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On a transmission problem for the time-harmonic Maxwell equations

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RIASSUNTO: *In questo lavoro si considera il problema di trasmissione per le equazioni di Maxwell armoniche nel tempo, per un diffusore a infiniti strati omogenei, costituiti da materiali diversi. Si dimostra l'esistenza e l'unicità della soluzione. Inoltre si costruisce una rappresentazione integrale del campo esterno totale e si esamina il comportamento asintotico dell'onda diffusa nella regione di radiazione.*

ABSTRACT: *The transmission for the time-harmonic Maxwell equations is studied for the case of an infinitely stratified, nested, bonded scatterer, whose homogeneous layers consist of different materials. The existence and uniqueness of solutions is proved. Moreover, an integral representation of the total exterior field is constructed, and the asymptotic behaviour of the scattered wave in the radiation region is studied.*

1 – Introduction

In this work we are studying the transmission problem for the time-harmonic Maxwell equations, in the case where a plane electromagnetic wave is incident upon a nested body consisting of an infinite number of homogeneous layers. On the surface that describe this tesselation are imposed appropriate (transmission) conditions, that express the continuity of the medium.

KEY WORDS AND PHRASES: *Maxwell equations – Scattering theory – Infinitely stratified scatterer – Transmission problem.*
A.M.S. CLASSIFICATION: 35P25 – 35Q60 – 78A45

For the mathematical electromagnetic theory, we refer to the books by Colton-Kress [7], and Fournet [11].

In Section 2, we, first, state the necessary elements of the electromagnetic theory, and formulate the transmission problem. Then we prove that the associated homogeneous transmission problem has as its only classical solution the trivial one. Next, we show that the initial non-homogeneous transmission problem has a generalized solution, which, by a regularity argument, turns to be classical. Such an approach has been used by the authors in [4], for transmission problems in acoustics. As far as the study of the transmission problem for the vector Helmholtz equation is concerned we refer to Wilde [17], while the conductive boundary problem for the Maxwell equations, has been studied by Angell and Kirsch [1].

In the Section 3, we construct an integral representation of the total exterior field; this representation incorporates all the information about the transmission and the radiation conditions. In addition, we study the asymptotic behaviour of the scattered wave in the radiation region (far-field pattern). The scattered electric and magnetic fields are expressed in terms of the electric far-field pattern, and the magnetic far-field pattern, respectively, in a form analogous to that of Colton-Paivarinta [8], for the case of a non-homogeneous scatterer.

For the proof of the existence and uniqueness of solutions of the transmission problem, we use a generalized solutions approach. In the case where the scatterer is not tesselated, but consist of a non-homogeneous material, such an approach has been used by Bihovski [6], for the interior problem of the time-dependent Maxwell equations. For the exterior problem of these equations, we refer to Barucq and Hanouzet [5]. One can consult, as well, the books by Dautray and Lions [9], and by Duvaut and Lions [10]. The standard approach, *i.e.* the implementation of potential theory, leads —in the case of our stratified scattered— to an infinite system of integral equations. Even in the case of a finite number of layers, the generalized solutions method does not present disadvantages as far as the length of the proof is concerned, in comparison to the standard method. For the standard method we refer to the work of Stevenson [15], Gray and Kleinman [12], Knauff and Kress [13], and Werner [16].

For the quantitative study, at low-frequencies, of the transmission problem, for a multi-layered scatterer, we refer to [2] and [3].

2 – Statement and solvability of the transmission problem

Consider electromagnetic wave propagation in an isotropic medium in $\mathbb{R}^3$, with space independent electric permittivity $\varepsilon \in \mathbb{R}$, magnetic permeability $\mu \in \mathbb{R}$, and electric conductivity $\sigma \in \mathbb{R}$. The electromagnetic wave is described by the electric field $\mathbb{E}$, and the magnetic field $\mathbb{H}$, satisfying the Maxwell equations

$$\operatorname{curl} \mathbb{E} + \mu \frac{\partial \mathbb{H}}{\partial t} = 0\,, \tag{2.1}$$

$$\operatorname{curl} \mathbb{H} - \varepsilon \frac{\partial \mathbb{E}}{\partial t} = \sigma \mathbb{E}\,. \tag{2.2}$$

For time-harmonic electromagnetic waves of the form

$$\mathbb{E}(x,t) = \operatorname{Re}\left\{ \left(\varepsilon + i\frac{\sigma}{\omega}\right)^{-\frac{1}{2}} E(x) e^{-i\omega t} \right\}\,, \tag{2.3}$$

$$\mathbb{H}(x,t) = \operatorname{Re}\left\{ \mu^{-\frac{1}{2}} H(x) e^{-i\omega t} \right\}\,, \tag{2.4}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts E and H, satisfy the reduced Maxwell equations:

$$\operatorname{curl} E - ikH = 0\,, \tag{2.5}$$

$$\operatorname{curl} H + ikE = 0\,, \tag{2.6}$$

where the wave number k is a constant given by

$$k^2 = \left(\varepsilon + i\frac{\sigma}{\omega}\right) \mu\omega^2\,, \tag{2.7}$$

where the sign of k chosen such that

$$\operatorname{Im} k \geq 0 \tag{2.8}$$

Remark 2.1. Any solution $\{E, H\}$ of (2.5), (2.6) is divergence free, *i.e.* $\operatorname{div} E = \operatorname{div} H = 0$. This follows immediately, with the use of the vector identity $\operatorname{div} \operatorname{curl} F = 0$.

Let $\tilde{\Omega}$ be a bounded, convex subset of $\mathbb{R}^3$, containing the origin, and having a C^2-boundary S_0. The exterior, Ω_0, of $\tilde{\Omega}$ is a homogeneous isotropic medium, with vanishing conductivity $\sigma_0 = 0$, and wave number k_0 given by $k_0^2 = \varepsilon_0\mu_0\omega^2 \in \mathbb{R}$. A core Ω_c, within which lies the origin, is contained in $\tilde{\Omega}$. We actually work in $\Omega = \tilde{\Omega} - \Omega_c$; we suppose that the boundary, S_c, of Ω_c is a C^2-surface. Ω is considered to be a bonded, nested, piecewise homogeneous body, consisting of annuli-like regions Ω_j, divided by C^2-surface S_j, $j = 1, 2, \ldots$ Each surface S_j surrounds S_{j+1} and S_c for all j. We assume that $\mathrm{dist}(S_{j-1}, S_j) > 0$ for $j = 1, 2, \ldots$, and that $\lim_{j\to\infty} S_j = S_c$. Let ε_j, μ_j, $\sigma_j \in \mathbb{R}$, be the electric permittivity, magnetic permeability, and electric conductivity, respectively, in Ω_j, *i.e.* Ω is a scatterer with piecewise constant electric permittivity, magnetic permeability, and electric conductivity; for the use of such scatterers, we refer to Fournet [11].

By the adjective "bonded" it is meant that the tangential components of the time independent electric and magnetic fields are continuous across each S_j, $j = 1, 2, \ldots$. Moreover, we assume that $\sum_{j=1}^{\infty} |S_j| < +\infty$, where $|S_j|$ is the measure of S_j. Such an Ω will be referred to as an infinitely stratified scattered.

We shall consider the scattering for time-harmonic waves by an infinitely stratified scatterer. Let E^{inc}, H^{inc} be the set of incident fields of the form

$$H^{\mathrm{inc}}(x) = \hat{b}\exp\{ik_0\hat{k}\cdot x\} \quad , \quad E^{\mathrm{inc}}(x) = -\frac{1}{ik_0}\,\mathrm{curl}\,H^{\mathrm{inc}}(x)\,,$$

where $\hat{k}$ is the propagation unit vector, $\hat{b}$ is the polarization unit vector; $\hat{b}\cdot\hat{k} = 0$.

The incoming wave E^{inc}, H^{inc} is scattered by Ω, resulting to the emanation of a scattered wave E_0, H_0. The total wave E^{tot}, H^{tot} in Ω_0 is given by

$$E^{\mathrm{tot}} = E^{\mathrm{inc}} + E_0, \qquad H^{\mathrm{tot}} = H^{\mathrm{inc}} + H_0\,. \tag{2.9}$$

The pairs E^{tot}, H^{tot} and E_0, H_0 satisfy the reduced Maxwell equations in Ω_0.

Moreover, E_0, H_0 must satisfy the Silver-Muller radiation conditions

$$\lim_{r\to\infty}(H_0 \times x - rE_0) = 0\,, \tag{2.10}$$

or

$$\lim_{r\to\infty}(E_0 \times x - rH_0) = 0\,, \tag{2.11}$$

where $r = |x|$, and the limit is assumed to hold uniformly in all directions.

Let k_j be the wave number in Ω_j, $j = 1, 2, \dots$, given by

$$k_j^2 = \left(\varepsilon_j + \frac{\sigma_j}{\omega}\right)\mu_j\omega, \qquad \operatorname{Im} k_j \geq 0\,.$$

The mathematical description of the diffraction of an incident time-harmonic wave, as considered above, by an infinitely stratified scattered, leads to a transmission problem of the following form:

Find E, H satisfying

$$(2.12)\quad \begin{cases} \left.\begin{aligned} \operatorname{curl} E_j - ik_jH_j = 0 \\ \operatorname{curl} H_j + ik_jE_j = 0 \end{aligned}\right\} & \text{in } \ \Omega_j,\ j = 0, 1, 2, \dots \quad \begin{aligned}&\text{(i)}\\&\text{(ii)}\end{aligned} \\[2ex] \left.\begin{aligned} n \times (E_0 + E^{\mathrm{inc}}) = n \times E_1 \\ n \times (H_0 + H^{\mathrm{inc}}) = n \times H_1 \end{aligned}\right\} & \text{on } \ S_0 \quad \begin{aligned}&\text{(iii)}\\&\text{(iv)}\end{aligned} \\[2ex] \left.\begin{aligned} n \times E_j = n \times E_{j+1} \\ n \times H_j = n \times H_{j+1} \end{aligned}\right\} & \text{on } \ S_j,\ j = 0, 1, 2, \dots \quad \begin{aligned}&\text{(v)}\\&\text{(vi)}\end{aligned} \\[2ex] E_0\,,\ H_0 \text{ satisfy (2.10)} & \text{(vii)} \end{cases}$$

where E_j, H_j denote the restriction of E, H in Ω_j, $j = 1, 2, \dots$. Moreover, the boundary behaviour, on the surface of the core, of the desired solution must be specified. We assume that

$$n \times E = n \times H = 0, \qquad \text{on} \quad S_c\,. \tag{2.13}$$

In the remaining of this section, we shall study the following non-homogeneous model mathematical transmission problem: Find vector fields E_j, $H_j \in C^{1,a}(\Omega_j) \cap C(\bar{\Omega}_j)$, $j = 0, 1, 2, \dots$ where $a \in (0, 1)$, satisfying the equations

$$
\left.\begin{array}{lll}
(2.14) & \begin{cases} \operatorname{curl} E_j - i\lambda_j H_j = 0 \\ \operatorname{curl} H_j + i\lambda_j E_j = 0 \end{cases} & \text{in } \ \Omega_j, \ j = 0,1,2,\dots \\
(2.15) & \begin{cases} p_0 n \times E_0 = p_1 n \times E_1 + f \\ q_0 n \times H_0 = q_1 n \times H_1 + g \end{cases} & \text{on } \ S_0 \\
(2.16) & \begin{cases} p_j n \times E_j = p_{j+1} n \times E_{j+1} \\ q_j n \times H_j = q_{j+1} n \times H_{j+1} \end{cases} & \text{on } \ S_j, \ j = 1,2,\dots \\
 & \quad (2.10) \text{ and } (2.13) &
\end{array}\right\} (NHTP)
$$

where $\lambda_j,\ p_j,\ q_j \in \mathbb{C} - \{0\}$, $j = 0,1,2,\dots$, and $f, g \in T_d^{0,a}(S_0)$; for the definition of $T_d^{0,a}$ we refer to [7].

The corresponding homogeneous transmission problem (*i.e.* when $f = g \equiv 0$ on S_0) will be denoted by (HTP).

We are now in a position to prove

THEOREM 2.1. *Suppose that the following conditions hold:*

$$
\sup_j \left| \frac{p_j \bar{q}_j}{p_0 \bar{q}_0} \lambda_j \right| < +\infty \tag{2.17}
$$

and

$$
\left| \operatorname{Im}\left(\frac{p_j \bar{q}_j}{p_0 \bar{q}_0}\right) \operatorname{Re}\lambda_j \right| \le \operatorname{Re}\left(\frac{p_j \bar{q}_j}{p_0 \bar{q}_0}\right) \operatorname{Im}\lambda_j, \quad j = 0,1,2,\dots \tag{2.18}
$$

Then (HTP) has only the trivial solution.

PROOF. We consider a ball B_R, with boundary S_R, centered at the origin with radius R, large enough to include $\bar{\Omega}$ in its interior. We denote by $\Omega_{0,R}$ the set $\Omega_0 \cap B_R$. By the radiation condition (2.10) we have

$$
\begin{aligned}
&\int_{S_R} \{|H_0 \times n|^2 + |E_0|^2\} ds - 2 \operatorname{Re} \int_{S_R} (n \times E_0) \cdot \bar{H}_0 \, ds = \\
&= \int_{S_R} |H_0 \times n - E_0|^2 ds = o(1), \quad \text{as} \quad R \longrightarrow \infty\,.
\end{aligned} \tag{2.19}
$$

Applying the divergence theorem on the vector field $E_0 \times \bar{H}_0$ in $\Omega_{0,R}$ with $\partial\Omega_{0,R} = S_R \cup S_0$, using that E_0, H_0 are solutions of the Maxwell equations in Ω_0, and introducing the boundary conditions on S_0, we get

$$\begin{aligned}\int_{S_R} (n \times E_0) \cdot \bar{H}_0 \, ds = \frac{p_1 \bar{q}_1}{p_0 \bar{q}_0} \int_{S_0} (n \times E_1) \cdot \bar{H}_1 \, ds + \\ + \int_{\Omega_{0,R}} \{ i\lambda_0 |H_0|^2 - i\bar{\lambda}_0 |E_0|^2 \} dx\end{aligned} \tag{2.20}$$

Repeating the above procedure successively on each region Ω_j, $j = 1, 2, \ldots$ and taking into account (2.13) we obtain

$$\begin{aligned}\int_{S_R} (n \times E_0) \cdot \bar{H}_0 \, ds = i\lambda_0 \int_{\Omega_{0,R}} |H_0|^2 dx - i\bar{\lambda}_0 \int_{\Omega_{0,R}} |E_0|^2 \, dx + \\ + \sum_{j=1}^{\infty} i \frac{p_j \bar{q}_j}{p_0 \bar{q}_0} \lambda_j \int_{\Omega_j} |H_1|^2 dx - \sum_{j=1}^{\infty} i \frac{p_j \bar{q}_j}{p_0 \bar{q}_0} \lambda_j \int_{\Omega_j} |E_j|^2 dx \, .\end{aligned} \tag{2.21}$$

By the structure of Ω, we have that

$$\sum_{j=1}^{\infty} \int_{\Omega_j} |H_j|^2 dx = \|H\|^2_{(L^2(\Omega))^3}, \quad \sum_{j=1}^{\infty} \int_{\Omega_j} |E_j|^2 dx = \|E\|^2_{(L^2(\Omega))^3} \, . \tag{2.22}$$

Hence, from (2.17) and (2.22) we conclude that the two series appearing in the RHS of (2.21) are (uniformly) convergent. We now insert the real part of (2.21) into (2.19) to get

$$\begin{aligned}&\frac{1}{2} \int_{S_R} \{ |H_0 \times n|^2 + |E_0|^2 \} ds + o(1) = \\ &= \mathrm{Im}(\bar{\lambda}_0) \int_{\Omega_{0,R}} |E_0|^2 dx - \mathrm{Im}(\lambda_0) \int_{\Omega_{0,R}} |H_0|^2 dx + \\ &+ \sum_{j=1}^{\infty} \mathrm{Im} \left(\frac{p_j \bar{q}_j}{p_0 \bar{q}_0} \bar{\lambda}_j \right) \int_{\Omega_j} |E_j|^2 dx - \sum_{j=1}^{\infty} \mathrm{Im} \left(\frac{p_j \bar{q}_j}{p_0 \bar{q}_0} \lambda_j \right) \int_{\Omega_j} |H_j|^2 dx\end{aligned} \tag{2.23}$$

From (2.18) we take

$$\operatorname{Im}\left(\frac{p_j \bar{q}_j}{p_0 \bar{q}_0}\lambda_j\right) \geq 0 \text{ and } \operatorname{Im}\left(\frac{\bar{p}_j q_j}{\bar{p}_0 q_0}\lambda_j\right) \geq 0 \quad j = 0, 1, 2, \dots \tag{2.24}$$

which, in view of (2.23) yield

$$(\operatorname{Im}\lambda_0)\int_{\Omega_{0,R}} |E_0|^2 dx = (\operatorname{Im}\lambda_0)\int_{\Omega_{0,R}} |H_0|^2 dx = 0, \quad R \to \infty\,. \tag{2.25}$$

If $\operatorname{Im}\lambda_0 > 0$, it follows that

$$E_0 = H_0 = 0, \quad \text{in} \quad \Omega_0\,. \tag{2.26}$$

On the other hand, if $\operatorname{Im}\lambda_0 = 0$, (2.21) and (2.24) yield

$$\operatorname{Re}\left(\int_{S_R} (n \times E_0) \cdot \bar{H}_0 \, ds\right) \leq 0\,, \tag{2.27}$$

whereby, implementing Theorem 6.10 of [7], we obtain again (2.26).

We proceed to showing that $E_1 = H_1 = 0$ in Ω_1.

By (2.26) and the transmission conditions on S_0, we obtain that

$$n \times E_1 = n \times H_1 = 0, \quad \text{on} \quad S_0\,. \tag{2.28}$$

Rewriting the Maxwell equations in Ω_1 as a first order system of six equations (via the components of E_1 and H_1), and doing the same with the initial data (2.28), we are led to a Cauchy problem for the referred to system. By Holmgren's uniqueness theorem [14], which is easily seen to apply in this case, we conclude that $E_1 = H_1 = 0$ in $\Omega_1 \cap V$, where V is a neighborhood of any point of S_0. Now, by the unique continuation principle for the Maxwell equations, ([7], Theorem 9.3), we obtain that $E_1 = H_1 = 0$ in Ω_1, as desired. By the same argument, E_2 and H_2 are shown to be vanishing in Ω_2, etc. We thus conclude that (HTP) has only the trivial solution. □

We now proceed to consider the solvability of $(NHTP)$. We need the following function spaces:

$$
\begin{aligned}
X^0(\Omega) &= (L^2(\Omega))^3 \\
X^1(\Omega) &= \{h \in X^0(\Omega) : \operatorname{curl} h \in X^0(\Omega)\} \\
R^m(\Omega_0) &= \Big\{ \begin{bmatrix} u \\ w \end{bmatrix} : u, w \in X^m_{\rm loc}(\Omega_0) : w(x) \times \frac{x}{|x|} - u(x) = 0 \left(\frac{x}{|x|} \right), \\
&\qquad \text{uniformly in all directions} \quad \frac{x}{|x|} \Big\}, \quad m = 0, 1 \\
Y^0_T(S_0) &= \{h \in X^0(S_0) : n \cdot h = 0 \quad \text{on} \quad S_0\} \\
Y^0_d(S_0) &= \{h \in Y^0_T(S_0) : \mathrm{Div} h \in H^{-1/2}(S_0)\} \\
Y^{1/2}_T(S_0) &= \{h \in (H^{1/2}(S_0))^3 : n \cdot h = 0 \quad \text{on} \quad S_0\} \\
Y^{1/2}_d(S_0) &= \{h \in (Y^{1/2}(S_0) : \mathrm{Div} h \in L^2(S)\} \,,
\end{aligned}
$$

where $\mathrm{Div}\, h$ is the surface divergence of h; for a definition [7].

We may rewrite (2.14) in the following unified way:

$$
\begin{aligned}
\operatorname{curl} E(x) &= i\lambda(x) H(x) \\
\operatorname{curl} H(x) &= -i\lambda(x) E(x)
\end{aligned} \tag{2.29}
$$

where

$$
\begin{aligned}
&E(x) = E_j(x), \quad H(x) = H_j(x), \\
&\lambda(x) = \lambda_j, \quad x \in \Omega_j, \quad j = 0, 1, 2, \dots
\end{aligned} \tag{2.30}
$$

Let, moreover,

$$
p(x) = p_j, \quad q(x) = q_j, \quad x \in \Omega_j, \quad j = 0, 1, 2 \dots \tag{2.31}
$$

$$
F(x) = \begin{cases} f(x) & x \in S_0 \\ 0 & x \in S_j \end{cases}, \quad G(x) = \begin{cases} g(x) & x \in S_0 \\ 0 & x \in S_j \end{cases}, \; j = 1, 2, \dots \tag{2.32}
$$

The transmission condition (2.15), (2.16) may, also, be rewritten as

$$
\left.
\begin{aligned}
&[p(x) n \times E(x)]^+_- = F(x) \\
&[q(x) n \times H(x)]^+_- = G(x)
\end{aligned}
\right. \qquad x \in S_j \,, \tag{2.33}
$$

where $[u(x)]_-^+ := u^+(x) - u^-(x)$ and $u^+(x)(u^-(x))$ denotes the limit of u on S_j from $\Omega_j(\Omega_{j+1})$.

DEFINITION 2.1. A function $\begin{bmatrix} E \\ H \end{bmatrix} \in (X^0)^2 \cap R^0(\Omega_0)$ is called a generalized solution of (2.29), (2.33) for F, $G \in Y_T^0(S_0)$, iff

$$\begin{aligned} &\int_{\mathbb{R}^3-\Omega_c} \begin{bmatrix} q(x) & 0 \\ 0 & p(x) \end{bmatrix} \begin{bmatrix} 0 & -\operatorname{curl}\varphi(x) \\ \operatorname{curl}\psi(x) & 0 \end{bmatrix} \cdot \begin{bmatrix} E(x) \\ H(x) \end{bmatrix} dx + \\ &- i \int_{\mathbb{R}^3-\Omega_c} \lambda(x) \begin{bmatrix} q(x) & 0 \\ 0 & p(x) \end{bmatrix} \begin{bmatrix} \varphi(x) & 0 \\ 0 & \psi(x) \end{bmatrix} \cdot \begin{bmatrix} E(x) \\ H(x) \end{bmatrix} dx = \\ &= \int_{S_0} \begin{bmatrix} \varphi(s) & 0 \\ 0 & \psi(s) \end{bmatrix} \cdot \begin{bmatrix} F(s) \\ G(s) \end{bmatrix} ds\,, \end{aligned} \tag{2.34}$$

for every $\varphi, \psi \in \{h \in X^1_{loc}(\mathbb{R}^3) : n \times h = 0$ on S_c, and $h(x) = o(\frac{1}{|x|}, \ |x| \to \infty\}$.

In relation to $\{(2.29), (2.33)\}$ we have the following regularity result; its proof is omitted for the sake of brevity, and may be performed by standard regularity arguments. See [4], [5], [6].

THEOREM 2.2. *Let* $\begin{bmatrix} E \\ H \end{bmatrix}$ *be a generalized solution of* $\{(2.29), (2.33)\}$.

(i) *Assume that* F, $G \in Y_T^{1/2}(S_0)$. *Then* $\begin{bmatrix} E \\ H \end{bmatrix} \in \left(X^1(\bar{\Omega})\right)^2 \cap R^1(\Omega_0)$.

(ii) *Assume that* F, $G \in T_d^{0,a}(S_0)$. *Then* $\begin{bmatrix} E \\ H \end{bmatrix} \in \left(C^{1,a}(\Omega_j) \cap C(\bar{\Omega}_j)\right)^2$, $j = 1, 2, \ldots$, *and* E, H *satisfy the radiation condition* (2.10).

REMARK 2.3. If $\begin{bmatrix} E \\ H \end{bmatrix}$ satisfies (2.34), and has the regularity properties of the conclusions of either Theorem 2.2. (i), or Theorem 2.2. (ii), then (2.33) is satisfied.

We may now state and prove the existence result for $(NHTP)$.

THEOREM 2.4. *Consider* $(NHTP)$ *with its parameters satisfying* (2.17) *and* (2.18). *Let, moreover,*

$$\sum_{j=0}^{\infty} p_j \lambda_j \neq 0 \quad \textit{and} \quad \sum_{j=0}^{\infty} q_j \lambda_j \neq 0\,. \tag{2.35}$$

Then $(NHTP)$ *has a unique solution.*

PROOF. Setting $u = \begin{bmatrix} E \\ H \end{bmatrix}$, $\mathbb{F}$ to be the extension of $\begin{bmatrix} F \\ G \end{bmatrix}$ into $X^0(\mathbb{R}^3)$, and introducing as in [9] the Maxwell operator

$$A := \begin{bmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{bmatrix} \tag{2.36}$$

we see that (2.29) may be written as

$$Au - i\lambda(x)u = \mathbb{F}\,, \tag{2.37}$$

while the corresponding homogeneous equation is

$$Au - i\lambda(x)u = 0\,. \tag{2.38}$$

Employing a line of argumentation analogous to that of [9], chapter IV (see also [6]), we may see that provided (2.35) is satisfied, the Fredholm alternative may be implemented for (2.37), (2.38). By the uniqueness of the trivial solution for (2.38) we conclude that (2.27) has a unique generalized solution, which —by Theorem 2.2 (ii)— is classical, obtaining thus the solvability of $(NHTP)$. ∎

REMARK 2.4. We note that the transmission problem $\{(2.12), (2.13)\}$, arising from the diffraction of an incident plane time-harmonic electromagnetic wave, by an infinitely stratified scatterer is a special case of $(NHTP)$ for $\lambda_j = k_j$, $p_j = q_j = 1$, $j = 0, 1, 2, \ldots$, and $f = E^{\text{inc}} \times n$, $g = H^{\text{inc}} \times n$. The conditions (2.17), (2.18) take in this case the form

$$\sup_j |k_j| < +\infty, \qquad j = 0, 1, 2, \ldots\,, \tag{2.39}$$

$$\operatorname{Im} k_j \geq 0, \qquad j = 0, 1, 2, \ldots\,, \tag{2.40}$$

while the conditions (2.35) become

$$\sum_{j=0}^{\infty} k_j \neq 0\,. \tag{2.41}$$

Let us note that (2.40) has been assumed already (in the definition of the k_j, after (2.11)), and that, provided the series in (2.41) converges, its sum cannot be zero since the k_j are wave numbers. As for (2.39), it is physically meaningfull.

REMARK 2.5. In the case that there is no core Ω_c inside Ω, and Ω is not supposed to be stratified (*i.e.* it consists of only one layer), then conditions (2.17), (2.18), (2.35) are slightly more general (as to that p_0 and p_1 are not equal to 1) of those appearing in [1] for the non-conductive case, and in [17].

3 – Integral representations of the exterior fields

In this section we shall construct integral representations, which contain all the information about the transmission and radiation conditions. One representation will be evaluated for the near exterior field and another for the far scattered field, which is known as the scattering amplitude, or far field pattern.

The total exterior field E^{tot}, H^{tot}, is the superposition of the incident and the scattered field, cf. (2.9). As it is well known, [7], [15], the scattered field E_0, H_0 has the following Stratton-Chu representation:

$$\begin{aligned} E_0(x) = \operatorname{curl}\int_{S_0} n' \times E_0(x')\phi(x,x')ds' + \\ - \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\int_{S_0} n' \times H_0(x')\phi(x,x')ds',\ x\in\Omega_0\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} H_0(x) = \operatorname{curl}\int_{S_0} n' \times H_0(x')\phi(x,x')ds' + \\ + \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\int_{S_0} n' \times E_0(x')\phi(x,x')ds',\ x\in\Omega_0\,, \end{aligned} \tag{3.2}$$

where n' denotes the unit normal vector to the surface S_0 directed into the exterior of Ω, and

$$\phi(x,x') = \frac{1}{4\pi}\,\frac{e^{ik_0|x-x'|}}{|x-x'|}, \quad x \neq x' \tag{3.3}$$

is the fundamental solution to the Helmholtz equation. As always, the observation vector x is assumed to have measure $|x|$ greater than the characteristic dimension of the scatterer, that is the radius of the smallest circumscribable sphere around the scatterer. Hence, there exists $\theta > 0$ such that $|x - x'| \geq \theta^{-1}$ and consequently

$$|\phi(x, x')| \leq \frac{\theta}{4\pi} \tag{3.4}$$

LEMMA 3.1. *The series*

$$\sum_{j=1}^{\infty} \int_{\Omega_j} \mathrm{curl}_{x'}(E_j(x')\phi(x, x'))dx' \tag{3.5}$$

$$\sum_{j=1}^{\infty} \int_{\Omega_j} \mathrm{curl}_{x'}(H_j(x')\phi(x, x'))dx' \tag{3.6}$$

converge uniformly.

PROOF. For the solutions to the Maxwell equations E_j, $H_j \in C^{1,a}(\Omega_j) \cap C(\bar{\Omega}_j)$, there exists $M > 0$ such that

$$|E_j(x')| \leq M, \quad |H_j(x')| \leq M, \quad x \in \Omega_j, \quad j = 1, 2, \dots \tag{3.7}$$

From a known vector formula we take

$$\begin{aligned} &\mathrm{curl}_{x'}(E_j(x')\phi(x, x')) = \\ &\qquad = \mathrm{grad}_{x'}\, \phi(x, x') \times E_j(x') + \phi(x, x')\, \mathrm{curl}_{x'}\, E_j(x')\,. \end{aligned} \tag{3.8}$$

Also we have

$$\mathrm{grad}_{x'}\, \phi(x, x') = \left(\frac{1}{|x - x'|} - ik_0\right) \frac{e^{ik_0|x-x'|}}{|x - x'|}\, \frac{x - x'}{|x - x'|}\,, \tag{3.9}$$

$$|\,\mathrm{grad}_{x'}\, \phi(x, x')| \leq (\theta + |k_0|)\frac{\theta}{4\pi}\,. \tag{3.10}$$

From the Maxwell equations we take

$$|\,\mathrm{curl}\, E_j(x')| \leq |k_j| M \leq k^* M\,, \tag{3.11}$$

where $k^* = \sup_j |k_j|$.

Using (3.4), (3.7)-(3.11) we get the following estimate

$$|\operatorname{curl}_{x'}(E_j(x')\phi(x,x'))| \leq \frac{\theta M}{4\pi}(\theta + |k_0| + k^*)\,, \tag{3.12}$$

$$\left|\int_{\Omega_j} \operatorname{curl}_{x'}(E_j(x')\phi(x,x'))dx'\right| \leq \frac{\theta M}{4\pi}(\theta + |k_0| + k^*)|\Omega_j|\,, \tag{3.13}$$

where $|\Omega_j|$ is the measure of the volume of Ω_j. Since, by the structure of the scatterer, we have $\sum_{j=1}^{\infty}|\Omega_j| = |\Omega|$, the series (3.5) converges uniformly. It is clear that also the series (3.6) converges uniformly. □

We denote by $\psi^E(x)$ and $\psi^H(x)$ the series (3.5) and (3.6) respectively. Then we can prove the following theorem.

THEOREM 3.1. *The total exterior field of the transmission problem* $(NHTP)$ *has the integral representation*

$$E^{\text{tot}}(x) = E^{\text{inc}}(x) + \operatorname{curl}\psi^E(x) - \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\psi^H(x)\,, \tag{3.14}$$

$$H^{\text{tot}}(x) = H^{\text{inc}}(x) + \operatorname{curl}\psi^H(x) - \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\psi^E(x)\,. \tag{3.15}$$

PROOF. We shall work with E^{tot}; the same argument is applied for H^{tot}, as well. From (2.9) and (3.1) taking into account that E^{inc}, H^{inc} is a solution to the Maxwell equations, we conclude that

$$\begin{aligned} E^{\text{tot}}(x) = E^{\text{inc}}(x) + \operatorname{curl}\int_{S_0} n' \times E^{\text{tot}}(x')\phi(x,x')ds' + \\ - \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\int_{S_0} n' \times H^{\text{tot}}(x')\phi(x,x')ds' \end{aligned} \tag{3.16}$$

Inserting the transmission conditions on the surface S_0, given by (2.12, (iii), (iv)), to (3.16) we obtain

$$\begin{aligned} E^{\text{tot}}(x) = E^{\text{inc}}(x) + \operatorname{curl}\int_{S_0} n' \times E_1(x')\phi(x,x')ds' + \\ - \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\int_{S_0} n' \times H_1(x')\phi(x,x')ds'\,. \end{aligned} \tag{3.17}$$

Applying successively the divergence theorem on the vector field $E_j(x')$ $\phi(x,x')$ in Ω_j, with $\partial\Omega_j = S_{j-1} - S_j$, and using the transmission conditions (2.12, (v), (iv)), we get, fro $j = 1, 2, \dots, N$

$$
\begin{aligned}
E^{\rm tot}(x) &= E^{\rm inc}(x) + \operatorname{curl}\int_{S_N} n' \times E_N(x')\phi(x,x')ds' + \\
&+ \operatorname{curl}\sum_{j=1}^{N}\int_{\Omega_j} \operatorname{curl}_{x'}(E_j(x')\phi(x,x')dx' + \\
&- \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\int_{S_N} n' \times H_N(x')\phi(x,x')ds' + \\
&- \frac{1}{ik_0}\operatorname{curl}\operatorname{curl}\sum_{j=1}^{N}\int_{\Omega_j} \operatorname{curl}_{x'}(H_j(x')\phi(x,x'))ds'\,.
\end{aligned}
\tag{3.18}
$$

Now, letting $N \to \infty$ and taking into account the boundary conditions on the core (2.13), and the convergence of the series (3.5), (3.6), we complete the proof. ∎

In the sequel, we study the far field patterns. Using the asymptotic form

$$
|x - x'| = |x| - \hat{x}\cdot x' + O\left(\frac{1}{|x|}\right),\ |x| \to \infty\,,
\tag{3.19}
$$

where $\hat{x} = \frac{x}{|x|}$, we derive

$$
\begin{aligned}
\phi(x,x') &= \frac{1}{4\pi}\,\frac{e^{ik_0|x-x'|}}{|x-x'|} = \\
&= \frac{1}{4\pi}\,\frac{e^{ik_0|x|}}{|x|}\left[e^{-ik_0\hat{x}\cdot x'} + O\left(\frac{1}{|x|}\right)\right], |x| \to \infty\,.
\end{aligned}
\tag{3.20}
$$

Inserting (3.20) to (3.5) and (3.6), we obtain, for $|x| \to \infty$

$$
\psi^E(x) = \frac{1}{4\pi}\,\frac{e^{ik_0|x|}}{|x|}\psi^E_\infty(\hat{x})\,,
\tag{3.21}
$$

$$
\psi^H(x) = \frac{1}{4\pi}\,\frac{e^{ik_0|x|}}{|x|}\psi^H_\infty(\hat{x})\,,
\tag{3.22}
$$

where the vector fields ψ_∞^E and ψ_∞^H are defined on the sphere and are given by the uniformly convergent series

$$\psi_\infty^E(\hat{x}) = \sum_{j=1}^{\infty} \int_{\Omega_j} \mathrm{curl}(E_j(x')e^{-ik_0\hat{x}\cdot x'})dx' + O\left(\frac{1}{|x|}\right), \tag{3.23}$$

$$\psi_\infty^H(\hat{x}) = \sum_{j=1}^{\infty} \int_{\Omega_j} \mathrm{curl}(H_j(x')e^{-ik_0\hat{x}\cdot x'})dx' + O\left(\frac{1}{|x|}\right). \tag{3.24}$$

Substituting (3.21) and (3.22) into (3.14) and (3.15), the scattered field admits the following form, as $|x| \to \infty$.

$$\begin{aligned} E_0 = \frac{e^{ik_0|x|}}{|x|}\Big[&\frac{ik_0}{4\pi}\hat{x} \times (\psi_\infty^E(\hat{x}) - \hat{x} \times \psi_\infty^H(\hat{x}))+ \\ &+ \frac{1}{4\pi}(\mathrm{curl}\,\psi_\infty^E(\hat{x}) - \hat{x} \times \mathrm{curl}\,\psi_\infty^H(\hat{x})+ \\ &- \mathrm{curl}(\hat{x} \times \psi_\infty^H(\hat{x}) - \frac{1}{ik_0}\mathrm{curl}\,\mathrm{curl}\,\psi_\infty^H(\hat{x})) + O\left(\frac{1}{|x|}\right)\Big], \end{aligned} \tag{3.25}$$

$$\begin{aligned} H_0(x) = \frac{e^{ik_0|x|}}{|x|}\Big[&\frac{ik_0}{4\pi}\hat{x} \times (\psi_\infty^H(\hat{x}) + \hat{x} \times \psi_\infty^E(\hat{x}))+ \\ &+ \frac{1}{4\pi}(\mathrm{curl}\,\psi_\infty^H(\hat{x}) + \hat{x} \times \mathrm{curl}\,\psi_\infty^E(\hat{x})+ \\ &+ \mathrm{curl}(\hat{x} \times \psi_\infty^E(\hat{x}) + \frac{1}{ik_0}\mathrm{curl}\,\mathrm{curl}\,\psi_\infty^E(\hat{x})) + O\left(\frac{1}{|x|}\right)\Big]. \end{aligned} \tag{3.26}$$

After lengthy calculations we obtain

$$E_0(x) = \frac{e^{ik_0|x|}}{|x|}\left[E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right],\ |x| \to \infty, \tag{3.27}$$

$$H_0(x) = \frac{e^{ik_0|x|}}{|x|}\left[H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right],\ |x| \to \infty, \tag{3.28}$$

where

$$E_\infty(\hat{x}) = \frac{ik_0}{4\pi}\hat{x} \times (\psi_\infty^E(\hat{x}) - \hat{x} \times \psi_\infty^H(\hat{x})), \tag{3.29}$$

$$H_\infty(\hat{x}) = \frac{ik_0}{4\pi}\hat{x} \times (\psi_\infty^H(\hat{x}) + \hat{x} \times \psi_\infty^E(\hat{x})), \tag{3.30}$$

are the electric far field pattern and the magnetic far field pattern, respectively, [7]. Also, if n is the unit outward normal on the unit sphere, then (3.29) and (3.30) imply that

$$H_\infty = n \times E_\infty, \quad n \cdot E_\infty = n \cdot H_\infty = 0\,. \tag{3.31}$$

The above may be summarized in the following

THEOREM 3.2. *The asymptotic form, as $|x| \to \infty$, of the scattering field of the transmission problem* $(NHTP)$ *is given by* (3.27)*,* (3.28) *and satisfies the relations* (3.31)*.*

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