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# Two classes of ideals determined by integer-valued polynomials

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RIASSUNTO: *Sia dato un dominio $D$ con campo dei quozienti $K$, e sia* $\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(d) \in D \forall d \in D\}$ *l'anello dei polinomi a valori interi su $D$. È noto che i polinomi binomiali* $\binom{X}{n} = \frac{X(X-1)\ldots(X-n+1)}{n!}$ *formano una base di* $\mathrm{Int}(\mathbb{Z})$ *e che per ogni numero primo $p$, il polinomio di Fermat* $f_p(X) = \frac{1}{p}(X^p - X)$ *appartiene ad* $\mathrm{Int}(\mathbb{Z})$. *Se il dominio $D$ contiene $\mathbb{Z}$, poniamo, per ogni intero non negativo $n$,* $C(n) = \{\alpha \in K \mid \alpha \cdot \binom{X}{n} \in \mathrm{Int}(D)\}$, *e per ogni numero primo $p$,* $E(p) = \{\alpha \in K \mid \alpha \cdot f_p(X) \in \mathrm{Int}(D)\}$. *$C(n)$ e $E(p)$ sono ideali di $D$, essi vengono determinati esplicitamente nel caso in cui $D$ sia un dominio di Dedekind.*

ABSTRACT: *If $D$ is a domain with quotient field $K$, let* $\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(d) \in D \text{ for every } d \in D\}$ *be the ring of integer-valued polynomials over $D$. It is well known that the binomial polynomials* $\binom{X}{n} = \frac{X(X-1)\ldots(X-n+1)}{n!}$ *form a basis of* $\mathrm{Int}(\mathbb{Z})$ *as a free $\mathbb{Z}$-module and that for every prime integer $p$, the Fermat polynomials* $f_p(X) = \frac{1}{p}(X^p - X)$ *are in* $\mathrm{Int}(\mathbb{Z})$. *If the domain $D$ contains $\mathbb{Z}$, for each nonnegative integer $n$, set* $C(n) = \{\alpha \in K \mid \alpha \cdot \binom{X}{n} \in \mathrm{Int}(D)\}$, *and for every prime integer $p$, set* $E(p) = \{\alpha \in K \mid \alpha \cdot f_p(X) \in \mathrm{Int}(D)\}$. *Each $C(n)$ and $E(p)$ is an ideal of $D$ which we explicitly determine when $D$ is a Dedekind domain.*

KEY WORDS AND PHRASES: *Integer-valued polynomial – Dedekind domain*
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## – Introduction

Let $D$ be a domain with quotient field $K$ and set

$$\mathrm{Int}(D) = \{f(X) \in K[X] | f(d) \in D \quad \text{for every} \quad d \in D\}$$

to be the ring of integer-valued polynomials over $D$. In [8], Pólya showed that the *binomial polynomials*, defined by

$$\binom{X}{0} = 1, \quad \text{and} \quad \binom{X}{n} = \frac{X(X-1)\dots(X-(n-1))}{n!}, \quad \text{for} \quad n \geq 1$$

form a basis of $\mathrm{Int}(\mathbb{Z})$ as a free $\mathbb{Z}$-module. In a later paper, Pólya asked which rings of algebraic integers $D$ possess similar bases for $\mathrm{Int}(D)$ as a $D$-module (*i.e.* a free basis $\{g_i(X)\}_{i=0}^{\infty}$ where each $g_i(X)$ is a degree $i$ polynomial) and solved this question for a quadratic number ring [9]. His argument centered on the fractional ideals of $D$ of the form

$$A(n) = \{0\} \bigcup \{\alpha \in K | \exists f \in \mathrm{Int}(D), \deg(f) = n, f = \alpha X_n + \dots\}$$

where $n$ is a nonnegative integer. Such a free basis exists if and only if these ideals are principal. Ostrowski [7] generalized Pólya's result to other rings of integers and Cahen [1] offered a complete description of the ideals $A(n)$ in the Dedekind case. Zantema [10] described a cohomological solution to Pólya's original problem.

Some recent papers have explored a topic related to the discussion above. If $D$ is a domain of characteristic 0 (and thus containing $\mathbb{Z}$) they asked whether the binomial polynomials $\binom{X}{n}$ serve themselves as a free basis of $\mathrm{Int}(D)$. By [3, Proposition 1] $\mathrm{Int}(D)$ is contained in the $D$-module generated by the binomial polynomials $\binom{X}{n}$. Hence, these polynomials form a basis of $\mathrm{Int}(D)$ if and only if they are contained in $\mathrm{Int}(D)$. Chabert and Gerboud [2] have given various other characterizations in the case of a ring of algebraic integers $D$ and Halter-Koch and Narkiewicz [5] considered the case of an arbitrary domain of characteristic 0. One characterization is that the polynomials of the form $f_p(X) = \frac{1}{p}(X^p - X)$, where $p$ is a prime integer (known as the *Fermat polynomials*), are contained in $\mathrm{Int}(D)$. Another is that, for any nonnegative integer $n, n!A(n) = D$.

In this paper, we let

$$C(n) = \left\{ \alpha \in K | \alpha \cdot \binom{X}{n} \in \mathrm{Int}(D) \right\},$$

and

$$E(p) = \{ \alpha \in K \mid \alpha \cdot f_p(X) \ \in \mathrm{Int}(D) \}.$$

We completely determine these ideals for a Dedekind domain $D$ (of characterisitic 0). We note that $C(n)$ is contained in $n!A(n)$. By definition, $C(n) = D$ if and only if $\binom{X}{n}$ belongs to $\mathrm{Int}(D)$ (and $E(p) = D$ if and only if $f_p(X)$ belongs to $\mathrm{Int}(D)$), hence $C(n) = D$ for all $n$ if and only if $n!A(n) = D$ for all $n$ (and also if and only if $E(p) = D$ for all $p$). However, we show that when this fails then eventually the inclusion of $C(n)$ in $n!A(n)$ is proper for some $n$.

Since any ideal of a Dedekind domain can uniquely be written as a product of maximal ideals, we determine the ideal $C(n)$ by computing the exponent of each maximal ideal $P$ of $D$ in such a decomposition. We show this exponent to be trivial unless $P$ contains a prime integer $p$ of $\mathbb{Z}$. We then proceed similarly for the ideals $E(p)$. Lastly, we conclude by an application to quadratic number rings giving some explicit examples.

Throughout, $\mathbb{Z}$ represents the integers, $\mathbb{N}$ the nonnegative integers, and $\mathbb{Q}$ the rationals. If $D$ is a Dedekind domain of characteristic 0 and $P$ a maximal ideal of $D$, let $v_P$ be the normalized valuation (*i.e.*, its value group is $\mathbb{Z}$) associated to $P$. If $P$ contains a prime integer $p$, the valuation $v_P$ extends the $p$-adic valuation of $\mathbb{Q}$ and we say that $P$ is *above* $p$. Throughout, we let $e_P$ be the *ramification index* of this extension (thus $e_P = v_P(p)$), and $f_P$ be its *residual degree*. Hence $f_P = [D/P : \mathbb{Z}/p\mathbb{Z}]$. We observe that $e_P$ is always finite, but that $f_P$ may be infinite (whenever the residue field $D/P$ is infinite). We say that $v_P$ is an *immediate extension* of the $p$-adic valuation if $e_P = f_P = 1$. If $\frac{a}{b}$ is in $\mathbb{Q}$ we let $\left[\frac{a}{b}\right]$ represent the greatest integer less than or equal to $\frac{a}{b}$. We use the symbol "$\subseteq$" to represent set containment, and "$\subset$" to represent proper set containment. For any other notation, the interested reader is referred to [6].

## 1 – Computation of the ideals C(n)

We open with some elementary observations concerning the ideals $C(n)$ and $A(n)$ for any domain $D$ of characteristic 0.

PROPOSITION 1.1. *Let $D$ be a domain of characteristic* 0.
1. *For each $n \in \mathbb{N}$, $C(n) \subseteq n!A(n) \subseteq D$.*
2. *For each $n \in \mathbb{N}$, $C(n) \subseteq C(n-1)$.*
3. $C(0) = C(1) = D$.

PROOF. 1. From the definition of $C(n)$, if $\alpha \in C(n)$ then $\frac{\alpha}{n!}$ is the leading coefficient of the degree $n$ polynomial $\alpha\binom{X}{n}$, hence $C(n) \subseteq n!A(n)$. The inclusion of $n!A(n)$ in $D$ follows from [3, Proposition 1].

2. Recall the well known binomial recursion

$$\binom{X}{n} = \binom{X-1}{n-1} + \binom{X-1}{n}.$$

If $\alpha \in C(n)$, then both $\alpha\binom{X}{n}$ and $\alpha\binom{X-1}{n}$ are in $\mathrm{Int}(D)$. Thus, so is $\alpha\binom{X-1}{n-1}$. Therefore $\alpha \in C(n-1)$.

3. This is obvious, since $\binom{X}{0} = 1$ and $\binom{X}{1} = X$. ∎

From here on we let $D$ be a Dedekind domain of characteristic 0. The computation of the ideal $C(n)$ will center around the polynomial

$$f_n(X) = X(X-1)\ldots(X-n+1) = n!\binom{X}{n}.$$

We denote by $f_n(D)$ the ideal generated by the elements $f_n(d)$ for every $d \in D$. By definition, the ideal $C(n)$ is the conductor in $D$ of the ideal $(\frac{1}{n!})f_n(D)$. With these hypotheses and notations, we immediately have the following.

LEMMA 1.2. 1. $C(n) = (n!)(f_n(D))^{-1}$.

2. *If $P$ is a prime ideal of $D$, then the exponent $c_P(n)$ of $P$ in the decomposition of $C(n)$ is equal to*

$$c_P(n) = v_P(n!) - \mathrm{Inf}_{x \in D}\{v_P(f_n(x))\}.$$

To compute $c_P(n)$, we then first determine the integers

$$i_P(n) = \mathrm{Inf}_{x \in D}\{v_P(f_n(x))\}.$$

We restrict ourselves to a prime ideal $P$ above a prime integer $p$ (otherwise, we shall see below that $c_P(n) = 0$). The valuation $v_P$ associated to $P$ is thus an extension of the $p$-adic valuation. Let $e_P = v_P(p)$ and $f_P = [D/P : \mathbb{Z}/p\mathbb{Z}]$ be respectively the ramification index and the residual degree of this extension.

LEMMA 1.3. *Let $P$ be a prime ideal of $D$ above a prime integer $p$.*
1. *If $f_P > 1$, then $i_P(n) = 0$.*
2. *If $f_P = 1$ and $e_P > 1$, then $i_P(n) = \left[\frac{n}{p}\right]$.*
3. *If $f_P = 1$ and $e_P = 1$, then $i_P(n) = v_P(n!)$.*

PROOF. 1. Clearly $i_P(n) \geq 0$, since the coefficients of $f_n$ are in $D$ (in fact in $\mathbb{Z}$). On the other hand, by definition of the residual degree, if $f_P > 1$, then $D/P$ strictly contains $\mathbb{Z}/p\mathbb{Z}$. Hence there exists $x_0 \in D$ such that, $\forall i \in \mathbb{Z}, (x_0 - i) \notin P$. Therefore

$$i_P(n) \leq v_P(f_n(x_0)) = \sum_{i=0}^{n-1} v_P(x_0 - i) = 0\,.$$

2. If $f_P = 1$, then $D/P \simeq \mathbb{Z}/p\mathbb{Z}$ and, for each $x \in D$, there exists $d \in \mathbb{Z}$ such that $(x - d) \in P$. Hence, for $0 \leq i \leq n-1,\ x - i \equiv d - i \pmod{P}$. Since $d, d-1, \ldots, d-n+1$ are $n$ consecutive integers, exactly $\left[\frac{n}{p}\right]$ are divisible by $p$ (or equivalently are in $P$). Therefore

$$v_P(f_n(x)) = \sum_{i=0}^{n-1} v_P(x - i) \geq \left[\frac{n}{p}\right].$$

On the other hand, if $e_P > 1$ and if $i \in \mathbb{Z}$, either $v_P(i) = 0$ or $v_P(i) > 1$ (equivalently $i \notin P$ or $i \in P^2$). Choosing $x_0$ in $P$ such that $v_P(x_0) = 1$, then $v_P(x_0 - i) = 0$, if $i$ is not divisible by $p$, and $v_P(x_0 - i) = 1$, if $i$ is divisible by $p$. Hence

$$v_P(f_n(x_0)) = \sum_{i=0}^{n-1} v_P(x_0 - i) = \left[\frac{n}{p}\right].$$

3. If $e_P = f_P = 1$, then $P = pD$ and the cardinality of $D/P$ is $p$. We could quote [2, Theorem 2.5] to conclude that the binomial polynomial $\binom{X}{n}$ is in $\mathrm{Int}(D_P)$, and hence that the exponent $c_P(n)$ of the decomposition of $C(n)$ is trivial (and thus that $i_P(n) = v_P(n!)$). But we give a direct proof. Since $n \in D$ and $f_n(n) = n!$, it is first clear that

$$i_P(n) \leq v_P(n!)\,.$$

On the other hand, if $x$ is a root of $f_n(X)$, then $v_P(f_n(x)) = \infty > v_P(n!)$. So suppose that $x$ is not a root of $f_n(X)$ and let $r = v_P(f_n(x)) + 1$. The map

$$\varphi_r : \mathbb{Z} \longrightarrow D_P/p^r D_P$$

is surjective, because its kernel is $\mathbb{Z}/p^r\mathbb{Z}$ and there are $p^r$ elements in both $\mathbb{Z}/p^r\mathbb{Z}$ and $D_P/p^r D_P$. Hence there is $d \in \mathbb{Z}$ such that $v_P(x - d) = r$. Since we have

$$v_P(f_n(x)) = \sum_{i=0}^{n-1} v_P(x - i)\,,$$

it is clear that, for $0 \le i \le n-1$, $v_P(x-i) < r$. Hence $v_P(x-i) = v_P(d-i)$ and therefore

$$v_P(f_n(x)) = \sum_{i=0}^{n-1} v_P(d - i) = v_P(f_n(d))\,.$$

Now the binomial polynomial $\binom{X}{n} = \frac{1}{n!} f_n(X)$ is integer-valued on $\mathbb{Z}$, thus $f_n(d)$ is divisible by $n!$ in $\mathbb{Z}$ and a fortiori in $D$. Therefore

$$i_P(n) \ge v_P(f_n(x)) = v_P(f_n(d)) \ge v_P(n!)\,. \qquad \square$$

We are now ready for the main result of the section.

PROPOSITION 1.4. *Let $D$ be a Dedekind domain of characteristic* 0 *and $n$ be a nonnegative integer. The ideal $C(n)$ is a product of maximal ideals of $D$, with any such maximal ideal $P$ being above a prime integer $p \le n$. Moreover, the exponent $c_P(n)$ of $P$ is given by the following formulae:*

1. *If $f_P > 1$, then $c_P(n) = v_P(n!)$.*
2. *If $f_P = 1$ and $e_P > 1$, then $c_P(n) = v_P(n!) - \left[\frac{n}{p}\right]$.*
3. *If $f_P = 1$ and $e_P = 1$, then $c_P(n) = 0$.*

PROOF. Since $C(n)$ is an ideal of $D$, then clearly $c_P(n) \ge 0$. Hence, if $v_P(n!) = 0$, it results from Lemma 1.2 that $c_P(n) = 0$. This is the case if $P \cap \mathbb{Z} = (0)$, since the valuation $v_P$ is then trivial on any integer, and also if $P$ is above a prime integer $p > n$. Lastly the formulae are a direct consequence of Lemma 1.2 and Lemma 1.3. $\square$

Remark 1.5. It is obvious that $c_P(n) > 0$ in the first case ($f_P > 1$). In fact, the same holds in the second case ($f_P = 1$ and $e_P > 1$). Indeed, from Legendre's well known formula, denoting by $v_p$ the $p$-adic valuation, then

$$v_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

and thus

$$c_P(n) = (e_P - 1)\left[\frac{n}{p}\right] + e_P \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] > 0\,.$$

On the other hand it is clear from the definition that the ideals $C(n)$ are trivial if and only if the binomial polynomials $\binom{X}{n}$ belong to $\mathrm{Int}(D)$. We thus recover one of the characterizations given by Chabert and Gerboud of the Dedekind domains $D$ such that the binomial polynomials $\binom{X}{n}$ form a basis of $\mathrm{Int}(D)$ [2, Theorem 2.5]: each maximal ideal $P$ of $D$ above a prime integer $p$ is such that the valuation $v_P$ is an immediate extension of the $p$-adic valuation (see also Remark 2.2 below).

From Proposition 1.1, if $C(n) = D$, then $C(n) = n!A(n)$. However, if some ideal $C(n)$ is not trivial, *i.e.* there is a maximal ideal $P$ in $D$ above a prime integer $p$ such that the valuation $v_P$ *is not* an immediate extension of the $p$-adic valuation, it results from next proposition that eventually $C(n) \subset n!A(n)$. This is in particular always the case for the ring of integers of an algebraic number field.

Proposition 1.6. *Let $D$ be a Dedekind domain containing $\mathbb{Z}$, $n$ be a nonnegative integer and $B(n)$ be the ideal such that $C(n) = B(n) \cdot n!A(n)$. Then $B(n)$ is a product of maximal ideals of $D$, with any such maximal ideal $P$ being above a prime integer $p \le n$. Moreover, the exponent $b_P(n)$ of $P$ is given by the following formulae:*

1. *If $f_P > 1$, then $b_P(n) = \sum_{k=1}^{\infty} \left[\frac{n}{p^{kf_P}}\right]$.*
2. *If $f_P = 1$ and $e_P > 1$, then $b_P(n) = \sum_{k=2}^{\infty} \left[\frac{n}{p^k}\right]$.*
3. *If $f_P = 1$ and $e_P = 1$, then $b_P(n) = 0$.*

Proof. Clearly, $b_P(n) = c_P(n) - a_P(n)$, where $a_P(n)$ is the exponent of $P$ in the decomposition of $n!A(n)$. Now, the results of [1, Section 2] yield that

$$A(n) = \prod P^{-\sum_{k=1}^{\infty}\left[\frac{n}{N(P)^k}\right]},$$

where $N(P) = pf_P$ is the norm of $P$ (i.e. the cardinality of $D/P$), hence

$$a_P(n) = v_P(n!) - \sum_{k=1}^{\infty} \left[\frac{n}{p^{kf_P}}\right].$$

From the previous proposition we thus get the following.

1. If $f_P > 1$, then $c_P(n) = v_P(n!)$, hence

$$b_P(n) = c_P(n) - a_P(n) = \sum_{k=1}^{\infty} \left[\frac{n}{p^{kf_P}}\right].$$

2. If $f_P = 1$ and $e_P > 1$, then

$$c_P(n) = v_P(n!) - \left[\frac{n}{p}\right]$$

and

$$a_P(n) = v_P(n!) - \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right],$$

hence

$$b_P(n) = c_P(n) - a_P(n) = \sum_{k=2}^{\infty} \left[\frac{n}{p^k}\right].$$

3. If $f_P = 1$ and $e_P = 1$, then both $C(n)$ and $n!A(n)$ are trivial. Hence, so is $B(n)$. □

## 2 – Computation of the ideals E(p)

We present now a computation of the ideals $E(p)$, perfectly similar in spirit to that of the last section. We state the main result.

Proposition 2.1. *Let $D$ be a Dedekind domain of characteristic* 0 *and $p$ be a prime number. The ideal $E(p)$ is a product of maximal ideals of $D$ above $p$ and the exponent $h_P(p)$ of such a maximal ideal $P$ is given by the following formulae:*

1. *If $f_P > 1$, then $h_P(p) = e_P$.*
2. *If $f_P = 1$, then $h_P(p) = e_P - 1$.*

Proof. By definition, the ideal $E(p)$ is the conductor in $D$ of the ideal generated by the values of the Fermat polynomial $f_p(X) = \frac{1}{p}(X^p -$

$X$). In a manner similar to Lemma 1.2, we thus have

$$h_P(p) = v_P(p) - \mathrm{Inf}_{x\in D}\{v_P(x^p - x)\}\,.$$

Since $E(p)$ is an ideal, then $0 \le h_P(p) \le v_P(p)$. Hence, if $v_P(p) = 0$, then $h_P(p) = 0$. This is clearly the case if $P$ is not above $p$. Since $v_P(p) = e_P$, the formulae will result from the computation of $\mathrm{Inf}_{x\in D}\{v_P(x^p - x)\}$:
1. If $f_P > 1$, the cardinality of the field $D/P$ is greater than $p$ and there exists some element $x_0 \in D$ such that $(x_0^p - x_0) \notin P$. Therefore

$$\mathrm{Inf}_{x\in D}\{v_P(x^p - x)\} = v_P(x_0^p - x_0) = 0\,.$$

2. If $f_P = 1$, then $D/P \simeq \mathbb{Z}/p\mathbb{Z}$ and $\forall x \in D, v_P(x^p - x) \ge 1$. On the other hand, if $x_0 \in D$ is such that $v_P(x_0) = 1$, then $v_P(x_0^p) = p$. Thus $v_P(x_0^p - x_0) = 1$ and

$$\mathrm{Inf}_{x\in D}\{v_P(x^p - x)\} = v_P(x_0^p - x_0) = 1\,. \qquad \square$$

Remark 2.2. It clearly results from this proposition that the ideals $E(p)$ are trivial if and only if, for any maximal ideal $P$ of $D$ above a nonzero prime $p$, the valuation $v_P$ is an immediate extension of the $p$-adic valuation. We thus recover another characterization given by Chabert and Gerboud of the Dedekind domains $D$ such that the binomial polynomials $\binom{X}{n}$ form a basis of $\mathrm{Int}(D)$ [2, Theorem 2.5].

## 3 – Application to quadratic fields

We now interpret Proposition 1.4 in the case of the ring of integers of a quadratic number field.

Proposition 3.1. *Let $D$ be the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$ where $d$ is a square free integer. Let $n$ be a positive integer and write*

$$n! = p_1^{b_1}\dots p_j^{b_j} q_1^{c_1}\dots q_k^{c_k} r_1^{a_1}\dots r_i^{a_i}$$

*where $p_t$ is a prime in $\mathbb{Z}$ which is inert in $D$, for $1 \le t \le j$, $q_t$ is a prime which splits, for $1 \le t \le k$ and $r_t$ is a prime in $\mathbb{Z}$ which ramifies in $D$,* *for* $1 \le t \le i$. *Then, letting* $r_tD = R_t^2$,

$$C(n) = p_1^{b_1} \dots p_j^{b_j} R_1^{2a_1 - \left[\frac{n}{r_1}\right]} \dots R_i^{2a_i - \left[\frac{n}{r_i}\right]}$$

PROOF. This is a direct application of the formulae given by Proposition 1.4.
1. If $p$ is inert, then $P = pD$ is a maximal ideal of $D$ such that $e_P = 1$ and $f_P = 2$. In this case $c_P(n) = v_P(n!) = v_p(n)$ is the exponent of $p$ in the decomposition of $n!$.
2. If $q$ splits in $D$, the maximal ideals above $q$ in $D$ do not appear in the decomposition of $C(n)$.
3. If $r$ ramifies, that is if $rD = R^2$, then $R$ is a maximal ideal of $D$ such that $e_R = 2$ and $f_R = 1$. In this case $c_R(n) = v_R(n!) - \left[\frac{n}{r}\right]$, where $v_R(n!) = 2v_r(n!)$ is twice the exponent of $r$ in the decomposition of $n!$ ∎

We illustrate the result 3.1 with two examples.

EXAMPLE 3.2. In the following chart, we list the prime factorization of the first 12 values of $n!$, followed by the prime factorizations of the ideals $C(n)$ and $n!A(n)$ when $D = \mathbb{Z}[i]$. Note that $D$ is a principal ideal domain, $(2) = (1+i)^2$ is the only ramified prime, and a prime $p$ is inert in $D$ if and only if $p \equiv 3 \pmod 4$ (see [6]).

| $n$ | $n!$ | $C(n)$ | $(n!)A(n)$ |
|---|---|---|---|
| 0 | 1 | $\mathbb{Z}[i]$ | $\mathbb{Z}[i]$ |
| 1 | 1 | $\mathbb{Z}[i]$ | $\mathbb{Z}[i]$ |
| 2 | 2 | $(1+i)$ | $(1+i)$ |
| 3 | $2 \cdot 3$ | $(1+i)(3)$ | $(1+i)(3)$ |
| 4 | $2^3 \cdot 3$ | $(1+i)^4(3)$ | $(1+i)^3(3)$ |
| 5 | $2^3 \cdot 3 \cdot 5$ | $(1+i)^4(3)$ | $(1+i)^3(3)$ |
| 6 | $2^4 \cdot 3^2 \cdot 5$ | $(1+i)^5(3)^2$ | $(1+i)^4(3)^2$ |
| 7 | $2^4 \cdot 3^2 \cdot 5 \cdot 7$ | $(1+i)^5(3)^2(7)$ | $(1+i)^4(3)^2(7)$ |
| 8 | $2^7 \cdot 3^2 \cdot 5 \cdot 7$ | $(1+i)^{10}(3)^2(7)$ | $(1+i)^7(3)^2(7)$ |
| 9 | $2^7 \cdot 3^4 \cdot 5 \cdot 7$ | $(1+i)^{10}(3)^4(7)$ | $(1+i)^7(3)^3(7)$ |
| 10 | $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ | $(1+i)^{11}(3)^4(7)$ | $(1+i)^8(3)^3(7)$ |
| 11 | $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | $(1+i)^{11}(3)^4(7)(11)$ | $(1+i)^8(3)^3(7)(11)$ |

EXAMPLE 3.3. We repeat the chart of the previous example, this time with $D = \mathbb{Z}[\sqrt{-5}]$. Here $D$ is not a principal ideal domain and $(2) = (2, 1+\sqrt{-5})^2$ and $(5) = (\sqrt{-5})^2$ are the only ramified primes. We let $P = (2, 1+\sqrt{-5})$.

| $n$ | $n!$ | $C(n)$ | $(n!)A(n)$ |
|---|---|---|---|
| 0 | 1 | $\mathbb{Z}[\sqrt{-5}]$ | $\mathbb{Z}[\sqrt{-5}]$ |
| 1 | 1 | $\mathbb{Z}[\sqrt{-5}]$ | $\mathbb{Z}[\sqrt{-5}]$ |
| 2 | 2 | $P$ | $P$ |
| 3 | $2 \cdot 3$ | $P$ | $P$ |
| 4 | $2^3 \cdot 3$ | $P^4$ | $P^3$ |
| 5 | $2^3 \cdot 3 \cdot 5$ | $P^4(\sqrt{-5})$ | $P^3(\sqrt{-5})$ |
| 6 | $2^4 \cdot 3^2 \cdot 5$ | $P^5(\sqrt{-5})$ | $P^4(\sqrt{-5})$ |
| 7 | $2^4 \cdot 3^2 \cdot 5 \cdot 7$ | $P^5(\sqrt{-5})$ | $P^4(\sqrt{-5})$ |
| 8 | $2^7 \cdot 3^2 \cdot 5 \cdot 7$ | $P^{10}(\sqrt{-5})$ | $P^7(\sqrt{-5})$ |
| 9 | $2^7 \cdot 3^4 \cdot 5 \cdot 7$ | $P^{10}(\sqrt{-5})$ | $P^7(\sqrt{-5})$ |
| 10 | $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ | $P^{11}(\sqrt{-5})^2$ | $P^8(\sqrt{-5})^2$ |
| 11 | $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | $P^{11}(\sqrt{-5})^2(11)$ | $P^8(\sqrt{-5})^2(11)$. |

In both examples we note that for some N, $C(n) \subset n!A(n)$ for $n \geq N$, as required by Proposition 1.6.

For the ideals $E(p)$, we similarly derive immediately the following from Proposition 2.1.

PROPOSITION 3.4. *Let $D$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$ where $d$ is a square free integer, and $p$ be a prime in $\mathbb{Z}$.*

1. *If $p$ splits in $D$, then $E(p) = D$.*
2. *If $p$ ramifies and $pD = R^2$, then $E(p) = R$.*
3. *If $p$ is inert, then $E(p) = pD$.*

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