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Methods Of Variational Analysis In Pessimistic Bilevel Programming

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METHODS OF VARIATIONAL ANALYSIS IN PESSIMISTIC BILEVEL PROGRAMMING

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DEDICATION

To My Parents

To My Teachers

To My Wife

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PREFACE

The dissertation is devoted to the study of the so-called pessimistic version of the bilevel programming problems in finite-dimensional spaces. Bilevel programming deals with a broad class of problems in *hierarchical optimization* that consist of minimizing some (upper level or leader's) objective function $F : \Re^n \times \Re^m \to \Re$ subject to upper-level constraints $x \in \Omega \subset \Re^n$ and $(x, y) \in \operatorname{gph} S \subset \Re^m \times \Re^n$. Here $\operatorname{gph} S$ stands for graph of the solution mapping to another (lower-level or follower's problem) parametric optimization problem given by $S(x) := \operatorname{argmin}\{\, f(x, y) \mid y \in K(x)\,\}$ with the cost function $f : \Re^n \times \Re^m \to \Re$ and the constraint set $K(x) \subset \Re^m$. This problem can be ill posed since the follower's problem may have many solutions and responses for every (or some) fixed leader's variables. In the so-called "pessimistic" case, or "strong" case, the leader has no possibility to influence the follower's choice, i.e., the leader has to minimize the damage resulting from an unwelcome selection of the follower by considering the worst solution in $S(x)$ with respect to the leader's goal:

$$\min \varphi_p(x) \quad \text{subject to } x \in \Omega$$

where $\varphi_p(x) := \max\{\, F(x, y) \mid y \in S(x)\,\}$.

The purpose of this dissertation is to establish, by a variety of techniques from convex, nonconvex and nonsmooth analysis, first order necessary and sufficient optimality conditions for pessimistic bilevel programming problems (PBLPPs). Problems of this type intrinsically nonsmooth even for smooth initial data and can be treated using appropriate tools of modern variational and generalized differentiation developed by Mordukhovich.

This dissertation is organized as follows: In Chapter 1, we analyze the pessimistic bilevel programming problems. Here we present a literature review, formulation of the PBLPPs, and an illustration of two practical problems, which emphasize the study of PBLPPs. We study the existence of the optimality conditions and then we look for various reformulations of PBLPPs. Pessimistic bilevel programming problems can be reformulated as generalized semi infinite programs, static minimax programs in the parametric constraint case, implicit programs, and mathematical programs with equilibrium constraints. Pessimistic version of bilevel programming problems consists of three level optimization and using the imposed assumption, the so-called reformulation assumption (RA), PBLPPs can be reduced to two level optimization problems, which indeed help us to obtain necessary optimality conditions.

Staring with Chapter 2, we study optimality conditions for PBLPPs. Chapter two is devoted to obtaining the optimality conditions using the techniques of the so-called implicit programming approach. We begin with assuming, so-called key implicit assumption (KIA) that there exists a Lipschitz selection $y(x) \in M(x) := \operatorname{argmax}\{\, F(x,y) \mid y \in S(x)\,\}$ near the optimal solution $(\bar{x}, \bar{y})$. Then we formulate the PBLPP as implicit programming problem and hence we obtain optimality conditions using tools of modern variational analysis. We also obtain the optimality conditions for continuous and Fréchet differentiable selection relaxing the Lipschitz continuity. We study constructions of the coderivative of selection mapping and present sufficient conditions for our basic assumptions.

Chapter 3 consists of three main sections for which we consider PBLPPs with differentiable data, convex data and Lipschitz data. For each case we study constructions or estimations of subdifferentials of marginal function and obtain necessary optimality conditions treating the PBLPP as static minimax programs with parametric constraints, i.e., by writing $\min_{x\in\Omega}\left(\max_{y\in S(x)} F(x,y)\right)$ which is equivalent to $\min_{x\in\Omega}\left(\max_{y\in K(x)} [F(x,y)-\mu f(x,y)]+\mu v(x)\right)$ provided that the assumption (RA) holds near the optimal point $(\bar{x},\bar{y})$.

In Chapter 4, we obtain necessary optimality conditions for PBLPPs using the well-known technique the so-called duality programming approach. Here we relax the convexity assumption and consider the non convex PBLPPs. Imposing the invexity type assumption, we reformulate the PBLPPs as mathematical programs with equilibrium constraints and hence we obtain necessary optimality conditions.

When the rational reaction set has finite cardinality, but not singleton, we can equivalently reformulate the PBLPP as $\min_{x\in\Omega}\left(\max_{i=\overline{1,r}}\left[\tilde{F}_i(x):=F(x,y_i(x)):=(F\circ y_i)(x)\right]\right)$ or $\min_{(x,\omega)\in\Omega\times\Re}\omega$ subject to $\tilde{F}_i(x)-\omega\le 0\quad i=1,2,\ldots,r$ where $S(x)=\{\,y_i(x)\mid i=1,2,\ldots,r$ where $r>1;\ y_i(x)$ solves the follower's problem $(l_x)\,\}$. Then using the tools of modern variational analysis and generalized differentiation we present necessary optimality conditions. In the last part of Chapter 5, we present some special classes of PBLPPs, which guarantee the existence of a finite number of rational responses.

Pessimistic formulation for the bilevel programs results in three level optimization and existence of the optimal solutions and optimality conditions depend on the value function, feasible set, and solution set to the lower level problems. For the reader's convenience in the appendices, we present some properties of marginal function, feasible set, and solutions set to the lower level problems from the nonsmooth optimization and stability theory.

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> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
>
> ∼ George Polya

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Chapter 1

Analysis of Pessimistic Bilevel Programming Problem

1.1 Problem Formulation and Basic Concepts

1.1.1 Motivation and Introduction

Bilevel programming problems are of growing interest both from theoretical and practical points of view. Optimization problems are encountered in a wide variety of domains such as experts and their guesses (as in the case of expert evaluation analysis), members of a group and their votes (voting models), various indicators of quality of a system and their values (decision making based on many criteria), and starting characteristics as well as partitions of objective into classes that they generate (classification problems). Generally speaking, hierarchical optimization problems or multilevel optimization problems play an important role in optimization decision problems including the engineering and experimental natural science regional planning, management, and economics problems, network design, and etc. Bilevel optimization problems are the first step to study hierarchical optimization problems. The framework of bilevel optimization problems is that there are both leader-level and follower-level problems, where each problem has its own objective function, and interactions between the two problems. Two players seek to optimize their individual objective functions. The first player (also called the leader) must take into account the reaction (or any possible reaction when non unique) of the second player (also known as the follower). Such a problem can be ill posed, since the lower level (follower's problem) may have many solutions and responses for every (or some) fixed leader's variables. In the so-called "optimistic case," many optimal reactions of the follower are possible and the follower is assumed to choose in favor of the leader. In this case, the upper level problem can be modeled using a bilevel formulation. These programs are quite difficult nonconvex optimization problems. The reader is referred to monographs; [5], [8], [13], [57], [79], [80], surveys; [6], [7], [9], [86], journal papers; [10], [11], [12], [22], [34], [46], [67], [68], [87], [88], [90], [91], and the references therein, for comprehensive studies, solutions methods, techniques, algorithms, optimality conditions, and various applications.

Alternatively, the so-called *pessimistic* formulation refers to the case where the leader protects himself/herself against the worst possible situation. See also [9], [8], [84], and the references therein.

The scope of this dissertation is mainly related to pessimistic bilevel programming problems (PBLPPs). To be more precise, the problem framework can be described as the following: Let the follower make his/her decision by solving a parametric optimization problem:

$$\min_{y} f(x, y) \qquad \text{subject to} \quad y \in K(x), \tag{1.1.1}$$

where the decision variable of leader-level is $x \in \Omega \subset \Re^n$, the decision variable of follower-level is $y \in \Re^m$, $f : \Re^n \times \Re^m \to \Re$ is a follower's objective function and $K(x) \subset \Re^m$ is the constraint set (or the feasible set) for the followers problem. The lower-level problem is parameterized by x. Define the solutions set mapping of the follower's problem $S : \Re^n \rightrightarrows \Re^m$ by

$$S(x) := \text{argmin}\{\, f(x,y) \mid y \in K(x)\,\}. \tag{1.1.2}$$

Then the leader's problem consists in minimizing

$$\text{`}\min_x\text{'}\, F(x,y) \qquad \text{subject to} \quad y \in S(x) \text{ and } x \in \Omega, \tag{1.1.3}$$

where $F : \Re^n \times \Re^m \to \Re$ and $\Omega \subset \Re^n$ is a closed set.

Since the leader controls only the variable x, the problem (1.1.3) is well defined only in the case when the optimal solution set $S(x)$ is uniquely determined for all parameter values $x \in \Omega$. Quotation marks have been used to express the ambiguity in the definition of the leader's problem in the case of multiple optimal solutions in the follower's problem for some parameter values. An interesting example of this matter can be found in [9].

To overcome this ambiguity, at least two approaches, known as *optimistic* and *pessimistic*, have been suggested. Both approaches rest on the introduction of a new lower level problem:

$$\begin{array}{lcl} & \nearrow & S(x) \text{ is uniquely determined for all } x \\ \text{BLPP} & \longrightarrow & \text{Optimistic Bilevel Programming Problem} \\ & \searrow & \text{Pessimistic Bilevel Programming Problem} \end{array}$$

Optimistic bilevel programming can be applied if the leader assumes that the follower will always take an optimal solution, which is the best one from the leader's point of view. The optimistic position seems not to be possible without any trouble at least in the cases when cooperation is not allowed, (e.g., in games of human being against nature), or if the follower's seriousness of keeping the agreement is not granted. Then when the leader is not able to influence the follower's choice, he/she has the way out of this unpleasant situation to bound the damage resulting from an undesirable selection of the follower. This leads to the pessimistic bilevel programming problem (PBLPP):

$$\min \varphi_p(x) \quad \text{subject to } x \in \Omega, \tag{1.1.4}$$

where

$$\varphi_p(x) := \max \{ F(x,y) \mid y \in S(x) \}. \tag{1.1.5}$$

1.1.2 Problem Formulations and Terminology

The pessimistic bilevel programming problem (PBLPP), with which we are concerned, is the following:

$$(PBLPP)\begin{cases} \min \varphi_p(x) \quad \text{subject to } x \in \Omega, \quad \text{where} \\ \varphi_p(x) := \max\{\, F(x,y) \mid y \in S(x) \,\}, \quad \text{and} \\ S(x) := \operatorname{argmin} \{\, f(x,y) \mid y \in K(x) \,\}. \end{cases}$$

Terminology:

Other important pessimistic bilevel programming problems terminology, definitions and notation are itemized below.

- The *relaxed feasible region* (or *constraint region*),

$$\Psi = \{\, (x,y) \in \Re^{n+m} \mid x \in \Omega, y \in K(x) \,\}.$$

- For a given (fixed) vector $x \in \Omega$, the *lower-level feasible set is*

$$K(x) = \{\, y \in \Re^m \mid g(x,y) \leq 0, h(x,y) = 0 \,\}.$$

- For a given $x \in \Omega$, the follower's problem (or lower level problem is),

$$\min_y f(x,y) \quad \text{such that } y \in K(x).$$

- For each $x \in \Omega$, the *lower-level reaction set* (or *rational reaction set is*),

$$S(x) = \{\, y \in \Re^m \mid y \in \operatorname{argmin}\{\, f(x,y) \mid y \in K(x) \,\} \,\}.$$

Every $y \in S(x)$ is a *rational response.* For a given x, $S(x)$ is an implicitly defined multivalued function of x that may have empty values for some values of its argument.

- The feasible set of the leader, also known as the *induced region* (or *inducible region*)

$$IR = \{ (x, y) \in \Re^{n+m} \mid x \in \Omega, y \in S(x) \}.$$

- When cooperation of the leader and the follower is not allowed, or if the leader is risk-averse and wishes to bound the damage resulting from and undesirable selection, we have

$$\varphi_p(x) := \max \{ F(x, y) \mid y \in S(x) \}$$

denotes the worst upper level objective function value achievable on the rational reaction set.

- The follower's problem or pessimistic bilevel programming problem is,

$$\min \varphi_p(x) \quad \text{subject to } x \in \Omega.$$

1.2 Existence of Optimality Conditions

This bilevel programming problem has an implicitly determined set of feasible solutions and it is usually a nonconvex optimization problem with a nondifferentiable objective function. Hence local optimal solutions may exist. To define the notion of a locally optimal solution, usual tools from optimization are applied to the PBLPP.

Definition 1.2.1 (pessimistic optimal solution). *A point* $(\bar{x}, \bar{y})$ *with* $\bar{y} \in S(\bar{x})$ *and* $\bar{x} \in \Omega$ *is a* local pessimistic optimal solution *if*

$$F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \qquad \textit{for all } y \in S(\bar{x}) \qquad \textit{and}$$

there exist $\epsilon > 0$ *such that*

$$\varphi_p(\bar{x}) \leq \varphi_p(x) \qquad \textit{for all } x \textit{ such that} \quad x \in \Omega, \quad \textit{with} \quad \| x - \bar{x} \| < \epsilon.$$

It is called a global pessimistic optimal solution *if* $\epsilon = \infty$ *can be selected.*

Remark 1.2.2. $(\bar{x}, \bar{y})$ is a local pessimistic optimal solution to PBLPP if and only if $\bar{x}$ is a local optimal solution to $\min_{x \in \Omega} \varphi_p(x)$ and $\bar{y}$ is a globally optimal solution to $\max_{y \in S(\bar{x})} F(\bar{x}, y)$. Equivalently, $(\bar{x}, \bar{y})$ is a local pessimistic optimal solution to PBLPP if and only if $\bar{x}$ is a local optimistic solution to $\min_{x \in \Omega} F(x, \tilde{y}(x))$, where $\tilde{y}(x)$ solves(globally) $\max_{y \in S(x)} F(x, y)$ for each x in a neighborhood of $\bar{x}$ with $\tilde{y}(\bar{x}) = \bar{y}$.

Sensitivity theory of parametric non-linear programming (see for more details, [14], [15], [17], [18], [19], [23], [38], [52], [70], [72], [73], and [95]) provides us with the existence, characterization, and related properties of $\tilde{y}(x)$ near $\bar{x}$.

Remark 1.2.3. A pessimistic optimal solution does not necessarily exist, even if the objective functions are continuous and Ω and $K(x) \equiv K$ are compact spaces. The following example is evidence.

Example 1.2.4 (nonexistence of pessimistic optimal solutions [47]). Let $f(x, y) = (x - \frac{1}{2})y$ and $F(x, y) = -xy$ be the corresponding objective functions. Also let $K(x) \equiv K = [0, 1]$ and $\Omega = [0, 1]$. Then with the previous notations, we have:

$$S(x) = \begin{cases} \{1\} & \text{if } 0 \leq x < \frac{1}{2} \\ [0,1] & \text{if } x = \frac{1}{2} \\ \{0\} & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \qquad \varphi_p(x) = \begin{cases} \{-x\} & \text{if } 0 \leq x < \frac{1}{2} \\ \{0\} & \text{if } x = \frac{1}{2} \\ \{0\} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Also, $\inf_{x \in \Omega} \varphi_p(x) = -\frac{1}{2}$ but $\varphi_p(x)$ is not lower semicontinuous at $x = -\frac{1}{2}$ and, hence the optimal solution does not exist.

Theorem 1.2.5 (existence of pessimistic optimal solutions [9]). *Pessimistic bilevel programming problem (PBLPP) has an optimal solution if the following conditions are satisfied.*

(i) F *is lower semicontinouse,*

(ii) $S : \Re^n \rightrightarrows \Re^m$ *is lower semicontinuous at all points with* $x \in \Omega$*, and*

(iii) the set $\{ (x, y) \mid y \in K(x),\ x \in \Omega \}$ *is nonempty and compact.*

Remark 1.2.6. Sufficient conditions for the lower semicontinuity of the rational reaction set are available in the optimization literature. See the appendices and the references therein for sufficient conditions for the lower semicontinuity of the solution set $S(x)$.

In addition to nonconvex and nondifferentiable behavior of PBLPP, another major drawback arises when the lower level problem is nonconvex. If the lower level problem is nonconvex for fixed parameter value x, the computation of a globally optimal solution in the lower level problem can be computationally intractable (especially tracing the global solution set mapping for a varying parameter). In this case it can be considered as being helpful to modify the bilevel program such that a locally optimal solution in the lower level problem is searched for instead of a globally optimal solution. But as it is shown in an example in [9], this can completely change the existence of optimal solutions of the PBLPP. Let's consider the following example.

Example 1.2.7 (global versus local). Let $f(x,y) = y^3 - 3y$ and $F(x,y) = (x+1)^2$. Also let $\Omega = [-3, 2]$ and $K(x) = \{\, y \mid y - x \geq 0 \,\}$. Then, if we consider $S(x)$ to be the set of global optimal solutions to the lower level problem, then pessimistic optimal solution does not exist. But, if $S(x)$ denotes the set of locally optimal solutions to the lower level problem, we have

$$\varphi_p(x) = \begin{cases} 4 & \text{if } x \in [-3, 1] \\ (x+1)^2 & \text{if } x \in \ (1, 2] \end{cases}$$

So, $\inf\limits_{x \in \Omega} \varphi_p(x) = 4$ and all points $x \in [-3, 1]$ are optimal solutions.

The reason for this behavior is that the point-to-set mapping of globally optimal solutions of a parametric problem is generally not lower semicontinuous if the convexity assumption is dropped. Being aware of the difficulties arising when global optimal solutions of nonconvex optimization problems are to be computed, some authors use optimality definitions, which determine whole sets of optimal solutions. To circumvent this unpleasant situation, Vogel (2002) has defined a weaker notion of pessimistic solutions.

Definition 1.2.8 (weak pessimistic optimal solution). *Let us define*

$$\rho(x) := sup\,\{\, F(x,y) \mid y \in S(x)\,\},$$

where $S(x)$ *is the solution set of the lower level problem (local or global), and*

$$\varrho(x) := inf\{\rho(x) \mid x \in \Omega\,\}.$$

A point $\bar{x}$ *with* $\bar{x} \in \Omega$ *is a* weak pessimistic solution *to the bilevel programming problem if, there exist* $\bar{y} \in S(\bar{x})$ *and sequence* $\{\, x_k \,\}_{k=1}^{\infty} \subset dom\, S$ *such that*

$$\lim_{k\to\infty} x_k = \bar{x}, \quad \lim_{k\to\infty} \rho(x_k) \to \varrho, \quad \textit{and} \quad F(\bar{x},\bar{y}) = \varrho.$$

Theorem 1.2.9 (existence of weak pessimistic optimal solutions [9]). *Let* $\{\,(x,y) \mid y \in S(x),\ x \in \Omega\,\}$ *be nonempty and bounded. Then the PBLPP has a weak pessimistic solution.*

The existence of solutions of pessimistic bilevel programming problem has been intensively investigated by Loridan and Morgan and their co-authors. The interested reader can find the corresponding results and discussions in Aboussoror [1], Aboussoror and Loridan [2, 3], Lignola and Morgan [45], Loridan and Morgan [47, 48, 49], Marhfour [56], and Tian and Zhou [83]. Henceforth we assume the existence and look at necessary optimality conditions, which is the main aim of this work.

1.3 Applications of Pessimistic Bilevel Programs

Pessimistic programs naturally arise in economics, engineering, optimization under uncertainty, and several other application areas. Among other practical applications, in what follows, we consider two examples that are formulated as pessimistic bilevel programs.

Example 1.3.1 (principle-agent problem [84]). The principle-agent problem illustrates the difficulty to design contracts under incomplete and asymmetric information. Assume that agent A is supposed to perform a task for principal P. The generated profit ρ of that task is a function of the effort $\phi \in [0, 1]$ that A puts into the task. Since P cannot observe ϕ, he/she designs an incentive-based contract, i.e., apart from a fixed salary, he/she pays A a fraction β of the observable task profit $\rho(\phi)$. P wants to choose β in such a way that his/her part of the profit is maximized. The setting can be cast as a Stackelberg game: P, the leader, moves first by deciding on the contract parameter β. After he/she has observed β, A (the follower) chooses the effort ϕ^* that maximizes his utility $u(\beta, \phi)$. Anticipating A's decision, P wants to choose a contract parameter β that maximizes his/her part of the profit, i.e., $(1 - \beta)\rho[\phi^*(\beta)]$.

In the following, assume that A's utility is determined by $u(\beta, \phi) = s(\beta, \phi) - r(\phi)$, where $s(\beta, \phi) = 10 + \beta\rho(\phi)$ denotes his/her salary and $r(\phi) = \phi + (5[\phi - 0.5]^+)^2$ represents A's reluctance to work. The profit generated by the task outcome is determined by $\rho(\phi) = 5\phi^2$. With this notation, P's optimization problem can be described as follows:

$$\max_{\beta \in [0,1]} \min_{\phi \in \phi^*(\beta)} \{5(1 - \beta)\phi^2\},$$

where $\phi^* = \operatorname*{argmax}_{\phi \in [0,1]} \{10 + 5\beta\phi^2 - \phi - (5[\phi - 0.5]^+)^2\}$ denotes the set of optimal efforts for a given contract design β. The model maximizes P's worst-case profit anticipating that A will choose any of his optimal responses.

Example 1.3.2 (production planing [84]). Two companies A and B produce the same product and need to decide on their production quantities. Company A is the market leader and has decided first on a quantity $x_A \in [0, 200]$. Company B, on the other hand, is a follower and decides on a quantity $x_B \in [0, 100]$ after observing A's decision. The market price for the product depends on the aggregated supply $x_\Sigma = x_A + x_B$ and is determined by $p(x_\Sigma) = 100 - 0.3x_\Sigma$, while the (constant) unit production costs are 50 for both companies. A and B want to maximize their profits.

The profits of companies A and B amount to $\rho_i(x_A, x_B) = \{p(x_A + x_B) - 50\}x_i$, $i \in \{A, B\}$. Company A's decision problem can be written as follows:

$$\max_{x_A \in [0,200]} \min_{x_B \in x_B^*(x_A)} \{[50 - 0.3(x_A + x_B)]x_A\},$$

where $x_B^*(x_A) = \operatorname*{argmax}_{x_B \in [0,100]} \{[50 - 0.3(x_A + x_B)]x_B\}$ denotes B's set of optimal responses to a given production quantity of A. The objective is to maximize company A's worst-case profit.

1.4 Reformulations, Assumptions and Properties

1.4.1 Representation of Optimal Solutions to the Lower Level Problem

Geometric Representation

Recall the lower level problem

$$l(x): \qquad \min_y f(x,y) \qquad \text{subject to} \quad y \in K(x),$$

and let $K(x) = \{\, y \in \Re^m \mid g(x,y) \leq 0, \quad h(x,y) = 0 \,\}$, where the perturbation parameter x runs over Ω that is a closed subset of $\Re^n$.

Definition 1.4.1.

Partially Convex Parametric Optimization Problem: By a partially convex parametric programming problem we mean that $y \to f(x,y)$ is convex in y for each x and the set $K(x)$ is convex for each x. (i.e., $g_i(x,.)$, $i = 1,\ldots,p$ are convex and $h_j(x,.)$, $j = 1,\ldots,q$, are affine linear for each fixed x).

Fully Convex Parametric Optimization Problem: By a fully convex parametric programming problem we mean that $f(x,y)$ is convex with respect to both x and y, and that gph$K$ is a convex set.

Let us assume the following:

- f, $g := (g_1, g_2, \ldots, g_p)^T$ and $h := (h_1, h_2, \ldots, h_q)^T$ are twice continuously differentiable on $\Omega \times \Re^m$;

- for each $x \in \Omega$, $l(x)$ is a partially convex programming problem.

Then the $S(x)$ is closed and convex (but may be empty), and we have the representation:

$$S(x) = \{\, y \in \Re^m \mid 0 \in \nabla_y f(x,y) + N(y, K(x)) \,\}$$
$$= \{\, y \in \Re^m \mid 0 \in \nabla_y f(x,y) + \partial_y \delta_K(x,y) \,\}, \tag{1.4.1}$$

where $\delta_K(x,y) = \delta_{K(x)}(y)$ denotes the indicator function of the set $K(x)$.

Variational Inequality Representation

Definition 1.4.2 (parametric variational inequality). *Let* $F : \Re^n \times \Re^m \to \Re^m$ *be a function and* $C : \Re^n \rightrightarrows \Re^m$ *be a set-valued map with (possibly empty) closed and convex values; i.e., for each* $x \in \Re^n$*,* $C(x)$ *is a closed and convex subset of* $\Re^m$*. The* parametric variational inequality *with parameter* x*, denoted by* $VI(F(x,.), C(x))$*, is defined by*

$$\textit{find } y \in C(x) \textit{ such that } (v-y)^T F(x,y) \geq 0 \textit{ for all } v \in C(x).$$

And let $SOL(F(x,.), C(x)) := \{\, y \in C(x) \mid (v-y)^T F(x,y) \geq 0 \ \forall\ v \in C(x) \,\}$ *be the* solution set *to the parametric variational inequality* $VI(F(x,.), C(x))$*.*

Now if $f : \Re^n \times \Re^m \to \Re$ is C^1, and $K(x)$ is convex, we have

$$\operatorname{argmin}\{\, f(x,y) \mid y \in K(x) \,\} \subseteq SOL(\nabla_y f(x,y), K(x)).$$

Furthermore, if $f(x,.)$ is convex, then $S(x) = SOL(\nabla_y f(x,y), K(x))$.

KKT Representation

Now fix some $\bar{x} \in \Omega$ and assume the *extended Slater constraint qualification* (ESCQ) holds at $\bar{x}$; i.e., there exists $\bar{y} := y(\bar{x}) \in \Re^m$ such that $g(\bar{x}, \bar{y}) < 0$ and $h(\bar{x}, \bar{y}) = 0$. Then we have

$$N(y, K(x)) = \{ \sum_{i=1}^{p} u_i \nabla_y g(x, y) + \sum_{i=1}^{q} v_i \nabla_y h(x, y) \mid u(x) \in \Re^p_+,\ v(x) \in \Re^q,$$
$$u_i(x) g_i(x, y) = 0, \quad \text{for} \quad i = 1, 2, \ldots, p \}. \qquad (1.4.2)$$

So,

$$S(x) = \{ y(x) \in K(x) \mid 0 \in \nabla_y f(x, y) + N(y, K(x)) \} \qquad (1.4.3)$$
$$= \{ y(x) \in \Re^m \mid \exists\, u(x) \in \Re^p_+ \text{ and } v(x) \in \Re^q \text{ such that}$$
$$\nabla_y L(x, y(x), u(x), v(x)) = 0,\ u(x)^T g(x, y) = 0,\ y(x) \in K(x) \}$$
$$= \left\{ y(x) \in \Re^m \;\middle|\; 0 \in \begin{pmatrix} \nabla_y L(x, y(x), u(x), v(x)) \\ -g(x, y(x)) \\ h(x, y(x)) \end{pmatrix} + \right.$$
$$\left. N_{\Re^m \times \Re^p_+ \times \Re^q}(y(x), u(x), v(x)) \right\}, \qquad (1.4.4)$$

where $\nabla_y L(x, y(x), u(x), v(x)) := \nabla_y f(x, y) + \sum_{i=1}^{p} u_i \nabla_y g(x, y) + \sum_{i=1}^{q} v_i \nabla_y h(x, y)$. For more details, see [70].

Another Simple Representation [54]

Suppose that for each fixed $x \in \Omega$. We have

- $f(x, .)$ is twice continuously differentiable and convex function on $y \in \Re^m$;
- $K(x)$ is a convex subset of $\Re^m$;
- $\bar{y} \in S(x)$.

Then we have

$$S(x) = \left\{ y \in K(x) \;\middle|\; \begin{array}{c} \langle \nabla_y f(x, \bar{y}), y - \bar{y} \rangle = 0 \\ \nabla_y^2 f(x, \bar{y})(y - \bar{y}) = 0 \end{array} \right\}. \tag{1.4.5}$$

Marginal Function Representation

We associate the lower level nonlinear parametric problem with a so-called value function or marginal function, $\mu : \Re^n \to \bar{\Re}$, defined by

$$\mu(x) := \begin{cases} \inf\{\, f(x, y) \mid y \in K(x) \,\} & \text{if } K(x) \neq \emptyset \\ \infty & \text{otherwise.} \end{cases}$$

Remark 1.4.3.

- Note that $\mu(\cdot)$ may take the value $-\infty$, because problem $l(x)$ may not have a solution. The value ∞ is assigned to $\mu(\cdot)$ by the convention that the infimum over the empty set is equal to ∞. In general the value function $\mu(\cdot)$ is not differentiable, neither convex, nor Lipschitz even if the functions f, g and h are. In spite of this eventual bad behavior of $\mu(\cdot)$, this function enables us to reformulate the bilevel programming problem as follows:

$$S(x) = \left\{ y \in \Re^m \;\middle|\; \begin{array}{c} f(x,y) - v(x) = 0, \\ y \in K(x) \end{array} \right\} \tag{1.4.6}$$

The reader is referred to [87] and the bibliographies therein for more details.

- By definition, S is a point-to-set valued mapping, which maps $x \in \Re^n$ to the set of global optimal solutions of problem $l(x)$. Also we define the local optimal solution set by

$$\begin{aligned} S_{\text{loc}}(x) := \{\, y \in K(x) \mid & \text{ there exists } \epsilon > 0 \text{ with} \\ & f(x,y) \le f(x,z) \text{ for all } z \in K(x) \textstyle\bigcap v_\epsilon(y) \,\}. \end{aligned}$$

  Then

  - ◇ $f(x,y) - \nu(x) = 0$ and $g(x,y) \le 0$ if and only if $y \in S(x)$;
  - ◇ $y \in S_{loc}(x)$ does not imply that $f(x,y) - \nu(x) = 0$;
  - ◇ $y \in S_{loc}(x)$ implies that $f(x,y) - \nu(x) \ge 0$ and $g(x,y) \le 0$ but the opposite is not true.

Penalty Function Representations

Definition 1.4.4 (semi-Lipschitz condition). *A function $h : \Re^n \to \Re$ is said to satisfy a* semi-Lipchitz condition *relative to $\Theta \subset \Re^n$ if there exists a constant (modulus) $l \ge 0$*

such that

$$h(x) - h(x') \leq l\|x - x'\| \quad \forall\, x \in \Theta, \quad x' \notin \Theta.$$

A function satisfying a semi-Lipschitz condition relative to Θ does not need to be Lipschitz continuous on Θ, although a (globally) Lipschitz continuous function always satisfies a semi-Lipschitz condition with respect to any set. In fact, an arbitrary function satisfies a semi-Lipschitz condition relative to any of its level sets.

Proposition 1.4.5 ([**91**])**.** *Suppose that for each $x \in \Omega \subset \Re^n$ we have*

- *$K(x)$ is a closed subset of $\Re^m$;*
- *$f(x,.)$ satisfies a semi-Lipschitz condition relative to the set $K(x)$ with modulus $l(x)$;*
- *let $\pi : \Re^n \times \Re^m \to \Re_+$ and $\beta : \Re^n \to \Re_+$ be functions satisfying $\beta(x) > l(x)$ and*

$$\pi(x,y) = \begin{cases} \geq \beta(x) d_K(x,y) & \text{if } y \notin K(x) \\ = 0 & \text{otherwise,} \end{cases}$$

where $d_K(x,y) := dist(y, K(x))$.

Then we have

$$S(x) = argmin\{\, f(x,y) + \pi(x,y) \mid y \in \Re^m \,\}. \tag{1.4.7}$$

Definition 1.4.6 (equi-Lipschitz continuous)**.** *The $f : \Re^n \times \Re^m \to \Re$ is said to be* *equi-Lipschitz continuous in y with constant l if $l > 0$ and for all $x \in \Omega$ we have*

$$|g(x,y) - g(x,y^{/})| \leq l \parallel y - y^{/} \parallel \quad \textit{for all} \quad y,\, y^{/}.$$

Proposition 1.4.7 ([91]). *If f is equi-Lipschitz continuous in y with constant l, then for any $\tau \geq l$ we have*

$$S(x) = argmin\{\, f(x,.) + \tau d_{\Omega}(x,.) \mid y \in \Re^{m}\,\}. \tag{1.4.8}$$

The reader is referred to Zhang [91, 92, 90] for comprehensive details.

In the optimization literature, the reader can find explicit and implicit representations of the rational reaction set. Among the marvelous techniques that are available in the literature, the reader is referred to [21], [34], [71] and [77] for representations of rational response and rational reaction set.

1.4.2 Reformulation and Basic Assumptions

A pessimistic bilevel programming problem is a three stage programming problem. So to derive the optimality conditions for PBLPP, it is wise to think about the reduction methods and basic assumption, which simplify the problem. Here we are going to make the following assumption that will help us to deriving optimality condition for PBLPP.

Assumption 1.4.8 (reformulation assumption (RA)). Let $x \in \Omega$ be fixed and $\tilde{y}(x)$ be a (global) optimistic solution to the lower level problem $l(x)$. Suppose that there exist $\bar{\mu} > 0$ such that

$$F(x, \tilde{y}(x)) - F(x, y) + \bar{\mu}(f(x, y) - v(x)) \geq 0 \quad \text{for all} \quad y \in K(x) \backslash S(x).$$

With this assumption one has the following result.

Theorem 1.4.9. *Let $\tilde{y}(x)$ be a (global) optimistic solution to the lower level problem $l(x)$. Suppose that the assumption (RA) holds at $\tilde{y}(x)$ for each $x \in \Omega$. Then $\tilde{y}(x)$ is a global solution to:*

$$\max F(x, y) \quad \text{subject to} \quad y \in S(x)$$

if and only if $\tilde{y}(x)$ is a global solution to:

$$\max\, [F(x, y) - \mu(f(x, y) - v(x))] \quad \text{subject to} \quad y \in K(x)$$

for all $\mu \geq \bar{\mu}$. Furthermore, the optimal values are equal.

Proof. Let $\tilde{y}(x)$ be a (global) optimistic solution to the lower level problem $l(x)$. Suppose that the assumption (RA) hold at $\tilde{y}(x)$ for each $x \in \Omega$.

If $\tilde{y}(x)$ is a global solution to:

$$\max F(x, y) \quad \text{subject to} \quad y \in S(x),$$

then $\tilde{y}(x) \in S(x)$ and $F(x,\tilde{y}) - F(x,y) \geq 0$ for all $y \in S(x)$. Let $\mu \geq \bar{\mu}$ and consider

$$\begin{aligned}
&F(x,\tilde{y}) - \mu(f(x,\tilde{y}) - v(x)) - (F(x,y) - \mu(f(x,y) - v(x)))\\
&= F(x,\tilde{y}) - F(x,y) + \mu(f(x,y) - v(x)) \qquad \text{as} \quad \tilde{y}(x) \in S(x)\\
&\geq F(x,\tilde{y}) - F(x,y) + \bar{\mu}(f(x,y) - v(x)) \qquad \text{as} \quad \mu \geq \bar{\mu}.
\end{aligned}$$

If $\quad y \in S(x), \quad$ then

$$\begin{aligned}
&\geq F(x,\tilde{y}) - F(x,y) \qquad \text{as} \quad f(x,y) - v(x) = 0\\
&\geq 0.
\end{aligned}$$

Now if $\quad y \in K(x), \quad$ then by the assumption (RA) we have

$$F(x,\tilde{y}) - F(x,y) + \bar{\mu}(f(x,y) - v(x)) \geq 0 \quad \text{for all} \quad y \in K(x).$$

Also, $\tilde{y}(x) \in S(x) \Rightarrow \tilde{y}(x) \in K(x)$.

So, $\tilde{y}(x)$ is a global solution of the problem

$$\max F(x,y) - \mu(f(x,y) - v(x)) \quad \text{subject to} \quad y \in K(x).$$

Conversely, suppose that $\tilde{y}(x)$ solve (globally):

$$\max F(x,y) - \mu(f(x,y) - v(x)) \quad \text{subject to} \quad y \in K(x)$$

for all $\mu \geq \bar{\mu}$.

Then

$$F(x,\tilde{y}) - \mu(f(x,\tilde{y}) - v(x)) - (F(x,y) - \mu(f(x,y) - v(x))) \geq 0 \;\forall\, y \in K(x)$$

$$F(x,\tilde{y}) - (F(x,y) - \mu(f(x,y) - v(x))) \geq 0 \quad \text{for all } y \in S(x)$$

$$F(x,\tilde{y}) - F(x,y) \geq 0 \quad \text{for all } y \in S(x).$$

So, $\tilde{y}(x)$ is a global solution to:

$$\max F(x,y) \quad \text{subject to} \quad y \in S(x).$$

□

Remark 1.4.10. If the assumption (RA) holds, then by the above theorem, $\tilde{y}(x)$ solves $\max\limits_{y \in S(x)} F(x,y)$ if and only if $\tilde{y}(x)$ solves $\min\limits_{y \in K(x)} [\mu f(x,y) - F(x,y)]$. But in this case the optimal values are not equal.

1.4.3 Sufficient Conditions for the Assumption(RA) and Lower Bound for $\bar{\mu}$

Remark 1.4.11. Assumption (RA) is clearly equivalent to the following:

$$0 < \sup_{\substack{y \in K(x) \setminus S(x) \\ x \in \Omega}} \frac{|F(x,y) - F(x,\tilde{y}(x))|}{f(x,y) - v(x)} < \infty.$$

Remark 1.4.12. Assumption (RA) is equivalently reduced to the following:

Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution to the PBLPP with $\tilde{y}(\bar{x}) = \bar{y}$. Let

$x \in N(\bar{x})$ be arbitrary and $\tilde{y}(x) \in S(x)$. Suppose that

- the functions f, and F are real valued;
- $F(x, \tilde{y}(x))$ and $v(x)$ are finite;
- $\exists\, \delta > 0$ and $\bar{\mu} > 0$ such that

$$F(x, y) - F(x, \tilde{y}(x)) \leq \bar{\mu}(f(x, y) - v(x))$$

  for all $y \in K(x) \backslash S(x)$ with $f(x, y) - v(x) < \delta$.

Then the assumption (RA) is satisfied.

Theorem 1.4.13. *Suppose that:*

- *$F(x, .)$ is Lipschitz continuous in y with uniformly in x with Lipschitz constant l.*

  i.e., $|f(x, y) - f(x, y^{/})| \leq l \parallel y - y^{/} \parallel$ for all $y,\, y^{/} \in K(x)$, and $x \in \Omega$.

- *The lower level problem $\{\, l(x) \mid x \in \Omega \,\}$ has a uniformly weak sharp minima.*

  i.e.,$\exists\, \alpha > 0$ such that $f(x, y) - v(x) \geq \alpha d(y, S(x))$ for all $y \in K(x)$ and $x \in \Omega$.

- *let $\tilde{y}(x)$ solve $\max\limits_{y \in S(x)} F(x, y)$.*

Then there exists $\bar{\mu} > 0$ such that

$$F(x, \tilde{y}(x)) - F(x, y) + \mu(f(x, y) - v(x)) \geq 0 \quad \textit{for all}\, y \in K(x) \textit{ and } \mu \geq \bar{\mu}.$$

Proof. Fix $x \in \Omega$. Let $\tilde{y}(x) \in S(x)$. Also let $\hat{y}(x) = \text{Proj}_{S(x)}(y)$. Then consider

$$\begin{aligned} F(x,\tilde{y}) - F(x,y) &\geq F(x,\hat{y}) - F(x,y) \qquad \text{as } \tilde{y}(x) \text{ solve } \max_{y \in S(x)} F(x,y) \\ &\geq -l \parallel y - \hat{y} \parallel \ \text{ as } F \text{ is u.Lipschitz continuous in } y \\ &= -ld(y, S(x)) \\ &\geq -l\alpha(f(x,y) - v(x)) \qquad \text{by weak sharp minima.} \end{aligned}$$

So, there exists $\bar{\mu} = l\alpha > 0$ such that,

$$F(x,\tilde{y}(x)) - F(x,y) + \mu(f(x,y) - v(x)) \geq 0 \quad \text{for all } y \in K(x) \text{ and } \mu \geq \bar{\mu}.$$

□

Theorem 1.4.14. *Suppose that $f(x,.)$ and $F(x,.)$ are convex and concave in y for all x respectively. Let $\tilde{y}(x) \in S(x)$ and assume the following qualification conditions hold true:*

(QC) $\qquad \partial_y f(x,\tilde{y}(x)) \bigcap \partial_y^+ F(x,\bar{y}(x)) \neq \emptyset.$

Then the assumption(A) holds at $\tilde{y}(x)$ with $\bar{\mu} = 1$.

Proof. Let $p \in \partial_y f(x,\tilde{y}(x)) \bigcap \partial_y^+ F(x,\tilde{y}(x))$.

Then $p \in \partial_y f(x,\tilde{y}(x))$ and $p \in \partial_y^+ F(x,\tilde{y}(x))$

This implies

$$\langle p, y - \tilde{y}(x) \rangle \leq f(x,y) - f(x,\tilde{y}(x)) \quad \text{for all } y$$

$$\langle p, y - \tilde{y}(x) \rangle \geq F(x,y) - F(x,\tilde{y}(x)) \quad \text{for all } y$$

So, $F(x,y) - F(x,\tilde{y}(x)) \leq f(x,y) - f(x,\tilde{y}(x)) \quad$ for all $y \in K(x)$.

i.e., $F(x,\tilde{y}(x)) - F(x,y) + (f(x,y) - v(x)) \geq 0 \quad$ for all $y \in K(x)$. □

1.5 Relations to Other Problems

1.5.1 Generalized Minimax Reformulations of PBLPP

Let us recall the pessimistic bilevel programming problem

$$\left.\begin{array}{l} \min\limits_{y} f(x,y) \qquad \text{subject to} \quad y \in K(x); \\[2ex] \varphi_p(x) := \max\{\, F(x,y) \mid y \in S(x) \,\}; \\[2ex] \min \varphi_p(x) \quad \text{subject to } x \in \Omega. \end{array}\right\} \qquad \text{PBLPP}$$

So PBLPP can be written as;

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x,y) \right). \tag{1.5.1}$$

A problem of this form is the *static minimax problem in the parametric constraint case.*

In particular, under the assumption (A), PBLPP can be equivalently written as

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} \left[F(x,y) - \mu f(x,y) + \mu v(x) \right] \right). \tag{1.5.2}$$

Now, if we assume that $K(x) = K$ (i.e., feasible set of the follower does not depends on x), we can formulate the problem as follows:

$$\min_{x\in\Omega}\left(\max_{y\in K}\left[F(x,y)-\mu f(x,y)+\mu v(x)\right]\right). \tag{1.5.3}$$

Then the problem becomes much simpler and is called *the static minimax problem in the non-parametric constraint case.*

Remark 1.5.1. The objective function is nonsmooth. That is a major drawback deriving the optimality condition using minimax technique.

1.5.2 Weak Stackelberg Problem

Now consider the following two-level optimization problem corresponding to a Stackelberg game (SG):

$$\text{Stackelberg Game:}\begin{cases}\text{Find}\,\bar{x}\in\Omega\subset\Re^n\,\text{such that}:\\ \sup\limits_{y\in S(\bar{x})}F(\bar{x},y)=\inf\limits_{x\in\Omega}\left(\sup\limits_{y\in S(x)}F(x,y)\right),\\ \text{where}\,S(x)\,\text{is the set of solutions to the lower level problem}\,l(x):\\ \min\limits_{y}f(x,y)\quad\text{subject to}\quad y\in Y\subset\Re^m.\end{cases}$$

Problem (SG) is called the weak Stackelberg problem and can be interpreted as a two-player nonzero-sum noncooperative game, see [56] and the references therein for more details. Let us introduce the following notation:

$$\varphi_p(x) := \sup_{y \in S(x)} F(x, y) \quad \text{and} \quad \varphi(x) := \inf_{x \in \Omega} (\phi_p(x)).$$

Definition 1.5.2 (**Stackelberg solution [47]**).

(i) *Any* $\bar{x} \in \Omega$ *verifying* $\varphi = \phi_p(\bar{x})$ *is called a* Stackelberg solution.

(ii) *Any pair* $(\bar{x}, \bar{y})$ *verifying* $\varphi = \phi_p(\bar{x})$ *and* $y \in S(\bar{x})$ *is called a* Stackelberg equilibrium pair.

Remark 1.5.3. Clearly, following relations hold. Any *global pessimistic optimal solution* is a *Stackelberg equilibrium point*. Conversely, any *Stackelberg solution* $(\bar{x}, \bar{y})$ need <u>not</u> be a *global pessimistic solution*, but if $(\bar{x}, \bar{y})$ is a *Stackelberg solution* with $F(\bar{x}, \hat{y}) = \sup_{y \in S(\bar{x})} F(\bar{x}, y)$, then $(\bar{x}, \hat{y})$ is a *pessimistic global solutions* to PBLBB.

1.5.3 Semi-Infinite Reformulations of PBLPP

Generalized Semi-Infinite Reformulations of PBLPP

Recall the PBLPP:

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x, y) \right).$$

This is equivalent to the following epi-graphical reformulation:

$$\min_{(x,\tau) \in \Omega \times \Re} \tau \qquad \text{subject to} \quad F(x, y) - \tau \leq 0 \quad \text{for all } y \in S(x),$$

i.e., $\min \tau \qquad \text{subject to } (x, \tau) \in M,$

where $M := \{ (x, \tau) \in \Omega \times \Re \mid F(x,y) - \tau \leq 0 \quad \text{for all}\, y \in S(x) \}$

with $S(x) = \{ y \in \Re^m \mid f(x,y) - v(x) \leq 0, \quad g(x,y) \leq 0 \}$, which is a generalized semi-infinite programming problem (GSIP).

Standard Semi-Infinite Reformulations of PBLPP

As opposed to a standard semi-infinite optimization problem (SIP), the possibly infinite index set of inequality constraints is independent of the parameter in a GSIP. The standard semi-infinite reformulation of PBLPP is possible under the assumption (RA). Under the assumption (RA), PBLPP is equivalent to:

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} \left[F(x,y) - \mu(f(x,y) - v(x)) \right] \right).$$

Let us assume that $K(x) \equiv K$, i.e., independent of x. Now look at the epi-graphical reformulation:

$$\min \tau \qquad \text{subject to } (x, \tau) \in M$$

where $M := \{ (x, \tau) \in \Omega \times \Re \mid F(x,y) - \mu(f(x,y) - v(x)) - \tau \leq 0 \quad \text{for all}\, y \in K \}$, which is a standard semi-infinite programming problem.

1.5.4 Implicit Programming Reformulations of PBLPP

Let $x \in \Omega$ be fixed and also let $\max_{y \in S(x)} F(x, y) = F(x, \tilde{y}(x))$, where $\tilde{y}(x)$ is a any globally solution to $\max_{y \in S(x)} F(x, y)$ for each $x \in \Omega$. Then the PBLPP can be written as:

$$\min F(x, \tilde{y}(x)) \qquad \text{subject to}\, x \in \Omega$$

where $\tilde{y}(x)$ solve(globally) $\max_{y \in S(x)} F(x, y)$ for each $x \in \Omega$.

Under the assumption (RA), this can be further simplified as:

$$\min F(x, \tilde{y}(x)) \qquad \text{subject to}\, x \in \Omega,$$

where $\tilde{y}(x)$ solve (globally) $\max_{y \in K(x)} [F(x, y) - \mu f(x, y)]$ for each $x \in \Omega$.

1.5.5 MPEC Reformulations of PBLPP

Consider the PBLPP:

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x, y) \right).$$

Under the assumption (RA) this equivalent to:

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} [F(x, y) - \mu(f(x, y) - v(x))] \right).$$

Now we consider:

$$\begin{aligned} M(x) &:= \text{argmax}\{\, F(x,y) \mid y \in S(x)\,\} \\ &= \text{argmax}\{\, F(x,y) - \mu f(x,y) + \mu v(x) \mid y \in K(x)\,\} \quad \text{by assumption (RA)} \\ &= \text{argmax}\{\, F(x,y) - \mu f(x,y) \mid y \in K(x)\,\} \qquad (1.5.4) \end{aligned}$$

if $K(x)$ is closed and $F(x,.) - \mu f(x,.)$ satisfies a semi-Lipschitz condition

$$= \text{argmax}\{\, F(x,y) - \mu f(x,y) - \pi(x,y) \mid y \in \Re^m\,\} \quad \text{(see section 1.4.1)}$$

Now write $-\zeta(x,y) := F(x,y) - \mu f(x,y) - \pi(x,y)$.

Assume that $\zeta(x,.)$ is convex in y for each x. Then

$$M(x) = \{\, y \in \Re^m \mid 0 \in \partial_y \zeta(x,y)\,\}$$

Consequently, the PBLPP can be written equivalently as

$$\left.\begin{aligned} &\min F(x,y), \\ &\text{subject to} \quad y \in M(x),\, x \in \Omega, \end{aligned}\right\} \qquad (1.5.5)$$

which is a MPEC problems.

Another MPEC reformulation of PBLPP is possible under some assumption.

Under the assumption (RA), PBLPP is equivalent to:

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} \left[F(x,y) - \mu(f(x,y) - v(x)) \right] \right).$$

Let $K(x) = \{ y \in Y \subset \Re^m \mid g(x,y) \leq 0 \}$ and write $\mathcal{F}(x,y) = F(x,y) - \mu(f(x,y) - v(x))$.

Now we make the following assumptions:

- for any fixed x, $\mathcal{F}(x,y)$ and $g(x,y)$ are concave and convex in y, respectively;
- for any fixed x, $\mathcal{F}(x,y)$ and $g(x,y)$ are continuous in y;
- Y is a convex set;
- for any fixed x, there exist $y \in Y$ such that $g(x,y) \leq 0$;.
- Y is non-empty and compact.

Under the above assumptions, PBLPP is equivalent to the following problem, which is a MPEC problem:

$$\left. \begin{array}{l} \min\limits_{(x,v)} \left(\max\limits_{y \in Y} \left(\mathcal{F}(x,y) - v^T g(x,y) \right) \right), \\ \text{subject to} (x,v) \in \Omega \times \Lambda, \end{array} \right\} \tag{1.5.6}$$

where $\Lambda = \{ v \mid v \geq 0 \}$.

1.6 Nonsmooth Analysis and Tools from Variational Analysis

It is clear that dealing with non smooth functions is unavoidable when we derive optimality conditions for PBLPP and thus requires appropriate tools of generalized differentiation. Various generalizations and constructions have been developed and are available in the literature of nonsmooth optimization. In this dissertation we use the basic/limiting constructions by Mordukhovich, which enjoy full robust calculus and turn out to be minimal among any constructions of this type satisfying natural properties employed in this paper. We refer the reader to the book by Mordukhovich [59] for more details, additional material, extensive comments, and bibliographies. Developing a geometric approach to the generalized differentiation, we begin with the definition of a normal cone to an arbitrary set in finite dimensions.

1.6.1 Normal Cones to Nonconvex Sets

Notation 1.6.1. For a multifunction $F : \Re^n \rightrightarrows \Re^m$, the *graph* of F denoted by $\operatorname{gph} F$ is defined as $\operatorname{gph} F := \{ (x, y) \in \Re^n \times \Re^m \mid y \in F(x) \}$. The *Kuratowski- Painleve outer/upper limit* of F as $x \to \bar{x}$ is

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \bigcap_{\epsilon>0} \operatorname{cl} \bigcup_{|x-\bar{x}|\leq\epsilon} F(x). \tag{1.6.1}$$

It is the set of all limiting points $\lim_{k\to\infty} y_k$, where $y_k \in F(x_k)$ and $x_k \to \bar{x}$ as $k \to \infty$. Let Ω be a closed subset of $\Re^n$ and let

$$P(x, \Omega) := \{ \omega \in \Omega \mid \| x - \omega \| = \operatorname{dist}(x, \Omega) \} = \bigcup_{\alpha>0} \alpha(x - \pi(x, A)) \tag{1.6.2}$$

be the set of best approximations of x in Ω with respect to the Euclidean distance function $\operatorname{dist}(x, \Omega)$.

Definition 1.6.2 (Mordukhovich (basic, limiting) normal cone). *Given* $\bar{x} \in \Omega$, *the following closed one*

$$N(\bar{x},\Omega) := \underset{x\to\bar{x}}{Lim\ sup}[cone(x - P(x,\Omega))] \tag{1.6.3}$$

is called the Mordukhovich (basic, limiting) normal cone *to* Ω *at* $\bar{x}$*. If* $\bar{x} \notin \Omega$*, we put* $N(\bar{x},\Omega) = \emptyset$.

It is clear from the definition that the normal cone is *robust* with respect to perturbations of $\bar{x}$; i.e., the multifunction $N(\bar{x},\Omega)$ always has a *closed graph*. Furthermore, $N(\bar{x},\Omega) \neq 0 \Leftrightarrow \bar{x} \in \partial\Omega$. If Ω is a convex set, then the normal cone introduced coincides with the classical normal cone of convex analysis. In general, the normal cone may be nonconvex in very simple situations, e.g., for the set $\Omega = \text{gph}\,|x|$ at $0 \in \Re^2$. For each $x \in \Omega$ let us consider the so-called set of the *prenormal/Fréchet normal cone* to Ω at x defined by

$$\widehat{N}(\bar{x},\Omega) := \{\, x^* \in \Re^n \mid \underset{x \xrightarrow{\Omega} \bar{x}}{\text{limsup}} \frac{\langle x^*, x - \bar{x}\rangle}{\|\ x - \bar{x}\ \|} \leq 0 \,\} \tag{1.6.4}$$

and put $\widehat{N}(\bar{x},\Omega) = \emptyset$ for $\bar{x} \notin \Omega$. Directly from the definition one can show that the set $\widehat{N}(\bar{x},\Omega)$ is a closed convex cone.

Proposition 1.6.3. *For any set* $\Omega \subset \Re^n$ *and point* $\bar{x} \in \Omega$ *we have*

$$N(\bar{x},\Omega) = \underset{x \xrightarrow{\Omega} \bar{x}}{limsup}\, \widehat{N}(\bar{x},\Omega). \tag{1.6.5}$$

It is clear that $\widehat{N}(\bar{x},\Omega) \subset N(\bar{x},\Omega)$ and the set Ω is called *regular* at $\bar{x}$ if $N(\bar{x},\Omega) = \widehat{N}(\bar{x},\Omega)$. It happens for all convex sets as well as for "smooth" sets which are locally described by equality and inequality type constraints with smooth functions under well-known Mangasarian-Fromovitz constraints qualification. On the other hand, this regularity is always broken for sets locally represented as graphs of nonsmooth Lipschitzian functions. But such sets naturally appear in the following *coderivative* constructions, which are realizations of the geometric (graphical) approach to differentiation going back to Fermat.

1.6.2 Coderivatives of Set-Valued Mappings

Let $F : \Re^n \rightrightarrows \Re^m$ be a multifunction of closed graph. The multifunction $D^*F(\bar{x},\bar{y}) : \Re^m \rightrightarrows \Re^n$ defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \{\, x^* \in \Re^n \mid (x^*,-y^*) \in N[(\bar{x},\bar{y}); \operatorname{gph} F] \,\} \tag{1.6.6}$$

is called the *coderivative* of F at the point $(\bar{x},\bar{y}) \in \operatorname{gph} F$. We put $D^*F(\bar{x},\bar{y})(y^*) = \emptyset$ if $(\bar{x},\bar{y}) \notin \operatorname{gph} F$. The symbol $D^*F(\bar{x})$ is used when F is single-valued at $\bar{x}$ and $\bar{y} = F(\bar{x})$. If we replace the basic normal cone in the above coderiavtive construction by the prenormal cone, we have the so-called *contingent coderivative* of F at $(\bar{x},\bar{y})$ defined by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \{\, x^* \in \Re^n \mid (x^*,-y^*) \in \widehat{N}[(\bar{x},\bar{y}); \operatorname{gph} F] \,\}. \tag{1.6.7}$$

The contingent coderivative is convex-valued and can be represented in the form

$$D^*F(\bar{x},\bar{y})(y^*) := \{\, x^* \in \Re^n \mid \langle x^*, v\rangle \le \langle y^*, u\rangle \forall (u,v) \in \operatorname{gph} \widehat{D}^*F(\bar{x},\bar{y}) \,\} \tag{1.6.8}$$

It is clear that $\widehat{D}^*F(\bar{x},\bar{y})(y^*) \subset D^*F(\bar{x},\bar{y})(y^*)$ for all $y^* \in \Re^m$ and if they coincide, then the multifunction F is said to be *differentially regular* at $(\bar{x},\bar{y})$. If F is single-valued and *strictly differentiable* at $\bar{x}$ with the gradient $\nabla F(\bar{x})$, in the classical sense that

$$\lim_{x,u \to \bar{x}} \frac{F(u) - F(x) - \langle \nabla S(\bar{x}), u - x \rangle}{\| u - x \|} = 0, \tag{1.6.9}$$

(this is automatic the case when F is *continuously differentiable* around $\bar{x}$), then

$$D^*F(\bar{x})(v) = \{\nabla F(\bar{x})^* v\} \quad \text{for all } v \in \Re^n. \tag{1.6.10}$$

This shows that the coderivative is a proper extension of the *adjoint* derivative operator to nonsmooth and set-valued mappings.

1.6.3 Subdifferentials of Nonsmooth Functions

Now let us consider an extended-real valued function $\varphi : \Re^n \to \bar{\Re} := [\infty, -\infty]$ and assume that $|\varphi(\bar{x})| < \infty$. Developing a geometric approach to the generalized differentiation of extended real-valued functions, we define the following subdifferential constructions.

(i) The *Mordukhovich (basic, limiting) subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) = \{\, x^* \mid (x^*, -1) \in N[(\bar{x}, \varphi(\bar{x})); \operatorname{epi} \varphi] \,\} \tag{1.6.11}$$

(ii) The *Mordukhovich (basic, limiting) upper subdifferential* of φ at $\bar{x}$ is defined by

$$\partial^+\varphi(\bar{x}) = \{ x^* \mid (x^*, -1) \in N[(\bar{x}, \varphi(\bar{x})); \operatorname{hypo}\varphi] \} \tag{1.6.12}$$

(iii) The *presubdifferential*(*Fréchet/viscosity subdifferential*) of φ at $\bar{x}$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) = \{ x^* \mid (x^*, -1) \in \widehat{N}[(\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi] \} \tag{1.6.13}$$

(iv) *Fréchet upper subdifferential* of φ at $\bar{x}$ is defined by

$$\widehat{\partial}^+\varphi(\bar{x}) = \{ x^* \mid (x^*, -1) \in \widehat{N}[(\bar{x}, \varphi(\bar{x})); \operatorname{hypo}\varphi] \} \tag{1.6.14}$$

One can easily observe that

$$-\widehat{\partial}^+(-\varphi)(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x}) = -\partial^+(-\varphi)(\bar{x}) \tag{1.6.15}$$

This inclusion is often strict, which may happen even for Fréchet differentiable function on $\Re$. We say that φ is *lower regular* at $\bar{x}$ if $\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x})$. Similarly we define *upper regularity* of φ at $\bar{x}$ by $\widehat{\partial}^+\varphi(\bar{x}) = \partial^+\varphi(\bar{x})$. The class of lower regular functions includes all convex functions, smooth functions, "max functions," etc. The basic subdifferential is nonconvex while the presubdifferetial is convex for an arbitrary function φ. If φ is convex, both $\partial\varphi(\bar{x})$ and $\widehat{\partial}\varphi(\bar{x})$ are reduced to the classical subdifferential of convex analysis. Also note that the sets $\widehat{\partial}\varphi(\bar{x})$ and $\widehat{\partial}^+\varphi(\bar{x})$ may be empty simultaneously for continuous on

$\Re$, e.g., for $\varphi(x) = x^{\frac{1}{3}}$ at $\bar{x} = 0$. Given a function $\varphi : \Re^n \to \bar{\Re}$ with $|\varphi(\bar{x})| < \infty$ is Fréchet differentiable at $\bar{x}$ if and only if $\widehat{\partial}\varphi(\bar{x}) \neq \emptyset$ and $\widehat{\partial}^+\varphi(\bar{x}) \neq \emptyset$, in which case $\widehat{\partial}\varphi(\bar{x}) = \widehat{\partial}^+\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$. Given a Lipschitz continuous function $\varphi : \Re^n \to \Re$ around $\bar{x}$, $\partial\varphi(\bar{x})$ is nonempty and compact. Also if φ happens to be strictly differentiable at $\bar{x}$, then $\partial\varphi(\bar{x}) = \partial^+\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$. Furthermore, we have the following representations. Consider a nonempty set $\Omega \subset \Re^n$ and its indicator function $\delta(\cdot, \Omega) : \Re^n \to \bar{\Re}$ defined by

$$\delta(x, \Omega) := 0 \quad \text{if } x \in \Omega \quad \text{and} \quad \delta(x, \Omega) := \infty \quad \text{if } x \notin \Omega. \tag{1.6.16}$$

Then for any $\bar{x} \in \Omega$ one has

$$\widehat{\partial}\delta(\bar{x}, \Omega) = \widehat{N}(\bar{x}, \Omega) \quad \text{and} \quad \partial\delta(\bar{x}, \Omega) = N(\bar{x}, \Omega). \tag{1.6.17}$$

Also one has the following analytical representation of subdifferential constructions

$$\widehat{\partial}\varphi(\bar{x}) = \Big\{\, x^* \;\Big|\; \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \,\Big\} \tag{1.6.18}$$

Given a lower semi continuous function around $\bar{x}$ we also have

$$\partial\varphi(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{\varphi} \bar{x}} \widehat{\partial}\varphi(x) \tag{1.6.19}$$

At the same time, for a given single-valued locally Lipschitzian mapping $f : \Re^n \to \Re^m$

the coderivative can be represented analytically as

$$D^*f(\bar{x}) = \partial\langle v, f\rangle(\bar{x}) \neq \emptyset \quad \text{for all} \quad v \in \Re^m, \tag{1.6.20}$$

where $\langle v, f\rangle)(x) := \langle v, f(x)\rangle$.

Most importantly the above noconvex constructions are rich in calculus. For comprehensive details, nice calculus rules and fruitful developments of nonsmooth analysis, the reader is referred to the monographs [59] and [60] by Mordukhovich.

Chapter 2

Implicit Programming Approach

2.1 Introduction

This chapter we devote to get necessary optimality conditions for the pessimistic bilevel programming problem (PBLPP) using the well-known techniques of the *implicit programming approach* (IPA). The procedure of obtaining necessary optimality conditions consists of two steps: namely, step one is to get the optimality conditions with selection function, and step two is to construct the corresponding derivative of the selection function. Having this in mind, we choose a selection function $y : \Re^n \to \Re^m$ by

$$y(x) \in M(x) := \operatorname{argmax} \{ F(x, y) \mid y \in S(x) \}.$$

Then the PBLPP can be written as

$$\min F(x, y(x)) \quad \text{subject to} \quad x \in \Omega.$$

The implicit programming approach to the solutions of PBLPP is motivated by this implicit form of the PBLP, which expresses this problem as an optimization problem in the upper-level variable x alone. In the following sections, we look at obtaining the optimality conditions for PBLPP under the Lipschitz selections, Fréchet differentiable selections and continuous selections. The IPA approach involves a deep understanding of the sensitivity analysis and stability theory of parametric nonlinear programming.

2.2 Optimality Conditions with Lipschitz Continuous Solution Selection

2.2.1 Basic Assumptions

Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution to the PBLPP. The key conceptual assumption within the IPA approach for PBLPPs near the point $(\bar{x}, \bar{y})$ is the following:

Assumption 2.2.1 (key implicit assumption (KIA)). Suppose there exists a neighborhood U of $\bar{x}$ and a locally Lipschitz continuous function $\tilde{y} : U \to \Re^m$ around $\bar{x}$ such that $\tilde{y}(\bar{x}) = \bar{y}$ and $\tilde{y}(x)$ solves (globally):

$$\max_{y} F(x, y) \quad \text{such that} \quad y \in S(x)$$

for each $x \in U$.

Under the assumption(KIA), we can reformulate the PBLPP as follows:

$$\min_{x \in \Omega} \varphi_p(x) := F(x, \tilde{y}(x)) := (F \circ \tilde{y})(x). \tag{2.2.1}$$

Remark 2.2.2.

1. We call the above function $\tilde{y}(x)$ an implicit (solution) function of the problem (2.2.1).

2. It happens that the PBLPP is equivalent to the above implicit programming problem (2.2.1) only in the case when $\tilde{y}(x)$ is a global optimal solution for each x. The computation of a globally optimal solution in the problem (2.2.1) can be computationally intractable (especially tracing the global solution set mapping for varying parameter). This will be a major drawback of this method. One of the natural ways to avoid this difficulty is to assume that the function $F(x,.)$ is concave in y for each x.

3. Note that the assumption (KIA) only ensures that the problems PBLPP and (2.2.1) are equivalent when x is near $\bar{x}$.

2.2.2 Optimality Conditions

Under the assumption (KIA), we can come up with the following optimality conditions for the pessimistic bilevel programming problem.

Theorem 2.2.3 (necessary optimality conditions). *Let $(\bar{x},\bar{y})$ be a local pessimistic optimal solution. Let $F:\Re^n\times\Re^m\to\Re$ be strictly differentiable at this point and the set $\Omega\subset\Re^n$ be closed at $\bar{x}$. If the assumption KIA is satisfies, then we have*

$$0\in\nabla_x F(\bar{x},\bar{y})+D^*\tilde{y}(\bar{x})(\nabla_y F(\bar{x},\bar{y}))+N(\bar{x},\Omega)$$

or

$$0\in\nabla_x F(\bar{x},\bar{y})+\partial\langle\nabla_y F(\bar{x},\bar{y}),\tilde{y}\rangle(\bar{x})+N(\bar{x},\Omega).$$

Proof. Assumption (KIA) implies that the PBLPP is equivalent to

$$\min_{x\in\Omega} \rho(x) := F(x, \tilde{y}(x)) := (F \circ \tilde{y})(x)$$

for each $x \in U$. Since F is strictly differentiable at $(\bar{x}, \bar{y})$, it is locally Lipschitz continuous around this point. Now $\tilde{y}(x)$ is locally Lipschitz continuous around $\bar{x}$ (by the assumption (KIA)) implies that the composition $F \circ \tilde{y}$ is also locally Lipschitz continuous around $\bar{x}$. Then by the Proposition 5.3 [60], we have

$$0 \in \partial(F \circ \tilde{y})(\bar{x}) + N(\bar{x}, \Omega).$$

Now by the subdifferentiation of composition, see the Theorem 1.110 [59] we have,

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + D^*\tilde{y}(\bar{x})(\nabla_y F(\bar{x}, \bar{y})) + N(\bar{x}, \Omega).$$

Also since $\tilde{y}(x)$ is Lipschitz continuous around $\bar{x}$, by the Theorem 1.90 in [59], $D^*\tilde{y}(\bar{x})(\nabla_y F(\bar{x}, \bar{y})) = \partial\langle\nabla_y F(\bar{x}, \bar{y}), \tilde{y}\rangle(\bar{x})$. Hence the results,

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + \partial\langle\nabla_y F(\bar{x}, \bar{y}), \tilde{y}\rangle(\bar{x}) + N(\bar{x}, \Omega).$$

□

2.2.3 Constructions of Coderivative of Selection Mappings

As we have seen in optimality conditions, we need to construct the coderivative of the implicit function. Throughout this section the differentiability properties of the solution map and such construction are considered. Let us start with some basic assumptions and a reformulation of the PBLPP.

Recall the assumption (RA): Let $x \in \Omega$ be fixed and $\bar{y}(x)$ be a (global) optimistic solution to the lower level problem $l(x)$. Suppose that there exists $\bar{\mu} > 0$ such that

$$F(x, \bar{y}(x)) - F(x, y) + \bar{\mu}(f(x, y) - v(x)) \geq 0 \quad \text{for all} \quad y \in K(x) \backslash S(x).$$

With this assumption, one has the following result.

Reformulation of PBLPP: With the above assumption RA, as shown in the Chapter 1, we have the following result. Let $(\bar{x}, \bar{y})$ be a pessimistic optimal solution and let $\bar{y}(x)$ be a (global) optimistic solution to the lower level problem $l(x)$ for each x near $\bar{x}$. Suppose that assumption (RA) holds at $\bar{y}(x)$ for each $x \in \Omega$.

Then $\bar{y}(x)$ is a global solution to the problem

$$\max F(x, y) \quad \text{subject to} \quad y \in S(x)$$

if and only if $\bar{y}(x)$ is a global solution to:

$$\max\ [F(x,y) - \mu(f(x,y) - v(x))] \quad \text{subject to} \quad y \in K(x) \qquad \text{for all } \mu \geq \bar{\mu}$$

if and only if $\bar{y}(x)$ is a global solution to:

$$\max\ [\mathcal{F}(x,y) := F(x,y) - \mu f(x,y)] \quad \text{subject to} \quad y \in K(x) \qquad \text{for all } \mu \geq \bar{\mu}$$

Now throughout this chapter we consider the following problem

$$\min_{y} \mathcal{F}(x,y) := \mu f(x,y) - F(x,y) \quad \text{s.t} \quad y \in K(x), \tag{P(x)}$$

where $x \in U(\in N(\bar{x})) \subset \Omega$ and the solution set mapping $M : \Re^n \to \Re^m$ is defined by

$$\begin{aligned} M(x) &:= \operatorname{argmax}\{\, F(x,y) \mid y \in S(x) \,\} \\ &= \operatorname{argmin}\{\, \mathcal{F}(x,y) := \mu f(x,y) - F(x,y) \mid y \in K(x) \,\}. \qquad \text{(by RA)} \end{aligned} \tag{2.2.2}$$

Now we make the following assumptions that we are going to impose throughout this section. The first assumption is indeed sufficient for KIA.

- We say that M possesses a *Lipschitz Single-Valued Localization*(LSVL) at $(\bar{x}, \bar{y})$, if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a Lipschitz selection $\tilde{y} : U \to \Re^m$ such that

$$\tilde{y}(x) = M(x) \cap V \quad \text{for all} \quad x \in U \qquad \text{and} \quad \tilde{y}(\bar{x}) = \bar{y}.$$

- The problem $P(x)$ is said to be *partially convex and smooth* programming problem if $F(x,\cdot)$ is smooth and convex in y and the set $K(x)$ is a closed and convex subset of $\Re^m$ for each x.

$$\text{Let} \quad \Psi\{f,F\} := \{\,(f,F)\,|f,F:\Re^n\times\Re^m\to\Re,\ \text{such that } f(x,\cdot)-F(x,\cdot)$$
$$\text{is smooth and convex function in } y \text{ for each } x \text{ near } \bar{x}\,\}.$$

In the following, we assume that the solution set $M(x)$ possess a Lipschitz single valued localization $\tilde{y}(x)$ for each x near $\bar{x}$ and study the differentiability of this vector valued function.

Case 1: We first consider $K(x)=K$, i.e., K is independent of the parameter x.

Theorem 2.2.4. *Let $(\bar{x},\bar{y})$ be a local pessimistic optimal solution to PBLPP and Let $P(x)$ be a partially convex and smooth problem for each x near $\bar{x}$. Assume also that $\nabla_y F:\Re^n\times\Re^m\to\Re^m$ is strictly differentiable at $(\bar{x},\bar{y})$ and M possess a Lipschitz single-valued localization $\tilde{y}(x)\in M(x)$ for each x near $\bar{x}$. Let $\bar{z}:=-\nabla_y F(\bar{x},\bar{y})$. Then we have*

$$D^*\tilde{y}(\bar{x})(v)\subseteq\{\,u\in\Re^n\mid\exists\, w\in\Re^m \textit{ such that } (u,-v)\in\ \nabla(\nabla_y F(\bar{x},\bar{y}))^T w$$
$$+\,D^*(N_K(\bar{y},\bar{z}))w\,\}$$

$$\textit{provided that}\ \ [-(\nabla^2_{xy}F(\bar{x},\bar{y}))^T z=0 \quad \textit{and} \quad w-(\nabla^2_{yy}F(\bar{x},\bar{y}))^T z=0$$
$$\textit{with } (w-z)\in N((\bar{y},\bar{z}),gph\,N_K)]\ \Rightarrow\ w=0 \textit{ and } z=0.$$

In particular, the above equality holds provided that the matrix $\nabla^2_{xy}F(\bar{x},\bar{y})$ *is of full raw rank.*

Proof. Since $P(x)$ is partially convex smooth problem, we have

$$M(x) = \{\, y \in K \mid 0 \in \nabla_y F(x,y) + N(y,K) \,\}.$$

Since K is closed subset, gph$N(\cdot,K)$ is closed. By applying the theorem 4.44(i) in [59], we get

$$\begin{aligned} D^*\tilde{y}(\bar{x})(v) &= D^*M(\bar{x},\bar{y})(v) \\ &\subseteq \{\, u \in \Re^n \mid \exists\, w \in \Re^m \text{ such that } (u,-v) \in \nabla(\nabla_y F(\bar{x},\bar{y}))^T w \\ &\qquad + D^*(N_K(\bar{x},\bar{y},\bar{z}))w \,\}. \end{aligned}$$

□

Case 2(a): Now we assume that the lower level feasible set K depends on the parameter x and has the following functional representation:

$$K(x) := \{\, y \in \Re^m \mid g_i(x,y) \le 0, i = 1,2,\dots,p,\ h_j(x,y) = 0, j = 1,2,\dots,q \,\}.$$

Let U be a neighborhood of $\bar{x}$ and assume the following:

- $\nabla_y F$ is continuously differentiable on $U \times \Re^m$;
- For each $x \in U$, $\nabla_y F(x,\cdot)$ is convex on $\Re^m$;

- $g = (g_1, g_2, \ldots, g_p)^T$ and $h = (h_1, h_2, \ldots, h_q)^T$ are twice continuously differentiable on $U \times \Re^m$;

- For each $x \in U$, $g(x, \cdot)$ is convex and $h(x, \cdot)$ is affine on $\Re^m$;

- Assume that the Extended Slater Constraint Qualification (ESCQ) holds for each $x \in U$.

  (Under the above assumptions ESCQ is equivalent to MFCQ.)

Then as we discussed in section (1.4.1) one has

$$\begin{aligned} M(x) &= \{\, y \in \Re^m \mid 0 \in \nabla_y F(x, y) + N(y, K(x)) \,\} \\ &= \left\{ y \in \Re^m \,\middle|\, 0 \in \begin{pmatrix} L(x, y, u, v) \\ -g(x, y) \\ h(x, y) \end{pmatrix} + N_{\Re^m \times \Re^p_+ \times \Re^q}(y, u, v) \right\}, \end{aligned}$$

where $L(x, y, u, v) := F(x, y) + \sum_{i=1}^p u_i g_i(x, y) + \sum_{j=1}^q v_j h_j(x, y)$ and u and v are Lagrange multipliers.

Under the above assumptions, Lagrangian multipliers u and v do not depend on $y \in M(x)$; see proposition 6.5.2[79], and it is clear that now the problem $P(x)$ is reduced to the case 1. That is K is independent of the parameter x. Hence one can arrive at the coderivative construction as above.

Case 2(b): Now let us relax the above qualification condition ESCQ and have the following theorem to cover the parameter dependent feasible set.

Theorem 2.2.5. *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution to PBLPP, and let $P(x)$ be a partially convex and smooth problem for each x near $\bar{x}$. Assume also that $\nabla_y F : \Re^n \times \Re^m \to \Re^m$ is strictly differentiable at $(\bar{x}, \bar{y})$ and that M possess a Lipschitz single-valued localization $\tilde{y}(x) \in M(x)$ for each x near $\bar{x}$. Let $\bar{z} := -\nabla_y F(\bar{x}, \bar{y})$. Then we have*

$$D^*\tilde{y}(\bar{x})(v) \subseteq \{\, u \in \Re^n \mid \exists\, w \in \Re^m \text{ such that } (u, -v) \in \nabla(\nabla_y F(\bar{x}, \bar{y}))^T w + D^*(N_K(\bar{x}, \bar{y}, \bar{z}))w \,\}$$

provided that $[\, v \in \Re^m \text{ with } 0 \in \nabla(\nabla_y F(\bar{x}, \bar{y}))^T v + D^*(N_K(\bar{x}, \bar{y}, \bar{z}))v] \;\Rightarrow\; v = 0.$

Proof. Since $P(x)$ is partially convex smooth problem, we have

$$M(x) = \{\, y \in K(x) \mid 0 \in \nabla_y F(x, y) + N((x, y); K) \,\},$$

where

$$N((x,y);K) := \begin{cases} N_{K(x)}(y) & \text{if} \quad K(x) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $K(x)$ is closed for each x near $\bar{x}$, gphN_K is closed around $(\bar{x}, \bar{y})$. Now apply the theorem 4.44(ii)[59], to get

$$D^*\tilde{y}(\bar{x})(v) \subseteq \{\, u \in \Re^n \mid \exists\, w \in \Re^m \text{ such that } (u, -v) \in \nabla(\nabla_y F(\bar{x}, \bar{y}))^T w + D^*(N_K(\bar{x}, \bar{y}, \bar{z}))w \,\}$$

□

Remark 2.2.6.

1 If the graph of N_K is locally convex around $(\bar{x}, \bar{y}, \bar{z})$, then N_K is N-regular at this point. And the above inequality holds as equality and $\tilde{y}(\cdot)$ is strictly differentiable at $\bar{x}$. However, it is important to mention that even though K is a convex set, the set gpf N is not a convex set.

2 *Calculation of Coderivative of Normal Cone to the Feasible Set K:*

In order to compute $D^*(N_K(\bar{x}, \bar{y}, \bar{z}))w$, one needs to compute $N[(\bar{x}, \bar{y}, \bar{z}); \operatorname{gph} N_k]$. The computation of this object appears to be rather involved. However if K has some special form, then one can have an explicit expression for the $D^*(N_K(\bar{x}, \bar{y}, \bar{z}))w$ and the reader is referred to [29] and [30] for such outstanding constructions.

3 Under more restrictive assumptions, it is possible to compute the derivative of selection mapping $\tilde{y}(x)$. In next section, we are going to briefly mention well-known results in classical nonlinear programming theory.

2.2.4 Existence of Implicit Solutions

In this section, we consider the existence of the implicit solution to the problem $P(x)$ and hence the sufficient conditions for the assumption (KIA). The following are some of the well-known results that provide the sufficient conditions for the assumption KIA, under the well-known assumptions. For proofs and further information, you may look at the corresponding references provided.

Recall the problem $P(x)$ with $K(x) = \{ y \in \Re^m \mid g(x,y) \leq 0, h(x,y) = 0 \}$.

Theorem 2.2.7 (**[23, 73]**)**.** *Consider the problem $P(x)$ and suppose that $(\bar{x}, \bar{y})$ is a solution to the problem $P(\bar{x})$. Suppose that the functions $f(x,y), F(x,y), g(x,y)$, and $h(x,y)$ and their partial derivatives with respect to y are continuously differentiable in a neighborhood of $(\bar{x}, \bar{y})$. Also suppose that one of the following condition is satisfied.*

(a) LICQ and SSOSC are satisfied at $\bar{y} \in M(\bar{x})$.

(b) LICQ, SCC, and SOSC are satisfied at $\bar{y} \in M(\bar{x})$.

(c) MFCQ, SSOSC, and CRCQ are satisfied at $\bar{y} \in M(\bar{x})$.

Then there exist open neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and uniquely determined locally Lipschitz function $\tilde{y} : U \to V$ such that $\tilde{y}(x)$ solve $P(x)$ for all $x \in U$.

Remark 2.2.8. Fiacco [23] also shows that in the settings of the above theorem with the assumption (b), the solution function $\tilde{y}(x)$ is C^1 and one arrive at the derivative the KKT equations. Now for simplicity we assume $K(x) = \{ y \in \Re^m \mid g(x,y) \leq 0 \}$. Also we assume that $\nabla^2_{yy} L$ is positive definite. Then one has

$$\begin{aligned}\nabla \tilde{y}(\bar{x}) = &- \nabla^2_{yy} L^{-1} \left(I - \nabla_y g_I^T \left(\nabla_y g_I \nabla^2_{yy} L^{-1} \nabla_y g_I^T \right)^{-1} \nabla_y g_I \nabla^2_{yy} L^{-1} \right) \nabla^2_{xy} L \\ &- \nabla^2_{yy} L^{-1} \nabla_y g_I^T \left(\nabla_y g_I \nabla^2_{yy} L^{-1} \nabla_y g_I^T \right)^{-1} \nabla_x g_I\end{aligned}$$

where $L(x,y,u) := F(x,y) - \mu f(x,y) \lambda^T$ and $g_I = g_I(\bar{x}, \bar{y})$ is the vector consisting of active constraints $g_i(\bar{x}, \bar{y})$, $i \in I(\bar{x}, \bar{y})$. See [23] and [79] for proofs and more details.

Theorem 2.2.9 (**[52]**)**.** *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution to PBLPP. Sup-*

pose that $f(x,y)$ and $F(x,y)$ are twice continuously differentiable at $(\bar{x},\bar{y})$ with $f,F \in \Psi\{f,F\}$. If $K(x)$ is a closed and convex subset of $\Re^m$ for each x near $\bar{x}$. Then the following are equivalent for any x near $\bar{x}$ and $y,v \in \Re^m$:

(i) y solves $P(x)$ and $v = y - \nabla_y F(x,y)$

(ii) v satisfies the parametric normal equation:

$$H(x,v) := \nabla_y(x,\pi(x,v)) + v - \pi(x,v) = 0 \quad \text{and} \quad y = \pi(x,v),$$

where $\pi(x,v) := Proj(v, K(x))$.

Remark 2.2.10. Once we obtain under the setting of theorem (2.4.1) a locally unique, Lipschitz solution $v = v(x)$ of $H(x,v) = 0$ near $(\bar{x},\bar{v})$, then as in the above theorem, this yields a locally unique Lipschitz solution $\tilde{y}(x) := \pi(x,v(x))$ of $P(x)$ near $\bar{x}$, as desired. Now let us look at the differentiability of the solution $\tilde{y}(x)$ near $\bar{x}$. Unfortunately under the above settings, this solution is not differentiable. If we assume the setting of the Theorem (2.4.1) with condition (b) and in addition to the strict complementarity slackness condition(SCS) holds at $(\bar{x},\bar{y})$, then $\tilde{y}(x)$ is Fréchet differentiable at $\bar{x}$. Of course the SCS is a very restrictive condition and the above coderivative estimate of the solution vector function enables us to relax the SCS.

2.2.5 More Sufficient Conditions for KIA

Clearly, if M is single valued near $\bar{x}$ and Lipschitz-like (also known as Aubin property) around $(\bar{x},\bar{y})$, then we have the LSL at $(\bar{x},\bar{y})$. We can formulate the following sufficient conditions for LSL and would like to direct interested readers to [17], [39], [40], [42], [44], [59], [73], [75] and [89] for sufficient conditions of Lipchitz-like property and single valuedness of M.

Theorem 2.2.11 (**[42, 44, 75]**). *For the solution set mapping $M : \Re^n \rightrightarrows \Re^m$ and $(\bar{x}, \bar{y}) \in gph\, M$ the following are equivalent:*

(a) M is a Lipschitz single-valued mapping near $\bar{x}$;

(b) M is hypomonotone near $(\bar{x}, \bar{y})$ and locally Lipschitz-like around $(\bar{x}, \bar{y})$;

(c) M is premonotone near $(\bar{x}, \bar{y})$ and locally Lipschitz-like around $(\bar{x}, \bar{y})$;

(d) M is maximal hypomonotone near $(\bar{x}, \bar{y})$ and locally Lipschitz-like around $(\bar{x}, \bar{y})$;

(e) M is locally Lipschitz-like around $(\bar{x}, \bar{y})$ with $D_*M(\bar{x}, \bar{y})(0) = \{0\}$

(f) M is single-valued near $\bar{x}$ and $D^*M(\bar{x}, \bar{y})(0) = \{0\}$.

Theorem 2.2.12 (**[39, 59, 73]**). *Assume that the $P(x)$ is partially convex smooth programming problem for each x near $\bar{x}$. Then*

$$M(x) = \{\, y \in \Re^m \mid 0 \in \nabla_y F(x, y) + N(y, K(x)) \,\}.$$

And M possesses LSL at $(\bar{x}, \bar{y})$ if one of the following conditions are satisfied:

(i) Suppose that $K(x) = K$.

(a) M satisfies the Strong Regularity Condition(SRC) at $(\bar{x}, \bar{y})$.

(b) $F(x,\cdot)$ *is strictly convex in* y *for each* x *near* $\bar{x}$, $\nabla^2_{xy}F(\bar{x},\bar{y}))$ *is surjective and the qualification condition:*

$$0 \in \nabla^2_{yy}(F(\bar{x},\bar{y}))^T z^* + D^*(N_K(\bar{y},\bar{z}))z^* \Rightarrow z^* = 0$$

holds.

(c) $K = \{\, y \in \Re^m \mid Ay - b \geq 0,\ Dy - c = 0 \,\}$ *is a non-empty polyhedral convex set,* $\nabla^2_{xy}F(\bar{x},\bar{y}))$ *exist and positive definite on the subspace* $L = C - C$ *where* $C = \{\, z \mid \nabla_y F(\bar{x},\bar{y})^T z = 0,\ Az \geq 0,\ Dz = 0 \,\}$

(ii) Now suppose that K *depend on parameter* x.

(d) N_K *is* N*-regular at* $(\bar{x},\bar{y},\bar{z})$ *and satisfies the qualification condition:*

$$(x^*,0) \in \nabla(\nabla_y F(\bar{x},\bar{y}))^T z^* + D^*(N_K(\bar{x},\bar{y},\bar{z}))z^* \Rightarrow x^* = z^* = 0$$

2.3 Optimality Conditions with Fréchet Differentiable Solution Selection

This section is devoted to obtaining the optimality conditions with Fréchet differentiable sections mapping. We can relax the Lipschitz continuity by imposing the Fréchet differentiability of the data. The following are optimality conditions.

2.3.1 Optimality Conditions

Theorem 2.3.1. *Recall the problem (2.2.1)*

$$\min_{x\in\Omega} \varphi_p(x) := F(x, \tilde{y}(x)) := (F \circ \tilde{y})(x).$$

Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution and let Ω be locally closed around $\bar{x}$. Let $\tilde{y}(x) \in M(x)$ for each x near $\bar{x}$ be Fréchet differentiable at $\bar{x}$ with $\tilde{y}(x) = \bar{y}$. Also assume that F is Fréchet differentiable at $(\bar{x}, \bar{y})$.

Then we have

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + \nabla \tilde{y}(\bar{x})^T (\nabla_y F(\bar{x}, \bar{y})) + N(\bar{x}, \Omega).$$

Proof. Since $\bar{x}$ is a local optimal solution to the problem (2.2.1),

$$0 \in \hat{\partial}[(F \circ \tilde{y})(\cdot) + \delta(\cdot, \Omega)](\bar{x})$$

F and $\tilde{y}$ are Fréchet differentiable at $(\bar{x}, \bar{y})$ and $\bar{x}$ respectively implies that $F \circ \tilde{y}$ is Fréchet differentiable at $\bar{x}$. Hence we have

$$\hat{\partial}[(F \circ \tilde{y})(\cdot) + \delta(\cdot, \Omega)](\bar{x}) = \hat{\partial}(F \circ \tilde{y})(\bar{x}) + N(\bar{x}, \Omega)$$

Now applying the Theorem 3.7 in [62] we have

$$\hat{\partial}(F \circ \tilde{y})(\bar{x}) = \nabla_x F(\bar{x}, \bar{y}) + \nabla \tilde{y}(\bar{x})^T (\nabla_y F(\bar{x}, \bar{y})).$$

Consequently we have

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + \nabla \tilde{y}(\bar{x})^T (\nabla_y F(\bar{x}, \bar{y})) + N(\bar{x}, \Omega).$$

□

2.3.2 Existence and Fréchet Differentiability of the Implicit Solutions

The following are some of the well-known results in classical nonlinear analysis that guarantee the existence and Fréchet differentiability of the implicit solution function $\tilde{y}(x)$ near $\bar{x}$.

Theorem 2.3.2. *Consider the problem $P(x)$. Suppose that $K(x) = K$, (i.e., the set K is independent of x) is a nonempty, closed, convex, polyhedral set, and the following assumptions hold:*

(a) Differentiability:

(i) $\nabla_y f$ and $\nabla_y F$ exist in a neighborhood of the point $(\bar{x}, \bar{y})$ and are continuous in (x, y) at $(\bar{x}, \bar{y})$,

(ii) $\nabla_x f$ and $\nabla_x F$ exist,

(b) Convexity:

(i) $f, F \in \Psi\{f, F\}$;

(ii) $K(x)$ is closed and convex for each x near $\bar{x}$;

(c) $\nabla_y F$ *is Lipschitz continuous in* (x, y) *in a neighborhood of* $(\bar{x}, \bar{y})$;

(d) LSL at $(\bar{x}, \bar{y})$*; i.e.,* M *possesses a Lipschitz single-valued localization at* $(\bar{x}, \bar{y})$;

(e) $K_0(\bar{y}) := \{ z \mid z \in \bigcup_{\lambda > 0} \lambda(K - \bar{y}), \nabla_y F(\bar{x}, \bar{y})^T z = 0 \}$ *is a subspace, i.e.,*$K_0 = K_0 \cap (-K_0)$.

Then there exist neighborhoods U_x *of* $\bar{x}$ *and* U_y *of* $\bar{y}$ *and a Lipschitz continuous function* $\tilde{y} : U_x \to U_y$*, which uniquely solves* $P(x)$ *in* U_x *and which satisfies* $\tilde{y}(x) \in \bar{y} + K_0(\bar{y})$ *for each* $x \in U_x$ *such that the function* $\tilde{y}(x)$ *is Fréchet differentiable at* $\bar{x}$ *and one has*

$$\nabla \tilde{y}(\bar{x}) = -Z[Z^T \nabla_y F(\bar{x}, \bar{y}) Z]^{-1} Z^T \nabla_x F(\bar{x}, \bar{y})$$

for any matrix Z *such that* $Z^T Z$ *is nonsingular and* $z \in K$ *if and only if* $z \in Zy$ *for all* $y \in \Re^l$ *where* l *is the dimension of* $K \cap (-K)$.

Furthermore, if in addition $\nabla_x f$ *and* $\nabla_x F$ *exist in a neighborhood of* $(\bar{x}, \bar{y})$ *and are continuous at* $(\bar{x}, \bar{y})$*, then* $\tilde{y}(x)$ *is continuously Fréchet differentiable at* $\bar{x}$.

Proof. By (a) and (b) we have

$$\begin{aligned} M(x) &= \{ y \in K \mid 0 \in \nabla_y F(x, y) + N_K(y) \} \\ &= \{ y \in K \mid 0 \in \nabla_y F(x, y)^T (z - y) \geq 0 \quad \text{for all} \quad z \in K \} \end{aligned}$$

Then applying Theorem 5.1 in [40] to the above variational Inequality we can obtain the above result. □

Theorem 2.3.3. *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution to PBLPP and Let $P(x)$ be a partially convex and smooth problem for each x near $\bar{x}$. Assume that $K(x) = K$ and let V and W be neighborhoods of $\bar{y}$ and $0 \in \Re^m$. Suppose also the following:*

(a) $\nabla_{\digamma}$ *is Lipschitz in x uniformly in y at $(\bar{x}, \bar{y})$ and $\nabla_{\digamma}(\cdot, \bar{y})$ is Fréchet differentiable at $\bar{x}$ with derivative $\nabla^2_{xy}\digamma(\bar{x}, \bar{y})$;*

(b) there exist a function g from V to $\Re^m$ with $g(\bar{y}) = \nabla_y \digamma(\bar{x}, \bar{y})$ which strongly approximates $\nabla_y \digamma$ in y at $(\bar{x}, \bar{y})$;

(c) there exist a function $\xi : W \to \Re^m$ such that $\xi(0) = \bar{y}$, $\xi(z) \in (g + N_K)^{-1}(z)$ for every $z \in W$, and ξ is Lipschiz and Fréchet differentiable at 0 with derivative $\nabla\xi(0)$.

Then there exist a neighborhood U of $\bar{x}$ and a function $\tilde{y} : U \to \Re^m$ such that $\tilde{y}(\bar{x}) = \bar{y}$ and $\tilde{y}(x) \in M(x)$ for each $x \in U$. Also we have $\tilde{y}(\cdot)$ is Lipschitz and Fréchet differentiable at $\bar{x}$ with derivative $-\nabla\xi(0)\nabla^2_{xy}\digamma(\bar{x}, \bar{y})$.

Proof. Since $P(x)$ with $K(x) = K$ is a partially convex smooth problem, we have

$$M(x) = \{\, y \in K \mid 0 \in \nabla_y \digamma(x, y) + N_K(y) \,\}.$$

Now by Corollary 2.7 in [17] one has the result.

□

2.4 Optimality Conditions with Continuous Solution Selection

We can further relax the assumption on the selection function. One can arrive at the following optimality conditions for PBLPP if the selection function $\tilde{y}(x)$ is continuous near $\bar{x}$.

2.4.1 Optimality Conditions

Theorem 2.4.1. *Consider the PBLPP. Suppose that $(\bar{x}, \bar{y})$ is a local pessimistic optimal solution and Ω is locally closed around $\bar{x}$. Suppose also that F is Fréchet differentiable at $(\bar{x}, \bar{y})$ and there exist a continuous selection $\tilde{y}(x) \in M(x)$ for each x near $\bar{x}$. We also impose the following qualification conditions:*

(a) $\partial^{\infty}(F \circ \tilde{y})(\bar{x}) \cap (-N(\bar{x}, \Omega)) = \{0\}$;

(b) $\partial^{\infty} F(\bar{x}, \bar{y}) \cap ker D^{*}\tilde{y}(\bar{x}) = \{0\}$.

Then we have

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + D^{*}\tilde{y}(\bar{x})(\nabla_y F(\bar{x}, \bar{y})).$$

Proof. Existence of solution selection function $\tilde{y}(x) \in M(x)$ for each x near $\bar{x}$ enable us to reformulate the PBLPP as following composite one level programming problem

$$\min_{x \in \Omega} F(x, \tilde{y}(x)) := (F \circ \tilde{y})(x)$$

Since $\bar{x}$ is a local optimal solution and under the above assumption, applying the proposition 5.3[60] we have

$$0 \in \partial(F \circ \tilde{y})(\bar{x}) + N(\bar{x}, \Omega)$$

Now applying the composition rule (Theorem 3.41 in [59]),

$$\partial(F \circ \tilde{y})(\bar{x}) \subseteq \nabla_x F(\bar{x}, \bar{y}) + D^* \tilde{y}(\bar{x})(\nabla_y F(\bar{x}, \bar{y})).$$

Thus we have

$$0 \in \nabla_x F(\bar{x}, \bar{y}) + D^* \tilde{y}(\bar{x})(\nabla_y F(\bar{x}, \bar{y})).$$

□

2.4.2 Existence and the Coderivative Constructions for the Implicit Solutions

Theorem 2.4.2 (**[38]**). *Consider the problem $P(x)$ and suppose that $(\bar{x}, \bar{y})$ is a solution to the problem $P(\bar{x})$. Suppose that the functions $f(x, y), F(x, y), g(x, y), h(x, y)$, and their partial derivatives with respect to y are continuously differentiable in a neighborhood of $(\bar{x}, \bar{y})$. Also suppose that the SSOSC and MFCQ holds at $\bar{y}$.*
Then there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and uniquely determined continuous vector function $\tilde{y} : U \to V$ such that $\tilde{y}(x)$ solve $P(x)$ for all $x \in U$ and $\tilde{y}(\bar{x}) = \bar{y}$.

Recall the problem $P(x)$ and the solution set $M(x)$. If partially convex smooth problem, then

$$\begin{aligned} M(x) &= \{\, y \in \Re^m \mid 0 \in \nabla_y F(x,y) + N(y, K(x)) \,\} \\ &= \{\, y \in \Re^m \mid -\nabla_y F(x,y) \in N((x,y);K) \,\} \\ &= \{\, y \in \Re^m \mid (x, y, -\nabla_y F(x,y)) \in \operatorname{gph} N_K \,\} \\ &= \{\, y \in \Re^m \mid q(x,y) \in \Theta \,\} \end{aligned} \tag{2.4.1}$$

where $\quad q(x,y) = (x, y, -\nabla_y F(x,y))$, and $\Theta = \operatorname{gph} N_K$.

Now consider the projection mapping $\pi : \Re^n \times \Re^m \to \Re^m$ defined by

$$\pi(x,z) := \operatorname{Proj}_{M(x)} z. \tag{2.4.2}$$

Then following properties hold.

- Since $P(x)$ is a partially convex smooth problem, $M(x)$ is closed and convex for each x near $\bar{x}$.

  Hence $\pi(x,z)$ is single-valued for each $(x,z) \in N(\bar{x}) \times \Re^m$.

- $M(x) = \bigcup_{z \in \Re^m} \{\, \pi(x,z) \,\}$ and $\tilde{y}(x) := \pi(x,z) \in M(x)$ for each $(x,z) \in \Re^n \times \Re^m$.

- Let $x \in \operatorname{dom} M$. Then $\pi(x,z)$ is continuous at (x,z) for all $z \in \Re^m$ if and only if M is continuous at x.

So if M is continuous near $\bar{x}$, $\pi(x,z)$ for all $z \in \Re^m$ provides a continuous selection for each x near $\bar{x}$. And we also have the following coderivative estimate for $\pi(x, z)$.

Theorem 2.4.3 (Coderivative of Projection Mapping [64]). *Let* $\bar{x} \in domM$*,* $\bar{z} \in \Re^m$ *and let* $\tilde{y}(\bar{x}) = \bar{y} := \pi(x, z)$*. Suppose that*

(a) Qualification Condition:

$$\left[(\nabla_y q(\bar{x}, \bar{y}))^T u,\ u \in N_\Theta(q(\bar{x}, \bar{y})) \right] \Rightarrow u = 0.$$

(b) Calmness Condition:

$$P(\omega) := \left\{ (x, y, d) \mid (q(x, y), d)^T + \omega \in gph\, N_\Theta \right\}$$

is calm at every point $(\bar{x}, \bar{y}, d)$ *with* $d \in N_\Theta(q(\bar{x}, \bar{y}))$ *and* $(\nabla_y q(\bar{x}, \bar{y}))^T d = \bar{z} - \bar{y}$*.*

Now one has the estimate

$$D^* \pi(\bar{z}, \bar{x}, \bar{y})(u) \subseteq \bigcup_{\substack{d \in N_\Theta(q(\bar{x}, \bar{y})) \\ (\nabla_y q(\bar{x}, \bar{y}))^T d = \bar{z} - \bar{y}}} \left\{ (t, v) \mid t = -w,\ (v, -u - w)^T \in D^* Q(\bar{x}, \bar{y}, \bar{z} - \bar{y})(w) \right\},$$

where $Q(x, y) := (\nabla_y q(x, y))^T N_\Theta(q(x, y))$*.*

Remark 2.4.4. For the proof of the above theorem and the construction of $D^* Q(\bar{x}, \bar{y}, \bar{z} - \bar{y})(w)$ where $Q(x, y) := (\nabla_y q(x, y))^T N_\Theta(q(x, y))$, the reader is referred to theorem 4.4 and theorem 3.1 respectively in [64].

The reader can find more interesting approaches for solving the problem $P(x)$ in nonlinear optimization theory. Exact Penalty Methods, Barrier Methods, Interior Point

Techniques, Smoothing Methods are some of the techniques also available in the literature solving the problems $P(x)$ and hence to obtain the solution selection function $\tilde{y}(x)$ and the corresponding derivative. Here we would like to direct the interested reader to the papers [21], [52], [71], and [77] for more details.

Chapter 3

Minimax Programming Approach

This chapter consists of four main sections. In Section 3.1, we consider the reformulation of the pessimistic bilevel programming problem into a programming problem which is called the *static minimax problem in the parametric constraint case.* The rest of the sections are devoted to exploring the various constructions of subdifferential of marginal functions and applying those to get the first order optimality conditions for PBLPP. More precisely, in Sections 3.2, 3.3 and 3.4 we consider differentiable pessimistic bilevel programming problems (i.e., with Fréchet differentiable original data), convex and concave PBLPPs and Lipschitzian PBLPPs respectively.

3.1 Generalized Minimax Reformulations of PBLPP

Let us recall the pessimistic bilevel programming problem (PBLPP).

$$l(x): \qquad \min_{y} f(x,y) \qquad \text{subject to} \quad y \in K(x),$$

$$m(x): \qquad \varphi_p(x) := \max\{\, F(x,y) \mid y \in S(x)\,\},$$

$$(\text{PBLPP}): \qquad \min \varphi_p(x) \quad \text{subject to } x \in \Omega.$$

So PBLPP can be written as:

$$\min_{x\in\Omega}\left(\max_{y\in S(x)} F(x,y)\right). \tag{3.1.1}$$

A problem of this form is called the *static minimax problem in the parametric constraint case*. Recall the Subsection (1.4.2) and the reformulation of the PBLPP. In particular, under the assumption (RA), PBLPP can be equivalently written as:

$$\begin{aligned}
&\min_{x\in\Omega}\left(\max_{y\in K(x)}\left[F(x,y)-\mu(f(x,y)-v(x))\right]\right)\\
&=\min_{x\in\Omega}\left(\max_{y\in K(x)}\left[F(x,y)-\mu f(x,y)+\mu v(x)\right]\right)\\
&=\min_{x\in\Omega}\left(\max_{y\in K(x)}\left[F(x,y)-\mu f(x,y)\right]+\mu v(x)\right)\\
&=\min_{x\in\Omega}\left[\mu v(x)-u(x)\right],
\end{aligned}\tag{3.1.2}$$

where $u(x) := \min\limits_{y\in K(x)}[\mu f(x,y) - F(x,y)]$.

3.2 Bilevel Programming Problem with Differentiable Data

Pessimistic bilevel programming problem with differentiable original data will be considered through out this section. We assume that the objective functions of lower level and upper level problems are Fréchet differentiable. The upper subdifferential of marginal function will be considered first and consequently applying that construction we will obtain the necessary optimality conditions for the PBLPP.

3.2.1 Upper Subdifferential of Marginal Function

Let us define $\phi : \Re^n \to \bar{\Re}$ by

$$\phi(x) := \begin{cases} \sup\{\, \rho(x,y) \mid y \in \Theta(x) \,\} & \text{if } y \in \Theta(x) \\ \\ -\infty & \text{if } y \notin \Theta(x) \end{cases}$$

Let $\theta(x,y) := \rho(x,y) - \delta((x,y), \operatorname{gph}\Theta)$ and the set valued mapping $\Phi(x) := \{\, y \in \Theta(x) \mid \rho(x,y) = \phi(x) \,\}$.

Theorem 3.2.1. *Let $\bar{y} \in \Phi(\bar{x})$ be such that $|\phi(\bar{x})| < \infty$ and Φ has a local upper Lipschtizian selection at $(\bar{x}, \bar{y}) \in gph\, \Phi$. Then we have*

(a) $\widehat{\partial}^+\phi(\bar{x}) = \{\, x^* \mid (x^*, 0) \in \widehat{\partial}^+\theta(\bar{x}, \bar{y}) \,\}$.

(b) Furthermore, if ρ is Fréchet differentiable at $(\bar{x}, \bar{y})$ we have

$$\widehat{\partial}^+\phi(\bar{x}) = \nabla_x \rho(\bar{x}, \bar{y}) - \widehat{D}^*\Theta(\bar{x}, \bar{y})(-\nabla_y \rho(\bar{x}, \bar{y})).$$

Proof. Let $x^* \in \widehat{\partial}^+\phi(\bar{x})$. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\phi(x) - \phi(\bar{x}) - \langle x^*, x - \bar{x} \rangle \le \epsilon \parallel x - \bar{x} \parallel \qquad \text{for all } x \in B(\bar{x}, \delta),$$

$$\phi(x) - \phi(\bar{x}) \le \langle (x^*, 0), (x,y) - (\bar{x}, \bar{y}) \rangle + \epsilon \parallel (x,y) - (\bar{x}, \bar{y}) \parallel \qquad \text{for all } (x,y) \in B((\bar{x}, \bar{y}), \delta),$$

$$\phi(x) - F(\bar{x}, \bar{y}) \le \langle (x^*, 0), (x,y) - (\bar{x}, \bar{y}) \rangle + \epsilon \parallel (x,y) - (\bar{x}, \bar{y}) \parallel \quad \text{for all } (x,y) \in B((\bar{x}, \bar{y}), \delta).$$

Now by the definition of ϕ and θ we have

$$\theta(x,y)-\theta(\bar{x},\bar{y})-\langle(x^*,0),(x,y)-(\bar{x},\bar{y})\rangle \leq \epsilon \parallel (x,y)-(\bar{x},\bar{y}) \parallel \text{ for all } (x,y) \in B((\bar{x},\bar{y}),\delta).$$

This implies that $(x^*,0) \in \widehat{\partial}^+\theta(\bar{x},\bar{y})$. So $\widehat{\partial}^+\phi(\bar{x}) \subset \{\, x^* \mid (x^*,0) \in \widehat{\partial}^+\theta(\bar{x},\bar{y}) \,\}$.

To prove the opposite inclusion, let $(x^*,0) \in \widehat{\partial}^+\theta(\bar{x},y^*)$ where $y^* \in M(\bar{x})$ be arbitrary. Then

$$\begin{aligned}
& \limsup_{(x,y)\to(\bar{x},y^*)} \frac{\theta(x,y)-\theta(\bar{x},y^*)-\langle(x^*,0),(x,y)-(\bar{x},y^*)\rangle}{\parallel (x,y)-(\bar{x},y^*) \parallel} \leq 0 \\
\Rightarrow \quad & \lim_{\delta\downarrow 0}\left(\sup_{(x,y)\in B((\bar{x},y^*),\delta)} \frac{\theta(x,y)-\phi(\bar{x})-\langle x^*,x-\bar{x}\rangle}{\parallel x-\bar{x} \parallel + \parallel y-y^* \parallel}\right) \leq 0.
\end{aligned}$$

Since Φ has a local upper Lipschitzian selection at $(\bar{x},\bar{y}) \in \operatorname{gph}\Phi$, i.e., there exists $\tilde{y}(x) \in \Phi$ and $l \geq 0$ such that $\parallel \tilde{y}(x)-\tilde{y}(\bar{x}) \parallel \leq l \parallel x-\bar{x} \parallel$ for all x near $\bar{x}$ with $\tilde{y}(\bar{x})=\bar{y}$, we have by taking $y^*=\bar{y}$,

$$\begin{aligned}
& \lim_{\delta\downarrow 0}\left(\sup_{x\in B(\bar{x},\delta)} \frac{\theta(x,\tilde{y}(x))-\phi(\bar{x})-\langle x^*,x-\bar{x}\rangle}{(l+1)\parallel x-\bar{x} \parallel}\right) \leq 0. \\
\Rightarrow \quad & \limsup_{x\to\bar{x}} \frac{\phi(x)-\phi(\bar{x})-\langle x^*,x-\bar{x}\rangle}{\parallel x-\bar{x} \parallel} \leq 0. \\
\Rightarrow \quad & x^* \in \widehat{\partial}^+\phi(\bar{x}).
\end{aligned}$$

Thus, $\{\, x^* \mid (x^*,0) \in \widehat{\partial}^+\theta(\bar{x},\bar{y}) \,\} \subseteq \widehat{\partial}^+\phi(\bar{x})$.

Consequently, we have the result (a), i.e., $\widehat{\partial}^+\phi(\bar{x}) = \{\, x^* \mid (x^*,0) \in \widehat{\partial}^+\theta(\bar{x},\bar{y}) \,\}$.

To prove the part (b), consider

$$\begin{aligned}\widehat{\partial}^{+}\theta(\bar{x},\bar{y}) &= -\widehat{\partial}\left[-\theta(\bar{x},\bar{y})\right] \\ &= -\widehat{\partial}\left[-F(\cdot,\cdot)+\delta((\cdot,\cdot),\operatorname{gph}\Theta)\right](\bar{x},\bar{y}).\end{aligned}$$

Since F is Fréchet differentiable at $(\bar{x},\bar{y})$,

$$\begin{aligned}\widehat{\partial}\left[-F(\cdot,\cdot)+\delta((\cdot,\cdot),\operatorname{gph}\Theta)\right](\bar{x},\bar{y}) &= -\nabla F(\bar{x},\bar{y})+\widehat{\partial}\left[\delta((\cdot,\cdot),\operatorname{gph}\Theta)\right](\bar{x},\bar{y}) \\ &= -\nabla F(\bar{x},\bar{y})+\widehat{N}\left[(\bar{x},\bar{y}),\operatorname{gph}\Theta\right].\end{aligned}$$

Consequently, we have

$$\begin{aligned}\widehat{\partial}^{+}\phi(\bar{x}) &= \{\, x^* \mid (x^*,0)\in\nabla\rho(\bar{x},\bar{y})-\widehat{N}\left[(\bar{x},\bar{y}),\operatorname{gph}\Theta\right]\} \\ &= \{\, x^* \mid (\nabla_x\rho(\bar{x},\bar{y})-x^*,\nabla_y\rho(\bar{x},\bar{y}))\in\widehat{N}\left[(\bar{x},\bar{y}),\operatorname{gph}\Theta\right]\} \\ &= \{\, x^* \mid \nabla_x\rho(\bar{x},\bar{y})-x^*\in\widehat{D}^*\Theta(\bar{x},\bar{y})(-\nabla_y\rho(\bar{x},\bar{y}))\,\} \\ &= \{\, x^* \mid x^*\in\nabla_x\rho(\bar{x},\bar{y})-\widehat{D}^*\Theta(\bar{x},\bar{y})(-\nabla_y\rho(\bar{x},\bar{y}))\,\}.\end{aligned}$$

Thus,

$$\widehat{\partial}^{+}\phi(\bar{x})=\nabla_x\rho(\bar{x},\bar{y})-\widehat{D}^*\Theta(\bar{x},\bar{y})(-\nabla_y\rho(\bar{x},\bar{y})).$$

□

3.2.2 Optimality Conditions

Now let us apply the above construction to obtain optimality conditions for the pessimistic bilevel programming problem.

Theorem 3.2.2. *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution such that $|\varphi_p(\bar{x})| < \infty$. Assume also that $M(x) := \text{argmax}\{\, F(x, y) \mid y \in S(x)\,\}$ has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$ and F is Fréchet differentiable at this point. Then one has*

(a) $\widehat{D}^* S(\bar{x}, \bar{y})(-\nabla_y F(\bar{x}, \bar{y})) \subseteq \nabla_x F(\bar{x}, \bar{y}) + \widehat{N}(\bar{x}, \Omega)$

(b) in addition suppose that the assumption (RA) holds, f is Fréchet differentiable and $\widehat{\partial} v(\bar{x}) \neq \emptyset$, then

$$0 \in \nabla_x F(\bar{x}, \bar{y}) - \widehat{D}^* K(\bar{x}, \bar{y})\, (\mu \nabla_y f(\bar{x}, \bar{y}) - \nabla_y F(\bar{x}, \bar{y})) + \widehat{D}^* K(\bar{x}, \bar{y})\, (\nabla_y \mu f(\bar{x}, \bar{y})) + \widehat{N}(\bar{x}, \Omega).$$

Proof. Since $\bar{x}$ is a local optimal solution to $\min\limits_{x \in \Omega} \varphi_p(\bar{x})$, and $|\varphi_p(\bar{x})|$ is finite, by the proposition 5.2[60] we have

$$-\widehat{\partial}^+ \varphi_p(\bar{x}) \subseteq \widehat{N}(\bar{x}, \Omega)$$

(a) Now by the above construction of the subdifferential of marginal function, we have

$$\widehat{D}^* S(\bar{x}, \bar{y})(-\nabla_y F(\bar{x}, \bar{y})) \subseteq \nabla_x F(\bar{x}, \bar{y}) + \widehat{N}(\bar{x}, \Omega).$$

(b) Under the assumption (RA), we have

$$\varphi_p(x) = \mu v(x) - u(x).$$

$$-\widehat{\partial}^+ \left(\mu v(\cdot) - u(\cdot)\right)(\bar{x}) \subseteq \widehat{N}(\bar{x}, \Omega),$$

$$0 \in \widehat{\partial}^+ \left(\mu v(\cdot) - u(\cdot)\right)(\bar{x}) + \widehat{N}(\bar{x}, \Omega),$$

$$0 \in -\widehat{\partial} \left(u(\cdot) - \mu v(\cdot)\right)(\bar{x}) + \widehat{N}(\bar{x}, \Omega).$$

Since $\varphi_p(\bar{x})$ is finite, this implies that $u(\bar{x})$ and $v(\bar{x})$ are finite and the assumption $\widehat{\partial} v(\bar{x}) \neq \emptyset$, we have

$$0 \in -\widehat{\partial} u(\bar{x}) + \widehat{\partial}(\mu v)(\bar{x}) + \widehat{N}(\bar{x}, \Omega), \qquad \text{by [62, Theorem 3.1(i)]}$$

$$0 \in \widehat{\partial}^+(-u)(\bar{x}) - \widehat{\partial}^+(-\mu v)(\bar{x}) + \widehat{N}(\bar{x}, \Omega),$$

$$0 \in \widehat{\partial}^+ \left[\max_{y \in K(x)} \left(F(\cdot, y) - \mu f(\cdot, y)\right)\right](\bar{x}) - \widehat{\partial}^+ \left(\max_{y \in K(x)} - \mu f(\cdot, y)\right)(\bar{x}) + \widehat{N}(\bar{x}, \Omega).$$

By assumption (RA), we have $M(x) = \operatorname{argmax}\{\, F(x, y) - \mu f(x, y) \mid y \in K(x) \,\}$.

Since $M(x) \subseteq S(x)$ and M has a local upper Lipschitzian selection at $(\bar{x}, \bar{y})$, so does S.

Then applying again the above subdifferential construction, we have

$$\begin{aligned} 0 \in \; & \nabla_x \left[F(\bar{x}, \bar{y}) - \mu f(\bar{x}, \bar{y})\right] - \widehat{D}^* K(\bar{x}, \bar{y}) \left(-\nabla_y \left[F(\bar{x}, \bar{y}) - \mu f(\bar{x}, \bar{y})\right]\right) \\ & - \left[-\mu \nabla_x f(\bar{x}, \bar{y}) - \widehat{D}^* K(\bar{x}, \bar{y}) \left(-\nabla_y - \mu f(\bar{x}, \bar{y})\right)\right] + \widehat{N}(\bar{x}, \Omega), \end{aligned}$$

which is equivalent to

$$\begin{aligned} 0 \in \; & \nabla_x F(\bar{x}, \bar{y}) - \mu \nabla_x f(\bar{x}, \bar{y}) - \widehat{D}^* K(\bar{x}, \bar{y}) \left(\mu \nabla_y f(\bar{x}, \bar{y}) - \nabla_y F(\bar{x}, \bar{y})\right) \\ & + \mu \nabla_x f(\bar{x}, \bar{y}) + \widehat{D}^* K(\bar{x}, \bar{y}) \left(\nabla_y \mu f(\bar{x}, \bar{y})\right) + \widehat{N}(\bar{x}, \Omega). \end{aligned}$$

Thus,

$$0 \in \nabla_x F(\bar{x},\bar{y}) - \widehat{D}^* K(\bar{x},\bar{y})\,(\mu \nabla_y f(\bar{x},\bar{y}) - \nabla_y F(\bar{x},\bar{y})) + \widehat{D}^* K(\bar{x},\bar{y})\,(\nabla_y \mu f(\bar{x},\bar{y})) + \widehat{N}(\bar{x},\Omega).$$

$\square$

Remark 3.2.3 (**Constructions of $D^*S(\bar{x},\bar{y})$ and $D^*K(\bar{x},\bar{y})$**)**.** The construction of coderivatives of feasible maps and solution set maps are known in optimization literature and one can find such nice construction in [29], [30], [43], [59], [64] and [65]. Let us consider for example the following constructions. For partially convex, smooth lower level problem, we have

$$S(x) = \{\, y \in \Re^m \mid 0 \in \nabla_y f(x,y) + \partial_y \delta_K(x,y) \,\}$$

Now let $Q(x,y) = \partial_y \delta_K(x,y)$. Then Q is closed-graph around $(\bar{x},\bar{y},\bar{z} := -\nabla_y f(x,y))$ and Q is normally regular at this point as $K(x)$ is convex for each x. Then we have

$$\begin{aligned} D^*S(\bar{x},\bar{y})(y^*) &= \hat{D}^*S(\bar{x},\bar{y})(y^*) \\ &= \{\, x^* \Re^n \mid \exists z^* \text{with} (x^* - \nabla_x f(\bar{x},\bar{y})^* z^*, -y^* - \nabla_y f(\bar{x},\bar{y})^* z^*) \in D^*Q(\bar{x},\bar{y},\bar{z})(z^*) \,\}, \end{aligned}$$

provided $0 \in \nabla f(\bar{x},\bar{y})^* z^* + D^*Q(\bar{x},\bar{y},\bar{z})(z^*) \Rightarrow z^* = 0$.

3.3 Fully Convex and Concave Bilevel Programming

In this we assume that the lower objective function is fully convex (i.e., jointly convex) while the upper level counter part is fully concave. First, let us look at the upper subdifferential construction of value functions.

3.3.1 Upper Subdifferential of Marginal Function

Let $\widehat{\phi} : \Re^n \to \bar{\Re}$ be given by

$$\widehat{\phi}(x) = \sup_{y \in \Re^m} \rho(x, y).$$

Theorem 3.3.1 (**[28, Lemma 5.3]**). *Let $\rho : \Re^n \times \Re^m \to \bar{\Re}$ be a concave function. If $\widehat{\Phi}(x) :=$ argmax$\{\, \rho(x, y) \mid y \in \Re^m \,\} \neq \emptyset$, then*

$$\widehat{\partial}^+ \widehat{\phi}(x) = \{\, x^* \in \Re^n \mid (x^*, 0) \in \widehat{\partial}^+ \rho(x, y) \,\},$$

where $y \in \widehat{\Phi}(x)$ is arbitrary and $0 \in \Re^m$.

Now let us turn to the general case, i.e., marginal function over the parametric constraint set. Recall marginal function defined in Subsection (3.2.1).

Theorem 3.3.2 (**[28, Theorem 5.11]**). *Let $\phi : \Re^n \to \bar{\Re}$ be defined in Subsection (3.2.1), where $\rho : \Re^n \times \Re^m \to \Re$ is a concave function and $gph\,\Theta \subseteq \Re^n \times \Re^m$ is a nonempty, convex set. Then we have*

(a) ϕ is concave,

(b) If $ri(gph\,\Theta) \cap ri(dom\,\rho) \neq \emptyset$ *and* $\Phi(x) := argmax\{\, \rho(x,y) \mid y \in \Theta(x) \,\} \neq \emptyset$, *then*

$$\widehat{\partial}^{+}\phi(x) = \{\, x^* \in \Re^n \mid (x^*, 0) \in \widehat{\partial}^{+}\rho(x,y) - \widehat{N}((x,y), gph\,\Theta) \,\},$$

(c) If ρ *is Fréchet differential and* $\Phi(x) \neq \emptyset$, *then*

$$\widehat{\partial}^{+}\phi(x) = \nabla_x \rho(x,y) - \widehat{D}^{*}\Theta(x,y)\left(-\nabla_y \rho(x,y)\right),$$

where $y \in \Phi(x)$ *be arbitrary.*

3.3.2 Optimality Conditions

Theorem 3.3.3. *Let* $(\bar{x}, \bar{y})$ *be a local pessimistic optimal solution with* $|\varphi_p(\bar{x})| < \infty$. *Assume the following holds:*

(a) Ω *is closed and convex subset of* $\Re^n$;

(b) $K : \Re^n \rightarrow \Re^m$ *is convex on* Ω, *uniformly bounded at the point* $\bar{x}$ *and closed around this point;*

(c) f *and* $-F$ *are jointly convex on the set* $\{\, (x,y) \mid y \in K(x),\ x \in \Omega \,\}$ *and lower semicontinuous.*

(d) $ri(gph\,K) \cap ri(domF) \neq \emptyset$ *and* $\widehat{\partial}(v(\bar{x})); \neq \emptyset$

(e) Assumption (RA) holds.

Then we have

$$0 \in x^* - u^* + \widehat{N}(\bar{x}, \Omega),$$

where $(x^*, 0) \in \widehat{\partial}^{+}[F(\cdot, y) - \mu f(\cdot, y)](\bar{x}) - \widehat{N}\left[(\bar{x}, \bar{y}), gph\,K\right]$ *and*

$(u^*, 0) \in \widehat{\partial}^{+}[-\mu f(\cdot, y)](\bar{x}) - \widehat{N}\left[(\bar{x}, \bar{y}), gph\,K\right].$

Proof. Since $\bar{x}$ is a local optimal solution, $((\bar{x},\bar{y})$ is a local pessimistic optimal solution) to $\min_{x\in\Omega}\varphi_p(\bar{x})$, and $|\varphi_p(\bar{x})|$ is finite, we have

$$-\widehat{\partial}^{+}\varphi_p(\bar{x}) \subseteq \widehat{N}(\bar{x},\Omega),$$

$$0 \in \widehat{\partial}^{+}\varphi_p(\bar{x}) + \widehat{N}(\bar{x},\Omega).$$

By the assumption (e), one has $\varphi_p(x) = \mu v(x) - u(x)$ where $u(x) := \min_{y\in K(x)}[\mu f(x,y) - F(x,y)]$ and $v(x) := \min_{y\in K(x)} f(x,y)$.

Since f and $-F$ are convex functions and K is convex mapping, $u(x)$ is concave and $v(x)$ is convex on Ω.

$$\begin{aligned}
\widehat{\partial}^{+}\varphi_p(\bar{x}) &= \widehat{\partial}^{+}(u+\mu v)(\bar{x})\\
&= -\widehat{\partial}((-u)-\mu v)(\bar{x})\\
&\subseteq -\widehat{\partial}(-u)(\bar{x}) + \widehat{\partial}(\mu v)(\bar{x}) \quad \text{since } u \text{ and } v \text{ are finite at } \bar{x} \text{ and } \widehat{\partial}(v(\bar{x})) \neq \emptyset\\
&= \widehat{\partial}^{+}u(\bar{x}) + \widehat{\partial}(\mu v)(\bar{x})\\
&= \widehat{\partial}^{+}(\max_{y\in K(\cdot)}[F(\cdot,y)-\mu f(\cdot,y)])(\bar{x}) - \widehat{\partial}^{+}(\max_{y\in K(\cdot)} -\mu f(\cdot,y))(\bar{x}).
\end{aligned}$$

Consequently, we have

$$0 \in \widehat{\partial}^{+}(\max_{y\in K(\cdot)}[F(\cdot,y)-\mu f(\cdot,y)])(\bar{x}) - \widehat{\partial}^{+}(\max_{y\in K(\cdot)} -\mu f(\cdot,y))(\bar{x}) + \widehat{N}.(\bar{x},\Omega)$$

Now by the Thereom (3.3.2)part (b), we have

$$0 \in \{\, x^* \in \Re^n \mid (x^*, 0) \in \widehat{\partial}^+[F(\cdot, y) - \mu f(\cdot, y)](\bar{x}) - \widehat{N}\,((\bar{x}, \bar{y}), \mathrm{gph}\, K)\,\}$$

$$\{\, u^* \in \Re^n \mid (u^*, 0) \in \widehat{\partial}^+[-\mu f(\cdot, y)](\bar{x}) - \widehat{N}\,[(\bar{x}, \bar{y}), \mathrm{gph}\, K]\,\} + \widehat{N}(\bar{x}, \Omega).$$

We have the following:

$$0 \in x^* - u^* + \widehat{N}(\bar{x}, \Omega),$$

where $(x^*, 0) \in \widehat{\partial}^+[F(\cdot, y) - \mu f(\cdot, y)](\bar{x}) - \widehat{N}\,[(\bar{x}, \bar{y}), \mathrm{gph}\, K]$ and

$(u^*, 0) \in \widehat{\partial}^+[-\mu f(\cdot, y)](\bar{x}) - \widehat{N}\,[(\bar{x}, \bar{y}), \mathrm{gph}\, K]\,.$

□

Remark 3.3.4. In addition to the assumptions of the above theorem, if F and f are Fréchet differentiable at $(\bar{x}, \bar{y})$, one can replace the condition $ri(\mathrm{gph}\, K) \cap ri(\mathrm{dom} F) \neq \emptyset$ and also by part (c) of the Theorem (3.3.2) we have the optimality conditions

$$0 \in \nabla_x F(\bar{x}, \bar{y}) - \widehat{D}^* K(\bar{x}, \bar{y})\,(\mu \nabla_y f(\bar{x}, \bar{y}) - \nabla_y F(\bar{x}, \bar{y})) + \widehat{D}^* K(\bar{x}, \bar{y})\,(\nabla_y \mu f(\bar{x}, \bar{y})) + \widehat{N}(\bar{x}, \Omega).$$

3.4 Lipschitzian Bilevel Programming

Now pessimistic bilevel problem with Lipschitzian data is considered. The following is the subdifferential estimate of marginal function.

3.4.1 Subdifferential Estimate of Marginal Function

Theorem 3.4.1. *Given $\bar{y} \in \Phi(\bar{x})$, assume that ρ is Lipschitz continuous around $(\bar{x}, \bar{y})$ and Θ is Lipschitz continuous around $\bar{x}$. Then*

(a) ϕ is Lipschitz continuous around $\bar{x}$.

(b) If Φ is inner semicontinuous at $(\bar{x}, \bar{y})$, then

$$\partial\phi(\bar{x}) \subset \text{-}co\left[\cup\{\, x^* + D^*\Theta(\bar{x}, \bar{y})(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, \bar{y})\,\}\right].$$

(c) If Φ is inner semicompact at $\bar{x}$, then

$$\partial\phi(\bar{x}) \subset \text{-}co\left[\cup\{\, x^* + D^*\Theta(\bar{x}, y^/)(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, y^/),\ y^/ \in \Theta(\bar{x})\,\}\right].$$

Proof. Since ρ is Lipschitz continuous around $(\bar{x}, \bar{y})$, and Θ is Lipschitz continuous around $\bar{x}$ by the Lemma 3.18(4) in [51], we have the Lipschitz continuity of ϕ around $\bar{x}$.

Now, since ϕ is Lipschitz continuous around $\bar{x}$, we have

$$\begin{aligned} \partial\phi(\bar{x}) &\subset \text{-co}\partial\left(-\phi\right)(\bar{x}) \\ &= \text{-co}\,\partial\left[-\sup_{y\in\Theta(\cdot)}\rho(\cdot, y)\right](\bar{x}) \\ &= \text{-co}\,\partial\left[\inf_{y\in\Theta(\cdot)} -\rho(\cdot, y)\right](\bar{x}). \end{aligned}$$

Now, if Φ is inner semicontinuous at $(\bar{x}, \bar{y})$, then by the Theorem 3.38(i) in [59], we have

$$\partial\left(\inf_{y\in\Theta(\cdot)} -\rho(\cdot, y)\right)(\bar{x}) \subset \cup\{\, x^* + D^*\Theta(\bar{x}, \bar{y})(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, \bar{y})\,\}.$$

Thus we arrive

$$\partial\phi(\bar{x}) \subset \text{-co}\left[\cup\{\, x^* + D^*\Theta(\bar{x}, \bar{y})(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, \bar{y})\,\}\right],$$

which proves the part (b). To prove the part (c), assume that Φ is inner semicompact at $\bar{x}$. Then by Theorem 3.38(ii) in [59], we have

$$\partial\left(\inf_{y\in\Theta(\cdot)} -\rho(\cdot, y)\right)(\bar{x}) \subset \cup\{\, x^* + D^*\Theta(\bar{x}, y^/)(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, y^/),\ y^/ \in \Theta(\bar{x})\,\}.$$

Hence the result,

$$\partial\phi(\bar{x}) \subset \text{-co}\left[\cup\{\, x^* + D^*\Theta(\bar{x}, y^/)(y^*) \mid (x^*, y^*) \in -\partial^+\rho(\bar{x}, y^/),\ y^/ \in \Theta(\bar{x})\,\}\right].$$

□

3.4.2 Optimality Conditions

Theorem 3.4.2. *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution, and let Ω be locally closed around $\bar{x}$. Assume the following:*

(a) F is locally Lipschitz continuous $(\bar{x}, \bar{y})$,

(b) S is Lipschitz-like around $(\bar{x}, \bar{y})$.

Then we have

(i) if $M(x) := argmax\{\, F(x,y) \mid y \in S(x)\,\}$ is inner semi continuous at $(\bar{x}, \bar{y})$, then

$$0 \in \text{-}co\,[\cup\{\, x^* + D^*S(\bar{x},\bar{y})(y^*) \mid (x^*, y^*) \in \partial F(\bar{x},\bar{y})\,\}] + N(\bar{x},\Omega)$$

(ii) if $M(x) := argmax\{\, F(x,y) \mid y \in S(x)\,\}$ is inner semi compact at $\bar{x}$, then

$$0 \in \text{-}co\left[\cup\{\, x^* + D^*S(\bar{x},y^{/})(y^*) \mid (x^*, y^*) \in \partial F(\bar{x},y^{/}),\ y^{/} \in M(\bar{x})\,\}\right] + N(\bar{x},\Omega)$$

Proof. The value function φ_p is locally Lipschitz continuous around $\bar{x}$ by (a),(b), and (c). Since $\bar{x}$ is a local optimal solution to $\min\limits_{x\in\Omega} \varphi_p(x)$ by Proposition 5.3 in [60], we have $0 \in \partial\varphi_p(\bar{x}) + N(\bar{x},\Omega)$. Now applying the above theorem, we arrive at the result.

□

Remark 3.4.3. Inner semicontinuity and inner semicompact assumption can be weakened by μ- inner semi continuity and μ- inner semi compactness; see [63] for more details.

Theorem 3.4.4. *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution and let Ω be locally closed around $\bar{x}$. Assume the following:*

(a) M is inner semi continuous $(\bar{x}, \bar{y})$;

(b) K is closed and Lipschitz-like around $(\bar{x}, \bar{y})$;

(c) Assumption (RA) satisfies;

(d) f and F are strictly differentiable at $(\bar{x}, \bar{y})$.

Then we have

$$0 \in \text{-}co\big[\nabla_x \mu f(\bar{x}, \bar{y}) - \nabla_x F(\bar{x}, \bar{y}) + D^* K(\bar{x}, \bar{y})(\nabla_y \mu f(\bar{x}, \bar{y}) - \nabla_y F(\bar{x}, \bar{y}))\big]$$
$$+ \nabla_x \mu f(\bar{x}, \bar{y}) + D^* K(\bar{x}, \bar{y})(\nabla_y \mu f(\bar{x}, \bar{y})) + N(\bar{x}, \Omega).$$

Proof. Consider the pessimistic bilevel programming problem. Then by the assumption (RA), PBLPP is equivalent to the following problem:

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} [F(x, y) - \mu f(x, y) - v(x)] \right)$$

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} [F(x, y) - \mu f(x, y)] + \mu v(x) \right)$$

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} [F(x, y) - \mu f(x, y)] - \max_{y \in K(x)} [-\mu f(x, y)] \right).$$

Since f and F are strictly differentiable at $(\bar{x}, \bar{y})$, f and F are locally Lipschitzian around $(\bar{x}, \bar{y})$. Also, since M is inner semicontinuous at $(\bar{x}, \bar{y})$ and $M \subset S$, S is inner semi continuous. Thus $\alpha(x) := \max\limits_{y \in K(x)} [F(x, y) - \mu f(x, y)]$ and $\beta(x) := \max\limits_{y \in K(x)} [-\mu f(x, y)]$ are locally Lipschitz continuous around $\bar{x}$; see Lemma 3.18(4) in [51]. Hence, $\alpha - \beta$ also Lipschitz continuous around $\bar{x}$. Since $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution, it is also an optimal solution to

$$\min_{x \in \Omega} (\alpha(x) - \beta(x))$$

under assumption (RA). Then apply Proposition 5.3 in [60] to get

$$
\begin{aligned}
0 &\in \partial\left(\alpha(x) - \beta(x)\right) + N(\bar{x}, \Omega) \\
&\subset \partial\alpha(\bar{x}) + \partial(-\beta)(\bar{x}) + N(\bar{x}, \Omega) && \text{(by the Theorem 3.36[59])} \\
&= \partial\left(\max_{y\in K(\cdot)}\left[F(\cdot, y) - \mu f(\cdot, y)\right]\right)(\bar{x}) + \partial\left(\min_{y\in K(\cdot)} \mu f(\cdot, y)\right)(\bar{x})
\end{aligned}
$$

Then applying the above theorem and Theorem 3.38(i) in [59], we have

$$
\begin{aligned}
0 \in \text{-co}\big[&\nabla_x \mu f(\bar{x}, \bar{y}) - \nabla_x F(\bar{x}, \bar{y}) + D^* K(\bar{x}, \bar{y})(\nabla_y \mu f(\bar{x}, \bar{y}) - \nabla_y F(\bar{x}, \bar{y}))\big] \\
&+ \nabla_x \mu f(\bar{x}, \bar{y}) + D^* K(\bar{x}, \bar{y})(\nabla_y \mu f(\bar{x}, \bar{y})) + N(\bar{x}, \Omega).
\end{aligned}
$$

□

Chapter 4

Duality Programming Approach

4.1 Introduction

Duality theory has fascinated many mathematicians since the underlying mathematical framework has been laid down in the context of convex analysis. Duality theorems establish typical connections between a constrained minimization problem and a constrained maximization problem. More precisely, if either problem has an optimal solution, then so has the other and both optimal values are equal under some well-known assumptions on original data. Over the years, many generalizations of this result relaxing the assumptions have been given in the literature. The reader is referred to [27], [35], [55], [76] and the references therein for more details. In this chapter, we study duality theorems and see the possible extension to nondifferentiable and nonconvex data. Also, duality theory is applied to the pessimistic bilevel programming problem to get the optimality conditions. The basic idea of the method is that the upper level maximization can be transformed into a minimization problem and consequently PLBPP can be equivalently written as mathematical program with equilibrium problem. To begin, with let us look at the following well-known duality theorems.

4.2 Duality Theory for Convex Programming Problems

Consider the following mathematical programming problem:

(**P**): $\quad \min f(x) \quad \text{subject to} \quad x \in X := \{\, x \in \Re^n \mid g_i(x) \le 0 \quad i = 1, 2, \ldots, p \,\}.$

Theorem 4.2.1 (Duality Theorem for Differentiable Convex Problem; Wolfe's Duality Theorem [53]). *Let f and g be differentiable convex function on $\Re^n$. Assume that the Slater's qualification condition is satisfied. If $\bar{x}$ is an optimal solution of P, then there exists a vector λ such that (x, λ) is optimal for the following problem:*

$$(\mathbf{D}) : \begin{cases} max\, \varphi(x, y) := f(x) + \lambda g(x) \\ subject\ to \quad (x, \lambda) \in Y := \{\, (x, \lambda) \in \Re^n \times \Re^p \mid \nabla_x \varphi(x, \lambda) = 0,\ \lambda \ge 0 \,\}. \end{cases}$$

Furthermore, the two problems have the same optimal value.

Theorem 4.2.2 (Duality Theorem for Nondifferentiable Convex Problem [76]). *Let f and g_i's be convex and non-smooth functions. Suppose that the Slater's qualification condition is satisfied. Then:*

(a) If $\bar{x}$ is feasible for problem P, then it is optimal for P if and only if there exists $\lambda \in \Re^p$ such that $\lambda \ge 0$, $\lambda^t g(x) = 0$, and

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{n} \lambda_i \partial g_i(\bar{x}).$$

(b) If $\bar{x}$ is optimal for problem P, then there exists $\bar{\lambda}$ such that $(\bar{x}, \bar{\lambda})$ is optimal for

problem:

$$\begin{cases} max\, \varphi(x,y) := f(x) + \lambda g(x) \\ subject\ to \quad (x,\lambda) \in Y := \{\,(x,\lambda) \in \Re^n \times \Re^p \mid 0 \in \partial f(x) + \sum_{i=1}^{p} \lambda_i \partial g_i(x),\ \lambda \geq 0\,\}. \end{cases}$$

Next, we will extend such duality results for nonconvex and nonsmooth mathematical programming problems.

4.3 Duality Theory for Nondifferentiable and Nonconvex Programming Problems

Definition 4.3.1 (Invex Functions and Invex Sets).

(a) Let $f : X \to \bar{\Re}$ and $\bar{x} \in X$ such that $|f(\bar{x})| < \infty$. Then f is called ρ- invex at $\bar{x} \in X$ with respect to some functions η, $\theta : X \times X \to \Re^n$, $\theta(x,u) \neq 0$ whenever $x \neq u$, if there exists a real number ρ such that for each $x^* \in \partial f(\bar{x})$,

$$f(x) - f(\bar{x}) \geq \langle\, x^*, \eta(x,\bar{x}) \rangle + \rho \|\theta(x,\bar{x})\|^2 \quad x \in X.$$

The function f is called ρ-invex, if the above condition holds for each $\bar{x} \in X$.

(b) Let $\Omega \subseteq X$ be closed and let $x \in \Omega$. We say that Ω is ρ-invex at $\bar{x}$ with respect to some functions η, $\theta : X \times X \to \Re^n$, $\theta(x,u) \neq 0$ whenever $x \neq u$, if there exists a real number ρ such that for each $x^* \in N(\bar{x}, \Omega)$,

$$\langle x^*, \eta(x, \bar{x}) \rangle \leq \rho \|\theta(x, \bar{x})\|^2 \qquad \text{for all } x \in X.$$

Consider the following mathematical programming problem:

(P): $\quad \min f(x) \quad$ subject to $\quad x \in \Omega.$

(D): $\quad \max f(x) \quad$ subject to $\quad x \in \{ x \mid 0 \in \partial f(x) + N(x, \Omega) \}.$

Theorem 4.3.2 (Duality Theorem for Nondifferentiable Nonconvex Problem). *Let $\bar{x}$ be an optimal solution to problem P and let $f : X \to \bar{\Re}$ be such that $|f(\bar{x})| < \infty$ and locally Lipschitz continuous around $\bar{x}$. Also let Ω be closed around $\bar{x}$. Suppose that f is ρ_0-invex and Ω is ρ_1-invex at each feasible point of the problem D with respect to the same function η and θ and that $\rho_0 - \rho_1 \geq 0$.*

Then one has $\bar{x}$ is an global optimal solution for D and optimal values are equal.

Proof. The proof is similar to the proof of the Theorem 4.1 in [35]. Let x and z be feasible for the problem P and D, respectively.

Then $0 \in \partial f(z) + N(z, \Omega)$ implies that there exists $v \in \partial f(z)$ and $w \in N(z, \Omega)$ such that

$v = -w$. Consider

$$\begin{aligned} f(x) - f(z) &\geq \langle v, \eta(x,z)\rangle + \rho_0\|\theta(x,z)\|^2 && \text{by invexity of } f \\ &= \langle -w, \eta(x,z)\rangle + \rho_0\|\theta(x,z)\|^2 \\ &- \rho_1\|\theta(x,z)\|^2 + \rho_0\|\theta(x,z)\|^2 && \text{as } N(z,\Omega) \text{ is invex} \\ &= (\rho_0 - \rho_1)\|\theta(x,z)\|^2 \\ &\geq 0 && \text{as } (\rho_0 - \rho_1) \geq 0 \end{aligned}$$

So $f(x) \geq f(z)$.

Now since $\bar{x}$ is an optimal solution to problem P, and since f is locally Lipschitz continuous around $\bar{x}$, we have

$$0 \in \partial f(\bar{x}) + N(\bar{x}, \Omega).$$

Thus $\bar{x}$ is feasible for problem D.

Now for any arbitrary feasible point z of D, one has $f(\bar{x}) \geq f(z)$. This implies that $\bar{x}$ is an optimal solution of D and optimal values are equal.

□

Now replacing Ω with the special structure, we have the following result.

Let $\Omega := \{\, x \in \Re^n \mid g_i(x) \leq 0 \quad i = 1, 2, \ldots, p \,\}$. Consider the problem P. The point (x, λ) is said to be a critical point for the problem P if, x is a feasible point for P, $\lambda \in \Re^p_+$ and

$$0 \in \partial f(x) + \sum_1^p \lambda_i \partial g_i(x) \quad \text{and} \quad \lambda_i g_i(x) = 0.$$

Consider the dual problem (D):

$$\max \varphi(x,y) := f(x) + \lambda g(x)$$
$$\text{subject to} \quad (x,\lambda) \in Y := \{\,(x,\lambda) \in \Re^n \times \Re^p \mid 0 \in \partial f(x) + \sum_{i=1}^{p} \lambda_i \partial g_i(x),\ \lambda \geq 0\,\}$$

Corollary 4.3.3. *Assume that the point $(\bar{x}, \bar{\lambda})$ is a critical point for (P). Assume also that f is ρ_0-invex, g_i is ρ_i-invex for each $i = 1, 2, \ldots, p$ with respect to the same functions η and θ, and $\rho_0 + \sum_1^p \geq 0$ for each feasible point (z, λ) of D. Then one has $\bar{x}$ and $(\bar{x}, \bar{\lambda})$ is a global optimal for P and D respectively. Also the optimal values are equal.*

4.4 Reformulation of PBLPP and Optimality Conditions

4.4.1 PBLPP with Convex Data

Recall the Pessimistic Bilevel Programming Problem:

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x,y) \right).$$

Under the reformulation assumption, this is equivalent to the following problem:

$$\min_{x \in \Omega} \left(\max_{y \in K(x)} \left[\mathcal{F}(x,y) := F(x,y) - \mu f(x,y) + \mu v(x) \right] \right).$$

Let $K(x) = \{\, y \in Y \subset \Re^m \mid g(x,y) \leq 0 \,\}$. Now let us make the following assumptions:

(i) For any fixed x, $-F(x,\cdot)$, $f(x,\cdot)$, and $g(x,\cdot)$ are convex for all $y \in Y$;

(ii) For any fixed x, there exists $y \in Y$ such that $g(x, y) < 0$. (i.e., Slater's qualification conditions holds);

(iii) Y is convex set.

Under the above assumptions (i) to (iii), by Duality Theorem one has for any fixed x, there exists an optimal solution $\bar{\lambda}(x) \geq 0$ to the problem:

$$\min_{\lambda} \left(H(x, v) := \max_{y \in Y} \left[F(x, y) - \lambda^t g(x, y) \right] \right)$$

$$\text{subject to} \lambda \in D := \{ \lambda \mid \lambda \geq 0; \exists \max \left[F(x, y) - \lambda^t g(x, y) \right] \}$$

Also, we have $F(x, \bar{y}(x)) = H(x, \bar{\lambda}(x))$, where $\bar{y}(x)$ solves the problem $\max_{y \in S(x)} F(x, y)$.

Furthermore, if we assume that

(iv) For any fixed x, $F(x, \cdot)$ and $g(x, \cdot)$ are lower semi continuous in y, and

(v) $Y \neq \emptyset$, compact,

then one has $D = \{ \lambda \mid \lambda \geq 0 \}$.

Consequently, under the above mentioned assumptions, PBLPP is equivalent to the following problem:

$$\min_{(x,\lambda) \in \Omega \times D} \left[\max_{y \in Y} \left(F(x, y) - \mu f(x, y) - \lambda^t g(x, y) + \mu v(x) \right) \right].$$

Again under the duality theorem, this is equivalent to the following MPEC:

$$\begin{cases} \min\left(F(x,y) - \mu f(x,y) - \lambda^t g(x,y) + \mu v(x)\right) \\ \text{subject to } 0 \in \partial_y \left(F(x,y) - \mu f(x,y) - \lambda^t g(x,y)\right) + N(y,Y), \quad (x,\lambda) \in \Omega \times D. \end{cases}$$

4.4.2 Nonconvex PBLPPs

Now look at another reformulation of PBLPP relaxing the above assumptions. Again using the Duality Theorem (4.3.2) mentioned above PBLPP can be equivalently written as a MPEC and hence the necessary optimality conditions can be obtained. Consider the programming problems:

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x,y) \right).$$

(P): $\quad \max\limits_{y \in S(x)} F(x,y)$

(D): $\quad \min F(x,y)$ such that $y \in \{\, y \mid 0 \in -\nabla_y F(x,y) + N(y,S(x)) \,\}$

Now we impose the following assumptions:

(i) Let $(\bar{x},\bar{y})$ be optimal solutions for pessimistic bilevel programming and let $\bar{z} = F(\bar{x},\bar{y})$ where $F : \Re^n \times \Re^m :\longrightarrow \bar{\Re}$ is finite at this point;

(ii) F is locally Lipschitz continuous and around $(\bar{x},\bar{y})$;

(iii) Ω is closed around $\bar{x}$;

(iv) For each x around $\bar{x}$, $-F(x,\cdot)$ is ρ_0-invex and $S(x)$ is ρ_1-invex at each feasible point (x,y) of D with respect to the same functions η and θ with the property that

$\rho_0 - \rho_1 \geq 0.$

Under the above assumption (i) to (iv), applying the above duality theorem we have,

$$\max\{\, F(x,y) \mid y \in S(x)\,\} = \min\{\, F(x,y) \mid y \in \;\{\, y \mid 0 \in -\partial_y F(x,y) + N(y, S(x))\,\}\,\}$$

for each x near $\bar{x}$. Then pessimistic bilevel programming problem is equivalent to the following MPEC

$$\begin{cases} \min F(x,y) \\ \text{subject to}\, y \in M(x) := \{\, y \mid 0 \in -\partial_y F(x,y) + N(y, S(x))\,\} \\ x \in \Omega \end{cases}$$

This is a mathematical program with equilibrium constraints (MPEQ). Now we are able to derive the following optimality conditions.

Theorem 4.4.1 (**Optimality Conditions**). *Suppose the above assumptions (i)-(iv) holds. In addition let us assume the following:*

(a) $F(x,y)$ is strictly differentiable at $(\bar{x},\bar{y})$ and $\nabla_y F(x,y)$ is strictly differentiable at this pint.

(b) $S(x)$ is closed for each x around $\bar{x}$.

(c) the following qualification conditions holds:

$$\left[(x^*,0) \in -\nabla\left(\nabla_y F(\bar{x},\bar{y})\right)^t (z^*) + D^*Q(\bar{x},\bar{y},\bar{z})(z^*),\; -x^* \in N(\bar{x},\Omega)\right] \Rightarrow x^* = 0$$

$\left[(x^*, y^*) = -\nabla \left(\nabla_y F(\bar{x}, \bar{y})\right)^t (z^*);\ (x^*, y^*) \in -D^* Q(\bar{x}, \bar{y}, \bar{z})(z^*)\right] \Rightarrow x^* = y^* = z^* = 0$ *where* $Q(x, y) := N(y, S(x))$.

Then there are $\tilde{x}^* \in N(\bar{x}, \Omega)$ *and* $z^* \in \Re^m$ *such that*

$$(-\nabla_x F(\bar{x}, \bar{y}) - \tilde{x}^*, -\nabla_y F(\bar{x}, \bar{y})) \in -\nabla \left(\nabla_y F(\bar{x}, \bar{y})\right)^t (z^*) + D^* \left(Q(\bar{x}, \bar{y}, \bar{z})\right)(z^*).$$

Proof. By the above duality theorem for each x in a neighborhood of $\bar{x}$, we have

$$\max \{ F(x, y) \mid y \in S(x) \} = \min \{ F(x, y) \mid y \in \ \{ y \mid 0 \in -\nabla_y F(x, y) + N(y, S(x)) \} \}$$

Then pessimistic bilevel programming problem is equivalent to the following MPEC

$$\begin{cases} \min F(x, y), \\ \text{subject to}\, y \in M(x) := \{ y \mid 0 \in -\nabla_y F(x, y) + N(y, S(x)) \}, \quad \text{and} \\ x \in \Omega. \end{cases}$$

Then by theorem 5.37[60] we find $\tilde{x}^* \in N(\bar{x}, \Omega)$ and $z^* \in \Re^m$ such that

$$(-\nabla_x F(\bar{x}, \bar{y}) - \tilde{x}^*, -\nabla_y F(\bar{x}, \bar{y})) \in -\nabla \left(\nabla_y F(\bar{x}, \bar{y})\right)^t (z^*) + D^* \left(Q(\bar{x}, \bar{y}, \bar{z})\right)(z^*)$$

□

4.5 Saddle Point Theorem

A point $(\bar{x}, \bar{y})$ is said to be a saddle point of $\phi(x, y)$ on the set $\Omega \times K$ if $\phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x, \bar{y})$ for all $x \in \Omega$ and $y \in K$.

Lemma 4.5.1. *Let $\Omega \subseteq \Re^n$ and $K \subseteq \Re^m$ be bounded and closed sets. Also assume that ϕ is continuous function defined on $\Omega \times K$. Then the function ϕ has a saddle point on $\Omega \times K$ if and only if*

$$\min_{x \in \Omega} \max_{y \in K} \phi(x, y) = \max_{y \in K} \min_{x \in \Omega} \phi(x, y).$$

Consider the pessimistic bilevel programming problem:

$$\min_{x \in \Omega} \left(\max_{y \in S(x)} F(x, y) \right).$$

Theorem 4.5.2. *Let $(\bar{x}, \bar{y})$ be a global pessimistic optimal solution. Suppose that the following holds:*

(a) Ω and $K(x) \equiv K$ are closed and convex.

(b) $\phi(x, y) := F(x, y) - \mu f(x, y) + \mu v(x)$ is continuous on $\Omega \times K$.

(c) RA holds.

(d) $\min_{x \in \Omega} \max_{y \in K} \phi(x, y) = \max_{y \in K} \min_{x \in \Omega} \phi(x, y)$

Then there exists y^0 such that $\phi(\bar{x}, \bar{y}) = \phi(\bar{x}, y^0)$ and

$$0 \in \partial_y \phi(\bar{x}, y^0) + N(y^0, K),$$

$$0 \in \partial_x \phi(\bar{x}, y^0) + N(\bar{y}, \Omega).$$

Proof. Since $(\bar{x}, \bar{y})$ is a global pessimistic optimal solution of PBLPP, one has

$$\phi(\bar{x}, \bar{y}) \leq \min_{x\Omega} \max_{y \in K} \phi(x, y) \tag{4.5.1}$$

and

$$\phi(\bar{x}, \bar{y}) = \max_{y \in K} \phi(\bar{x}, y). \tag{4.5.2}$$

Assumption (d) implies that there exits $(x^0, y^0) \in \Omega \times K$ such that

$$\phi(x^0, y) \leq \phi(x^0, y^0) \leq \phi(x, y^0) \qquad \text{for all } (x, y) \in \Omega \times K \tag{4.5.3}$$

and

$$\phi(x^0, y^0) = \min_{x \in \Omega} \max_{y \in K} \phi(x, y) \tag{4.5.4}$$

By (4.5.1)-(4.5.4) we have

$$\phi(\bar{x}, y^0) \leq \phi(\bar{x}, \bar{y}) \leq \phi(x^0, y^0) \leq \phi(\bar{x}, y^0),$$

which implies

$$\phi(\bar{x}, y^0) = \phi(\bar{x}, \bar{y}) = \phi(x^0, y^0). \tag{4.5.5}$$

Now by (4.5.2) and (4.5.5), $\phi(\bar{x}, y) \leq \phi(\bar{x}, y^0)$ for all $y \in K$ and by (4.5.3), $\phi(\bar{x}, y^0) \leq \phi(x, y^0)$ for all $x \in \Omega$

Thus,

$$\phi(\bar{x}, y) \leq \phi(\bar{x}, y^0) \leq \phi(x, y^0) \quad \text{for all}\,(x, y) \in \Omega \times K$$

Since $\phi(\bar{x}, y) \leq \phi(\bar{x}, \bar{y})$ for all $y \in K$ and $\phi(x, \cdot)$ is Lipschitz continuous around $\bar{y}$ for each $x \in \Omega$ we have

$$0 \in \partial_y \phi(\bar{x}, \bar{y}) + N(\bar{y}, K)$$

Since $\phi(\bar{x}, y^0) \leq \phi(x, y^0)$ for all $x \in \Omega$ and $\phi(\cdot, y)$ is Lipschitz continuous around $\bar{x}$ for each $y \in K$, we have

$$0 \in \partial_x \phi(\bar{x}, y^0) + N(\bar{x}, \Omega)$$

□

Chapter 5

Optimality Conditions for PBLPPs with Finite Rational Responses

5.1 Introduction and Basic assumptions

Let us recall the pessimistic bilevel programming problem PBLPP. Throughout this chapter, it is assumed that $(\bar{x}, \bar{y})$ is a local pessimistic optimal solution, and let $U \subset \Omega$ be a neighborhood of $\bar{x}$.

$$
(PBLPP): \begin{cases}
(l_x): & \min_y f(x,y) \quad \text{subject to } y \in K(x), \\
(m_x): & \max_y F(x,y) \quad \text{subject to } y \in S(x) := \operatorname{argmin}\{\, f(x,y) \mid y \in K(x) \,\}, \\
(pblpp): & \min \varphi_p(x) := \max_{y \in S(x)} F(x,y) \quad \text{subject to } x \in \Omega.
\end{cases}
$$

Assumption 5.1.1. $S(x)$ is finite but not singleton for each $x \in U$.

Throughout this chapter, it is assumed that the lower level problem has a finite number of solutions. If the solution set to lower level problem is singleton, then the optimistic and pessimistic versions coincide. Therefore, we may assume that the solution set to the lower level problem is not singleton.

Write $S(x) = \{\, y_i(x) \mid i = 1, 2, \ldots, r \text{ where } r > 1;\ y_i(x) \text{ solves } l(x) \,\}$

Then the PBLPP can be equivalently written as

$$\min_{x\in\Omega}\left(\max_{i=\overline{1,r}}\left[\tilde{F}_i(x) := F(x, y_i(x)) := (F\circ y_i)(x)\right]\right), \tag{5.1.1}$$

or epi-graphical formulation as

$$\min_{(x,\omega)\in\Omega\times\Re}\omega \quad \text{subject to } \tilde{F}_i(x)-\omega\leq 0 \quad i=1,2,\ldots,r. \tag{5.1.2}$$

Now let us define the following sets:

$I(\bar{x}) := \{\, i\in\{\,1,2,\ldots,r\,\} \mid \tilde{F}_i(\bar{x}) = \varphi_p(\bar{x}) := \max_{i\in\overline{1,r}}\tilde{F}_i(\bar{x})\,\}$ and

$\Lambda(\bar{x}) := \{\,(\lambda_1,\lambda_2,\ldots,\lambda_r) \mid \lambda_i\geq 0,\ \sum_{i=1}^r\lambda_i = 1,\ \lambda_i(F_i(\bar{x})-\varphi_p(\bar{x})) = 0\,\}.$

5.2 Necessary Optimality Conditions

This section is devoted to get the necessary optimality conditions for PBLPP under the above assumptions made. When the lower level problem has finite but not singleton number of solutions we can obtain the following Fritz-John type necessary optimality conditions. Then we study some special classes of pessimistic bilevel problems for which there are a finite number of rational responses. Then we can obtain the following type of necessary optimality conditions for those classes of pessimistic bilevel programming problems.

Theorem 5.2.1 (Necessary Optimality Conditions). *Let $(\bar{x}, \bar{y})$ be a local pessimistic optimal solution. Assume that Ω is locally closed around $\bar{x}$ and for each x near $\bar{x}$, $S(x)$ has finite cardinality, but not singleton. Assume also the followings:*

(a) F is strictly differentiable at $(\bar{x}, \bar{y})$ and $|F(\bar{x}, \bar{y})| < \infty$ where $\bar{y} = y_i(\bar{x})$ for all $i \in I(\bar{x})$;

(b) $y_i : \Re^n \to \Re^m$ is locally Lipschitz continuous around $\bar{x}$.

Then there exists $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, r$ and not all equals zero such that

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \nabla_x F(\bar{x}, \bar{y}) + \partial \langle \nabla_y F(\bar{x}, \bar{y}), \lambda_i y_i \rangle (\bar{x}) + N(\bar{x}, \Omega),$$

$$\lambda_i (F(\bar{x}, y_i(\bar{x})) - maxF(\bar{x}, y_i(\bar{x}))) = 0.$$

Proof. F is locally Lipschitz continuous around $(\bar{x}, \bar{y})$ as F is strictly differentiable at this point. Since $y_i(x)$ $i = 1, 2, \ldots, r$ is locally Lipschitz continuous around $\bar{x}$, the composition $F_i(x) = F(x, y_i(x)) := (F \circ y_i)(x)$ is also locally Lipschitz continuous around $\bar{x}$ for each $i = 1, 2, \ldots, r$. This implies that the maximum function $\varphi_p(x) := \max_{i \in \overline{1,r}} \tilde{F}_i(x)$ is locally Lipschitz continuous around $\bar{x}$. Now since Ω is locally closed around this point, we have

$$0 \in \partial\varphi_p(\bar{x}) + N(\bar{x}, \Omega)$$

(by [60, Proposition 5.3])

$$\subseteq \bigcup \left\{ \partial \left(\sum_{i \in I(\bar{x})} \lambda_i \tilde{F}_i \right) (\bar{x}) \mid (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda(\bar{x}) \right\} + N(\bar{x}, \Omega)$$

(subdifferentiation of maximum function [59, Theorem 3.46(ii)])

$$\subseteq \bigcup \left\{ \sum_{i \in I(\bar{x})} \lambda_i \partial \tilde{F}_i(\bar{x}) \mid (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda(\bar{x}) \right\} + N(\bar{x}, \Omega)$$

(sum rule for basic subdifferential [59, Theorem 3.36])

$$= \bigcup \left\{ \sum_{i \in I(\bar{x})} \lambda_i \left[\nabla_F(\bar{x}, \bar{y}) + D^* y_i(\bar{x})(\nabla_F(\bar{x}, \bar{y})) \right] \mid (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda(\bar{x}) \right\} + N(\bar{x}, \Omega)$$

(subdifferentiation of compositions [59, Theorem 1.110(ii)])

$$= \bigcup_{(\lambda_1, \ldots, \lambda_r) \in \Lambda(\bar{x})} \left[\sum_{i \in I(\bar{x})} \lambda_i \nabla_x F(\bar{x}, \bar{y}) + \partial \langle \nabla_y F(\bar{x}, \bar{y}), \lambda_i y_i \rangle (\bar{x}) \right] + N(\bar{x}, \Omega).$$

That implies that there exits $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, r$ with $\sum_i^r \lambda_i = 1$ such that

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \nabla_x F(\bar{x}, \bar{y}) + \partial \langle \nabla_y F(\bar{x}, \bar{y}), \lambda_i y_i \rangle (\bar{x}) + N(\bar{x}, \Omega),$$

$$\lambda_i (F(\bar{x}, y_i(\bar{x})) - \max F(\bar{x}, y_i(\bar{x}))) = 0.$$

□

5.3 Some Classes of Pessimistic Bilevel Programming Problems

In this section some particular classes of pessimistic bilevel programming problem will be considered. Indeed we will consider problems for which we can assure the finite number of optimal solutions exits. To begin with, we make the following assumptions throughout in this section.

Assumption 5.3.1.

(a) $K(x)$ is a polyhedral convex set for each x near $\bar{x}$ and suppose that $K(x)$ has the following form:

$$K(x) := \left\{ y \in \Re^m \mid \begin{array}{ll} \langle a_i(x), y\rangle + \alpha_i(x) \le 0; & i = 1, 2, \dots, p \\ \langle a_j(x), y\rangle + \alpha_j(x) \le 0; & j = p+1, p+2, \dots, p+q \end{array} \right\}; \tag{5.3.1}$$

(b) $a_1, \dots, a_{p+q} \in C^1(\Omega, \Re^m)$ and $\alpha_1, \dots, \alpha_{p+q} \in C^1(\Omega, \Re)$

(c) K is lower semi continuous closed at every point of $U \subseteq \Omega$;

(d) $K(\bar{x})$ posses at least one vertex for some $\bar{x}$.

Proposition 5.3.2 (**[50, Lemma 2.2]**)**.** *Under the above assumptions, there exists an open set $U_0 \subseteq U$ and k subsets $J_1, J_2, \dots, J_k \subseteq \{1, 2, \dots, p+q\}$ with $|J_1| = |J_2| \dots = |J_k| = m$ with $J_i := \{j_i^1, j_i^2, \dots, j_i^m\}$ such that for every $x \in U_0$, the polyhedral convex* *set $K(x)$ possess exactly k vertices $v_1(x), v_2(x) \ldots, v_k(x) \in C^1(U_0, \Re^m)$ defined by:*

$$v_i(x) = A_{J_i}^{-1}(x)\,(-\alpha_{J_i}(x)) \quad i = 1, 2, \ldots, k, \tag{5.3.2}$$

where $A_{J_i}(x)$ denotes the matrix whose rows are $a_{j_i^1}(x), \ldots, a_{j_i^m}(x)$, and $\alpha_{J_i}(x)$ denotes the vector whose components are $\alpha_{j_i^1}(x), \ldots, \alpha_{j_i^m}(x)$.

5.3.1 Bilevel Programming Problem with Smooth Upper Level Objective Functions and Smooth and Concave Lower Level Objective Functions

In this section we consider the following classes of pessimistic bilevel programming problems.

$$(PBLPP): \begin{cases} (l_x): & \min\limits_y f(x,y) \quad \text{subject to } y \in K(x) \\ (m_x): & \max\limits_y F(x,y) \quad \text{subject to } y \in S(x) := \operatorname{argmin}\{\, f(x,y) \mid y \in K(x)\,\} \\ (pblpp): & \min \varphi_p(x) := \max\limits_{y \in S(x)} F(x,y) \quad \text{subject to } x \in \Omega. \end{cases}$$

Assumption 5.3.3.

(a) F is strictly differentiable at $(\bar{x}, \bar{y})$ and $|F(\bar{x}, \bar{y})| < \infty$;

(b) $f \in C^1(\Omega \times \Re^m, \Re)$ and $f(x, \cdot)$ is concave for every $x \in U$;

(c) $\inf\limits_{y \in K(x)} f(x,y)$ is finite for each $x \in U$.

Then we have the following necessary optimality conditions for above-mentioned special classes of PBLPPs.

Theorem 5.3.4 (Necessary Optimality Conditions). *Under the above assumptions there exists $v_1(x), v_2(x), \ldots, v_r(x)$, $r \leq k$ vertices of $K(x)$ such that v_i $i = 1, 2, \ldots, r$ are continuously differentiable around $\bar{x}$, and also there are lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_r$ having the following optimality conditions:*

$$0 \in \Sigma_{i \in I(\bar{x})} \left[\lambda_i \nabla_y F(\bar{x}, \bar{y}) + \nabla v_i(\bar{x})^t \nabla_y F(\bar{x}, \bar{y})\right] + N(\bar{x}, \Omega),$$

$$\lambda_i(F(\bar{x}, y_i(\bar{x})) - maxF(\bar{x}, y_i(\bar{x}))) = 0,$$

$$\lambda_i \geq 0 \quad and \quad \sum_i^r \lambda_i = 1.$$

Proof. Since $f \in C(\Omega \times \Re^m, \Re)$, $f(x, \cdot)$ is concave for every fixed $x \in U$, and $\inf_{y \in K(x)} f(x, y)$ is finite for each $x \in U$, there exits a neighborhood $U_0 \subseteq U$ of $\bar{x}$ and a function $v \in C(U_0, \Re^m)$ such that $v(x)$ is a vertex of $K(x)$ and $v(x) \in S(x)$ for all $x \in U_0$; see [50, Proposition 4.1]. Now let $v_1(x), v_2(x), \ldots, v_r(x)$ be the vertices of $K(x)$ at which the lower level problem attains its minimum. So we may take now $S(x) = \{ v_1(x), v_2(x), \ldots, v_r(x) \}$ where $r \leq k$. Since $v_i(x)$ is C^1, it is locally Lipschitz continuous around $\bar{x}$. Then there are Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

$$0 \in \Sigma_{i \in I(\bar{x})} \left[\lambda_i \nabla_y F(\bar{x}, \bar{y}) + \nabla v_i(\bar{x})^t \nabla_y F(\bar{x}, \bar{y})\right] + N(\bar{x}, \Omega),$$

$$\lambda_i(F(\bar{x}, y_i(\bar{x})) - \max F(\bar{x}, y_i(\bar{x}))) = 0,$$

$$\lambda_i \geq 0 \quad \text{and} \quad \sum_i^r \lambda_i = 1.$$

□

Example 5.3.5. Let us consider the lower level problem to be the following quadratic programming problem:

$$\min_y f(x,y) := \frac{1}{2}\langle y, Qy\rangle + \langle y, Px\rangle + p^t x + q^t y \quad \text{subject to } y \in K(x).$$

Then if Q is negative semidefinite, then the function $f(x,\cdot)$ is concave for each fixed x. Then one has $S(x) \subset$ extreme points of $K(x)$, i.e., $S(x)$ is finite and one has the above necessary optimality condition of PBLPP with strictly differentiable cost functions and quadratic (negative semidefinite) lower level programming problems.

5.3.2 Bilevel Programming Problem with Convex and Smooth Upper Level Objective Functions

Consider the following problem:

$$\varphi_p(x) := \max F(x,y) \quad \text{subject to } y \in S(x). \tag{5.3.3}$$

Proposition 5.3.6 ([74, Corollary 32.3.4]). *For each $x \in \Omega$ assume the following:*

(a) $S(x)$ is non-empty, polyhedral convex set contained in dom$F$;

(b) $S(x)$ contains no lines (or equivalently $S(x)$ has at least one extreme points if it is closed and convex);

(c) $F(x,\cdot)$ is convex and bounded above on $S(x)$.

Then φ_p is attained at one of the (finitely many) extreme points of $S(x)$.

Remark 5.3.7. If the assumptions of the above proposition are satisfied, then the above problem can be equivalently expressed as:

$$\varphi_p(x) := \max F(x,y) \quad \text{subject to } y \in S_e(x) := \{\, y_1(x), y_2(x), \ldots, y_r(x) \,\}, \tag{5.3.4}$$

where S_e is the set of extreme points or vertices of $S(x)$.

(2.A) Linear Lower Level Problems

Now let us consider the pessimistic bilevel programming problem with linear lower level problem and the upper level objective function is convex and smooth. To be more precise, let us assume the following:

Assumption 5.3.8.

(a) F is strictly differentiable at $(\bar{x}, \bar{y})$, $|F(\bar{x}, \bar{y})| < \infty$, and $F(x, \cdot)$ is convex and bounded above on $S(x)$ for each x near $\bar{x}$;

(b) $f(x,y) = \langle c(x), y \rangle$ where $c \in C^1(\Omega, \Re^m)$;

(c) $\inf\limits_{y \in K(x)} f(x,y) > -\infty$ for all $x \in U$.

Theorem 5.3.9 (Necessary Optimality Conditions). *Under the above assumptions there exists $v_1(x), v_2(x), \ldots, v_k(x)$, vertices of $S(x)$ such that v_i, $i = 1, 2, \ldots, k$, are continuously differentiable around $\bar{x}$, and also there are lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_k$*

having the following optimality conditions:

$$0 \in \Sigma_{i\in I(\bar{x})} \left[\lambda_i \nabla_y F(\bar{x},\bar{y}) + \nabla v_i(\bar{x})^t \nabla_y F(\bar{x},\bar{y})\right] + N(\bar{x},\Omega),$$

$$\lambda_i(F(\bar{x},y_i(\bar{x})) - max\, F(\bar{x},y_i(\bar{x}))) = 0,$$

$$\lambda_i \geq 0 \quad and \quad \sum_i^r \lambda_i = 1.$$

Proof. Under the above-mentioned assumptions on lower level problem, by Proposition 3.1 in [50] one has the following. There exists a neighborhood $U_0 \subseteq U$ of $\bar{x}$, $v_1, v_2, \ldots, v_l \in C^1(U_0, \Re^m)$, and an integer k; $0 < k \leq l$ such that

$$S(x) = \{\, y \in \Re^m \mid y = \Sigma_1^l\, \lambda_i v_i(x),\, \Sigma_1^k\, \lambda_i = 1,\, \lambda_i \geq 0;\, i = 1, 2, \ldots, l\,\}$$

i.e., for each $x \in U_0$, $S(x)$ is polyhedral convex set.

Also, since $S(x)$ is closed and has at least one vertex, $S(x)$ contains no lines. Now since $F(x,\cdot)$ is convex and bounded above on $S(x)$, we have

$$\max_{y\in S(x)} F(x,y)$$

is equivalent to the following problem:

$$\max_{y\in S_e(x)} F(x,y),$$

where $S_e(x) = \{\, v_1(x), v_2(x), \ldots, v_k(x)$ are the extreme points (or vertices) of $S(x)$. Then by the above Theorem (5.3.9) one has the following necessary optimality conditions: there are Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_k$ having the following optimality conditions:

$$0 \in \Sigma_{i\in I(\bar{x})} \left[\lambda_i \nabla_y F(\bar{x}, \bar{y}) + \nabla v_i(\bar{x})^t \nabla_y F(\bar{x}, \bar{y})\right] + N(\bar{x}, \Omega),$$

$$\lambda_i (F(\bar{x}, y_i(\bar{x})) - \max F(\bar{x}, y_i(\bar{x}))) = 0,$$

$$\lambda_i \geq 0 \quad \text{and} \quad \sum_i^r \lambda_i = 1.$$

□

(2.B) Quadratic Lower Level Problems

Consider the following quadratic programming problem.

$$(l_x): \qquad \min_y f(x,y) := \tfrac{1}{2}\langle y, Qy\rangle + \langle y, Px\rangle + p^t x + q^t y \quad \text{subject to } y \in K(x).$$

Assumption 5.3.10.

(a) D is a symmetric positive semidefinite matrix;

(b) $S(x)$ is not singleton or empty.

Proposition 5.3.11 ([41, Theorem 4.4], [54, Theorem 1]). *Let D be a symmetric, positive semidefinite matrix. Then $f(x,\cdot)$ is convex for each fixed x and $S(x)$ is closed and convex. Furthermore we have*

$$S(x) = K(x) \cap \{\, y \in \Re^m \mid \langle \nabla_y f(x, \bar{y}), y - \bar{y}\rangle = 0 \,\} \cap \{\, y \in \Re^m \mid \nabla_y^2 f(x, \bar{y})(y - \bar{y}) = 0 \,\},$$

i.e., $S(x)$ is polyhedral convex set. Also $S(x)$ is finite if and only if it is a singleton or it is empty.

Now if the upper level objective function F is strictly differentiable at $(\bar{x}, \bar{y})$, $|F(\bar{x}, \bar{y})| < \infty$, and $F(x, \cdot)$ is convex and bounded above on $S(x)$ for each x near $\bar{x}$ and lower level cost function is a quadratic function as above, we can obtain the necessary optimality conditions as above Theorem 5.3.9.

(2.C) Parametric Polyhedral Lower Level Problems

Let us consider the following parametric polyhedral lower level problem:

$$l(x): \qquad \min_{y} f(x, y) :=$$
$$\max\{\langle a_i(x), y\rangle + \alpha_i(x) \mid i = 1, 2, \ldots, s\} + \delta_N(y) \quad \text{subject to } y \in K(x),$$

where $a_1(x), a_2(x), \ldots, a_s(x) \in C^1(\Omega, \Re^m)$, $\alpha_1(x), \alpha_2(x), \ldots, \alpha_s(x) \in C^1(\Omega, \Re)$, $\delta_N(x)$ is the indicator function of a polyhedral set, and $K(x)$ is convex polyhedral defined as above.

Proposition 5.3.12 (**[50, Proposition 5.1(iii)]**). *Suppose that the optimal value function $\inf_{y \in K(x)} f(x, y)$ is finite. Then there exist an open set $U_0 \subset U$ such that*

$$S(x) = \{\, y \in \Re^m \mid y = \sum_{i=1}^{l} \lambda_i v_i(x),\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0,\ i = \overline{1, l}\,\} \quad \textit{for every} \, x \in U_0$$

where $v_1(x), v_2(x), \ldots, v_k(x) \in C^1(U_0, \Re^m)$. i.e., $S(x)$ is polyhedral convex set.

Remark 5.3.13. Now if the upper level objective function F is strictly differentiable at $(\bar{x}, \bar{y})$, $|F(\bar{x}, \bar{y})| < \infty$, and $F(x, \cdot)$ is convex and bounded above on $S(x)$ for each x near $\bar{x}$, one can arrive at the above necessary optimality conditions.

Appendix A

Overview of Classical Nonlinear Theory

We consider the following parametric nonlinear optimization problem.

$$\min_{y} f(x,y) \quad \text{subject to} \quad y \in k(x), \tag{l(x)}$$

where $y \in \Re^m$ is the decision variable, $x \in \Re^n$, the perturbation parameter, $f : \Re^n \times \Re^m \rightarrow \Re$, and $K : \Re^n \rightrightarrows \Re^m$ is a constraint map or a feasible solution map such that, for each $x \in \Re^n$, $K(x)$ is a subset of $\Re^m$.

In particular, our main interest lies in the inequality and equality constrained parametric nonlinear programming problem with:

$$K(x) = \{\, y \in Y \mid g(x,y) \leq 0, \quad h(x,y) = 0 \,\},$$

where $Y \subset \Re^m$, $g : \Re^n \times \Re^m \rightarrow \Re^p$, $h : \Re^n \times \Re^m \rightarrow \Re^q$.

Basic Assumptions in Sensitivity Analysis and Stability Theory

The Lagrangian and the set of Karush-Kuhn-Tucker vectors for the problem $l(x)$ are defined as follows: Let $L(x,y,u,v) = f(x,y) + u^T g(x,y) + v^T h(x,y)$ be the corresponding *Lagrange function* and the *vectors of Lagrange multipliers* that correspond to the optimal solution $y(x)$ of $l(x)$ are defined as follows:

$$\Lambda(x,y) = \{\,(u,v) \in \Re^p_+ \times \Re^q \;|\nabla_y L(x,y(x),u(x),v(x)) = 0,\; u_i(x) \geq 0,$$
$$u_i(x)g_i(x,y(x)) = 0 \; i = 1,\ldots,p,\; y(x) \in K(x)\,\}$$

Henceforth we assume that $l(x)$ has an optimal solution for the true parameter $\bar{x}$. Let $\bar{y} = y(\bar{x})$ and $I(\bar{x},\bar{y}) = \{\, i \in \{1,2,\ldots,p\} \mid g_i(\bar{x},\bar{y}) = 0\,\}$ be so-called *indices of active inequality constraints.*

Mangasarian Fromowits Constraint Qualification(MFCQ): We say that MFCQ is satisfied at $(\bar{x},\bar{y})$ if there exist a direction $d \in \Re^m$ satisfying:

1 $\nabla_y g_i(\bar{x},\bar{y})d < 0$ for each $i \in I(\bar{x},\bar{y})$, $\quad \nabla_y h_j(\bar{x},\bar{y})d = 0$ for each $j = 1,2,\ldots,q$, and,

2 the gradient $\{\,\nabla_y h_j(\bar{x},\bar{y}) \mid j = 1,2,\ldots,q\}$ are linearly independent.

Linear Independence Constraint Qualification(LICQ): We say that LICQ is satisfied at $(\bar{x},\bar{y})$ if the set $\{\nabla_y g_i(\bar{x},\bar{y}), i \in I(\bar{x},\bar{y}), \nabla_y h_j(\bar{x},\bar{y}), j = 1,2,\ldots,q\}$ is lineally independent.

Strict Complementarity Slackness (SCS): We say that SCS is satisfied at $(\bar{x},\bar{y})$ if for $i = 1,2,\ldots,p$, $u_i(\bar{x}) > 0$ if $g_i(\bar{x},\bar{y}) = 0$ and $u_i(\bar{x}) = 0$ if $g_i(\bar{x},\bar{y}) < 0$.

Second Order Sufficient Condition(SOSC): We say that SOSC satisfied at $(\bar{x},\bar{y})$ if, the inequality; $z^T\nabla^2_{xx}L(y(\bar{x}),u(\bar{x}),v(\bar{x}),\bar{x})z > 0$ holds true for each $z \neq 0$ such that

$$z^T \nabla_x g_i(\bar{x}, \bar{y}) = 0 \quad \text{for all } i \in I(\bar{x}, \bar{y}) \quad \text{for which } u_i(\bar{x}) > 0,$$

$$z^T \nabla_x g_i(\bar{x}, \bar{y}) \geq 0 \quad \text{for all } i \in I(\bar{x}, \bar{y}) \quad \text{for which } u_i(\bar{x}) = 0,$$

$$z^T \nabla_x h_j(\bar{x}, \bar{y}) = 0 \quad \text{for all } j = 1, 2, \ldots, q.$$

Strong Second Order Sufficient Condition(SSOSC) The SSOSC is said to satisfied at $(\bar{x}, \bar{y})$ if, for each $(u, v) \in \Lambda(\bar{x}, \bar{y})$ and for each $d \neq 0$ satisfying $\nabla_x g_i(\bar{x}, \bar{y})d = 0$ for all $i \in I(\bar{x}, \bar{y})$ for which $u_i(\bar{x}) > 0$, and $z^T \nabla_x h_j(\bar{x}, \bar{y}) = 0$ for all $j = 1, 2, \ldots, q$, we have

$$d^T \nabla^2_{xx} L(y(\bar{x}), u(\bar{x}), v(\bar{x}), \bar{x}) z > 0$$

Constant Rank Constraint Qualification(CRCQ): The CRCQ is said to satisfied for problem $l(x)$ at a point $(\bar{x}, \bar{y})$ if there exists an open neighborhood $B((\bar{x}, \bar{y}); \epsilon)$, $\epsilon > 0$ such that for each subsets $I \subseteq I(\bar{x}, \bar{y})$ and $J \subseteq \{1, 2, \ldots, q\}$, the family of gradient vectors

$$\{\nabla_y g_i(x, y) \mid i \in I\} \bigcup \{\nabla_y h_j(x, y) \mid j \in J\}$$

has the same rank for all $(x, y) \in B((\bar{x}, \bar{y}); \epsilon)$

Remark

- It is well-known that, under an appropriate constraint qualification(e.g (MFCQ) or (LICQ)[(LICQ) implies (MFCQ)]), if, $\bar{y}$ is a local optimal solutions of $l(x)$, then the Karush-Kuhn-Tuker Condition(KKT) at $\bar{x}$ with $(\bar{u}, \bar{v})$ holds: i.e., $\bar{y} \in K(\bar{x})$ and $(\bar{u}, \bar{v}) \in \Lambda(\bar{x}, \bar{y})$.
- Let functions f, g and h be continuously differentiable on $\{\bar{x}\} \times \Re^m$ and $\bar{y}$ be a locally optimal solution with $x = \bar{x}$.
  (a) If (MFCQ) holds at $\bar{y}$, then $\Lambda(\bar{x}, \bar{y})$ is a nonempty, convex compact set.
  (b) If (LICQ) holds at $\bar{y}$, then $\Lambda(\bar{x}, \bar{y})$ is a nonempty, singleton set.
  (c) If the problem $l(x)$ is partially convex and (MFCQ) holds at $\bar{y}$, then $\Lambda(\bar{x}, \bar{y})$ is not empty for any $\bar{y} \in S(x)$, and $\Lambda(\bar{x}, \bar{y})$ does not depend on $\bar{y} \in S(x)$. Also we have $S(x) = \{\, y \in K(x) \mid \Lambda(\bar{x}, \bar{y}) \neq \emptyset \,\}$.

Historical Note on Sensitivity Analysis and Stability Theory: Overview

Sensitivity and stability analysis has its origin in parametric nonlinear programming which was pioneered by Fiacco and McCormick. Under the LICQ, the SCS and SOSC, Fiacco and McCormick considered the KKT system of a parametric nonlinear problem and employed the classical implicit function theorem to obtain the existence of a locally unique, F-differentiable solution function. See [23] for more details. Robinson [73, and the reference therein] introduced the generalized equations and used them for the study of parametric nonlinear problem. Among the prominent contributions in these papers was the concept of a strongly regular solution of a generalized equation which has dominated much of the research in this area since its introduction. This theory made no use of the SCS required by Fiacco and McCormick. Thus the implicit function was not F-differentiable but B-differentiable. Robinson's work [73] and Kojimas's fundamental paper [38] introduced the strongly stable stationary points of parametric nonlinear programming problem and provide a complete characterization of such points under the MFCQ and SOSC. For more details and development of sensitivity analysis and stability theory, reader is referred to [8, 14, 17, 18, 23, 37, 40, 59, 73, 75, 95].

Appendix B

Properties of Marginal Functions

Let $K : \Re^n \rightrightarrows \Re^m$ be a multivalued mapping and let $f : \Re^n \times \Re^m \to \Re$ be a function. We define so-called *marginal functions* (or *optimal value functions*) $\nu : \Re^n \to \Re \bigcup \{\pm\infty\}$ and $\mu : \Re^n \to \Re \bigcup \{\pm\infty\}$ defined by,

$$\nu(x) := \begin{cases} \inf\{\, f(x,y) \mid y \in K(x)\,\} & \text{if } K(x) \neq \emptyset \\ \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\mu(x) := \begin{cases} \sup\{\, f(x,y) \mid y \in K(x)\,\} & \text{if } K(x) \neq \emptyset \\ \\ -\infty, & \text{otherwise.} \end{cases}$$

Together with we define the solution set mapping $S, M : \Re^n \rightrightarrows \Re^m$ by

$$S(x) := \operatorname{argmin}\{\, f(x,y) \mid y \in K(x)\,\}, \quad \text{and} \quad M(x) := \operatorname{argmax}\{\, f(x,y) \mid y \in K(x)\,\}.$$

Continuity of Marginal Functions

Proposition [51, Lemma 3.18]

1 Let the mapping K be *lsc* at the point $\bar{x}$ and

(a) the function f be *usc*, then the function ν is *usc* at $\bar{x}$;

(b) the function f be *lsc*, then the function μ is *lsc* at $\bar{x}$.

2 Let the mapping K be *usc* and uniformly bounded at the point $\bar{x}$ and

(a) the function f be *lsc*, then the function ν is *lsc* at $\bar{x}$;

(b) the function f be *usc*, then the function μ is *usc* at $\bar{x}$.

3 If the mapping K is continuous and uniformly bounded at the point $\bar{x}$ and the function f is continuous, then the functions ν and μ are continuous at $\bar{x}$.

Proposition[26, Theorem 3.3]

For $l(x)$, suppose that f is continuous on $N(\bar{x}) \times \Re^m$, g and h are continuously differentiable on $N(\bar{x}) \times \Re^m$. Also assume that $K(x)$ is non-empty and uniformly compact near $\bar{x}$. Then $\mu(x)$ is continuous at $\bar{x}$, provided that the MFCQ holds at $\bar{y} \in K(\bar{x})$.

Lipschitz Properties of Marginal Functions

Proposition

[51] If the mapping K is Lipschitz continuous on the set $\Omega \subset \Re^n$ (with Lipschitz constant l_1) and the function f is Lipschitz continuous on $\Omega \times K(\Omega)$ (with constant l_2), then the functions ν and μ are Lipschitz continuous on Ω with the constant $l = (l_1 + 1)l_2$.

[26] For problem $l(x)$, suppose that the functions f, g and h are continuously differentiable on $\{\bar{x}\}$ and the set $\{\, (x, y) \in \Re^n \times \Re^m \mid g(x, y) \leq 0, h(x, y) = 0 \,\}$ is nonempty and compact as well as MFCQ are satisfied at all points (x, y) with $y \in K(\bar{x})$. Then the optimal value function $\nu(x)$ is locally Lipschitz continuous at $\bar{x}$.

The above compactness assumption can be replaced by the inf-compactness assumption.

Inf-Compactness Assumption: There exist a number α, a compact set $C \subset \Re^m$, and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $\nu(\bar{x}) < \alpha$ and the set $\{\, y \mid f(x,y) \leq \alpha$ and z is feasible for $l(x) \,\} \subset C$ for all x in $N(\bar{x})$.

Proposition [33, Proposition 1] Suppose the inf-compactness assumption holds and LICQ holds at each $y \in S(\bar{x})$. Then the optimal value function $\nu(x)$ is Lipschitz continuous near $\bar{x}$.

We can further weaken the above assumption and obtain the Lipschitz continuity of value functions.

Inner-Semicontinuous: Given a set valued mapping $S : \Re^n \rightrightarrows \Re^m$ and a function $\mu : \Re^n \to \bar{\Re}$, we say that S is *μ-inner-semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ if for any sequence $x_k \xrightarrow{\mu} \bar{x}$, there is a sequence $y_k \in S(x_k)$ that contains a subsequence converging to $\bar{y}$.

Inner-Semicompact: Given a set valued mapping $S : \Re^n \rightrightarrows \Re^m$ and a function $\mu : \Re^n \to \bar{\Re}$, we say that S is *μ-inner-semicompact* at $\bar{x}$ with $S(\bar{x}) \neq \emptyset$ if for any sequence $x_k \xrightarrow{\mu} \bar{x}$, there is a sequence $y_k \in S(x_k)$ containing a convergent subsequence.

See [61, Theorem 5.2] for Lipschitz continuity of the marginal function under the above inner semicontinuity assumption.

Convexity of Optimal Value Function

convex: The point-to-set mapping $R : \Re^n \rightrightarrows \Re^m$ is called *convex* on a convex set $S \subset \Re^n$ if the set gph $R \cap (S \times \Re^m)$ is convex, or equivalently if for all x_1, $x_2 \in S$ and $\lambda \in (0, 1)$, one has $\lambda R(x_1) + (1 - \lambda)R(x_2) \subset R(\lambda x_1 + (1 - \lambda)x_2)$.

essentially convex: We introduce a slight extension of the previous notion and call $R : \Re^n \rightrightarrows \Re^m$ *essentially convex* on a convex set $S \subset \Re^n$ if for all x_1,$x_2 \in S$, $x_1 \neq x_2$, and $\lambda \in (0, 1)$, one has $\lambda R(x_1) + (1 - \lambda)R(x_2) \subset R(\lambda x_1 + (1 - \lambda)x_2)$.

It is clear that if R is convex on S, then it is essentially convex on S. Also a convex map R on S is convex-valued on S, i.e., its value $R(x)$ at $x \in S$ is a convex set.

Proposition: [24, Proposition 2.1] Consider the general parametric optimization problem $l(x)$. If f is jointly convex on the set $\{\, (x, y) \mid y \in K(x), x \in \Omega\}$, K is essentially convex on S, and Ω is convex, then μ is convex on Ω.

Concavity of Optimal Value Function

Definition: The point-to-set mapping $R : \Re^n \rightrightarrows \Re^m$ is called concave on a convex set $S \subset \Re^n$ if the set gph$R \cap (S \times \Re^m)$ is concave, or equivalently if for all x_1, $x_2 \in S$ and $\lambda \in (0, 1)$, one has $\lambda R(x_1) + (1 - \lambda)R(x_2) \supset R(\lambda x_1 + (1 - \lambda)x_2)$.

Proposition [24, Proposition 3.1] Consider the general parametric optimization problem (P_x). If f is jointly concave on $\Re^n \times \Omega$, K is convex on S, and Ω is a convex set, then μ is concave on Ω.

Appendix C

Properties of Feasible Sets and Solution Sets

Continuity of Set-Valued Mappings

(B-l.s.c) A set valued mapping $S : X \rightrightarrows Y$ is called lower semicontinuous at $\bar{x}$ in the sense of Berge (B-l.s.c) if for any open set $V \subset Y$ such that $V \cap S(x) \neq \emptyset$, there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $S(x) \cap V \neq \emptyset$. Or equivalently, $S(\bar{x}) \subset \liminf_{x \to \bar{x}} S(x)$, where

$$\liminf_{x \to \bar{x}} S(x) := \{\, y \in Y \mid \forall\, x_k \to \bar{x},\ \exists y_k :\ y_k \in F(x_k), k = 1, 2, \ldots,\ y_k \to y\}.$$

(H-l.s.c) A set valued mapping $S : X \rightrightarrows Y$ is called lower semicontinuous at $\bar{x}$ in the sense of Hausdorff (H-l.s.c) if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $S(\bar{x}) \subset S(x) + \epsilon B$ for all $x \in \bar{x} + \delta B$.

(B-u.s.c) A set valued mapping $S : X \rightrightarrows Y$ is called upper semicontinuous at $\bar{x}$ in the sense of Berge (B-u.s.c) if for any open set $V \subset Y$ satisfying $S(\bar{x}) \subset V$, there exists $\delta > 0$ such that, for every $x \in B(\bar{x}, \delta)$, $S(x) \subset V$. Or equivalently, $\limsup_{x \to \bar{x}} S(x) \subset S(\bar{x})$, where

$$\limsup_{x \to \bar{x}} S(x) := \{\, y \in Y \mid \exists\, x_k \to \bar{x},\ \exists y_k :\ y_k \in F(x_k), k = 1, 2, \ldots,\ y_k \to y\}.$$

(H-u.s.c) A set valued mapping $S : X \rightrightarrows Y$ is called upper semicontinuous at $\bar{x}$ in the sense of Hausdorff (H-u.s.c) if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $S(x) \subset S(\bar{x}) + \epsilon B$ for all $x \in \bar{x} + \delta B$.

Remark

- If S is B-u.s.c at every point $x \in X$, then it is called a closed mapping. As can be easily proved, the closedness of mapping S is equivalent to the closedness of gph$S$ in $X \times Y$.

- From the definitions it follows that if S is H-l.s.c at $\bar{x}$ then S is B-l.s.c at $\bar{x}$.

Proposition Let the multivalued mapping $S : X \to CS(Y)$ be uniformly bounded at the point $\bar{x}$, where $CS(Y)$ denote the family of all non-empty compact subsets of the space Y. Then the mapping S is H-l.s.c (H-u.s.c) at the point $\bar{x}$ if and only if it is B-l.s.c (B-u.s.c) at this point.

For more detail see [36], [51], and references therein.

Properties of Feasible Sets

Proposition

- If functions g and h are continuous on $\{\bar{x}\} \times \Re^m$, then the constraint map $K(x)$ is a closed map at $\bar{x}$.

- [79, Remark 6.3.1] Suppose that the functions g and h are continuous on $\{\bar{x}\} \times \Re^m$ and h is continuously differentiable in y on $N(\bar{x}) \times \Re^m$, where $N(\bar{x})$ is a neighborhood of $\bar{x}$. Also assume that the rank $\nabla_y h(\bar{x}, y) = q$ for any y satisfying $h(\bar{x}, y) = 0$ and $K(\bar{x}) \neq \emptyset$, and $\text{cl}K^{-}(\bar{x}) = K(\bar{x})$, where $K^{-}(\bar{x}) = \{\, y \in Y \mid g(x, y) < 0, \quad h(x, y) = 0 \,\}$. Then the constraint map $K(x)$ is continuous at $\bar{x}$.

- Let the multivalued mapping K be nonempty, convex and uniformly bounded at the point $\bar{x}$. Then K is Lipschitz continuous in a neighborhood of $\bar{x}$.

Definition: (Uniformly Compactness) The set valued mapping $K : \Re^n \rightrightarrows \Re^m$ is said to be *uniformly compact* near $\bar{x} \in \Re^n$, if for any $\bar{x} \in \Re^n$, there exist a ball $B(\bar{x}, \epsilon) = \{\, x \in \Re^n \mid \| \, x - \bar{x} \, \| \leq \epsilon \,\}$, such that $\text{cl}(\cup\{\, K(x) \mid x \in B(\bar{x}, \epsilon) \,\})$ is compact.

Proposition [26, Proposition 3.1] Let K is uniformly compact near $\bar{x} \in \Re^n$. Then K is closed at $\bar{x}$ if and only if $K(x)$ is compact and K is upper semicontinuous at $\bar{x}$.

Definition: (Quasimonotonic) The function ϕ is called *quasimonotonic* on a convex set $M \in \Re^n$, if ϕ is both quasiconvex and quasiconcave on M, i.e., if the sets $E(\alpha) = \{\, x \in M \mid \phi(x) = \alpha \,\}$ are convex for all $\alpha \in \bar{\Re}$.

Proposition [24, Proposition 2.3] Let $K(x) = \{\, y \in Y \mid g(x, y) \leq 0, \quad h(x, y) = 0 \,\}$. If g is jointly quasiconvex on $Y \times \Omega$, h is jointly quasimonotonic on $Y \times \Omega$, and Y and Ω are convex sets, then K is convex on Y.

Properties of Optimal Solutions Map

Proposition

[79, Theorem 6.3.5] If f is continuous on $\{\bar{x}\} \times \Re^m$, and if $K(\bar{x})$ is continuous map at $\bar{x}$, then $S(\bar{x})$ is a closed map at $\bar{x}$.

[79, Theorem 6.3.4] If f is continuous on $\{\bar{x}\} \times \Re^m$, and if $K(\bar{x})$ is a nonempty compact set, then $S(\bar{x})$ is a nonempty compact set.

Proposition [51]

- Let the multivalued mapping K be continuous and uniformly bounded at the point $\bar{x}$, and let the function f be continuous. Then the mappings $S(\cdot)$ and $M(\cdot)$ are *usc* at $\bar{x}$.

- Let the multivalued mapping K be *usc* at the point $\bar{x}$, let ν be *usc* at $\bar{x}$ (μ be *lsc* at $\bar{x}$ resp.) and f be *lsc*.(*usc*., resp.). Then the mappings $S(\cdot)$ and $M(\cdot)$ are *usc* at $\bar{x}$.

- [8, Theorem 4.3] If the set $K(x) = \{\, (x, y) \in \Re^n \times \Re^m \mid g(x, y) \leq 0, \quad h(x, y) = 0 \,\}$ is non-empty and compact, then $S(x) \neq \emptyset$ as well as $S_{loc}(x) \neq \emptyset$ are compact sets for each $x \in \{\, z \mid K(x) \neq \emptyset \,\}$. In addition if MFCQ is satisfied, then $S(x)$ is usc.

Proposition [36]

Let $U(A, \epsilon) := \{\, x \in X \mid \exists\, a \in A,\ \|a - x\| < \epsilon \,\}$ and let us make the following assumption. Assumption (H1): for any $\epsilon > o$ there exists $\alpha > 0$ and $\delta > 0$ such that for all $x \in B(\bar{x}, \delta)$ and $y \in \Delta(x, \epsilon) := (K(x) - U(S(x), \epsilon))$, one has $f(x, y) \geq \nu(x) + \alpha$.

- Suppose that $K(\cdot)$ is uniformly closed, bounded, and nonempty near $\bar{x}$. Also assume that $K(\cdot)$ is B-l.s.c at $\bar{x}$. Then $S(\cdot)$ is H-l.s.c at $\bar{x}$ if and only if (H1) holds.

- Suppose that the lower level problem (l_x) is partially convex for every $x \in \Omega$ and the

Slater conditions holds at $\bar{x}$. Also assume that $K(\cdot)$ is uniformly closed, bounded and nonempty near $\bar{x}$. Then $S(\cdot)$ is H-l.s.c at $\bar{x}$ if and only if (H1) holds.

For more details reader is referred to [36], [93], and the references therein.

Proposition [8]

- For convex parametric optimization problems, $S(x)$ is closed and convex, but possibly empty. Also $\operatorname{gph} S$ need not be convex; see [13, pp:58-59].

- Consider the problem $l(x)$. If MFCQ is satisfied at each point (x, y) with $y \in S_{loc}(x)$, then $S(x) \subset S_{loc}(x) \subset \{\, y \in K(x) \mid \Lambda(x, y) \neq \emptyset \,\}$. The equality hold, i.e., $S(x) = \{\, y \in K(x) \mid \Lambda(x, y) \neq \emptyset \,\}$, if in addition P_x is a convex parametric optimization problem.

Lipschitz Property and Coderivative of Feasible set map and Solution set Map

The reader is referred to [59] and [75] for the Lipschitz property of solution set map and the coderivative constructions.

Definition (strict derivative [75])

For a mapping $M := \Re^n \rightrightarrows \Re^m$, the *strict derivative mapping* $D_*M(\bar{x}, \bar{y}) : \Re^n \rightrightarrows \Re^m$ for M at $\bar{x}$ for $\bar{y}$, where $\bar{u} \in M(\bar{x})$, is defined by

$$D_*M(\bar{x}, \bar{y})(u) := \{\, z \mid \exists \tau_n \downarrow 0, (x_n, y_n) \underset{gphM}{\longrightarrow} (\bar{x}, \bar{y}), u_n \to u,$$

$$\text{with } z_n \in [M(x_n + \tau_n u_n) - y_n]/\tau_n, z_n \to z \,\}$$

When the multifunction happens to be a Lipschitzian mapping $\tilde{y} : \Re^n \to \Re^m$ near $\bar{x}$, the strict derivative at $\bar{x}$ is written as $D_*\tilde{y}(\bar{x})$ and can be obtained by

$$D_*\tilde{y}(\bar{x})(w) = \left\{ z \mid \exists\, x_n \to \bar{x}, \tau_n \downarrow 0 \text{ with } \frac{F(x_n + \tau_n w) - F(x_n)}{\tau_n} \to z \right\}$$

For Lipschitzian mapping $\tilde{y}$, single-valuedness of the strict derivative mapping $D_*\tilde{y}(\bar{x})$ corresponds to the classical notion of strict differentiability (hence the name "strict derivative mapping"), and for continuously differentiable mappings, the strict derivative coincides with the Jacobian.

> Most practical questions can be reduced to problems of largest and smallest magnitudes...and it is only by solving these problems that we can satisfy the requirements of practice which always seeks the best, the most convenient.
>
> ∼ P.L. Čebyšew

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ABSTRACT

METHODS OF VARIATIONAL ANALYSIS IN PESSIMISTIC BILEVEL PROGRAMMING

by

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Advisor: Boris S. Mordukhovich

Major: Mathematics

Degree: Doctor of Philosophy

Bilevel programming problems are of growing interest both from theoretical and practical points of view. These models are used in various applications, such as economic planning, network design, and so on. The purpose of this dissertation is to study the pessimistic (or strong) version of bilevel programming problems in finite-dimensional spaces. Problems of this type are intrinsically nonsmooth (even for smooth initial data) and can be treated by using appropriate tools of modern variational analysis and generalized differentiation developed by B. Mordukhovich.

This dissertation begins with analyzing pessimistic bilevel programs, formulation of the problems, literature review, practical application, existence of the optimal solutions, reformulation and related to the other programming. The mainstream in studying optimization problems consists of obtaining necessary optimality conditions for optimality, and the main focus of this dissertation is to obtain necessary optimality conditions for pessimistic bilevel programming problems. Optimality conditions for the optimistic ver-

sion of bilevel programming are extensively discussed in the literature. However, there are just a few papers devoted to the pessimistic version of bilevel programming problems and most of these papers concern the existence of optimal solutions. This dissertation is devoted to establish, by a variety of techniques from convex and nonsmooth analysis, several versions of first order necessary and sufficient optimality conditions for pessimistic bilevel programming problems.

To achieve our goal, we first use the implicit programming techniques, and depending on the continuous, Lipschitz, and Fréchet differentiable selections, we obtain necessary optimality conditions. The value function technique plays a central role in sensitivity analysis, controllability, and even in establishing necessary optimality conditions. We consider constructions or estimations of the subdifferential of value functions and come up with the optimality conditions using minimax programming approach treating the cases: convex data, differentiable (strict) data, and Lipschitz data separately. We also use the duality programming approach and obtain optimality conditions extending the convex case to the nonconvex case. In the last chapter, we produce the necessary optimality conditions for pessimistic bilevel programming with the rational reaction (optimal solutions set of the lower level problem) set of finite cardinality, but not singleton. We present then some classes of pessimistic bilevel programs for which there are finite rational responses.

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