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On Some Uniform Bounds for Smooth Algebraic Functions

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RIASSUNTO: *In questo lavoro si dimostrano alcune disuguaglianze relative a funzioni algebriche C^∞ (cioè soluzioni C^∞ di equazioni polinomiali) che sono cruciali per provare proprietà di scala di medie e massimi delle suddette funzioni, tipiche nel caso polinomiale. Si ottiene inoltre che $x \longmapsto (y - f(x))^2$, dove f è una funzione algebrica liscia, si comporta come un polinomio (relativamente alle proprietà di scala di medie e massimi).*

ABSTRACT: *In this work we prove some inequalities for smooth algebraic functions (smooth solutions to polynomial equations) which are crucial for proving some scaling properties of their averages and maxima, that are typical in the case of polynomials. As a byproduct, it is shown that $x \longmapsto (y - f(x))^2$, where f is a smooth algebraic function, behaves like a polynomial (in terms of scaling properties of averages and maxima).*

1 – Introduction

The purpose of this paper is to establish some polynomial-like properties of smooth real-valued algebraic functions, i.e. smooth solutions to polynomial equations. The properties we are interested in are similar to those stated in Fefferman [1]:
if $P(x)$ is a polynomial of degree $\leq d$ then, with Q a (closed) cube of $\mathbb{R}^n$

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and constants depending only on d and the dimension n :

$$\text{Av}_{x\in Q}|P| \le \max_{x\in Q}|P| \le C\text{Av}_{x\in Q}|P|; \tag{a}$$

$$\max_{x\in Q}|\nabla P| \le C(\text{diam}Q)^{-1}\max_{x\in Q}|P|; \tag{b}$$

If $P \ge 0$ on Q then $\exists$ a subcube $Q' \subset Q$ with $(diamQ') \ge c(diamQ)$ on which

$$\min_{x\in Q'} P \ge \frac{1}{2}\max_{x\in Q} P. \tag{c}$$

(Notice that (c) is a consequence of (b).) We shall refer to these properties as *polynomial-like scaling properties.*

The work of Stein and his collaborators (see for instance Nagel-Stein-Wainger [9]) brought to light that a subelliptic differential operator is governed by a family of non-Euclidean balls. In Parmeggiani [10] and [11] a family of non-Euclidean balls in the cotangent bundle of $\mathbb{R}^n$ is attached to the (total) symbol $p(x,\xi)$, supposed nonnegative, of a subelliptic pseudodifferential operator, by embedding the unit cube through canonical transformations satisfying suitable estimates on the derivatives(1). A crucial step in this construction is an extension of the above properties (a), (b) and (c) (and a few more) to smooth real-valued algebraic functions and to polynomials evaluated on graphs of smooth algebraic functions. These results have been generalized by C.Fefferman and R.Narasimhan in [6] and [7], works in which they prove also similar properties for polynomials evaluated on higher codimensional smooth algebraic varieties.

We start by proving some "ellipticity" properties of the average, with respect to one of the variables, of a nonnegative polynomial. We then prove two theorems about scaling properties of averages and maxima of functions whose gradients are "controlled" by the function itself (in terms of $L^\infty-$norm). Afterwards we show that the aforementioned properties extend to smooth algebraic functions and to polynomials of the kind $(y-X)^d$, where $y \in \mathbb{R}$ is a parameter and X is a real variable, when evaluated at $X = f(x)$ with f a smooth algebraic function, and when

(1)This results in necessary and sufficient conditions for L^2-a-priori bounds for subelliptic operators. See Fefferman [1], Fefferman-Phong [2,3,4,5].

evaluated at $X = f(x_1, x') - (\mathrm{Av}_{x_1} f)(x')$, with f a smooth algebraic function in x_1, polynomial (of a-priori bounded degree) in x'. Loosely speaking, one has to study that in order to understand the geometry of one of the main "normal forms" (after a symplectomorphism) of $p(x, \xi)$ on a box of fixed size:

$$p(x, \xi) = \xi_1^2 + (\xi_2 - \theta(x_1, x_2))^2 + V(x_1, x_2)\,,$$
$$V(x_1, x_2) = p(x_1, x_2, 0, \theta(x_1, x_2))$$

where, upon rescaling to the unit cube, θ is a smooth algebraic function in x_1, polynomial in x_2, and $p(x_1, x_2, 0, \xi_2)$ is a polynomial, both of a-priori bounded degree and maximum norms. Here we study the estimates relative to the "quadratic" part of p. The much more difficult case of the estimates relative to V are treated in [6] and [7].

We address the interested reader to [10] and [11] for more details about the use of these polynomial-like properties.

It should be stressed once more the novelty here is that also in the case of smooth algebraic functions we have a complete control on the size of the regions in terms of the size of the functions (when one wants to get informations about maxima and averages; a typical example is property (c) above), and on the scaling properties of the $L^\infty-$norms in terms of the sizes of the regions on which the norms are taken.

2 – The Results.

We start by studying some scaling properties of averages, with respect to one of the variables, of polynomials and of maxima of smooth functions with "controlled" gradient[(2)].

PROPOSITION 2.1. *Suppose $0 \le f(x_1, x_2)$ is a nonnegative polynomial of degree d for $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^N$. Take $(x_1, x_2) \in I \times Q$, with $diam I \sim \rho$, $diam Q \sim \delta$, $0 < \delta < 1$, $\delta \le \rho \le 1$, and take $x_2^0 \in Q$. Suppose $\Big(Av_{x_1 \in I} f\Big)(x_2) \sim \delta^4$ $\forall x_2 \in Q^*$ (the double of Q, as usual). Then there*

[(2)]In the sequel, every constant $C, c, c(n, d), c_1, c_2, c_3, c_4, c_5, C_\alpha$, is a universal constant. For $A, B \ge 0$, $A \sim B$ means that $\exists C, c > 0$, universal constants, such that $cB \le A \le CB$.

exist $Q_1 \subset Q$, $I_1 \subset I$, *center*$(Q_1) = x_2^0$, *diam*$Q_1 \sim \delta \sim$ *diam*I_1, *such that*

$$f(x_1, x_2) \sim \delta^4, \qquad \forall (x_1, x_2) \in I_1 \times Q_1 \,.$$

PROOF. We have,

$$f(x_1^0, x_2^0) = \max_{x_1 \in I} f(x_1, x_2^0) \geq c_1 \delta^4$$

(since f is a nonnegative polynomial and $\Big(\mathrm{Av}_{x_1 \in I} f\Big)(x_2^0) \sim \delta^4$) for some $x_1^0 \in I$ and $\max_{(x_1,x_2) \in I \times Q} f(x_1, x_2) \sim \delta^4$.

Choose now $\tilde{I}_1$, $x_1^0 \in \tilde{I}_1$, with $\mathrm{diam}\tilde{I}_1 = c_0 \delta$. Since $\tilde{I}_1 \subset I$, $\max_{\tilde{I}_1 \times Q} f \leq c_2 \delta^4$.

Also, with a universal constant $c = c(N+1, d)$, f being a polynomial ≥ 0,

$$\max_{\tilde{I}_1 \times Q} |\nabla f| \leq \frac{c c_2}{\delta} \delta^4 = c_3 \delta^3 \,.$$

We can then find $I_1 \subset \tilde{I}_1$, $Q_1 \subset Q$, Q_1 centered at x_2^0, with $\mathrm{diam} I_1 = c_4 \delta = \mathrm{diam} Q_1$, so that $\mathrm{diam}\big(I_1 \times Q_1\big) \sim \mathrm{diam}\big(\tilde{I}_1 \times Q\big)$ and

$$c_5 \max_{\tilde{I}_1 \times Q} f \leq \min_{I_1 \times Q_1} f$$

for a universal constant c_5 : for $(x_1, x_2) \in I_1 \times Q_1$,

$$f(x_1, x_2) = f(x_1^0, x_2^0) + \int_0^1 < (\nabla f)\big(x^0 + t(x - x^0)\big), (x - x^0) > dt \geq$$

$$\geq f(x_1^0, x_2^0) - 2c_4 c_3 \delta^4 \geq (c_1 - 2c_4 c_3)\delta^4 = \Big(\frac{c_1}{2c_2}\Big) c_2 \delta^4 \geq c_5 \max_{I_1 \times Q_1} f$$

when $c_4 = c_1/(4c_3)$ and $c_5 = c_1/(2c_2)$. □

The meaning of the above Proposition is that if the average of a nonnegative polynomial with respect to one of the variables is "elliptic", then, in a smaller box (whose size we have control on), the polynomial is "elliptic" in all the variables.

THEOREM 2.2. *Let* $Q \subset \mathbb{R}^n$ *be a (closed) cube and* $Q^* = 2Q$. *Let* $f \in C^\infty(Q^*)$ *be such that*

$$\|\nabla f\|_{L^\infty(Q)} \leq \frac{c}{diam Q} \|f\|_{L^\infty(Q)} \,.$$

Then

$$(1) \qquad Av_{x\in Q}|f| \sim \max_{x\in Q}|f|$$

(i.e., as always, the two quantities are equivalent by universal constants independent of f, depending only on n and c).

PROOF. We can suppose Q to be the unit cube centered at the origin. $\exists \bar{x} \in Q$ such that $\|f\|_{L^\infty(Q)} = |f(\bar{x})|$.

Recall that if $\mathrm{diam}Q = \alpha$ then $|Q| = |\mathrm{side}(Q)|^n = (\alpha/\sqrt{n})^n$.

So, choose a cube $Q_1 \subset Q$ with $\bar{x} \in Q_1$ and $\mathrm{diam}Q_1 = 1/(2\tilde{c})$, where $\tilde{c} = \max\{1, c\}$. Hence $|Q_1| \sim |Q|$. We have,

$$\begin{aligned} f(x) &= f(\bar{x}) + <\int_0^1 (\nabla f)\big(\bar{x} + t(x-\bar{x})\big)dt, (x-\bar{x})>= \\ &= f(\bar{x}) + <F(x,\bar{x}), (x-\bar{x})> \ . \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Av}_{x\in Q}|f| &\geq \frac{|Q_1|}{|Q|}\mathrm{Av}_{x\in Q_1}|f| \geq c\Big|\frac{1}{|Q_1|}\int_{Q_1} f(x)dx\Big| = \\ &= c\Big|\frac{1}{|Q_1|}\int_{Q_1}\Big\{f(\bar{x}) + <F(x,\bar{x}), (x-\bar{x})>\Big\}dx\Big| \geq \\ &\geq c\Big\{|f(\bar{x})| - \frac{1}{2}\|f\|_{L^\infty(Q)}\Big\} = \frac{1}{2}c\|f\|_{L^\infty(Q)}\,, \end{aligned}$$

since

$$\int_{Q_1} |<F(x,\bar{x}), (x-\bar{x})>|dx \leq c\|f\|_{L^\infty(Q)}\Big(\mathrm{diam}Q_1\Big)|Q_1|\,,$$

whence

$$c\|f\|_{L^\infty(Q)} \leq \mathrm{Av}_{x\in Q}|f| \leq \|f\|_{L^\infty(Q)}\,. \qquad \square$$

THEOREM 2.3. *Suppose $f \in C^\infty(Q_0^*)$ and $\forall Q \subset Q_0$ (all the cubes considered are closed cubes)*

$$\|\nabla f\|_{L^\infty(Q)} \leq \frac{c}{diamQ}\|f\|_{L^\infty(Q)} \ .$$

Suppose $diam Q \sim diam Q_0$*, then*

$$\|f\|_{L^\infty(Q)} \sim \|f\|_{L^\infty(Q_0)}. \tag{2}$$

Proof. As usual, we can suppose Q_0 to be the unit cube in $\mathbb{R}^n$ centered at the origin. Let $x^0 := x_0^0 \in Q_0$ be such that $|f(x^0)| = \max_{x \in Q_0} |f(x)|$. Set $\tilde{c} = \max\{1, c\}$.

If $x^0 \in Q$ we are done.

Suppose $x^0 \notin Q$. Let x_Q be the center of Q and let L_0 be the line through x^0 and x_Q and consider, on L_0, the segment $[x^0, x_Q]_{L_0}$. Let $F_1'(Q), F_1''(Q)$ be two parallel faces of Q, opposite with respect to x_Q, at which L_0 meets Q transversally, such that the point $F_1' \cap L_0$ is closer to x^0 along L_0, than the point $F_1''(Q) \cap L_0$.

(In case L_0 intersects Q in a corner or vertex, we just choose one of the possible faces).

On $[x^0, x_Q]_{L_0}$ choose x_1 such that $\mathrm{dist}_{L_0}(x^0, x_1) = 1/(2\tilde{c})$, where dist_{L_0} is the distance on the line L_0.

We have that $F_1''(Q) \subset H_1$, a hyperplane. Choose $H(x_1)$ to be the hyperplane parallel to H_1 through x_1. By convexity of Q_0, the segment of L_0, $[x^0, L_0 \cap F_1''(Q)]_{L_0} \subset Q_0$, and $\forall t \in [x^0, L_0 \cap F_1''(Q)]_{L_0}$, $H(t) \cap Q_0 \neq \emptyset$, where $H(t)$ is the hyperplane parallel to H_1 through t. Denote by P the band between the boundaries $H(x_1)$ and H_1. Then $Q \subset P \cap Q_0$.

Notice that $\mathrm{side}(Q) \leq \mathrm{dist}(H(x_1), H_1) < \mathrm{side}(Q_0)$.

Hence there exists Q_1 with $F_1''(Q) \subset F_1''(Q_1) \subset H_1$ and $x_1 \in \partial Q_1$, so that $Q \subset Q_1 \subset P \cap Q_0$. Let $x_1^0 \in Q_1$ be such that $\|f\|_{L^\infty(Q_1)} = |f(x_1^0)|$. If $x_1^0 \in Q$ we stop here, otherwise consider the line L_1 through x_1^0 and x_Q, and a point $x_2 \in [x_1^0, x_Q]_{L_1}$ with $\mathrm{dist}_{L_1}(x_1^0, x_2) = \mathrm{diam} Q_1/(2\tilde{c})$.

Consider now, with obvious notations, $H(x_2)$ parallel to H_2. Then $Q \subset P_1 \cap Q_1 \subset P \cap Q_0$ and $\mathrm{side}(Q) \leq \mathrm{dist}(H(x_2), H_2) < \mathrm{side}(Q_1)$.

Therefore there exists Q_2, $x_2 \in \partial Q_2$, $F_2''(Q) \subset F_2''(Q_2) \subset H_2$, and $Q \subset Q_2 \subset Q_1 \cap P_1$, $F_2''(Q)$ being the farthest face of Q, along L_1, with respect to x_1^0.

Notice that $\mathrm{diam} Q \leq \mathrm{diam} Q_2 \leq Q_0$.

Suppose we constructed the Q_j's, $j = 0, 1, 2, \ldots, k$ (so, in particular, $x_j^0 \notin Q$, $\forall j$, x_j^0 a point of maximum for $|f|$ on Q_j), we want to construct Q_{k+1}.

Recall that, for $j = 1, 2, \dots, k$, we have

$$Q \subset Q_j \subset Q_{j-1} \cap P_{j-1}$$

(P_j's are determined by pairs of hyperplanes parallel to the coordinate-hyperplanes. Notice that $P_j \cap Q_j$ is an $n-$dimensional parallelepiped).

Consider $x_k^0 \in Q_k$ such that $\|f\|_{L^\infty(Q_k)} = |f(x_k^0)|$.

If $x_k^0 \in Q$ we stop here the construction of the sequence of cubes, otherwise let L_k be the line through x_k^0 and x_Q. Take $x_{k+1} \in [x_k^0, x_Q]_{L_k}$ with

$$\text{dist}_{L_k}(x_k^0, x_{k+1}) = \frac{\text{diam} Q_k}{2\tilde{c}} \,.$$

Then, with the obvious notations, consider $H(x_{k+1})$ and H_{k+1} (chosen as above). Then $Q \subset P_k \cap Q_k \subset P_{k-1} \cap Q_{k-1}$. Again

$$\text{side}(Q) \leq \text{dist}(H(x_{k+1}), H_{k+1}) < \text{side}(Q_k) \implies$$

$$\implies \exists Q_{k+1}, \quad F''_{k+1}(Q) \subset F_{k+1}(Q_{k+1}) \subset H_{k+1}$$

such that $x_{k+1} \in \partial Q_{k+1}$, $Q \subset Q_{k+1} \subset Q_k \cap P_k$.

Notice that, $\forall j$, $\text{diam} Q \leq \text{diam} Q_j \leq Q_0$.

Since, at each step, we shrink the region by an amount $> \text{diam} Q/(2\tilde{c})$, there exists N such that $N \leq 2\tilde{c}(\text{diam} Q_0/\text{diam} Q)$, and the construction stops at x_{N+1}, i.e. $x_{N+1} \in Q$.

Then we have, $\forall j, \; j = 0, 1, \dots, N$:

$$f(x_j^0) = f(x_{j+1}) + \langle F(x_j^0, x_{j+1}), (x_j^0 - x_{j+1}) \rangle,$$

so that,

$$\|f\|_{L^\infty(Q_j)} \leq \|f\|_{L^\infty(Q_{j+1})} + \frac{1}{2}\|f\|_{L^\infty(Q_j)},$$

i.e.

$$\|f\|_{L^\infty(Q_j)} \leq 2\|f\|_{L^\infty(Q_{j+1})},$$

whence

$$\|f\|_{L^\infty(Q_0)} \leq 2^N \|f\|_{L^\infty(Q)} \,. \qquad \square$$

Our aim is now to show that algebraic functions, i.e. solutions to a polynomial equation, satisfy the hypotheses of Theorem 2.2 and 2.3 (and hence enjoy a polynomial-like scaling property).

Theorem 2.4. *Let* $Q = Q_1 \times I$ *be the unit cube, centered at the origin, in* $\mathbb{R}^{n+1}$, *with coordinates* $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. *Let* $P(x, y)$ *be a polynomial of a-priori bounded degree* d, *with* $|\partial_y P| \geq C > 0$, $\forall (x, y) \in Q^*$, *and* $\|P\|_{L^\infty(Q^*)} \leq C_*$, *for fixed constants* $C, C_* > 0$. *Let* $y = f(x)$ *be the solution to* $P(x, y) = 0$ *on* Q^*, *with* $f \in C^\infty(\frac{1}{2}Q_1^*)$, $\|f\|_{L^\infty(Q_1)} \leq 2$. *Then, for* $|\alpha| \leq 2$ *(and actually* $\forall \alpha$*)*

$$\|\partial_x^\alpha f\|_{L^\infty(Q_1)} \leq C_\alpha \|P\|_{L^\infty(Q^*)}(M - m) \leq C_* C_\alpha (M - m) \tag{3}$$

where $M = \max_{x \in Q_1} f(x)$, $m = \min_{x \in Q_1} f(x)$ *and the* C_α*'s depend only on* $n + 1$ *and* d.

Proof. Notice that, by hypothesis, $J = [m, M] \subset I^*$. We have

$$P(x, y) = \int_0^1 (\partial_y P)(x, f(x) + t(y - f(x)))dt\Big(y - f(x)\Big)$$

so that, $\forall x \in Q_1$, $\forall y \in J$, P being a polynomial,

$$\max_{(x,y) \in Q_1 \times J} |P(x, y)| \leq c(d, n) \max_{(x,y) \in Q_1 \times J} |y - f(x)| = c(d, n)|M - m|\,.$$

It follows that, $\forall y \in J$,

$$\begin{aligned} |\partial_x P(x, y)| &\leq c(d, n) \max_{x \in Q_1} |P(x, y)| \leq \\ &\leq c(d, n)\|P\|_{L^\infty(Q^*)}|M - m| \leq c(d, n)C_*|M - m|\,, \end{aligned}$$

so that,

$$\max_{(x,y) \in Q_1 \times J} |\partial_x P(x, y)| \leq c(d, n)\|P\|_{L^\infty(Q^*)}|M - m| \leq c(d, n)C_*|M - m|\,.$$

Hence, since $|(\partial_x P)(x, f(x))| \leq \max_{(x,y) \in Q_1 \times J} |\partial_x P(x, y)|$, we obtain, using the formula from the Implicit Function Theorem, for $|\alpha| = 1$,

$$\partial_x^\alpha f(x) = -\frac{(\partial_x^\alpha P)(x, f(x))}{(\partial_y P)(x, f(x))},$$

that, $|\alpha| = 1$,

$$\|\partial_x^\alpha f\|_{L^\infty(Q_1)} \leq \frac{c(d,n)}{C}\|P\|_{L^\infty(Q^*)}|M-m| \leq \frac{c(d,n)}{C}C_*|M-m|\,.$$

Now, for $|\alpha| = 2$, $\alpha = \alpha_1 + \alpha_2$, $|\alpha_1| = |\alpha_2| = 1$,

$$\partial_x^\alpha f(x) = -\Big\{\frac{(\partial_x^\alpha P)\big(x,f(x)\big) + (\partial_x^{\alpha_1}\partial_y P)\big(x,f(x)\big)\partial_x^{\alpha_2} f(x)}{(\partial_y P)\big(x,f(x)\big)} + \\ - (\partial_x^{\alpha_1} P)\big(x,f(x)\big)\frac{(\partial_x^{\alpha_2}\partial_y P)\big(x,f(x)\big) + (\partial_y^2 P)\big(x,f(x)\big)\partial_x^{\alpha_2} f(x)}{(\partial_y P)\big(x,f(x)\big)^2}\Big\}\,.$$

As we already know, $|(\partial_x^{\alpha_1} P)\big(x,f(x)\big)| \leq C(M-m)$.

For $(\partial_x^\alpha P)\big(x,f(x)\big)$ we have: $\forall y \in J$

$$|\partial_x^\alpha P(x,y)| \leq \max_{x\in Q_1}|\partial_x^{\alpha_1} P(x,y)| \leq C(M-m),$$

whence $\max_{(x,y)\in Q_1\times J}|\partial_x^\alpha P(x,y)| \leq C(M-m)$ and

$$|(\partial_x^\alpha P)\big(x,f(x)\big)| \leq \max_{(x,y)\in Q_1\times J}|\partial_x^\alpha P(x,y)| \leq C(M-m),$$

with, clearly, $C = c(d,n)$.

Now,

$$|(\partial_x^{\alpha_2}\partial_y P)\big(x,f(x)\big)| \leq \max_{(x,y)\in Q^*}|\partial_x^{\alpha_2}\partial_y P(x,y)| \leq \\ \leq c(d,n)\|P\|_{L^\infty(Q^*)} \leq c(d,n)C_*\,,$$

$$|(\partial_y^2 P)\big(x,f(x)\big)| \leq \max_{(x,y)\in Q^*}|\partial_y^* P(x,y)| \leq \\ \leq c(d,n)\|P\|_{L^\infty(Q^*)} \leq c(d,n)C_*\,.$$

It follows, for $|\alpha| = 2$,

$$\|\partial_x^\alpha f\|_{L^\infty(Q_1)} \leq c(d,n)\|P\|_{L^\infty(Q^*)}|M-m| \leq c(d,n)C_*|M-m| \qquad \square$$

Corollary 2.5. *Same hypotheses as in Theorem 2.4.* $\forall Q_1' \subset Q_1$, *we have, for* $|\alpha| \leq 2$,

$$\|\partial_x^\alpha f\|_{L^\infty(Q_1')} \leq \frac{c(d,n)C_*}{(diamQ_1')^{|\alpha|}}\Big(\max_{x\in Q_1'} f(x) - \min_{x\in Q_1'} f(x)\Big)\,. \tag{4}$$

Proof. The proof follows from the proof of Theorem 2.4 considering

$$J(Q_1') = [\min_{Q_1'} f, \max_{Q_1'} f] = [m(Q_1'), M(Q_1')],$$

and noticing the following facts: $\forall y \in J(Q_1')$, $\forall x \in Q_1'$,

$$|(\partial_x P)(x, f(x))| \leq \frac{c(d,n)}{\mathrm{diam}Q_1'} C_* |M(Q_1') - m(Q_1')|,$$

since, $\forall y \in J(Q_1')$, $\forall \alpha$:

$$\begin{aligned}\max_{x\in Q_1'} |\partial_x^\alpha P(x,y)| &\leq c(d,n) \max_{x\in Q_1'} |P(x,y)| (\mathrm{diam}Q_1')^{-|\alpha|} \leq \\ &\leq \frac{c(d,n)\|P\|_{L^\infty(Q^*)}}{(\mathrm{diam}Q_1')^{|\alpha|}} |M(Q_1') - m(Q_1')| \leq \\ &\leq \frac{c(d,n)C_*}{(\mathrm{diam}Q_1')^{|\alpha|}} |M(Q_1') - m(Q_1')|\end{aligned}$$

and

$$\begin{aligned}|(\partial_y^\alpha P)(x, f(x))| &\leq \max_{(x,y)\in Q^*} |\partial_y^\alpha P(x,y)| \leq \\ &\leq c(d,n)\|P\|_{L^\infty(Q^*)} \leq c(d,n)C_*\,.\end{aligned}$$

∎

A very nice consequence is the following:

Corollary 2.6. *Same hypotheses as in Theorem 2.4. Then:*

$$\|\nabla f\|_{L^\infty(Q)} \sim \Big(\max_{x\in Q} f(x) - \min_{x\in Q} f(x)\Big). \tag{5}$$

This allows us to prove the

Corollary 2.7. *Let f be a smooth algebraic function satisfying the conditions in Corollary 2.6. Consider, for fixed $y \in \mathbb{R}$, the polynomial in $X \in \mathbb{R}$, $P_y(X) = (y-X)^2$, and the associate function $p_y(x) = (y-f(x))^2$. Then:*

$$Av_{x\in Q} p_y(x) \sim \max_{x\in Q} p_y(x)\,, \tag{6}$$

and

$$\|\partial_x p_y\|_{L^\infty(Q)} \le C\|p_y\|_{L^\infty(Q)}\,, \tag{7}$$

where C and the constants in the equivalence do not depend on y.

Proof. Let $J = [\min_{x\in Q} f, \max_{x\in Q} f]$. Consider $\partial_X P_y(X) = -2(y-X)$. Recall that $\mathrm{diam} Q \sim 1$. Then, for a universal constant C (as usual all the constants C are **universal** constants),

$$\max_{X\in J} |\partial_X P_y(X)| \le \frac{C}{|J|} \max_{X\in J} |P_y(X)|\,.$$

Hence, since: $x \in Q \implies f(x) \in J$,

$$\begin{aligned}|\partial_x p_y(x)| &\le 2 \max_{X\in J} |\partial_X P_y(X)| \|\partial_x f\|_{L^\infty(Q)} \le \\ &\le \frac{2C}{|J|} \max_{X\in J} |P_y(X)| \|\partial_x f\|_{L^\infty(Q)} \le \\ &\le \frac{C'}{|J|}\Big(\max_Q f - \min_Q f\Big) \|P_y\|_{L^\infty(I)} = C' \max_{x\in Q} |p_y(x)|\,,\end{aligned}$$

whence

$$\|\partial_x p_y\|_{L^\infty(Q)} \le C\|p_y\|_{L^\infty(Q)}\,,\,,$$

and

$$\mathrm{Av}_{x\in Q} p_y \sim \|p_y\|_{L^\infty(Q)}\,.$$

□

Remark 2.8. Of course the Corollary holds true for $P_y(X) = (y-X)^d$, $d \ge 1$. We stated it for $d = 2$ since this is what is needed in [10] and [11].

Given now the algebraic function $f(x_1, x_2)$, we have to examine the scaling properties of $g(x_1, x_2) = f(x_1, x_2) - (\mathrm{Av}_{x_1 \in I} f)(x_2) := f(x_1, x_2) - \bar{f}(x_2)$.

LEMMA 2.9. *Suppose f, P satisfy the hypotheses of Theorem 2.4. Suppose now $Q = Q_1 \times I = I \times Q_2 \times I$, Q_2 the unit cube in $\mathbb{R}^{n-1}$, $(x_1, x_2) \in I \times Q_2$. Define $\bar{f}(x_2) = (Av_{x_1 \in I} f)(x_2)$. Then, for a constant C independent of f and x_2, we have, $\forall x_2$ fixed,*

$$(i) \qquad \|\partial_{x_1} g(., x_2)\|_{L^\infty(I)} \leq C(\max_{x_1 \in I} g(x_1, x_2) - \min_{x_1 \in I} g(x_1, x_2));$$

$$(ii) \qquad \|\partial_{x_1} (g(., x_2)^2)\|_{L^\infty(I)} \leq C \|g(., x_2)^2\|_{L^\infty(I)};$$

$$(iii) \qquad \|\partial_{x_1} g\|_{L^\infty(Q)} \leq C \|g\|_{L^\infty(Q)} .$$

PROOF. Define

$$M(g)(x_2) = \max_{x_1 \in I} g(x_1, x_2) = \Big(\max_{x_1 \in I} f(x_1, x_2)\Big) - \bar{f}(x_2) = M(f)(x_2) - \bar{f}(x_2),$$

and, with $m(f)(x_2) = \min_{x_1 \in I} f(x_1, x_2)$,

$$m(g)(x_2) = \min_{x_1 \in I} g(x_1, x_2) = m(f)(x_2) - \bar{f}(x_2) .$$

$\forall x_2 \in Q_2$ fixed, consider $J(x_2) = [m(f)(x_2), M(f)(x_2)]$. Then

$$|P(x_1, x_2, y)| \leq C \|P\|_{L^\infty(Q^*)} \max_{(x_1, y) \in I \times J(x_2)} |y - f(x_1, x_2)| \leq$$

$$\leq C \|P\|_{L^\infty(Q^*)} \big|M(f)(x_2) - m(f)(x_2)\big| ,$$

C **independent of** x_2. It follows that

$$|\partial_{x_1} f(x_1, x_2)| \leq C |(\partial_{x_1} P)\big(x_1, x_2, f(x_1, x_2)\big)| \leq$$

$$\leq C \|P\|_{L^\infty(Q^*)} \big|M(f)(x_2) - m(f)(x_2)\big| ,$$

being $\mathrm{diam} I \sim 1$. Since

$$M(g)(x_2) - m(g)(x_2) = M(f)(x_2) - \bar{f}(x_2) - m(f)(x_2) + \bar{f}(x_2) =$$

$$= M(f)(x_2) - m(f)(x_2)$$

and $\partial_{x_1} g(x_1, x_2) = \partial_{x_1} f(x_1, x_2)$, point (i) and (iii) follow at once.

About (ii) :

$$\partial_{x_1}(g(x_1, x_2)^2) = 2g(x_1, x_2)\partial_{x_1} g(x_1, x_2),$$

therefore

$$|\partial_{x_1}(g(x_1, x_2)^2)| \leq 2C\|g(., x_2)\|^2_{L^\infty(I)} = 2C\|g(., x_2)^2\|_{L^\infty(I)}\,. \qquad \square$$

Corollary 2.10. *Under the same hypotheses, suppose further that $f(x_1, x_2)$ is a polynomial of bounded degree D in x_2. Then the Bernstein's inequality holds for $g(x_1, x_2) := f(x_1, x_2) - \bar{f}(x_2)$:*

$$\|\nabla g\|_{L^\infty(Q)} \leq C\|g\|_{L^\infty(Q)}\,. \tag{8}$$

Corollary 2.11. *Same hypotheses of Corollary 2.10. Consider the function*

$$p_y(x_1, x_2) = \big(y - g(x_1, x_2)\big)^2\,.$$

Then

$$Av_{x \in Q} p_y \sim \max_{x \in Q} p_y(x) \tag{9}$$

and

$$\|\partial_x p_y\|_{L^\infty(Q)} \leq C\|p_y\|_{L^\infty(Q)}\,, \tag{10}$$

for universal constants **independent of** *y.*

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