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Examples of differential geometric behaviour of projective varieties in positive characteristic

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Riassunto: *Studiamo tre esempi di proprietà di carattere geometrico-differenziale delle varietà proiettive in caratteristica* p*: (1) classificazione di superfici in* $\mathbf{P}^{2n+1}$ *il cui* m*-esimo spazio osculatore ha sempre dimensione* $2m$ $(1 \leq m \leq n)$*; (2) ipersuperfici con rango Hessiano* 0*; (3) ipersuperfici singolari di spazi proiettivi pesati con fascio tangente localmente libero.*

Abstract: *Here we study three examples of differential geometric behaviour of projective varieties in positive characteristic: (1) the classification of smooth surfaces in* $\mathbf{P}^{2n+1}$ *whose* m*-th osculating spaces have everywhere dimension* $2m$ $(1 \leq m \leq n)$*; (2) hypersurfaces with Hessian rank* 0*; (3) singular hypersurfaces in weighted projective spaces whose tangent sheaf is locally free and a subbundle of the restricted tangent bundle.*

1 – Introduction

In the last few years an active field of research was the study of projective properties of subvarieties of $\mathbf{P}^n$ under the assumption that the algebraically closed base field $\mathbf{K}$ has positive characteristic. In this paper we give three independent results on this topic. In the second section we

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show how to extend to positive characteristic the classification theorem in [6] on surfaces in $\mathbf{P}^{2n+1}$ with extremal osculating behaviour. Most of section three is devoted to a refinement of the non enumerative part of [18] about hypersurfaces with "pathological" (i.e. impossible in characteristic 0) differential geometric behaviour (see Theorems 3.1, 3.2, 3.3, 3.4 and 3.5). The proofs of these five theorems (i.e. a reduction to the case of plane curves) are simpler the the ones used in [18]. At the end of this section (just to give another example of funny behaviour of the derivatives in positive characteristic) we give the very easy extension of [4], Th. 0.1, from the case of hypersurfaces of $\mathbf{P}^n$ to the case of hypersurfaces of a weighted projective space (see Theorem 3.6). In each case we use freely background and proofs of the quoted references.

2 – Surfaces with extremal osculating behaviour

In this section we show that only very minor changes (given below) will be sufficient to extend the theorem of [6] in positive characteristic under the assumption that $p := \mathrm{char}(\mathbf{K}) \geq n$, i.e. to prove the following result.

THEOREM 2.1. *Assume $p := \mathrm{char}(\mathbf{K}) \geq n$, and $p \geq 7$ if $n = 2$. Let $X \subset \mathbf{P}^{2n+1}$, $n \geq 2$, be a smooth projective surface not contained in a hyperplane and such that for every $x \in X$ and every $m \leq n$ the m-th osculating space $\mathrm{Osc}^m_X(x)$ at x has dimension $2m$. Then X is a balanced rational normal scroll.*

The projective part of the proof of 2.1 is trivial, but the second part related to the classification of surfaces in positive characteristic is less trivial. The characteristic 0 proof of Theorem 2.1 given in [4] was mainly based on the intermediate results and calculations made in [13] to prove particular cases of theorem. Everything in [13] used in [4] works verbatim under our assumption on p until we arrive at [13], Corollary at page 219, i.e. to the statement that X has Kodaira dimension $\kappa(X) < 0$ and that $X \neq \mathbf{P}^2$. It is very easy to check that $X \neq \mathbf{P}^2$. Hence to find a contradiction we may assume $\kappa(X) \geq 0$. Let S be the minimal model of X and b the number of blowing ups of points needed to pass from S to X. Note that $c_2(X) = c_2(S) + b$. The proof of corollary works verbatim in positive characteristic if one can prove that $c_2(S) \geq 0$. By the positive characteristic classification of surfaces (see [8] or [7]) we have $\kappa(X) \neq 0$. By [12], we have $c_2(X) \geq 0$ if $\kappa(X) = 1$. Hence we assume $\kappa(X) = 2$. Thus by [3], Th. A and 0.3.1, or [1] or [2], Th. A, a multiple of $K + H$ is still spanned by global sections and we have ([2], table at page 179) $H(K+H) > 0$ and $K(K+H) > 0$. Furthermore, since $\kappa(X) = 2$ we have $KH > 0$. Since $\kappa(X) = 2$, by [9], Prop. 4.5, we have $pc_2(S)+c_1(S)^2 \geq 0$. Since $p \geq n$ by assumption we still have $(n-1)^2c_2(S) + nc_1(S)^2 \geq 0$. Hence if $n \geq 3$ the numerical part of the proof of [13], Corollary at page 219, works verbatim and we get a contradiction. Thus we may assume $n = 2$. For $n = 2$ we have $c_2(X) + 2(K+H)^2 + 2KH + H^2 = 0$. Hence we may assume $c_2(X) + K^2 < 0$, i.e. by Noether formula $\chi(\mathbf{O}_X) < 0$. By [15], Th. 8, we have $p \leq 7$. Hence we may assume $p = 7$, $n = 2$, $c_2(X) < 0$. By [9], Prop. 4.5, we have $p(c_2(X)-b)+(K^2+b) \geq 0$. Hence $11c_2(X)+2K^2 \geq 0$. By [13], eq. (2), we have $11c_2(X)+2K^2+10KH = 0$, contradiction. Now we need to look at the proofs in [4]. By the part of the proof of [13] after the corollary just extended to our setting we may assume $n \geq 5$, $\kappa(X) < 0$ and that X is not a relatively minimal model. Since $\kappa(X) \neq 2$ we still have Bogomolov's criterion of instability ([14], Th. 7). Hence we may use Reider's method (see for instance [14], Cor. 8) to obtain the very ampleness of $(K+H)$ except exactly the characteristic 0 cases. Hence, using the very ampleness of $K+H$ as in [4], the numerical part of [4] at page 206 works verbatim and concludes the proof of 2.1. □

3 – Hypersurfaces with Hessian rank 0

Here we extended the non enumerative part of [18] to the case in which the hypersurface has not too many singularities. In particular all the results will be proven for every normal hypersurface; this result was previously known, since it is contained in the unpublished part of the thesis [10]. We stress that the proofs (i.e. the reduction to the case of plane curves) are completely different and much simpler that the proofs in [10], [16], [17] and [18]. For background and definitions (e.g. coordinate gap number b_2) see [18] and references therein. At the end of the section we give an easy example (see Theorem 3.6) of extension to the case of a weighted projective space of a positive characteristic funny behaviour known for an ordinary projective space.

We fix an algebraically closed base field $\mathbf{K}$ with $p := \mathrm{char}(\mathbf{K}) > 0$. We fix homogeneous coordinates $x_0, \dots, x_n$ on a projective space $\mathbf{P}^n$; if U is a homogeneous polynomial, let U_i or $D_i(U)$ (resp. U_{ij} or $D_{i,j}(U)$ and so on) its partial derivative with respect to x_i (resp. x_i and x_j). We fix an integral hypersurface $X \subset \mathbf{P}^n$ with degree d and let G be its homogeneous equation. We will give a total weight $\mathrm{wtsg}(X)$ for the contributions of the singularities of X to give numerical bounds among the assumptions in all the theorems (except 3.6) proven in this section. Let $\{S_b\}_{n\in B}$ be the family of all irreducible components of dimension $n-2$ of $\mathrm{Sing}(X)$ (with its reduced structure); note that $B = \emptyset$ if X is normal. We take a general plane $\prod \subset \mathbf{P}^n$ and look at the integral curve $C := X \cap \prod$. For each singular point, P, of X let m_P be the multiplicity of C and e_P the multiplicity of the Jacobian ideal of C (i.e. the minimum of the intersection multiplicities at P of C with its partial derivative loci). Set

$$\mathrm{wtsg}(X) := \mathrm{wtsg}(C) = \sum_{P\in \mathrm{Sing}(C)} \left(e_P - m_P(m_P-2)\right). \tag{1}$$

Now we give the statements of the theorems which will be proved here.

Theorem 3.1. *Assume $p > 2$. Let $X := \{G = 0\}$ be a degree d hypersurface of $\mathbf{P}^n$ with coordinate gap number $b_2(X) > 2$. Assume* $\mathrm{wtsg}(X) < d$. *Then the second order partial derivatives $G_{i,j}$ of G vanish identically.*

Theorem 3.2. *Assume $p > 2$. Let $X := \{G = 0\}$ be a degree d hypersurface of $\mathbf{P}^n$ with $b_2(X) = q = p^e$ with $e > 0$. Assume $G_{i,j} = 0$ for all i, j and that for a general plane section C of X we have*

$$\sum_{P\in \mathrm{Sing}(C)} e_P < (1-(1/p))d^2. \tag{2}$$

The all the partial derivatives $D_i^q(G)$ of order q $(0 \le i \le n)$ are identically 0.

Theorem 3.3. *Assume $p > 2$. Let $X := \{G = 0\}$ be a degree d hypersurface of $\mathbf{P}^n$ with $b_2(X) = q = p^e$ with $e > 0$. Assume* $\mathrm{wtsg}(X) <$

d. Then $d = kq + 1$ for some k and there are $n + 1$ degree k polynomials $P_0, \ldots, P_n$ such that

$$G = \sum_i x_i P_i^q \,. \tag{3}$$

Now we consider the case $p = 2$.

THEOREM 3.4. *Assume $p = 2$. Let $X := \{G = 0\}$ be a degree d hypersurface of $\mathbf{P}^n$ with $b_2(X) = q = 2^e$ with $e \geq 2$. Assume that for a general plane section C of X we have*

$$\sum_{P \in \mathrm{Sing}(C)} e_P < (d/2)\,.$$

The $b_2(X) = 2^e$ with $e \geq 2$ if and only if $G_{i,j} = 0$ for all i, j and $D_i^t(G) = 0$ for all t with $t = 2^a$ and $1 \leq a < e$.

THEOREM 3.5. *Assume $p = 2$. Let $X := \{G = 0\}$ be a degree d hypersurface of $\mathbf{P}^n$ with $b_2(X) = q = 2^e$ with $e \geq 2$. Assume that for a general plane section C of X we have*

$$\sum_{P \in \mathrm{Sing}(C)} e_P < (d/2)\,.$$

Then $d = kq + 1$ for some k and there are $n + 1$ degree k polynomials $P_0, \ldots, P_n$ such that

$$G = \sum_i x_i P_i^q \,.$$

PROOF OF 3.1 By linear algebra to show that G has all second order partial derivatives $G_{i,j}$ identically 0, it is sufficient to show the corresponding vanishing in any system of coordinates and at a general point. Hence it is sufficient to restrict G to a general plane Π. Then $C := C \cap \Pi$ is non reflexive and 3.1 follows from [5], Th. 3. ∎

By the last remark in [5] the bound in Theorem 3.1 can be improved under certain arithmetic conditions; for instance if p divides m_P (resp. $m_P - 1$), then in the rigid hand side of (1) one can put m_{P^2} (resp. $m_P(m_P - 1)$) instead of $m_P(m_P - 2)$.

Proof of 3.2 As in the previous proof we reduce to the case of a general plane saection $C := X \cap \Pi$. Note that the bound (2) is much weaker than the bound in the statement of [11], Th. 5.5. Hence 3.2 follows quoting [5] instead of [11], Th. 5.1, in the proof of [11], Th. 5.5. ∎

Proof of 3.4 Reduce as in the previous proofs to the case of a plane curve and use [11], Th. 5.11. ∎

Proof of 3.5 Note that as in [18], proof of Th. 2.2 (or in [11], proof of Cor. 5.10) one obtains in a formal way the canonical form (3) as soon as one has proved the vanishing of all partial derivatives $G_{i,j}$ and D_i^q. Use 3.2 if $p < 2$; use 3.4 and [11], Cor. 5.16, if $p = 2$ ∎

In the last part of this paper we extend from the case of $\mathbf{P}^n$ to the case of a weighted projective space $W = \mathbf{P}(w_0, \ldots, w_n)$ the classification theorem [4], Th. 0.1, on singular hypersurface whose tangent sheaf is a subbundle of the restriction of $T\mathbf{P}^n$. Here we allow the case $p = 0$.

Theorem 3.6. *Let $\mathbf{K}$ be an algebraically closed field; set $p :=$ $\mathrm{char}(\mathbf{K}) \geq 0$. Let X be an integral hypersurface of a tame weighted projective space $W := \mathbf{P}(w_0, \ldots, w_n)$ (i.e. if $p > 0$ assume that all the weights of W are coprime to p). Let $\pi : \mathbf{P}^n \to W$ be the canonical cover. Assume $\mathrm{Sing}(X) \neq \emptyset$, $\pi^1(X)$ integral and not smooth and that the tangent sheaf TX is a subbundle of $TW|X$ with $\mathbf{O}_X(k)$ as quotient sheaf; assume that $\mathbf{O}_W(k)$ is locally free in a neighborhood of X. Then $p > 0$ and there are weighted homogeneous polynomials u, h, v such that the weighted homogeneous polynomial f of X is of the form*

$$f = u^p h + v^p \,. \tag{4}$$

Viceversa, if $p > 0$, is given by (4) with $k := \deg(h) > 0$ and $\mathbf{O}_W(k)$ is invertible in a neighborhood of $X := \{f = 0\}$, $\mathrm{Sing}(X) \neq \emptyset$ and at each point of X one of the “weighted” partial derivatives of h does not vanish, then $(TW|X)/TX \cong \mathbf{O}_X(k)$.

The only problem to prove 3.6 is to find the "right" set up (tame weighted projective spaces and the local freeness of $\mathbf{O}_W(k)$ in a neighborhood of X). After that, the proof of [4], Th. 0.1, works verbatim.

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