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Semilinear cooperative elliptic systems on $\mathbb{R}^n$

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RIASSUNTO: *Si studia il sistema ellittico cooperativo semilineare* (1 – a,b,c) *definito in* $\mathbb{R}^n$ *con* $n > 2$. *In esso* a, b, c, d *sono delle costanti con* b, $c > 0$; ρ, f *e* g *sono funzioni assegnate e* ρ *è non negativa ed infinitesima all'infinito. Si stabiliscono in primo luogo le condizioni necessarie e sufficienti sui coefficienti affinché sussista un principio di massimo. Si riconosce poi che queste condizioni assicurano l'esistenza di soluzioni nel caso lineare e quando le funzioni* f *e* g *verifichino certe condizioni di "sublinearità". Con certe ipotesi aggiuntive si ottiene anche l'unicità. Infine si estendono i risultati al caso in cui le incognite siano in numero maggiore di 2.*

ABSTRACT: *We study here the following semilinear cooperative elliptic system defined on* $\mathbb{R}^n$, $n > 2$:

(1 – a) $$-\Delta u = a\rho(x)u + b\rho(x)v + f(x,u,v) \quad x \in \mathbb{R}^n\,,$$

(1 – b) $$-\Delta v = c\rho(x)u + d\rho(x)v + g(x,u,v) \quad x \in \mathbb{R}^n\,,$$

(1 – c) $$u \longrightarrow 0\,,\ v \longrightarrow 0 \quad as \quad |x| \longrightarrow +\infty\,.$$

Here a, b, c, d *are constants such that* $b, c > 0$; ρ, f *and* g *are given functions;* ρ *is nonnegative and tends to* 0 *at* ∞. *We first establish necessary and sufficient conditions on the coefficients for having a Maximum Principle for the linear System. Then we show that these conditions ensure existence of solutions for the linear System and for the semilinear System when* f *and* g *satisfy some "sublinear" condition . Under some additional assumption we also derive uniqueness of the solutions. Finally we show that our results can be extended to* $N \times N$ *systems,* $N > 2$.

KEY WORDS AND PHRASES: *Cooperative elliptic systems – Weighted Sobolev spaces – Unbounded domains*
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1 – Introduction

It is well known that the Maximum Principle plays an important role in the theory of nonlinear equations (see e.g. [18]). An analogous theory has been established for semilinear systems by [10-12], [19], [7], [13,14] and [1].

In [11,12] the authors consider System (1) with $\rho(x) = 1$ defined on a bounded open set Ω with Dirichlet boundary conditions. They show that the necessary and sufficient condition for having Maximum Principle is:

$$a < \Lambda, \qquad d < \Lambda, \qquad (\Lambda - a)(\Lambda - d) > bc\,, \tag{2}$$

where Λ is the first eigenvalue of the Dirichlet Laplacian defined on Ω.

Here, we extend this result to System (1) when f and g are independent of u and v. We make use of an earlier result by [4] and [6] who have studied the eigenvalues of

$$-\Delta u = \lambda \rho(x) u, \quad x \in \mathbb{R}^n\,, \quad u(x) \to 0 \text{ as } |x| \to \infty\,. \tag{3}$$

They show that for $n > 2$, if

$$\exists\, k > 0\,,\, r > 1 \quad \text{such that} \quad 0 < \rho < \frac{k}{(1 + |x|^2)^r} \tag{4}$$

then (3) admits an infinite sequence of positive eigenvalues; the first one, which we will denote by λ_ρ, is simple and is associated with a positive eigenfunction ψ_ρ .

We show in Section 3 that the Maximum Principle holds if and only if (2_ρ) holds:

$$a < \lambda_\rho \quad , \quad d < \lambda_\rho\,. \tag{$2_\rho - 1$}$$

$$(\lambda_\rho - a)(\lambda_\rho - d) > bc\,. \tag{$2_\rho - 2$}$$

Then, we prove existence of solutions for $f, g \in L^2_{\frac{1}{\rho}}(\mathbb{R}^n)$ in Section 4. In Section 5 we study semilinear problems with f, g satisfying some "sublinear" condition; we adapt the method of sub-super solutions for proving existence of non negative solutions. Moreover, under some further assumptions on f, g, we prove uniqueness of the non negative solutions. Finally we extend some of our results to $N \times N$ systems in Section 6.

To establish our results we adapt the proofs of [13,14] and [5].

We recall that throughout the paper, $n > 2$.

2 – The scalar case

2.1 – Some technical results

To prove our theorems we use some notations and results which are established e.g. in [6], Section 4, and that we recall briefly .

Let us introduce

$$V = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \int_{\mathbb{R}^n} (|\nabla u|^2 + (1 + |x|^2)^{-1} u^2) dx < \infty \right\}$$

with inner product

$$(u, v) = \int_{\mathbb{R}^n} (\nabla u . \nabla v + (1 + |x|^2)^{-1} uv) dx\,.$$

Since $n > 2$, it follows from Hardy's inequality that:

Lemma 1. *The integrodifferential form*

$$l(u, v) = \int_{\mathbb{R}^n} \nabla u . \nabla v dx$$

is an inner product for V which is equivalent to the original one:

$$\int_{\mathbb{R}^n} (1 + |x|^2)^{-1} u^2 dx \le \gamma \int_{\mathbb{R}^n} |\nabla u|^2 dx\,.$$

Moreover, if we denote by $\|u\|_V = \Big(\int\limits_{\mathbb{R}^n} |\nabla u|^2\Big)^{\frac{1}{2}} dx$, then:

Lemma 2. *The quantity* $\|u\|_{\text{-}\rho} = \Big\{\int\limits_{\mathbb{R}^n} (|\nabla u|^2 + \rho u^2) dx\Big\}^{\frac{1}{2}}$ *is also a norm on* V *which is equivalent to the previous one* $\|u\|_V$.

If we denote by $(\ ,\ )_\rho$ the inner product in $\mathcal{H} := L^2_\rho(\mathbb{R}^n)$:

$$(u,v)_\rho = \int\limits_{\mathbb{R}^n} \rho u v dx\,,$$

and by τ the operator defined by Riesz representation theorem by:

$$(u,v)_\rho = l(\tau u, v)\ \forall (u,v) \in V \times V\,,$$

then:

Lemma 3. *For* ρ *satisfying* (4), τ *is compact in* V.

2.2 – The eigenvalue problem

The following lemma is also proved in [6], Section 4:

Lemma 4. *For* ρ *satisfying* (4), *the eigenvalue problem* (3) *admits a positive principal eigenvalue* λ_ρ *which is associated with a positive eigenfunction* $\psi_\rho \in V$; *moreover* λ_ρ *is characterized by*

$$\lambda_\rho \int\limits_{\mathbb{R}^n} \rho u^2 dx \leq \int\limits_{\mathbb{R}^n} |\nabla u|^2 dx \qquad \forall u \in V. \tag{5}$$

The equality in (5) *holds if and only if* u *is proportional to* ψ_ρ.

2.3 – The scalar case

We study now the scalar case ($N = 1$):

$$\text{(E)} \qquad -\Delta u = \mu\rho(x)\,u + f \quad \text{in } \mathbb{R}^n\,, \quad u(x) \to 0 \ \text{ as } |x| \to \infty\,.$$

We establish exactly as when Ω is bounded:

Proposition 1. *For $f \in L^2_{\frac{1}{\rho}}(\mathbb{R}^n)$, the Maximum Principle holds for (E) if and only if $\mu < \lambda_\rho$.*

Proposition 2. *For $0 \leq f \in L^2_{\frac{1}{\rho}}(\mathbb{R}^n)$, there exists a unique positive solution $u \in V$ for (E) if and only if $\mu < \lambda_\rho$.*

Proof of Proposition 1.
The condition is necessary: Assume that $f \in L^2_{\frac{1}{\rho}}(\mathbb{R}^n), f \geq 0$ and that the Maximum Principle holds for (E), i.e. any $u \in V$ solution of (E) is nonnegative. Then, multiplying (E) by ψ_ρ, the principal eigenfunction defined in $II.B$ and integrating, we obtain:

$$\int_{\mathbb{R}^n} -\Delta u\psi_\rho dx = -\int_{\mathbb{R}^n} u\Delta\psi_\rho dx = \mu\int_{\mathbb{R}^n} \rho u\psi_\rho dx + \int_{\mathbb{R}^n} f\psi_\rho dx\,;$$

Hence, by (5):

$$(\lambda_\rho - \mu)\int_{\mathbb{R}^n} \rho u\psi_\rho dx = \int_{\mathbb{R}^n} f\psi_\rho dx\,;$$

since u, ρ and ψ_ρ are nonnegative, then $\lambda_\rho > \mu$.

The condition is sufficient: Suppose that $f \geq 0$ and that $\mu < \lambda_\rho$. We multiply (E) by $u^- = max(0, -u)$ and we get:

$$\int_{\mathbb{R}^n} -\Delta u u^- = \int_{\mathbb{R}^n} \nabla u\nabla u^- dx = \mu\int_{\mathbb{R}^n} \rho u u^- dx + \int_{\mathbb{R}^n} f u^- dx =$$

$$= -\int_{\mathbb{R}^n} |\nabla u^-|^2 dx = -\mu\int_{\mathbb{R}^n} \rho|u^-|^2 dx + \int_{\mathbb{R}^n} f u^- dx\,;$$

by (5) we derive:

$$0 \leq (\lambda_\rho - \mu) \int_{\mathbb{R}^n} \rho |u^-|^2 dx \leq - \int_{\mathbb{R}^n} f u^- dx \leq 0$$

which implies that $u^- = 0$ i.e. $u \geq 0$. ∎

Proof of Proposition 2. If $u \geq 0$ is the unique solution of (E), then necessarily by Proposition (1), $\mu < \lambda_\rho$. Let us show now that this condition is sufficient.
Assume that $\mu < \lambda_\rho$; the sesquilinear form

$$a(u,v) = \int_{\mathbb{R}^n} (\nabla u . \nabla v - \mu \rho u v) dx$$

is obviously continuous on V; moreover it is coercive.
Choose $m \geq 1$ such that $\mu + m > 0$ and define on V the equivalent norm

$$\|u\|^2_{m,\rho} = \int_{\mathbb{R}^n} (|\nabla u|^2 + m \rho u^2) \, dx \, . \tag{6}$$

Then from (5) we have

$$a(u,u) = \int_{\mathbb{R}^n} (|\nabla u|^2 + m\rho u^2) dx - (\mu + m) \int_{\mathbb{R}^n} \rho u^2 dx \geq (1 - \frac{\mu + m}{\lambda_\rho + m}) \|u\|^2_{m,\rho} \, .$$

Hence by Lax Milgram lemma (see e.g.[16]), (E) admits a solution in V which is non-negative by Proposition (1). ∎

3 – Maximum principle for linear systems

Now we establish necessary and sufficient conditions for having a Maximum Principle for the following system defined in $\mathbb{R}^n, n \geq 3$

(S – 1) $$-\Delta u = a\rho u + b\rho v + f(x) \qquad x \in \mathbb{R}^n$$

(S – 2) $$-\Delta v = c\rho u + d\rho v + g(x) \qquad x \in \mathbb{R}^n$$

(S – 3) $$u \longrightarrow 0 \, , \; v \longrightarrow 0 \text{ as } |x| \longrightarrow \infty \, ,$$

where:

$$f\,,\, g \in \mathcal{H}' = L^2_{\frac{1}{\rho}}(\mathbb{R}^n)\,. \tag{7}$$

$$a, b, c \text{ and } d \text{ are constants such that } b, c > 0 \tag{8}$$

In this section, we prove that if f and g are non-negative, then any pair $(u, v) \in V \times V$ satisfying (S) in the weak sense is non-negative if and only if (2_ρ) is satisfied. More precisely :

THEOREM 3. *Assume that* (4), (7) *and* (8) *hold. System* (S) *satisfies Maximum Principle if and only if inequalities* (2_ρ) *are satisfied.*

PROOF.
The condition is necessary: Assume that $f \geq 0\,,\, g \geq 0$ and that the Maximum Principle holds, i.e. if (u, v) is a pair of solutions then $u \geq 0$, $v \geq 0$. $(2_\rho - 1)$ is established as for the scalar case, considering succesively (S–1) and (S–2). Now, multiplying (S–1) by ψ_ρ and integrating over $\mathbb{R}^n$, we obtain by Green's formula:

$$\int_{\mathbb{R}^n} -\Delta u.\psi_\rho dx = \lambda_\rho \int_{\mathbb{R}^n} \rho u.\psi_\rho dx = a \int_{\mathbb{R}^n} \rho u.\psi_\rho dx + b \int_{\mathbb{R}^n} \rho v.\psi_\rho dx + \int_{\mathbb{R}^n} f.\psi_\rho dx$$

i.e.

$$(\lambda_\rho - a) \int_{\mathbb{R}^n} \rho u.\psi_\rho dx - b \int_{\mathbb{R}^n} \rho v.\psi_\rho dx \leq \int_{\mathbb{R}^n} f.\psi_\rho dx\,. \tag{9}$$

Similarly

$$(\lambda_\rho - d) \int_{\mathbb{R}^n} \rho v.\psi_\rho dx - c \int_{\mathbb{R}^n} \rho u.\psi_\rho dx \leq \int_{\mathbb{R}^n} g.\psi_\rho dx\,. \tag{9'}$$

(9) and (9′) is a Cramer System in $X = \int_{\mathbb{R}^n} \rho u.\psi_\rho dx$ and $Y = \int_{\mathbb{R}^n} \rho v.\psi_\rho dx$; since by hypothesis the right-hand side member is non-negative as well as X and Y, we obtain $(2_\rho - 2)$.

The condition is sufficient: Multiplying (S–1) by u^- and integrating over $\mathbb{R}^n$, we obtain:

$$\int_{\mathbb{R}^n} -\Delta u . u^- dx = \int_{\mathbb{R}^n} \nabla u . \nabla u^- dx = -\int_{\mathbb{R}^n} |\nabla u^-|^2 dx =$$

$$= a \int_{\mathbb{R}^n} \rho u u^- dx + b \int_{\mathbb{R}^n} \rho v u^- dx + \int_{\mathbb{R}^n} f u^- dx\,;$$

we change the signs and, since $f, u^- \geq 0$, we deduce from (5):

$$\lambda_\rho \int_{\mathbb{R}^n} |\sqrt{\rho} u^-|^2 dx \leq a \int_{\mathbb{R}^n} |\sqrt{\rho} u^-|^2 dx + b \int_{\mathbb{R}^n} \rho v^- u^- dx\,.$$

By Cauchy-Schwarz inequality:

$$(\lambda_\rho - a) \int_{\mathbb{R}^n} |\sqrt{\rho} u^-|^2 dx \leq b \Big(\int_{\mathbb{R}^n} |\sqrt{\rho} u^-|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} |\sqrt{\rho} v^-|^2 dx \Big)^{\frac{1}{2}}\,.$$

Similarly:

$$(\lambda_\rho - d) \int_{\mathbb{R}^n} |\sqrt{\rho} v^-|^2 dx \leq c \Big(\int_{\mathbb{R}^n} |\sqrt{\rho} u^-|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} |\sqrt{\rho} v^-|^2 dx \Big)^{\frac{1}{2}}.$$

We multiply the two inequalities and combine the result with $(2_\rho - 2)$; therefore $u^- = 0$ or $v^- = 0$; hence $u \geq 0$ or $v \geq 0$, and by proposition (2), $u \geq 0$ and $v \geq 0$. ∎

4 – Existence of solutions for linear systems

By Lax-Milgram lemma, we prove the existence of a solution for System (S) under the same conditions and the same hypotheses (4), (7) and (8) when $\mathcal{H}' \ni f$, $\mathcal{H}' \ni g$; moreover, if $f \geq 0, g \geq 0$, this solution is non negative.

Theorem 4. *If* (2_ρ), (4), (7) *and* (8) *are satisfied, then System* (S) *has a unique solution* $(u, v) \in V \times V$ *for* $f, g \in \mathcal{H}'$*; moreover, if* $f, g \geq 0$*, then* $u, v \geq 0$.

Proof. We first notice that if (S) has a unique positive solution, then inequalities (2_ρ) are satisfied by System (3).
Assume now that (2_ρ) holds. Choose $m \geq 0$ such that $a+m \geq 0$, $d+m \geq 0$ and use again the equivalent norm on V defined by (6): $\|u\|_{m,\rho}$.

Let us consider the bilinear form $a : V^2 \times V^2 \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} a\big((u,v),(w,z)\big) = &\frac{1}{b}\Big(\int_{\mathbb{R}^n} (\nabla u.\nabla w + m\rho uw)dx\Big) + \\ &+ \frac{1}{c}\Big(\int_{\mathbb{R}^n} (\nabla v.\nabla z + m\rho vz)dx\Big) - \frac{a+m}{b}\int_{\mathbb{R}^n} \rho uw\,dx + \\ &- \int_{\mathbb{R}^n} \rho vw\,dx - \int_{\mathbb{R}^n} \rho uz\,dx - \frac{d+m}{c}\int_{\mathbb{R}^n} \rho vz\,dx \end{aligned}$$

Obviously a is continuous on $V \times V$. Moreover, we can show that it is coercive:
By Cauchy-Schwarz inequality and by (7), we get:

$$\begin{aligned} a\big((u,v),(u,v)\big) = &\frac{1}{b}\Big(\int_{\mathbb{R}^n} |\nabla u|^2 + m\rho u^2\Big)dx + \frac{1}{c}\Big(\int_{\mathbb{R}^n} |\nabla v|^2 + m\rho v^2\Big)dx + \\ &- \frac{a+m}{b}\int_{\mathbb{R}^n} \rho u^2 dx - \frac{d+m}{c}\int_{\mathbb{R}^n} \rho v^2 dx - 2\int_{\mathbb{R}^n} \rho uv\,dx \geq \\ &\geq \frac{1}{b}(1 - \frac{a+m}{\lambda_1+m})\|u\|^2_{m,\rho} + \frac{1}{c}(1-\frac{d+m}{\lambda_1+m})\|v\|^2_{m,\rho} + \\ &- \frac{2}{\lambda_1+m}\|u\|_{m,\rho}\|v\|_{m,\rho}\,. \end{aligned}$$

It is clear by (2_ρ) that a is coercive. Hence by Lax-Milgram lemma, there exists a unique solution $(u,v) \in V\times V$ for (S). Moreover, if $f, g \geq 0$, this solution is non-negative by the Maximum Principle. ∎

5 – Positive solution for semilinear systems

5.1 – Existence

In this section we adapt the method of sub and super solutions [17] to establish the existence of positive solutions for System (1). Since we work on $\mathbb{R}^n$, we can't consider a larger domain for constructing a supersolution.

We assume that:

(10) For any $u \in \mathcal{H}$, $v \in \mathcal{H}$, $x \longrightarrow f(x,u,v)$ (resp.$g(x,u,v)$) is a Caratheodory function;

(11 – a) $$0 \leq f(x,u,v) \leq \frac{u}{\alpha}\rho(x) \quad \forall\, u,v \geq 0\,,\ \forall\, x \in \mathbb{R}^n,$$

(11 – b) $$0 \leq g(x,u,v) \leq \frac{v}{\beta}\rho(x) \quad \forall\, u,v \geq 0\ ,\ \forall\, x \in \mathbb{R}^n,$$

where, α and β are (positive) solutions of the following linear system:

(12 – a) $$(\lambda_\rho - a)\alpha - b\beta = 1 > 0$$

(12 – b) $$-\, c\alpha + (\lambda_\rho - d)\beta = 1 > 0$$

Condition (11) is analoguous to conditions of sublinearity when Ω is bounded.

THEOREM 5. *Suppose that* (4), (8), (10) *and* (11) *are satisfied. Then, if* (2_ρ) *holds, there exists a positive solution for System* (1).

PROOF. We claim that:
(i) $(u_\circ, v_\circ) = (0,0)$ and $(u^\star, v^\star) = (\alpha\psi_\rho, \beta\psi_\rho)$ is a coupled sub-supersolution.

Obviously, by (11):

$$-\,\Delta u_\circ - a\rho u_\circ - b\rho v_\circ - f(x,u_\circ,v) \leq 0 \quad \forall v \in [v_\circ, v^\star];$$

$$-\,\Delta v_\circ - c\rho u_\circ - d\rho v_\circ - g(x,u,v_\circ) \leq 0 \quad \forall u \in [u_\circ, u^\star]\,.$$

We show now that:

(13–a) $$0 \leq -\Delta u^\star - a\rho u^\star - b\rho v^\star - f(x,u^\star,v) \ \ \forall v \in [v_\circ, v^\star];$$

(13–b) $$0 \leq -\Delta v^\star - c\rho u^\star - d\rho v^\star - g(x,u,v^\star) \ \ \forall u \in [u_\circ, u^\star];$$

By definition of $u^\star$, $\psi_\rho > 0$, and by (11–a), and (12–a):

$$-\Delta u^\star - a\rho u^\star - b\rho v^\star = ((\lambda_\rho - a)\alpha - b\beta)\rho\psi_\rho = \rho\psi_\rho \geq f(x, u^\star, v) \qquad \forall v \in [0, v^\star].$$

Similarly we derive (13-b).
(ii) Definition of the operator T:

We introduce $T : (u, v) \in \mathcal{H} \times \mathcal{H} \longrightarrow (w, z) := T(u, v) \in V \times V$, where (w, z) is the unique solution of

$$\text{(14–a)} \qquad -(\Delta + m\rho)w = (a + m)\rho u + b\rho v + f \qquad \in \mathbb{R}^n$$

$$\text{(14–b)} \qquad (-\Delta + m\rho)z = c\rho u + (d + m)\rho v + g \qquad \in \mathbb{R}^n.$$

Here $m > 0$ is chosen as above such that $a + m > 0$, $d + m > 0$.
Note that by (11.a), for $u \in \mathcal{H} = L^2_\rho$, $x \longrightarrow f(x, u, v)$ is in $\mathcal{H}' = L^2_{\frac{1}{\rho}}$.

Equation (14–a) can be rewritten as $-\Delta w = -m\rho w + F$, with $F = (a + m)\rho u + b\rho v + f > 0$, $F \in \mathcal{H}'$. By Proposition 2, this equation possesses a solution $w \in V$.

Analogously we show the existence of $z \in V$ and hence T is well defined. We prove now that:
(iii) $K = [u_\circ, u^\star] \times [v_\circ, v^\star]$ is invariant by T.

For $V \ni u \geq 0$ and $V \ni v \geq 0$, it follows from Proposition (2) that w and z are non-negative.

We show now that if $u \leq u^\star$ and $v \leq v^\star$ then $w \leq u^\star$ and $z \leq v^\star$. We substract $(14 - a)$ from $(13 - a)$ and we obtain

$$(-\Delta+m\rho)(u^\star-w)=(a+m)\rho(u^\star-u)+b\rho(v^\star-v)-f(x,u,v)+\frac{\rho}{\alpha}u^\star = H > 0\,,$$

or equivalently

$$-\Delta(u^\star - w) = -m\rho(u^\star - w) + H\,.$$

Then by Proposition 1, $u^\star - w \geq 0$. Analogously we have: $z \leq v^\star$.

(iv) Finally we show that $T : V \times V \longrightarrow V \times V$ is completely continuous:

Let $u_k \to u$ and $v_k \to v$ in $\mathcal{H}$; by (10), (11), $f(x, u_k, v_k) \to f(x, u, v)$ in $\mathcal{H}'$. Let us denote by w_k and z_k the sequences associated with u_k and v_k by (14); it follows that:

$$(-\Delta+m\rho)(w-w_k) = (a+m)\rho(u-u_k)+b\rho(v-v_k)+f(x,u,v)-f(x,u_k,v_k)\,.$$

Multiplying by $w - w_k$ and integrating we have

$$\int_{\mathbb{R}^n} |\nabla(w-w_k)|^2 + m\int_{\mathbb{R}^n} \rho(w-w_k)^2 =$$

$$= (a+m)\int_{\mathbb{R}^n} \rho(u-u_k)(w-w_k) + b\int_{\mathbb{R}^n} \rho(v-v_k)(w-w_k) +$$

$$+ \int_{\mathbb{R}^n} (f(x,u,v) - f(x,u_k,v_k))(w-w_k)\,.$$

By Cauchy-Schwarz inequality, $\|w - w_k\|_V \to 0$ as $\|u - u_k\|_V \to 0$, $\|v - v_k\|_V \to 0$. Similarly $z_k \to z$ in V.
Now we prove the compactness of T. We multiply $(14 - a)$ by w and integrate:

$$\int_{\mathbb{R}^n} |\nabla w|^2 + m\int_{\mathbb{R}^n} \rho w^2 = (a+m)\int_{\mathbb{R}^n} \rho uw + b\int_{\mathbb{R}^n} \rho vw + \frac{1}{\alpha}\int_{\mathbb{R}^n} \frac{\alpha f(x,u,v)}{\rho u}\rho uw.$$

Hence, by (11–a):

$$\|w\|_{\rho,m} \leq C(\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}).$$

Analogously:

$$\|z\|_{\rho,m} \leq C(\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}).$$

Therefore if u_j and v_j are bounded sequences in V, the associated sequences w_j and z_j are bounded in V.

We show now that w_j and z_j are Cauchy sequences in V.

Suppose that $\|u_j\|^2_V \leq M$ and $\|v_j\|^2_V \leq M$. Let $\varepsilon > 0$ be fixed. Choose R large enough so that $(1 + R^2)\rho(R) < \dfrac{\varepsilon M}{16\gamma}$, where γ is defined in Lemma 1.

Let $B = \{x \in \mathbb{R}^n \mid |x| < R\}$ and $B' = \{x \in \mathbb{R}^n \mid |x| > R\}$; since u_j is bounded in V, u_j is bounded in $H^1(B)$; but B is bounded and therefore the embedding $H^1(B)$ into $L^2(B)$ is compact; hence there exists a convergent subsequence, still denoted by $(u_j)_{j \in \mathbb{N}}$, which is a Cauchy sequence and we can choose j and k large enough so that

$$\int_B \rho |u_j - u_k|^2 dx \leq \int_B |u_j - u_k|^2 dx < \frac{\varepsilon}{4}\,.$$

Moreover

$$\int_{B'} \rho |u_j - u_k|^2 dx = \int_{B'} (1 + |x|^2)\rho(x) \frac{1}{(1 + |x|^2)} |u_j - u_k|^2 dx \leq$$

$$\leq \frac{\varepsilon M}{16\gamma} \gamma \|u_j - u_k\|_V^2 \leq \frac{\varepsilon}{4}\,.$$

Since analogous inequalities hold for v_j , we can deduce that w_j is a Cauchy sequence in V. Hence it converges towards w . The same holds for z_j and therefore, T is compact in $V \times V$.

We can apply Schauder fixed point theorem and we deduce that there exists at least one positive solution $(u, v) \in V \times V$ of System (1) satisfying $u_\circ \leq u \leq u^\star$, $v_\circ \leq v \leq v^\star$. ∎

5.2 – Uniqueness

For proving uniqueness, we assume additional assumption on f , g :

(15) We assume that there exists a concave function $(x, u, v) \longrightarrow H(x, u, v)$ such that:

$$f(x, u, v) = b \frac{\partial H}{\partial u}(x, u, v) \ \textit{ and } \ g(x, u, v) = c \frac{\partial H}{\partial v}(x, u, v).$$

Then,we have:

THEOREM 6. *Assume that* (8), (2_ρ) *and* (15) *are satisfied, then there exists a unique solution of System* (1).

Proof. Assume that (u_1, v_1) and (u_2, v_2) are solutions of (1). If we set:

$$w = u_1 - u_2 \quad and \quad z = v_1 - v_2$$

then

$$-\Delta w = aw\rho + bz\rho + b\Big(\frac{\partial H}{\partial u}(x, u_1, v_1) - \frac{\partial H}{\partial u}(x, u_2, v_2)\Big) \qquad \text{in } \mathbb{R}^n$$

$$-\Delta z = cw\rho + dz\rho + c\Big(\frac{\partial H}{\partial v}(x, u_1, v_1) - \frac{\partial H}{\partial v}(x, u_2, v_2)\Big) \qquad \text{in } \mathbb{R}^n\,,$$

$$w \longrightarrow 0\,,\; z \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty\,.$$

Multiplying the first equation by $\frac{w}{b}$ and the second by $\frac{z}{c}$ and integrating over $\mathbb{R}^n$, we get:

$$b^{-1}\int_{\mathbb{R}^n}|\nabla w|^2 dx + c^{-1}\int_{\mathbb{R}^n}|\nabla z|^2 dx = \frac{a}{b}\int_{\mathbb{R}^n}\rho w^2 dx + 2\int_{\mathbb{R}^n}\rho wz\, dx + \frac{d}{c}\int_{\mathbb{R}^n}\rho z^2 dx +$$

$$+\int_{\mathbb{R}^n}\Big[\Big(\frac{\partial H}{\partial u}(x, u_1, v_1) - \frac{\partial H}{\partial u}(x, u_2, v_2)\Big)(u_1 - u_2) +$$

$$+\Big(\frac{\partial H}{\partial v}(x, u_1, v_1) - \frac{\partial H}{\partial v}(x, u_2, v_2)\Big)(v_1 - v_2)\Big] dx\,,$$

and from (5) and (15), we derive:

$$b^{-1}(\lambda_\rho - a)\int_{\mathbb{R}^n}\rho w^2 dx + c^{-1}(\lambda_\rho - d)\int_{\mathbb{R}^n}\rho z^2 dx \leq 2\int_{\mathbb{R}^n}\rho wz dx\,.$$

Let us choose δ such that:

$$\frac{(\lambda_\rho - a)}{b} > \delta^2 > \frac{c}{(\lambda_\rho - d)}\,;$$

then, we have:

$$b^{-1}(\lambda_\rho - a)\int_{\mathbb{R}^n}\rho w^2 dx + c^{-1}(\lambda_\rho - d)\int_{\mathbb{R}^n}\rho z^2 dx \leq 2\int_{\mathbb{R}^n}\rho\delta w\frac{1}{\delta}z\, dx \leq$$

$$\leq \int_{\mathbb{R}^n}\rho\delta^2 w^2 dx + \int_{\mathbb{R}^n}\rho\frac{z^2}{\delta^2}dx \leq b^{-1}(\lambda_\rho - a)\int_{\mathbb{R}^n}\rho w^2 dx + c^{-1}(\lambda_\rho - d)\int_{\mathbb{R}^n}\rho z^2 dx$$

which implies $w = z = 0$, i.e. $(u_1, v_1) = (u_2, v_2)$. □

REMARK 1. When $n = 2$ it is proved in [6] that there is no positive eigenvalue for problem (3); in fact the non-negative eigenvalue is 0 and we can extend our previous theorem with $\lambda_\rho = 0$.

REMARK 2. We can apply the method in [12] for studying some non-cooperative systems where (7) is replaced by (16) $b < 0$, $c > 0$, and $(d-a)^2 + 4bc > 0$.

6 – The case of N × N Systems

We can also extend our results to the case of a system with N equations:

(S′) $$-\Delta U = A\rho U + \mathcal{F},\ in\mathbb{R}^n,\ U \to 0\, as\, |U| \to +\infty,$$

where U (resp. $\mathcal{F}$) is a column matrix with elements u_i (resp. f_i) and where

(17) A is a $N \times N$ matrix with constant coefficients

(17′) The coefficients outside the diagonal are positive.

For such a system analogous results (Maximum Principle and existence of solutions) hold with (2_ρ) replaced by

(C_ρ) $$\mathcal{B} := (\lambda_\rho I - A) \text{ is a nonsingular M-matrix.}$$

THEOREM 7. *Assume that* (4) *and* (17) *hold, and that* $F \in \mathcal{H}'^N$. *Then System* (S') *satisfies Maximum Principle if and only if inequalities* (C_ρ) *are satisfied.*

The proof of Theorem 7, is very similar to that of [14], with the same change of spaces as above so that we only sketch the proof. We recall some technical results concerning M-matrices (which can be found e.g. in [3], or [14], Lemmas 1 and 2).

6.1 – Matricial calculus lemmas

First recall that a nonsingular square matrix $\mathcal{B} = (b_{ij})$ is a M-matrix if $b_{ij} \le 0$ for $i \ne j$, $b_{ii} < 0$ and if all principal minors extracted from $\mathcal{B}$ are positive.

We introduce the following notation:
For $1 \le k \le N$, we denote by B_k the matrix obtained by taking the last $(N-k)$ rows and columns out of the matrix $B := (\lambda_\rho I - A)$. Then:

Lemma 4. *Assume that all principal minors of order $j \le k$, extracted from B_N are positive. If $\det B_{k+1} < 0$, then for all $Y \in \mathbb{R}^{k+1}, Y > 0$, the solution $X \in \mathbb{R}^{k+1}$ of the equation $B_{k+1}X = Y$ is negative.*

Lemma 5. *Assume that B_N is a nonsingular M-matrix; then for all $Y \in \mathbb{R}^N, Y \le 0$(resp.$\ge 0$), the solution $X \in \mathbb{R}^N$ of $B_N X = Y$ is non positive (resp. non negative).*

Proof of theorem 7.
The condition is necessary: We assume that the Maximum Principle is satisfied; we prove by induction on k and by contradiction that all the principal minors of order k, extracted from B_N, are positive.

We know from Section 2 -scalar case- that the result holds for $k = 1$.

Assume now that it holds for k.

i) If $\det B_{k+1} < 0$, we can choose $Y \in \mathbb{R}^N$, with components $Y_i = 1$ for $1 \le i \le k+1$, $X \in \mathbb{R}^N$ with components $X_i = 0$ for $k+2 \le i \le N$ such that $Y = B_N X$. Then $X' \in \mathbb{R}^{k+1}$ (resp. $Y' \in \mathbb{R}^{k+1}$) with components X_i (resp. Y_i), $1 \le i \le k+1$ satisfies $B_{k+1}X' = Y'$.

By Lemma 3, $X_i < 0$ for $1 \le i \le k+1$.

Hence, since the system is cooperative, $Y_i = \sum_{j=1}^{j=k+1} (-a_{i,j})X_j \ge 0$ for $k+1 \le i \le N$.

Then $U = X\psi_\rho \le$ satisfies

$$-\Delta U = \lambda_\rho X \rho \psi_\rho = A\rho U + F$$

with $F = Y\rho\psi_\rho \ge 0$, which contradicts the Maximum Principle.

ii) If $\det B_{k+1} = 0$, we consider —with the same notations as above— X, such that $B_{k+1}X' = 0$. Since $\det B_k \ne 0, X_{k+1} \ne 0$ and we can take

$X_{k+1} = -1$. Then $\sum_{p=1}^{p=k}(-a_{j,p}) = (-a_{j,k+1}), 1 \le j \le k+1$. Since the system is cooperative, $B_k X'' \le 0$ with $X'' \in \mathbb{R}^k$, with components $X_i, i \le k$. By Lemma 3, $X_i \le 0, i \le k$, and as above, we contradict the Maximum Principle.

It follows from i) and ii) that all the principal minors of order $k+1$, extracted from B_N are positive. Hence $\lambda_\rho I - A$ is a nonsingular M-matrix.

The condition is sufficient: This proof is almost the same than when $N = 2$. We multiply the i-th equation by u_i^- and integrate. Finally we obtain: $(\lambda_\rho I - A)Z \le 0$ where Z is the column matrix with elements $[\int_{\mathbb{R}^n} \rho \mid u_i^- \mid^2]^{1/2}$.

By Lemma 2 this implies $Z \le 0$ and hence $U \ge 0$. □

We can also extend Theorem 5 and prove the existence of a nonnegative solution for a semilinear system:

$$(1') \qquad -\Delta U = A\rho U + \mathcal{F}(x, U),\ \in \mathbb{R}^n\ , \quad U \to 0 \text{ as } |x| \to +\infty.$$

Assume that

(18) For any $u_i \in \mathcal{H}$, for any $1 \le j \le N$ $U \to f_j(x, U)$ is a Caratheodory function;

(19) For any $1 \le i \le N$, $0 \le f_i(x, U) \le \dfrac{u_i}{\alpha_i}\rho(x)$ $\forall u_i \ge 0$, $\forall x \in \mathbb{R}^n$,

where α is a column matrix with components α_i such that $(\lambda_\rho I - A)\alpha = \mathbf{1}$; here $\mathbf{1}$ is the column matrix with N elements equal to 1.

THEOREM 8. *Assume that* (4) *and* (17) *to* (19) *are satisfied; if* (C_ρ) *holds, then* (1′) *possesses one non negative solution in* V^N.

The proof of this theorem is completely similar to that of Theorem 5 and we omit it here. □

We also can extend Theorem 1 in [5]:

THEOREM 9. *Assume that* (4), (17) *and* (C_ρ) *are satisfied and that* $\mathcal{F} \in \mathcal{H}'^N$. *Then* (S′) *has a unique solution* $U \in V^N$.

We follow here the proof of [2], which is derived from [5] but a bit shorter.

Proof. If necessary, we first choose m such that $a_{ii}+m>0$ for any $1\leq i\leq N$ and we write equation (1_i) as:

$$(-\Delta+m)u_i=(a_{ii}+m)\rho u_i+\sum_{j\neq i}a_{ij}\rho u_j+f_i. \tag{20}$$

For any $\varepsilon\in\,]0;1[$, we derive from (20):

$$\begin{aligned}(-\Delta+m)u_i^\varepsilon&=(a_{ii}+m)\rho u_i^\varepsilon[1+\mid \varepsilon u_i^\varepsilon\mid]^{-1}+\\&\quad+\sum_{j\neq i}a_{i,j}\rho u_j^\varepsilon[1+\mid\varepsilon u_j^\varepsilon\mid]^{-1}+f_i.\end{aligned} \tag{21}$$

We will prove that u_i^ε is bounded in V and hence we can deduce from Schauder fixed point theorem that such $(u_i^\varepsilon)_{1\leq i\leq n}$ exist.

We first prove:

i) *$\varepsilon u_i^\varepsilon$ is bounded in V and tends to 0 strongly in $\mathcal{H}$ and weakly in V.* We multiply (21) by $\varepsilon^2u_i^\varepsilon$ and integrate over $\mathbb{R}^n$. Since $[1+\mid\varepsilon u_i^\varepsilon\mid]^{-1}<1$, we have:

$$\begin{aligned}\int_{\mathbb{R}^n}\mid\nabla\varepsilon u_i^\varepsilon\mid^2+m\int_{\mathbb{R}^n}\rho\mid\varepsilon u_i^\varepsilon\mid^2&\leq(a_{ii}+m)\int_{\mathbb{R}^n}\rho\mid\varepsilon u_i^\varepsilon\mid^2+\\&\quad+\sum_{j\neq=i}a_{ij}\int_{\mathbb{R}^n}\rho\mid\varepsilon^2u_i^\varepsilon u_j^\varepsilon)\mid+\int_{\mathbb{R}^n}\varepsilon^2u_i^\varepsilon f_i\,.\end{aligned} \tag{22}$$

Set $\|u_i^\varepsilon\|_{\mathcal{H}}:=\Big[\int_{\mathbb{R}}^n\rho\mid\varepsilon u_i^\varepsilon\mid^2\Big]^{1/2}$. It follows from (5) and from Cauchy-Schwarz inequality that:

$$(\lambda_\rho-A)\|u_i^\varepsilon\|_{\mathcal{H}}\leq\varepsilon[\int_{\mathbb{R}^n}(\rho)^{-1}\mid f_i\mid^2]^{1/2}. \tag{23}$$

Hence $\varepsilon u_i^\varepsilon$ and tends to 0 strongly in $\mathcal{H}$ as ε tends to 0. This result combined with (22) implies that $\varepsilon u_i^\varepsilon$ and tends to 0 weakly in V.

ii) *u_i^ε is bounded in V.* Here we follow [5]. Assume that

$$t_\varepsilon:=\max(\|u_i^\varepsilon\|_V)\to+\infty\text{ as }\varepsilon\to0\,. \tag{24}$$

Set

$$z_i^\varepsilon := u_i^\varepsilon \cdot t_\varepsilon^{-1}$$

Of course $\|z_i^\varepsilon\|_V \leq 1$; therefore, there exists z_i such that z_i^ε tends to z_i as $\varepsilon \to 0$, strongly in $\mathcal{H}$ and weakly in V. Moreover, by (24), as $\varepsilon \to 0, f_i \cdot t_\varepsilon^{-1} \to 0$ *a.e.*

Hence, dividing (21) by t_ε and passing through the limit, we deduce:

$$(-\Delta + m)z_i = (a_{ii} + m)\rho(z_i) + \sum_{j \neq i} a_{ij}\rho(z_j).$$

Hence $z_i = 0$ which contradicts the fact that there exists a sequence $(\varepsilon_k)_{k\in\mathbb{R}}$ such that for one index i, $\|{z_i}^{\varepsilon_k}\| = 1$.
Therefore, passing through the limit, $u_i^\varepsilon \to u_i^0$ and u_i^0 satisfies (S′). □

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