{n} a_k \mu * \phi(x - b_k), \qquad x \in \mathbb{R}^d, \tag{2.11}$$

which implies the equation $\mu * s(b_j) = a_j \mu * \phi(0)$. Hence

$$\begin{aligned} |a_j| &\le \left| \frac{\mu * s(b_j)}{\mu * \phi(0)} \right| \\ &\le \left(\frac{\|\mu\|}{|\mu * \phi(0)|} \right) \max\{|s(x)| : \|x - b_j\| \le 1\}. \end{aligned} \tag{2.12}$$

Of course, we want (2.12) to provide the least upper bound that can be achieved with this technique. Thus we obtain the optimization problem:

$$\begin{aligned} \text{Maximize } |\mu * \phi(0)| &= \left| \int_{-1}^{1} |t| \, d\tilde{\mu}(t) \right| \\ \text{subject to } \int_{-1}^{1} p(t) \, d\tilde{\mu}(t) &= 0, \qquad p \in \mathbb{P}_d, \text{ and } \|\tilde{\mu}\| = 1. \end{aligned} \tag{2.13}$$

We complete the proof that this maximum value is Δ_d in Lemma 2.6 below. □

Lemma 2.6. *For every signed measure ν on $[-1,1]$ having total variation $\|\nu\| = 1$ and satisfying $\int_{-1}^{1} p(t) \, d\nu(t) = 0$, when p is a polynomial of degree d, we have*

$$\left| \int_{-1}^{1} |t| \, d\nu(t) \right| \le \Delta_d. \tag{2.14}$$

Furthermore, this bound can be attained.

Proof. For any polynomial p of degree d we have the relations

$$\left| \int_{-1}^{1} |t| \, d\nu(t) \right| = \left| \int_{-1}^{1} |t| - p(t) \, d\nu(t) \right| \le \| \, |\cdot| - p\|_\infty.$$

Hence $\left| \int_{-1}^{1} |t| \, d\nu(t) \right| \le \Delta_d$. To show that the bound is sharp, we introduce the quotient space $X := C[-1,1]/\mathbb{P}_d$ endowed with the quotient norm

$$\|f + \mathbb{P}_d\| = \inf_{p \in \mathbb{P}_d} \|f - p\|_\infty. \tag{2.15}$$

Then the dual space X^* is the vector space of signed measures annihilating $\mathbb{P}_d$, that is

$$X^* = \{\nu \in C[-1,1]^* : \int_{-1}^{1} p(t) \, d\nu(t) = 0, \quad p \in \mathbb{P}_d\}. \tag{2.16}$$

Now, given any function $f \in C[-1,1]$, the Hahn–Banach theorem implies the existence of $\nu^* \in X^*$ such that $\|\nu^*\| = 1$ and $\nu^*(f + \mathbb{P}_d) = \|f + \mathbb{P}_d\|$, or

$$\int_{-1}^{1} f(t) \, d\nu^*(t) = \Delta_d. \tag{2.17}$$

Setting $f = |\cdot|$ and noting that ν^* can be chosen to be even in this case, the proof is complete. □

The result of Theorem 2.5 may remind several readers of the norm estimate for the inverse of the Euclidean distance matrix found by Ball [1]. Ball's proof was based on an ingenious modification of the Schoenberg integral representation of the Euclidean norm, but it is straightforward to use our technique to obtain the result, and our proof generalizes to all polyharmonic RBFs. Thus the analyses [7] and [1], two entirely independent, and seemingly unrelated papers, can be derived using the Fourier technique described here. Indeed, we recall from Proposition 3.3 of [2] that

$$\Big|\sum_{j,k=1}^{n} y_j y_k \phi(x_j - x_k)\Big| = (2\pi)^{-d} \int_{\mathbb{R}^d} \Big|\sum_{k=1}^{n} y_k e^{i x_k^T \xi}\Big|^2 |\widehat{\phi}(\xi)|\, d\xi \tag{2.18}$$

when $(y_k)_{k=1}^n$ is a real sequence for which $\sum y_k = 0$.

Theorem 2.7. *Let d be an odd positive integer and let $(y_k)_{k=1}^n$ be a real sequence for which $\sum_{k=1}^n y_k = 0$. If $(x_k)_{k=1}^n$ are points in $\mathbb{R}^d$ satisfying $\|x_j - x_k\| \geq 1$ for $j \neq k$, then*

$$\Big|\sum_{j,k=1}^{n} y_j y_k \phi(x_j - x_k)\Big| \geq \Delta_d \sum_{k=1}^{n} y_k^2. \tag{2.19}$$

Proof. Let μ be a rotation-invariant measure of total variation $\|\mu\| = 1$ satisfying (2.6). Then

$$\begin{aligned}\Big|\sum_{j,k=1}^{n} y_j y_k \phi(x_j - x_k)\Big| &\geq (2\pi)^{-d} \int_{\mathbb{R}^d} \Big|\sum_{k=1}^{n} y_k e^{i x_k^T \xi}\Big|^2 |\widehat{\mu}(\xi)\widehat{\phi}(\xi)|\, d\xi \\ &= |\mu * \phi(0)| \sum_{k=1}^{n} y_k^2, \qquad (2.20)\end{aligned}$$

using the fact that $|\widehat{\mu}(\xi)| \leq 1$ if $\|\mu\| = 1$. Thus the greatest lower bound attainable with this method occurs when $\mu * \phi(0) = \Delta_d$, by Lemma 2.6. □

Another use of these convolutions is to generate a function $\mu * \phi$ suitable for use as an approximate identity, that is $\mu * \phi$ is to be an absolutely integrable function for which $\int_{\mathbb{R}^d} \mu * \phi(x)\, dx \neq 0$. Any rotation-invariant measure μ satisfying the conditions of Theorem 2.4 yields a continuous function $\mu * \phi$ of compact support, but to be an approximate identity we require $\lim_{\xi \to 0} \widehat{\mu}(\xi)\widehat{\phi}(\xi) \neq 0$, that is $\int_{-1}^{1} t^{d+1}\, d\tilde{\mu}(t) \neq 0$. Such measures obviously exist in profusion. However, one constructive approach would be to choose μ to be a measure with transform (1.8).

There is a marked change when the dimension d is even.

Theorem 2.8. *Let the dimension d be an even positive number and let μ be a rotation-invariant compactly supported measure on $\mathbb{R}^d$. Then $\mu * \phi$ cannot be absolutely integrable if*

$$\int_{\mathbb{R}^d} \mu * \phi(x)\, dx \neq 0.$$

Proof. If $\mu * \phi$ is an absolutely integrable function, then its Fourier transform is continuous. Using the expression for $\widehat{\mu * \phi}$, we deduce the necessary conditions

$$\int_{-1}^{1} t^{2l}\, d\tilde{\mu}(t) = 0 \text{ for } l = 0, 1, \ldots, d/2.$$

However, because d is even, we have

$$\widehat{\mu * \phi}(\xi) = -\lambda_d \Big(c_{(d+2)/2} \|\xi\| \int_{-1}^{1} t^{(d+2)/2}\, d\tilde{\mu} + O(\|\xi\|^3) \Big)$$

for small $\|\xi\|$, whence

$$0 = \widehat{\mu * \phi}(0) = \int_{\mathbb{R}^d} \mu * \phi(x)\, dx,$$

as required. □

Given the compactly supported convolution $\widehat{\psi}(\xi) = \widehat{\mu}(\xi)\widehat{\phi}(\xi)$, it is natural to form new compactly supported positive definite functions via the inverse Fourier transform of $|\widehat{\psi}(\xi)|^2$, and initial results are highly promising.

3. Acknowledgement

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