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Embedding Theorems for Sobolev-Besicovitch spaces $W^{k,1}_{ap}(\mathbb{R}^s)$

G. DELL'ACQUA – P. SANTUCCI

RIASSUNTO: *Si dimostrano teoremi di inversione di tipo Sobolev per spazi di Sobolev-Besicovitch $W^{k,q}_{ap}$ di funzioni quasi-periodiche con $q \in [1,2]$. Lo strumento fondamentale per la dimostrazione del teorema principale è il teorema di Hausdorff-Young per funzioni quasi-periodiche*

ABSTRACT: *We show embedding theorems of Sobolev type for Sobolev-Besicovitch spaces $W^{k,q}_{ap}$ of almost periodic functions with $q \in [1,2]$. The fundamental tool for the proof of the main theorem is the Hausdorff-Young theorem for a.p. functions.*

1 – Introduction

In this paper, we prove embedding theorems for Sobolev-Besicovitch spaces $W^{k,q}_{ap}(\mathbb{R}^s)$ of almost periodic B^q_{ap}-functions, $\forall\, q \in [1,2]$. This subject was already dealt with, in the case $1 < q \leq 2$, as a consequence of embedding theorems for Sobolev-Besicovitch spaces $H^{k,q}_{ap}(\mathbb{R}^s)$ (see [9]). Here we prove embeddings for $W^{k,q}_{ap}(\mathbb{R}^s)$ spaces in a direct way, involving not the $H^{k,q}_{ap}(\mathbb{R}^s)$ spaces, but the Hausdorff-Young theorem. We remark that the case $q = 1$, stated in this paper, is not included in [9]. Indeed,

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the embedding theorem we prove for $W^{k,1}_{ap}(\mathbb{R}^s)$ spaces cannot be obtained via the H-spaces, as the latter are defined for $q > 1$ only.

In section 2 we recall some basic notations, definitions and properties of $B^q_{ap}(\mathbb{R}^s)$ and $W^{k,q}_{ap}(\mathbb{R}^s)$ spaces.

In section 3 we prove the main theorem, and make some remarks.

2 – Notations and definitions

For any $s \in \mathbb{N}$ let $\mathcal{P}(\mathbb{R}^s)$ denote the complex vector space of all trigonometric polynomials in s variables, that is $P \in \mathcal{P}(\mathbb{R}^s) \Leftrightarrow \exists\, \omega \in \mathbb{N}$, $\exists\, c_1, \ldots, c_\omega \in \mathbb{C}$ and $\exists\, \lambda^1, \ldots, \lambda^\omega \in \mathbb{R}^s$ such that $\lambda^1, \ldots, \lambda^\omega$ are distinct and

$$P(x) = \sum_{j=1}^{\omega} c_j e^{i\lambda^j \cdot x} \qquad \forall\, x \in \mathbb{R}^s\,, \tag{2.1}$$

where "•" represents the usual inner product in $\mathbb{R}^s$.

If every $c_j (j = 1, \ldots, \omega)$ is different from zero, the set

$$\sigma(P) := \{\lambda^1, \ldots, \lambda^\omega\}$$

is called the spectrum of P, and the map

$$\lambda \to a(\lambda; P) := \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} P(x) e^{-i\lambda\cdot x} dx = \begin{cases} c_j & \text{if } \lambda = \lambda^j \text{ for some } j \\ 0 & \text{if } \lambda \notin \sigma(P) \end{cases}$$

is called the Bohr-transform of P. Here $Q_T = [-T, T]^s$.

For any fixed $q \in [1, +\infty[$ we shall denote by $B^q_{ap}(\mathbb{R}^s)$ the completion of $\mathcal{P}(\mathbb{R}^s)$ with respect to the norm defined by

$$\|P\|_q := \lim_{T\to\infty} \Big(\frac{1}{|Q_T|} \int_{Q_T} |P(x)|^q\, dx\Big)^{1/q}, \qquad \forall\, P \in \mathcal{P}(\mathbb{R}^s)\,.$$

An element $f \in B^q_{ap}(\mathbb{R}^s)$ is defined by a sequence of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$ such that

$$f = \lim_n P_n \quad \text{in} \quad B^q_{ap}(\mathbb{R}^s)$$

and

$$\|f\|_q := \lim_{T\to\infty}\Big(\frac{1}{|Q_T|}\int_{Q_T}|f(x)|^q\,dx\Big)^{1/q} = \lim_{n\to\infty}\|P_n\|_q\,.$$

Recall that the space $B^\infty_{ap}(\mathbb{R}^s) := C^0_{ap}(\mathbb{R}^s)$ of all uniformly almost periodic (u.a.p.) functions is the completion of $\mathcal{P}(\mathbb{R}^s)$ with respect to the norm

$$\|P\|_\infty := \sup_{x\in\mathbb{R}^s}|P(x)|, \qquad \forall\, P\in\mathcal{P}(\mathbb{R}^s)\,. \tag{2.2}$$

For these spaces we have the following chain of continuous embeddings and inequalities, where $q_1, q_2 > 1$ and $q_1 < q_2 < +\infty$:

$$\begin{gathered} C^0_{ap}(\mathbb{R}^s) = B^\infty_{ap}(\mathbb{R}^s) \subset B^{q_2}_{ap}(\mathbb{R}^s) \subset B^{q_1}_{ap}(\mathbb{R}^s) \subset B^1_{ap}(\mathbb{R}^s)\,,\\ \|f\|_\infty \ge \|f\|_{q_2} \ge \|f\|_{q_1} \ge \|f\|_1\,. \end{gathered} \tag{2.3}$$

For any $f\in B^q_{ap}(\mathbb{R}^s)$ the map

$$\lambda \to a(\lambda; f) := \lim_{T\to\infty}\frac{1}{|Q_T|}\int_{Q_T} f(x)e^{-i\lambda\cdot x}\,dx$$

will be called the Bohr-transform of f.

We will call the subset of $\mathbb{R}^s$, $\sigma(f)$, defined by

$$\sigma(f) := \{\lambda\in\mathbb{R}^s | a(\lambda; f)\neq 0\} \tag{2.4}$$

the spectrum of the function $f\in B^q_{ap}(\mathbb{R}^s)$. The members of $\sigma(f)$ will be called the Fourier exponents of f.

For any $f\in B^q_{ap}(\mathbb{R}^s)$ one has:

$$\lim_{|\lambda|\to+\infty} a(\lambda; f) = 0\,; \tag{2.5}$$

$$\sigma(f) \quad \text{is at most countable}\,; \tag{2.6}$$

$$\sigma(f) = \emptyset \Leftrightarrow a(\lambda; f) = 0\ \forall\,\lambda\in\mathbb{R}^s \Leftrightarrow f = 0 \in B^1_{ap}(\mathbb{R}^s)\,. \tag{2.7}$$

Let us recall Hausdorff-Young theorem for B^q_{ap} spaces, which will be used later (for a proof, see [3], [4], [7]).

THEOREM 2.1 (Hausdorff-Young). *If* $f \in B^q_{ap}(\mathbb{R}^s)$ *then*

$$\Big(\sum_{\lambda \in \sigma(f)} |a(\lambda; f)|^{q'} \Big)^{1/q'} \leq \|f\|_q \quad \text{if} \quad q \in]1, 2] \tag{2.8}$$

and

$$\|f\|_q \leq \Big(\sum_{\lambda \in \sigma(f)} |a(\lambda; f)|^{q'} \Big)^{1/q'} \quad \text{if} \quad q \in [2, +\infty[\,. \tag{2.9}$$

Here $q' = \frac{q}{q-1}$, *and the series in* (2.9) *need not converge.*

In the subsequent sections we will use the following inequalities as well:

A) $\sum_{i=1}^{\nu} a_i^r \leq \Big(\sum_{i=1}^{\nu} a_i \Big)^r \leq 2^{(\nu-1)(r-1)} \Big(\sum_{i=1}^{\nu} a_i^r \Big) \quad \forall\, r \geq 1, a_i \geq 0.$

B) For any multi-index $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$ set $|\alpha| := \alpha_1 + \dots + \alpha_s$; moreover, set $x_j^{\alpha_j} = 1$ if $x_j = \alpha_j = 0$ and $(x)^\alpha := x_1^{\alpha_1} \dots x_s^{\alpha_s}$ for any $x \in \mathbb{R}^s$. Then $\exists\, p_0, p_1 \in \mathbb{R}_+$ s.t. $\forall\, \lambda \in \mathbb{R}^s$, $\nu \in \mathbb{N}_0$

$$p_0 |\lambda|^{2\nu} \leq \sum_{|\alpha| = \nu} |(\lambda)^\alpha|^2 \leq p_1 |\lambda|^{2\nu} \,.$$

DEFINITION 2.1. (i) *For any* $q \in [1, +\infty]$, $k \in \mathbb{N}_0$ *and* $P \in \mathcal{P}(\mathbb{R}^s)$ *we set*

$$\begin{aligned} \|P\|_{W^{k,q}} &:= \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha P\|_q^q \Big)^{1/q} \,, \\ \|P\|_{W^{k,\infty}} &:= \sum_{|\alpha| \leq k} \|\partial^\alpha P\|_\infty \,. \end{aligned} \tag{2.10}$$

Here $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_s^{\alpha_s}$ *and* $\partial_j = \frac{\partial}{\partial x_j}$. *Equations* (2.10) *define norms on* $\mathcal{P}(\mathbb{R}^s)$ *and we have* $\|P\|_{W^{0,q}} = \|P\|_q$.

(ii) *For any* $q \in [1, +\infty]$ *we shall denote by* $W^{k,q}_{ap}(\mathbb{R}^s)$ *the completion of* $\mathcal{P}(\mathbb{R}^s)$ *with respect to the norm* $\| \cdot \|_{W^{k,q}}$. *These spaces are called Sobolev-Besicovitch spaces of order* k *and type* B^q.

We define a norm in the space $W^{k,q}_{ap}(\mathbb{R}^s)$ *in the following way:*

$$\|f\|_{W^{k,q}} := \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_q^q \Big)^{1/q} \,. \tag{2.11}$$

One can easily prove that the norm (2.11) is equivalent to

$$\sum_{|\alpha|\le k} \|\partial^\alpha f\|_q \,. \tag{2.12}$$

In what follows, we shall use both norms (2.11) and (2.12).

Clearly, $W^{k,q}_{ap}(\mathbb{R}^s) \subseteq B^q_{ap}(\mathbb{R}^s)\ \forall\, k \ge 0,\ \forall\, q \ge 1$.

According to definition (ii), an element f of $W^{k,q}_{ap}(\mathbb{R}^s)$ is defined by a sequence $(P_n)_{n\in\mathbb{N}}$ of trigonometric polynomials converging to f in $B^q_{ap}(\mathbb{R}^s)$, such that $(\partial^\alpha P_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B^q_{ap}(\mathbb{R}^s)$ for any multi-index α with $|\alpha| \le k$. Since the space $B^q_{ap}(\mathbb{R}^s)$ is complete, we can define an element f_α in $B^q_{ap}(\mathbb{R}^s)$ by

$$f_\alpha := \lim_n \partial^\alpha P_n \,, \tag{2.13}$$

and we call f_α the strong α-derivative of f and write

$$\partial^\alpha f := f_\alpha \,.$$

Observe that for any $\varphi \in C^\infty_{ap}(\mathbb{R}^s)$ an integration by parts gives

$$\lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} (\partial^\alpha P_n(x))\varphi(x)dx = (-1)^{|\alpha|} \lim_{T\to\infty} \frac{1}{|Q_T|} \int_{Q_T} P_n(x)\partial^\alpha \varphi(x)dx \,,$$

since an u.a.p. function is bounded on $\mathbb{R}^s$, so that the asymptotic means on the boundary vanish.

To each function $f \in B^q_{ap}(\mathbb{R}^s)$ we associate formally the Bohr-Fourier series

$$f \sim \sum_{\lambda\in\sigma(f)} a(\lambda; f) e^{i\lambda\cdot x} \,. \tag{2.14}$$

Let us show the relationship between the Bohr-Fourier series of f and f_α. Let $f \in W^{k,q}_{ap}(\mathbb{R}^s)$, $P_n \to f$ in B^q_{ap} and $|\alpha| \le k$. For any $\lambda \in \mathbb{R}^s$ we can write

$$a(\lambda; f_\alpha) = \lim_n a(\lambda; \partial^\alpha P_n) = \lim_n \lim_{T\to+\infty} \frac{1}{|Q_T|} \int_{Q_T} (\partial^\alpha P_n(x) e^{-i\lambda\cdot x}) dx =$$

$$= \lim_n \lim_{T\to\infty} (-1)^{|\alpha|} (\lambda)^\alpha \frac{1}{|Q_T|} \int_{Q_T} P_n(x)\partial^\alpha \varphi(x)dx = i^{|\alpha|}(\lambda)^\alpha a(\lambda; f) \,.$$

It follows that f_α has the same Fourier exponents as f, expect $\lambda = 0$ if 0 is in the spectrum of f. Moreover the Fourier coefficients of f and f_α are related by

$$a(\lambda, f_\alpha) = i^{|\alpha|}(\lambda)^\alpha a(\lambda; f), \qquad \forall\, \lambda \in \sigma(f)\,. \tag{2.15}$$

Therefore, we have

$$f_\alpha(x) \sim \sum_{\lambda\in\sigma(f)} i^{|\alpha|} a(\lambda; f) e^{i\lambda\cdot x}\,. \tag{2.16}$$

Observe that, when f_α represents the ordinary derivative of f, the Bohr-Fourier series of f_α coincides with the right-hand side of (2.16).

3 – Embedding theorems

Suppose $\Lambda \subseteq \mathbb{R}^s \setminus \{0\}$, card$\Lambda$ = card$\mathbb{N}$ and that Λ has only one limit point, the point at infinity. Let us restrict ourselves to the case where the elements of Λ can be ordered with respect to the absolute value, that is to say

$$\Lambda = \{\lambda^1, \lambda^2, \dots, \lambda^j, \dots\} \quad \text{with} \quad |\lambda^1| \le |\lambda^2| \le \ldots \le |\lambda^j| \le \ldots\,.$$

Finally, let us suppose that there exists $\beta \ge 0$ such that

$$\sum_{\lambda\in\Lambda} \frac{1}{|\lambda|^\gamma} < +\infty \qquad \forall\, \gamma > \beta\,. \tag{3.1}$$

Let us set

$$B^q_{ap}(\Lambda) := \{f \in B^q_{ap}(\mathbb{R}^s) : \sigma(f) \subseteq \Lambda\}\,.$$

We define analogously $W^{k,q}_{ap}(\Lambda)$, $C^0_{ap}(\Lambda)$ and $C^{0,\mu}_{ap}(\Lambda)$, where $C^{0,\mu}_{ap}(\mathbb{R}^s)$ is the space of the u.a.p. functions that are holderian with exponent μ, equipped with the usual norm.

THEOREM 3.1. *Let Λ satisfy the above conditions and let $q \in [1, 2]$. the following statements hold.*

i) *If* $kq > \beta$ *then*

$$W^{k,q}_{ap}(\Lambda) \hookrightarrow C^0_{ap}(\Lambda)\,.$$

Moreover, if $kq > \beta \geq (k-1)q$ *then*

$$W^{k,q}_{ap}(\Lambda) \hookrightarrow C^{0,\mu}_{ap}(\Lambda) \quad \forall\, \mu \in \Big[0, k - \frac{\beta}{q}\Big[\,.$$

ii) *If* $kq = \beta$ *then*

$$W^{k,q}_{ap}(\Lambda) \hookrightarrow B^r_{ap}(\Lambda) \quad \forall\, r \geq 1\,.$$

iii) *If* $kq < \beta < \frac{2kq}{2-q}$ *then*

$$W^{k,q}_{ap}(\Lambda) \hookrightarrow B^r_{ap}(\Lambda) \quad \forall\, r \in \Big[1, \frac{\beta q}{\beta - kq}\Big[\,.$$

Proof. We will prove the theorem only in the case $q = 1$.

Recall that $\sigma(f) \subseteq \Lambda$ means also that we consider functions with vanishing asymptotic mean.

i) Let us consider the Bohr-Fourier series (2.14) and (2.16) of f and f_α, for any multi-index α such that $|\alpha| \leq k$. Observe that

$$\begin{aligned} |a(\lambda; f)||(\lambda)^\alpha| &= \lim_{T\to+\infty} \frac{1}{|Q_T|}\Big| \int_{Q_T} f_\alpha(x) e^{-i\lambda\cdot x} dx\Big| \leq \\ &\leq \lim_{T\to+\infty} \frac{1}{|Q_T|} \int_{Q_T} |f_\alpha| dx = \|f_\alpha\|_1\,. \end{aligned} \tag{3.2}$$

By (3.2) and inequality (B), we get

$$\begin{aligned} \sum_{j=n+1}^{n+p} |a(\lambda^j; f)| &= \sum_{j=n+1}^{n+p} |a(\lambda^j; f)||\lambda^j|^k \cdot \frac{1}{|\lambda^j|^k} \leq \\ &\leq \frac{1}{p_0^{1/2}} \sum_{j=n+1}^{n+p} |a(\lambda^j; f)| \sum_{|\alpha|=k} |(\lambda^j)^\alpha| \frac{1}{|\lambda^j|^k} \leq \\ &\leq \frac{1}{p_o^{1/2}} \sum_{|\alpha|=k} \|f_\alpha\|_1 \sum_{j=n+1}^{n+p} \frac{1}{|\lambda^j|^k}\,. \end{aligned}$$

Since $k > \beta$ and (3.1) holds with $\gamma = k$, we obtain

$$\|f\|_\infty \leq \sum_{j=1}^{\infty} |a(\lambda; f)| \leq C_1 \|f\|_{W^{k,1}}$$

for some $C_1 \geq 0$ independent of f. Thus the Bohr-Fourier series of f is absolutely convergent and hence, as is well known, unconditionally uniformly convergent to an u.a.p. function f^* such that $\|f - f^*\|_{W^{k,1}} = 0$.

Therefore $f \in C^0_{ap}(\mathrm{I\!R}^s)$.

Recall that the usual norm of the space $C^{0,\mu}_{ap}(\mathrm{I\!R}^s)$ is given by

$$\|f\|_{C^{o,\mu}} = \|f\|_\infty + [f]_\mu \,,$$

where

$$[f]_\mu = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\mu}$$

and that the following inequality holds (see for example [3])

$$[f]_\mu \leq \sum_{j=1}^{\infty} |a(\lambda^j; f)| [e^{i\lambda^j(\cdot)}]_\mu \leq 2^{1-\mu} \sum_{j=1}^{\infty} |a(\lambda^j; f)| |\lambda^j|^\mu \,. \tag{3.3}$$

If μ satisfies $0 < \mu < k - \beta$, using (B) and (3.2) yields

$$\begin{aligned}\sum_{j=n+1}^{n+p} |a(\lambda^j; f)| |\lambda^j|^\mu &= \sum_{j=n+1}^{n+p} |a(\lambda^j; f)| |\lambda^j|^k \cdot \frac{1}{|\lambda^j|^{k-\mu}} \leq \\ &\leq \frac{1}{p_0^{1/2}} \sum_{j=n+1}^{n+p} |a(\lambda^j; f)| \sum_{|\alpha|=k} |(\lambda^j)^\alpha| \frac{1}{|\lambda^j|^{k-\mu}} \leq \\ &\leq \frac{1}{p_0^{1/2}} \sum_{|\alpha|=k} \|f_\alpha\|_1 \sum_{j=n+1}^{n+p} \frac{1}{|\lambda^j|^{k-\mu}} \leq C_2 \|f\|_{W^{k,1}} \,,\end{aligned}$$

for some $C_2 > 0$ independent of f.

As $\mu \in]0, m-\beta[\subset]0,1[$, (3.3) gives

$$[f]_\mu \leq \sum_{j=1}^{\infty} |a(\lambda^j; f)| [e^{i\lambda^j(\cdot)}]_\mu \leq \sum_{j=1}^{\infty} 2^{1-\mu} |a(\lambda^j; f)| |\lambda^j|^\mu \leq C_3 \|f\|_{W^{k,1}} ,$$

for some $C_3 > 0$ independent of f.

Therefore,

$$\|f\|_{C^{0,\mu}} = \|f\|_\infty + [f]_\mu \leq C_4 \|f\|_{W^{k,1}} ,$$

with $C_4 = C_2 + C_3$.

ii) Let us choose r such that $r > 2$, and let $r' = \frac{r}{r-1}$. Using inequalities (A), (B) and (3.2) we get

$$\begin{aligned}
\sum_{j=n+1}^{n+p} \left|a(\lambda^j; f)\right|^{r'} &= \sum_{j=n+1}^{n+p} \left|a(\lambda^j; f)\right|^{r'} \Big(\sum_{|\alpha|=k} |(\lambda^j)^\alpha| \Big)^{r'} \Big(\sum_{|\alpha|=k} |(\lambda^j)^\alpha| \Big)^{-r'} = \\
&= \sum_{j=n+1}^{n+p} \Big(\sum_{|\alpha|=k} |a(\lambda^j; f)| |(\lambda^j)^\alpha| \Big)^{r'} \Big(\sum_{|\alpha|=k} |(\lambda^j)^\alpha| \Big)^{-r'} \leq \\
&\leq \sum_{j=n+1}^{n+p} \Big(\sum_{|\alpha|=k} \|f_\alpha\|_1 \Big)^{r'} \Big(\sum_{|\alpha|=k} |(\lambda^j)^\alpha| \Big)^{-r'} \leq \\
&\leq C_5 \sum_{|\alpha|=k} \|f_\alpha\|_1^{r'} \Big(\sum_{j=n+1}^{n+p} \frac{1}{|\lambda|^{kr'}} \Big) ,
\end{aligned} \tag{3.4}$$

for some $C_5 > 0$ independent of f.

Since $kr' = \beta \frac{r}{r-1} > \beta$, we can apply (3.1) with $\gamma = kr'$. Inequality (3.4) and Hausdorff-Young theorem then give

$$\|f\|_r \leq \Big(\sum_{j=1}^{\infty} \left|a(\lambda^j; f)\right|^{r'} \Big)^{1/r'} \leq C \|f\|_{W^{k,1}}$$

for any $r > 2$ and for some $C > 0$ independent of f.

Now, the thesis follows from (2.3).

iii) Let $r > 2$ and $r' = \frac{r}{r-1}$ as before. Since $k < \beta < 2k$, it follows that

$$\Big[2, \frac{\beta}{\beta - k} \Big[\neq \emptyset .$$

As $kr' > \beta \Leftrightarrow r < \frac{\beta}{\beta-k}$, by (3.1), (3.4) and Hausdorff-Young theorem we get

$$\|f\|_r \le \Big(\sum_{j=1}^{\infty} \big|a(\lambda^j; f)\big|^{r'}\Big)^{1/r'} \le M\|f\|_{W^{k,1}}$$

for some $M > 0$ independent of f. Since (2.3) holds, the proof for the case $q = 1$ is complete.

The same technique works also when $1 < q \le 2$. However, this result has already been proved in [9] in a wider context, as a consequence of embedding theorems for the spaces $H^{k,q}_{ap}(\mathbb{R}^s)$.

REMARK 3.1. While proving the first part of Theorem 3.1, we have proved something more, *i.e.* that if $kq > \beta$ then

$$\sum_{j=1}^{\infty} |a(\lambda^j; f)| < +\infty\,.$$

This is a generalization of a result given by STEIN and WEISS [12, p.249], in the case $q = 2$, in the context of periodic functions of class C^k.

The condition $kq > \beta$ is sharp for the absolute convergence of a Fourier series. Indeed, if the dimension s is even, the series

$$\sum_{|j|>1} |j|^{-s}(\log|j|)^{-1} e^{ic|j|\log(|j|)^a} e^{2\pi i j\cdot x}$$

with $c \ne 0$ and $0 < a < \frac{2}{s}$, is the Fourier series of a function of class $C^{s/2}$, but is not absolutely convergent (see [12, p.282]).

Theorem 3.1 generalized also the classical result given in [13, p.242] for the periodic case with $s = 1$.

Under the same assumption for Λ as in Theorem 3.1, we have the following

COROLLARY 3.1. *If* $q \in [1, 2]$ *and* $k > \beta q$ *then, for any* $n \in \mathbb{N}$

$$W^{k+n,q}_{ap}(\Lambda) \subset C^n_{ap}(\Lambda)\,.$$

PROOF. For any $\alpha \in \mathbb{N}^s_0$ with $|\alpha| \le n$, f_α belongs to $W^{k,q}_{ap}(\Lambda)$, with $k > \beta q$. Hence, we have that $f \in C^n_{ap}(\Lambda)$ by Theorem 3.1.

REMARK 3.2. Under the hypothesis of Corollary 3.1, the Bohr-Fourier series of f_α is absolutely convergent, and therefore unconditionally uniformly convergent, for any α such that $|\alpha| \leq n$.

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Indirizzo degli Autori:

G. Dell'Acqua – Campo de' fiori, 22/B-00186 Roma, Italia

P. Santucci – Via Casentino, 6-50127 Firenze, Italia