Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 599-612

Property $\tilde{\Omega}$ and holomorphic functions with values in a pseudoconvex space having Stein morphism into a complex Lie group

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RIASSUNTO: *Si dimostra una condizione necessaria e sufficiente affinché uno spazio nucleare di Frechet E, dotato di una base di Schauder, abbia la proprietà $\widetilde{\Omega}$. La condizione è che esista in E un sottoinsieme B tale che, essendo E_B lo spazio di Banach generato da B e τ_E la topologia indotta su E_B dalla topologia di E, ogni funzione olomorfa definita in (E_B, τ_E), con valori in uno spazio pseudoconvesso che ammetta un morfismo di Stein su di un gruppo complesso di Lie, possa essere prolungata olomorficamente su E. Si tratta di una generalizzazione del risultato dimostrato da* MEISE *a* VOGT *in* [4] *nel caso scalare.*

ABSTRACT: *It is shown that a nuclear Frechet space E with a Schauder basis has the property $\widetilde{\Omega}$ if and only if there exists a compact balanced convex set B in E such that every holomorphic function on (E_B, τ_E), where E_B is the Banach space spanned by B and τ_E is the topology of E_B induced by the topology of E, with values in any pseudoconvex space having a Stein morphism into a complex Lie group, can be extended holomorphically to E. For the scalar case the proof was provided by* MEISE *and* VOGT [4].

Let E be a Frechet space with a fundamental system of semi-norms

KEY WORDS AND PHRASES: *Holomorphic function of uniform type, Pseudoconvex space, Stein morphism, Complex Lie group*
A.M.S. CLASSIFICATION: 22Exx, 32A10

$\|.\|_1 \leq \|.\|_2 \leq \dots$ For each $k \in \mathbb{N}$ define a norm

$$\|.\|_k^*\colon E^* \to [0, +\infty]$$

where E^* denotes the dual space of E, by

$$\|x^*\|_k^* = \sup\{|x^*(x)|\colon \|x\|_k \leq 1\}\,.$$

We say that E has the property $\widetilde{\Omega}$ if

$$\forall p \in \mathbb{N}\,, \quad \exists q \in \mathbb{N}\,, \quad d > 0\,, \quad \forall k \in \mathbb{N}\,, \quad \exists C > 0\colon$$

$$\|x^*\|_q^{*1+d} \leq \|x^*\|_k^* \|x^*\|_p^{*d}$$

for all $x^* \in E^*$.

The property $\widetilde{\Omega}$, together with other properties, were introduced and investigated by Vogt in [11] and [12]. Recently Meise and Vogt [4] gave an important characterization of a nuclear Frechet space having the property $\widetilde{\Omega}$. They proved that a nuclear Frechet space E has the property $\widetilde{\Omega}$ if and only if one of the following conditions holds:

MV_1) there exists a compact balanced convex subset B of E for which E_B is dense in E, where E_B is the Banach space spanned by B, such that every holomorphic function of E_B which has a holomorphic extension to a neighbourhood of zero in E, can be extended holomorphically to E.

MV_2) If in addition E has the bounded approximation property, there exists a compact balanced convex subset B of E for which E_B is dense E such that every holomorphic function on (E_B, τ_E), where τ_E is the topology of E_B induced by the topology of E, can be extended holomorphically to E.

In the present note we extend the result of Meise and Vogt [4] to the case of holomorphic maps with values in pseudoconvex spaces having a Stein morphism [1] into a complex Lie group. To obtain the main result (Theorem 2), in Section 1 we investigate the approximation of continuous plurisubharmonic functions on Frechet spaces namely the Bremermann–Noverraz Theorem in the Frechet case with a Schauder basis. Using the result obtained and the method of Meise and Vogt we shall prove the main result in Section 2.

1 – Approximation of continuous plurisubharmonic function in Frechet spaces

Let E be a topological vector space, and G be an open set in E. An upper-semicontinuous function $\varphi\colon G \to [-\infty, +\infty)$ is called plurisubharmonic on G if it is subharmonic on the interestion of G with every complex line in E. In [6] Noverraz proved that if G is a polynomially convex domain in a Banach space E with the approximation property then every continuous plurisubharmonic function φ on G can be written in the form

$$\text{(BN)} \qquad \varphi(z) = \lim_{b\to\infty} \max\{c_j^n \log|f_j^n(z)| : 1 \le j \le m_n\}$$

where f_j^n are holomorphic functions on E.

Moreover, the convergence is uniform on compact sets in G, i.e. is the result of Bremermann in the finite dimensional case. In this section we extend the above approximation to the Frechet space.

Theorem 1. *Let E be a Frechet space with a Schauder basis $\{e_j\}$ and let E_0 be a dense subspace of E containing $\{e_j\}$. Then E has a continuous norm if and only if every continuous plurisubharmonic function on a polynomially convex domain G in E_0 can be written in the form* (BN) *where f_j^n are holomorphic functions on E_0.*

Proof. First prove the necessity of the theorem. Assume that G is a polynomially convex domain in E_0. Write

$$G = \bigcup_{m\ge1} F_m = \bigcup_{m\ge1} \operatorname{Int} F_m$$

where F_m are closed sets in E_0.

Put for each $j \ge 1$

$$Q_j = \{z \in G\colon \|z\| < j\} \quad \text{and} \quad K_j = cl(F_j \cap Q_j \cap A_j(E))$$

where $\|.\|$ is a continuous norm on E and

$$A_j(z) = \sum_{1\le k\le j} e_k^*(z) e_k \quad \text{for every} \quad z \in E$$

where $\{e_j^*\}$ is the dual system of $\{e_j\}$.

Then $\quad K_j \subseteq F_j \cap A_j(E) \subseteq G \cap A_j(E) \quad \text{for} \quad j \geq 1\,.$

Since the topology of $A_j(E)$ is defined by $\|.\|_{|A_j(E)}$, it follows that K_j is compact in $A_j(E)$ for every $j \geq 1$. Thus, owing to the polynomial convexity of $G \cap A_j(E)$, according to the Bremermann Theorem there exist polynomials P_k^j on $A_j(E)$ and $c_k^j, 1 \leq k \leq m_j$ such that

$$\|\varphi - \psi_j\|_{K_j} < \frac{1}{j}$$

where $\psi_j(z) = \max\{c_k^j \log|P_k^j(z)|\colon 1 \leq k \leq m_j\}$.

Obviously ψ_j o A_j are plurisubharmonic functions on E. We prove that $\{\psi_j \text{ o } A_j\}_{j\geq 1}$ converges uniformly on every compact set in G to φ. Given K a compact set in G, take m_0 such that

$$K + V \subset K + cl(V) \subset \operatorname{Int} F_{m_0} \tag{1}$$

for some neighbourhood V of zero in E_0.

Since $A_j(z) \to z$ uniformly on a compact set in E, we get

$$A_j(K) \subset K + V \quad \text{for} \quad j > j_0\,. \tag{2}$$

From (1) and (2) we have

$$A_j(K) \subset F_{m_0} \subset F_j \quad \text{for} \quad j > j_1 = \max(j_0, m_0)\,. \tag{3}$$

On the other hand, since $\bigcup_{j\geq 1} A_j(K)$ is relatively compact in E and $\|.\|$ is continuous on E, it follows that

$$\bigcup_{j\geq j_1} A_j(K) \subset Q_{j_2} \quad \text{for some} \quad j_2 > j_1\,. \tag{4}$$

From (3) and (4) we have

$$A_j(K) \subset Q_j \cap F_j \cap A_j(E) \subset K_j \quad \text{for} \quad j > j_2\,.$$

Hence

$$\begin{aligned}\|\psi_j A_j - \varphi\|_K &\leq \|\psi_j A_j - \varphi A_j\|_K + \|\varphi A_j - \varphi\|_K \leq \\ &\leq \|\psi_j - \varphi\|_{A_j(K)} + \|\varphi A_j - \varphi\|_K \leq \\ &\leq \|\psi_j - \varphi\|_{K_j} + \|\varphi A_j - \varphi\|_K \leq \\ &\leq \frac{1}{j} + \|\varphi A_j - \varphi\|_K\end{aligned}$$

for $j > j_2$. This implies that $\|\psi_j A_j - \varphi\|_K \to 0$ as $j \to \infty$.

We now turn to the proof of sufficiency. Let us show that there exists a sequence $\{\lambda_j\} \subset \mathbb{C}$ such that

$$\lambda_{j_k} e_{j_k} \nrightarrow 0$$

for every subsequence $\{\lambda_{jk}\} \subset \{\lambda_j\}$.

Let D be a polynomially convex open set in $\mathbb{C}$ consisting of infinitely many components, $D = \bigcup_{j\geq 1} D_j$. We may assume that $0 \in D_1$. Put

$$G = (\bigcup_{j\geq 1} D_j e_1 + M) \cap E_0$$

where $M = cl(\mathrm{span}\{e_j\}_{j\geq 2})$.

Obviously G is polynomially convex in E_0. On G define a continuous plurisubharmonic function φ given by

$$\varphi(z) = |e_j^*(z)| \quad \text{for} \quad z \in D_j e_1 + M$$

By hypothesis, there exist constants $0 < c_k^n < 1$ and holomorphic functions $f_k^n, 1 \leq k \leq m_n, n \geq 1$ on E_0 such that the sequence of plurisubharmonic functions $\{\psi_n\}$ is uniformly convergent on every compact set in$G$ to φ, where

$$\psi_n(z) = \max\{c_k^n \log |f_k^n(z)| : 1 \leq k \leq m_n\}\,.$$

For each $j \geq 1$ consider the functions ψ_n on $D_j e_1 + \mathbb{C} e_j$, which are convergent uniformly on every compact set in $D_j e_1 + \mathbb{C} e_j$ to function

$$|e_j^*(z)| = |z_j|, z = z_1 e_1 + z_j e_j\,.$$

This implies that there exists n_j such that ψ_{n_j} depends on z_j. Thus there exists $z_1^j \in \mathbb{C}$ with $|z_1^j| < \dfrac{1}{j}$ such that $\psi_{n_j}(z_1^j, z_j)$ depends on z_j. Then there exists $\lambda_j \in \mathbb{C}$ such that

$$|\psi_{n_j}(z_1^j, \lambda_j)| > j \quad \text{for every} \quad j \geq 1\,.$$

We claim that the sequence $\{\lambda_j\}$ is the desired sequence. Indeed, we assume that there exists a subsequence $\{\lambda_{j_p}\} \subset \{\lambda_j\}$ such that $\lambda_{j_p} e_{j_p} \to 0$. Consider the compact set in E_0

$$K = \{z_1^{j_p} e_1 + \lambda_{j_p} e_j, 0\}_{j\geq 1}\,.$$

Since $0 \in D_1 e_1 + M$ for sufficiently large p we have

$$z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p} \in D_1 e_1 + M\,.$$

Hence

$$\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p}) = |e_1^*(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| = |z_1^{j_p}| \to 0$$

as $p \to \infty$ and

$$\begin{aligned}\|\varphi - \psi_{n_{j_p}}\|_K \geq & |\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p}) - \psi_{n_{j_p}}(z_1^{j_p} + \lambda_{j_p} e_{j_p})| \\ \geq & |\psi_{n_{j_p}}(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| - |\varphi(z_1^{j_p} e_1 + \lambda_{j_p} e_{j_p})| \\ > & j_p - |z_1^{j_p}| \to \infty\end{aligned}$$

as $p \to \infty$.

This is impossible.

Since $\{e_j\}$ is a Schauder basis, we have

$$0 = \lim_{j\to\infty} e_j^*(z) e_j = \lim_{j\to\infty} (e_j^*(z)/\lambda_j)\lambda_j e_j$$

for every $z \in E$. Put

$$\rho(z) = \sup\{|e_j^*(z)/\lambda_j| \colon j \geq 1\}\,.$$

Since E is a Frechet, by the Banach–Steinhaus theorem, ρ is a continuous norm on E. The theorem is proved.

2 – Holomorphic maps with values in a pseudoconvex space

First we recall [1] that a holomorphic map θ from a complex space X to a complex space Y is called a Stein morphism if for every $y \in Y$ there exists a neighbourhood V of y such that $\theta^{-1}(V)$ is a Stein space.

In this section we shall prove the following

Theorem 2. *Let E be a nuclear Frechet space with a Schauder basis $\{e_j\}$. The following conditions are thus equivalent:*

(i) *E has the property $(\widetilde{\Omega})$.*

(ii) *There exists a compact balanced convex set B in E containing all e_j such that every holomorphic function on (E_B, τ_B) with values in any pseudoconvex space G having a Stein morphism into a complex Lie group can be extended holomorphically to E.*

(iii) *There exists a compact balanced convex set B in E containing all e_j such that every holomorphic function on (E_B, τ_B) with values in any pseudoconvex space G having a Stein morphism into a complex Lie group which has a holomorphic extension to some neighbourhood of zero in E, is of uniform type.*

PROOF. (iii)⇒(ii). Given $f\colon (E_B, \tau_E) \to G$ a holomorphic function with values in a pseudoconvex space G having a Stein morphism into a complex Lie group, where B is as in (iii).

Take a neighbourhood U of zero in E such that $f(E_B \cap U)$ is contained in a coordinate neighbourhood of zero in E. Hence, using (iii) f is extended holomorphically to E.

(ii)⇒(i). Applying the result of MEISE and VOGT [4] to the case of $G = \mathbb{C}$ we get that E has the property $(\widetilde{\Omega})$.

(i)⇒(iii). Let us first assume that (iii) holds for complex Lie groups. Since E is a Frechet space, we may assume that E_B contains all e_j.

Given $f\colon (E_B, \tau_E) \to G$ a holomorphic function which can be extended holomorphically to a neighbourhood of zero in E with values in G as in (iii) where B is a compact balanced convex set in E such that

$$\forall p \in \mathbb{N}\,, \exists q \in \mathbb{N}\,, d > 0\,, C > 0\colon$$

$$\|x^*\|_q^{*1+d} \le C\|x^*\|_B^*\|x^*\|_p^{*d} \quad \text{for every} \quad x^* \in E^*\,.$$

Such a set exists (see [4]).

Let φ be a continuous plurisubharmonic exhaustion function on G and let $\theta\colon G \to S$ be a Stein morphism, where S is a complex Lie group. Assuming that (iii) holds for S, there exists a holomorphic map $\tilde{f}\colon E \to S$ such that

$$\tilde{f} \circ \psi_B = \theta f\,,$$

and there exists $p \in \mathbb{N}$ and a holomorphic map $g\colon E_p \to S$ such that

$$\tilde{f} = g \circ \omega_p$$

where $\psi_B\colon E_B \to E$, $\omega_p\colon E \to E_p$ are the canonical maps. Thus

$$\theta f = g \circ \omega_p \circ \psi_B\,.$$

1) First consider the case where E has a continuous norm.

a) Applying Theorem 1 to the continuous plurisubharmonic function $\psi = \varphi f$ we can write

$$\psi = \lim_{n\to\infty} \max\{c_j^n \log|f_j^n|\colon 1 \le j \le m_n\}$$

and the convergence is uniform on compact sets in (E_B, τ_E), where f_j^n are holomorphic functions on E and we can assume that $0 \le c_j^n < 1$ for every $1 \le j \le m_n$ and every $n \ge 1$. Using Hartogs Lemma, this implies that there exists $q_1 \ge p$ such that

$$M = \sup\{c_j^n \log|f_j^n(z)|\colon \|z\|_{q_1} \le 1\,; 1 \le j \le m_n\,; n \ge 1\} < \infty\,.$$

Thus

$$\sup\{|f_j^n(z)|\colon \|z\|_{q_1} \le 1\} \le \exp\Big(\frac{M}{c_j^n}\Big)$$

for all $1 \le j \le m_n$ and all $n \ge 1$.

Since E has the property $(\widetilde{\Omega})$ using the same argument as Meise and Vogt [4] we can find $q \ge q_1$ independent of $1 \le j \le m_n$, $n \ge 1$ and holomorphic functions g_j^n on E_q such that

$$f_j^n = g_j^n \omega_q \quad \text{for} \quad 1 \le j \le m_n \quad \text{and} \quad n \ge 1\,.$$

Moreover

$$\sup\{|g_j^n(\hat z)|\colon \|\hat z\| \le r, \hat z \in E_q\} = C_r \exp\Big(\frac{M}{c_j^n}\Big) < \infty$$

for every $1 \le j \le m_n$, $n \ge 1$ and every $r > 0$.

These inequalities imply that

$$\sup\{\psi(z)\colon \|z\|_q \le r\} \le C_r < \infty$$

for every $r > 0$.

Thus $f(z+\operatorname{Ker}\|.\|_q)$ is relatively compact in the Stein manifold $\theta^{-1}(g\omega_q\psi_B(z))$ for every $z\in E_B$. Hence by the Liouville theorem

$$f(z+\operatorname{Ker}\|.\|_q)=\quad\text{const for every}\quad z\in E_B$$

and the form

$$h(z+\operatorname{Ker}\|.\|_q)=f(z)\quad\text{for}\quad z\in E_B$$

defines a function h on $E_B/Ker\|.\|_q$ with values in G, where $E_B/\operatorname{Ker}\|.\|_q$ is the image of E_B under canonical projection $\omega_q\colon E\to E_q$. Given $z_0+\operatorname{Ker}\|.\|_q\in E_B/\operatorname{Ker}\|.\|_q$. By hypothesis, there exists a neighbourhood V of $g(z_0+\operatorname{Ker}\|.\|_q)$ such that $\theta^{-1}(V)$ is a Stein space. Take $\delta>0$ such that

$$g(z_0+\delta\widetilde{U}_q)\subset V\quad\text{with}\quad\widetilde{U}_q=\{z+\operatorname{Ker}\|.\|_q,\|z\|_q\le 1\}\,.$$

Since

$$\sup\{\psi(z_0+\delta z)\colon z\in E_B\,,\|z\|\le 1\}<\infty$$

it follows that

$$h(z_0+\delta\widetilde{U}_q)=f(z_0+\delta U_q)$$

is relatively compact in $\theta^{-1}(V)$. From the Steiness of $\theta^{-1}(V)$ and since h is the Gateaux holomorphism, we infer that h is holomorphic on $E_B/\operatorname{Ker}\|.\|_q$.

b) Extend h to a holomorphic function on a neighbourhood D of $E_B/\operatorname{Ker}\|.\|_q$ in E_q. This extension is also denoted by h. Consider the domain of existence D_h of h as a Riemann domain over E_q. Since $E_B/\operatorname{Ker}\|.\|_q$ is dense in E_q it follows that D_h is contained in E_q as an open subset. We show that D_h is pseudoconvex. It suffices to show that D_h satisfies the weak disc condition. This means that if a sequence $\{\sigma_n\}\subset H(\Delta,D_h)$, the space of holomorphic maps from the open unit disc Δ in $\mathbb{C}$ into D_h equipped with the compact–open topology, which is convergent to $\sigma\in H(\Delta^*,D_h)$ in $H(\Delta^*,D_h)$ with $\Delta^*=\Delta\backslash\{0\}$, the σ can be extended holomorphically to Δ and $\{\sigma_n\}$ is convergent to σ in $H(\Delta,D_h)$.

First let us observe that the complex Lie group S satisfies the weak disc condition. This follows from the fact that S is a holomorphic bundle over a commutative Lie group whose fibers are Stein manifolds. From the assumption that G has a Stein morphism into a complex Lie group, we conclude that G satisfies the weak disc condition. Hence $\{h\sigma_n\}$ is convergent to $h\sigma$ in $H(\Delta, D_h)$.

Take a Stein neighbourhood V which can be considered as a closed submanifold of $\mathbb{C}^m$ for some $m \geq 1$ of $h\sigma(0), \varepsilon > 0$ and $N > 0$ such that

$$h\sigma(\varepsilon\Delta) \subset V \quad \text{for every} \quad n > N\,.$$

For each $n > N$ define a holomorphic function

$$\hat{\sigma}\colon \varepsilon\Delta \to \limind_{k\geq 1} H^\infty(W_k, \mathbb{C}^m)$$

by

$$\hat{\sigma}(t)(x) = h(\sigma_n(t) + x)$$

and

$$\hat{\sigma}\colon \varepsilon\Delta^* \to \limind_{k\geq 1} H^\infty(W_k, \mathbb{C}^m)$$

by

$$\hat{\sigma}(t)(x) = h(\sigma(t) + x)$$

where $\{W_k\}_{k\geq 1}$ is the basis of neighbourhoods of $0 \in E_q$. It follows that the sequence $\{\hat{\sigma}_n\}$ converges the $\hat{\sigma}$ in $H(\varepsilon\Delta^*, H^\infty(W_k, \mathbb{C}^m))$. Indeed, given K a compact set in $\varepsilon\Delta^*$ and hence $\sigma(K)$ is a compact set in D_h. Then there exists $V \subset D$ such that h is uniform continuous on $\sigma(K)+V$, i.e. for every $\delta > 0$ there exists $V(\delta) \subset V$ such that for $x, y \in \sigma(K) + V\,; x - y \in V(\delta)$, we have

$$\|h(x) - h(y)\| < \delta\,.$$

For each $k \geq 1$ and $r > 0$ put

$$U_{kr} = \Big\{ f \in H^\infty(W_k, \mathbb{C}^m)\colon \|f\|_{W_k} \leq \frac{1}{r} \Big\}$$

and consider $\{U_l\}$ with $l\colon \mathbb{N} \to \mathbb{N}$, defined by

$$U_l = \overline{\text{conv}}\Big(\bigcup_{k\geq 1} j_k(U_{k,l(k)}) \Big)$$

where $j_k\colon H^\infty(W_k,\mathbb{C}^m)\to \operatorname{limind}_{k\ge 1} H^\infty(W_k,\mathbb{C}^m)$ are canonical embeddings. It is easy to see that $\{U_l\}$ is a basis of neighbourhood of 0 in $\operatorname{limind}_{k\ge 1} H^\infty(W_k,\mathbb{C}^m)$. Given a U_l in $\operatorname{limind}_{k\ge 1} H^\infty(W_k,\mathbb{C}^m)$. Take k_0 such that $W_{k_0}\subset V$ and N_0 sufficiently large so that

$$\sigma_n(t)-\sigma(t)\subset W_{k_0}$$

$$\sigma_n(t)-\sigma(t)\subset V\Big(\frac{1}{l(k_0)}\Big)$$

for every $n>N_0$ and all $t\in K$.

Thus for all $n>N_0$ we get $\sigma_n(t),\sigma(t)\in H^\infty(W_{k_0},\mathbb{C}^m)$ for all $t\in K$ and

$$\sup_{t\in K}\sup_{x\in W_{k_0}}\|h(\sigma_n(t)+x)-h(\sigma(t)+x)\|<\frac{1}{l(k_0)}\,.$$

i.e.

$$\sup_{t\in K}\sup_{x\in W_{k_0}}\|\hat\sigma_n(t)(x)-\hat\sigma(t)(x)\|<\frac{1}{(k_0)}\,.$$

Then $\hat\sigma_n(t)-\hat\sigma(t)\subset U_{k_0,l(k_0)}$ for all $t\in K$.

Thus we infer that $\{\hat\sigma_n\}$ converges to $\hat\sigma$ in $H(\varepsilon\Delta^*,\operatorname{limind}_{k\ge 1} H^\infty (W_k,\mathbb{C}^m))$ and hence $\hat\sigma$ can be extended holomorphically to $\varepsilon\Delta$ and $\{\hat\sigma_n\}$ converges to $\hat\sigma$ in $H(\varepsilon\Delta,\operatorname{limind}_{k\ge 1} H^\infty(W_k,\mathbb{C}^m))$.

Since $\{\hat\sigma_n(\frac{\varepsilon\Delta}{2})\}$ is bounded in $\operatorname{limind}_{k\ge 1} H^\infty(W_k,\mathbb{C}^m)$ and the inductive limit is regular [8] there exists k_1 such that

$$\hat\sigma_n(t)\in H^\infty(W_{k_1},\mathbb{C}^m)\quad\text{for every}\quad |t|\le\frac{\varepsilon}{2}$$

and every $n>N$.

Let us observe that σ can be extended holomorphically to $\varepsilon\Delta$ and $\sigma_n\to\sigma$ in $H(\Delta,E_q)$. It remains to check whether $\sigma(0)\in D_h$. We have $\hat\sigma_n(0)(x)=h(\sigma_n(0)+x)$ for every $x\in W_{k_1}$ and $n>1$. This yields $\sigma(0)\in D_h$.

c) Since the topology of E is defined by Hilbert semi–norms without loss of generality we may assume that E_q is a Hilbert space. Choose $q>p$ such that the canonical map $\omega_{qp}\colon E_q\longrightarrow E_p$ is compact. Let τ denote the linear metric topology on $H(D_h)$ generated by the uniform convergence on sets

$$K_r=\{\omega_{qp}(z)\colon \|z\|\le r\,,\omega_{qp}(z)\in D_h\,,\operatorname{dist}(\omega_{qp}(z)\,,\partial D_h)\ge 1/r\}\,.$$

Since the canonical map $[H(D_h), \tau] \to H(E)$ is continuous and since

$$H(E)_{bor} \cong \operatorname{limind}_k H_b(E_k)$$

(see [4]), where $H(E)_{bor}$ denotes the bornological space associated with $H(E)$ and for every $k \geq 1$, by $H_b(E_k)$ we denote the Frechet space of holomorphic functions on E_k which are bounded on every bounded set in E_k, we can find $k > q$ such that $H(D_h) \subseteq H_b(E_k)$. It remains to check whether $\operatorname{Im} \omega_{kp} \subset D_h$. In the converse case there exists $z \in E_k$ such that $\omega_{kp} \in \partial D_h$. Choose a sequence $\{z_n\} \subset E/\operatorname{Ker} \|.\|_k$ which converges to z. Since E_p is a Hilbert space we can find $f \in H(D_h)$ such that

$$\sup |f\omega_{kp}(z_n)| = \infty .$$

This is impossible because $f\omega_p \in H(E_k)$.

2) General case. Let $p_1 > p$ such that $f(U_{p_1} \cap E_B)$ is contained and relatively compact in a Stein open subset of G, where $U_{p_1} = \{z \in E \colon \|z\|_{p_1} \leq 1\}$. From the Liouville theorem, it follows that

$$f(z + \operatorname{Ker} \|.\|_{p_1}) = f(z) \quad \text{for} \quad z \in U_{p_1} \cap E_B .$$

Hence the unique principle implies that the relation holds for all $z \in E_B$. As in [7], define

$$J = \{j \in \mathbb{N} \colon \|e_j\|_{p_1} \neq 0\}$$

and write

$$E = E^1 \oplus E^2$$

where E^1 is the subspace of E with a Schauder basis $\{e_j, j \in J\}$ and a continuous norm $\|.\|_{p_1}|_{E^1}$, and $E^2 = \operatorname{Ker} \|.\|_{p_1}$.

Using 1) to $f|_{E^1_{B^1}}$ where $B^1 = B \cap E^1$, we can find $k > p_1$ and a holomorphic function h^1 on E^1_k with values in G such that $f|_{E^1} = h_1\omega_k|_{E^1}$. It is easy to see that

$$E_k = E^1_k \oplus E^2_k .$$

Consequently, by setting $h(z) = h^1(z^1)$ for $z = (z^1, z^2) \in E_k$ we get a holomorphic function h on E_k with values in G for which $f = h\omega_k$.

To complete the proof it remains to check that (iii) holds for every complex Lie group. We assume that G is a complex Lie group. By [10] there exists a Stein morphism θ from G onto a torus S. Since S has an universal cover which is a Euclidean space, from MEISE and VOGT (iii) holds for S and hence (iii) holds for G.

The theorem is proved.

Acknowledgements

The authors acknowledge the kind help received from Prof. Nguyen Van Khue during the preparation of the present paper and also the referee for his suggestions.

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Lavoro pervenuto alla redazione il 19 febbraio 1993
ed accettato per la pubblicazione il 6 dicembre 1995.
Bozze licenziate il 15 settembre 1996

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