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Low-frequency acoustic scattering by an infinitely stratified scatterer

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RIASSUNTO: *In questo lavoro si consida il problema della diffusione di un'onda acustica piana da un diffusore a infiniti strati con nucleo permeabile o rigido. Anzitutto si dimostra l'esistenza e l'unicità della soluzione; poi, usando l'approssimazione di bassa frequenza, si riduce il problema ad una successione iterativa di problemi con potenziale. Si ricavano infine limitazioni per i coefficienti di bassa frequenza e per l'ampiezza di diffusione normalizzata.*

ABSTRACT: *In this work we consider the problem of scattering of a plane acoustic wave by an infinitely stratified scatterer with a soft, a rigid or a resistive core. Firstly, we prove the existence and uniqueness of solutions of this problem. Then using the low-frequency approximation, we reduce it to an iterative sequence of potential problems. Bounds for the low-frequency coefficients and for the normalized scattering amplitude are derived.*

1 – Introduction

The low frequency approximation, appropriate for large, compared to the characteristic dimension of the scatterer, wavelengths, has been widely employed to investigate the problem of scattering of a plane acoustic wave.

KEY WORDS AND PHRASES: *Transmission problem – Infinitely stratified scatterer – Low-frequency theory – Scattering amplitude.*
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A systematic development for the case of a soft scatterer (Dirichlet boundary condition) of arbitrary shapes is given by MORSE and FESHBACH in [17]. AR and KLEINMAN in [1] formulated the Neuman boundary value problem for the Helmholtz equation (rigid scatterer). A significant contribution to acoustic scattering at low-frequencies has been made by DASSIOS [7].

TWERSKY in [19] has obtained the leading term approximation of the real part of the scattering amplitude by direct application of the general scattering theorem. The acoustic scattering by a multi layered ellipsoid is studied in [2]. In [18], SABATIER and DOLVECK-GUILBARD, and in [19], DUPUY and SABATIER, present the scattering theory corresponding to the impedance equation, with discontinuity surface corresponding to a jump in impedance and / or its normal derivative. Results for an infinitely stratified scatterer in low-frequency electromagnetic scattering theory are given in [3]. Existence, uniqueness and regularity results for diffraction problems for general second order, linear parabolic and hyperbolic equations, with coefficients having discontinuities of the first kind on an infinite number of smooth surfaces, are studied by the authors in [5].

In this work we describe a systematic and integrated theory for the scattering of a plane acoustic wave by an infinitely stratified scatterer with a soft, a rigid, or a resistive core. Such a scatterer consists of infinitely many closed penetrable surfaces S_j, $j = 1, 2, \ldots$ containing a soft or a rigid core. Every surface S_a is inside S_b for $a > b$. On these surfaces there must be imposed certain conditions, known as "transmission conditions" that physically express the continuity of the medium, and the equilibrium of the forces acting on it.

Such scatterers appear in several domains of physics. The sound diffraction by the human brain, corresponds to an acoustic transmission problem [10], similar to the one we are studying here. Other examples include a transmission problem for the impedance equation [18], and the composite materials, having an "onion" structure, in elasticity [16].

Our method is closely related to one of the approaches used to study problems for inhomogeneous media. The material parameters, which - for inhomogeneous media - are functions of the position vector, are approximated by piecewise constant functions. If a tesselation like the one described above is assumed, with constant material parameters in each layer, then the exact solution to the stratified problem, might be excepted to be a reasonable approximation to the solution of the problem for an inhomogeneous medium. (A similar case is the one of laminated media in acoustics [11]). A thorough study of such an approximation, is under consideration by the present authors.

In Section 2 we formulate the scattering problem. We define precisely the infinitely stratified scatterer with a core, and give the boundary, transmission, and radiation conditions. In Section 3 we first prove that the only classical solution of the homogeneous transmission problem for the Helmholtz equation is the trivial one. This extends a results of Kress and Roach referring to one interface [12], to our infinitely stratified structure. For the existence of solutions of the non homogeneous transmission problem we introduce a generalized solutions approach. Then by a regularity argument we prove that the weak solution is a classical one. The standard approach, i.e. the implementation of potential theory, [2], [6], [12], leads, in our case to an infinite system of integral equations.

Even in the case of finite number of layers, our generalized solutions method does not present disadvantages as far as the length of the proof is concerned, in comparison to the standard method. Integral representations for the total exterior field and the normalized scattering amplitude are given in Section 4. In Section 5, the low-frequency theory is applied to the scattering problem, and bounds for the low-frequency coefficients and the scattering amplitude are derived. We conclude this section with some comments on how related existing results follow as special cases of our general problem.

2 – Statement of the problem

Let $\widetilde{\Omega}$ be a bounded, convex and closed subset of $\mathbb{R}^3$ with boundary S_0. We assume that a core Ω_c, containing the origin of coordinates, with boundary S_c, lies in the interior of $\widetilde{\Omega}$. The set $\Omega = \widetilde{\Omega} - \Omega_c$ is divided into annuli-like regions Ω_j by surfaces S_j, $j = 1, 2, \ldots$, where S_j surrounds $S_{j+1} \cdot S_0$, S_c, S_j are 2-dimensional C^2 surfaces. We, also, suppose that $\operatorname{dist}(S_{j-1}, S_j) > 0$, $j = 1, 2, \ldots$ and that $\lim_{j\to\infty} S_j = S_c$. The set $\widetilde{\Omega}$, described above, is called an infinitely stratified scatterer. The exterior Ω_0 of $\widetilde{\Omega}$ as well as each Ω_j, $j = 1, 2, \ldots$, are homogeneous isotropic media. The radius a of the smallest sphere in $\mathbb{R}^3$, that circumscribes the scatterer is called the characteristic dimension of the scatterer.

The wave number k_j in each region Ω_j, is given by

$$k_j^2 = \frac{\omega}{c_j^2}(\omega + id_j)\,, \quad i^2 = -1\,, \quad j = 0, 1, 2, \dots \tag{2.1}$$

where ω is the angular frequency of the incident wave, c_j is the speed of sound, and d_j is the damping coefficient in Ω_j. We choose the sign of k_j, as usual, such that

$$\operatorname{Im} k_j \geq 0\,, \qquad j = 0, 1, 2, \dots\,.$$

It is obvious that $\operatorname{Re} k_j \neq 0$, $j = 0, 1, 2, \dots$.

We assume that a plane acoustic wave $u^{inc}(\mathbf{r})$ is incident upon the infinitely stratified scatterer. Suppressing the harmonic dependence exp $(-i\omega t)$, the incident wave takes the form

$$u^{inc}(\mathbf{r}) = \mathrm{e}^{ik_0 \hat{\mathbf{k}} \cdot \mathbf{r}}\,, \tag{2.2}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of propagation. Assume that the modulus of the position vector $\mathbf{r}$ is greater than the characteristic dimension of the scatterer a.

The total acoustic field $\psi^{(j)}$ in each interior region Ω_j satisfies the Helmholtz equation

$$\Delta\psi^{(j)}(\mathbf{r}) + k_j^2 \psi^{(j)}(\mathbf{r}) = 0\,, \quad j = 1, 2, \dots \tag{2.3}$$

while, since the total exterior field $u^{(0)}(\mathbf{r})$ given by

$$u^{(0)}(\mathbf{r}) = u^{inc}(\mathbf{r}) + \psi^{(0)}(\mathbf{r}) \tag{2.4}$$

where $\psi^{(0)}(\mathbf{r})$ is the scattered field, also satisfies the Helmoholtz equation in Ω_0, (2.3) is true for all $j = 0, 1, 2, \dots$.

Also, on the surface of the core, the total field satisfies an appropriate homogeneous boundary condition (Dirichlet or Neumann). The scattered field $\psi^{(0)}(\mathbf{r})$ is assumed to satisfy Sommerfeld's incoming radiation condition

$$\frac{\partial \psi^{(0)}(\mathbf{r})}{\partial n} - ik_0 \psi^{(0)}(\mathbf{r}) = 0\left(\frac{1}{r}\right), \quad r \to \infty\,. \tag{2.5}$$

As it is well known, [6], $\psi^{(0)}$ automatically satisfies

$$\psi^{(0)}(\mathbf{r}) = 0\left(\frac{1}{r}\right), \qquad r \to \infty\,. \tag{2.6}$$

Let ρ_j denote the mass density on the layer Ω_j.

On S_0 we have the following transmission conditions

$$\begin{aligned} &\psi^{(1)} - \psi^{(0)} = u^{inc} \\ &\frac{1}{\rho_1}\frac{\partial \psi^{(1)}}{\partial n} - \frac{1}{\rho_0}\frac{\partial \psi^{(0)}}{\partial n} = \frac{1}{\rho_0}\frac{\partial u^{inc}}{\partial n}\,. \end{aligned} \tag{2.7}$$

The transmission conditions on S_j, $j = 1, 2, \ldots$, are given by

$$\begin{aligned} &\psi^{(j+1)} - \psi^{(j)} = 0 \\ &\frac{1}{\rho_{j+1}}\frac{\partial \psi^{(j+1)}}{\partial n} - \frac{1}{\rho_j}\frac{\partial \psi^{(j)}}{\partial n} = 0 \end{aligned} \tag{2.8}$$

By a standard procedure, the homogeneous equations and nonhomogeneous transmission conditions of the above problem, can be transformed to

$$\Delta \psi^{(j)} + k_j^2 \psi^{(j)} = f_j \quad \text{in} \quad \Omega_j \tag{2.9}$$

$$\left.\begin{aligned} &\psi^{(j+1)} - \psi^{(j)} = 0 \\ &\frac{1}{\rho_{j+1}}\frac{\partial \psi^{(j+1)}}{\partial n} - \frac{1}{\rho_j}\frac{\partial \psi^{(j)}}{\partial n} = 0 \end{aligned}\right\}, \quad \text{on} \quad S_j \tag{2.10}$$

for all $j = 0, 1, 2, \ldots$, where $f_j \equiv 0$, $j = 1, 2, \ldots$, and f_0 is a known C^2 function depending on u^{inc} and ρ_0. For more details about the physical problem we refer to [11].

3 – Solvability of the scattering problem

In this section we prove the existence and uniqueness of solutions for the acoustic transmission problem. In what follows we shall make the following assumptions on the coefficients of (2.9), (2.10); the superposed bar denotes complex conjugation:

$$(3.1)\quad \begin{cases} \text{Let } k \in \mathbb{C}-\{0\} \text{ with } 0 \le \arg k \le \pi,\ j=0,1,2,\dots \text{ be such that} \\ \dfrac{\overline{k_j^2}}{\overline{k_0^2}} \in \mathbb{R}, \quad \text{with} \quad \sup \dfrac{\overline{k_j^2}}{\overline{k_0^2}} < +\infty . \\ \text{Moreover, let } \sup \dfrac{\rho_0}{\rho_j} < +\infty. \end{cases}$$

Let us denote by (H) the homogeneous transmission problem, consisting of the equations

$$(3.2)\qquad \Delta\psi^{(j)} + k_j^2\psi^{(j)} = 0\,, \quad \text{in} \quad \Omega_j\,, \quad j = 0,1,2,\dots$$

the transmission conditions (2.10), the radiation condition (2.5), the homogeneous Dirichlet boundary condition on S_c, and (3.1).

Remark 3.1 The homogeneous Neumann boundary condition on S_c may be considered instead, without any problem whatsoever in what follows.

Now we are in a position to prove

Theorem 3.1. *(H) has only the trivial solution.*

Proof. Let $\Omega_{0,R} = \{\mathbf{r} \in \Omega_0 : r < R\}$, $R > 0$. By Green's first theorem on $\Omega_{0,R}$, we obtain

$$(3.3)\quad \int\limits_{r=R} \psi^{(0)} \frac{\partial\bar{\psi}^{(0)}}{\partial n} ds = \int\limits_{\Omega_{0,R}} \psi^{(0)} \Delta\bar{\psi}^{(0)} du + \int\limits_{S_0} \psi^{(0)} \frac{\partial\bar{\psi}^{(0)}}{\partial n} ds + \int\limits_{\Omega_{0,R}} |\operatorname{grad}\psi^{(0)}|^2 du$$

which, again by Green's first theorem over Ω_1 and the transmission conditions (2.10), becomes

$$(3.4)\quad \begin{aligned} \int\limits_{r=R} \psi^{(0)} \frac{\partial\bar{\psi}^{(0)}}{\partial n} ds &= \int\limits_{\Omega_{0,R}} \psi^{(0)} \Delta\bar{\psi}^{(0)} du + \int\limits_{\Omega_{0,R}} |\operatorname{grad}\psi^{(0)}|^2 du + \\ &+ \frac{\rho_0}{\rho_1} \int\limits_{\Omega_1} \psi^{(1)} \Delta\bar{\psi}^{(1)} du + \frac{\rho_0}{\rho_1} \int\limits_{\Omega_1} |\operatorname{grad}\psi^{(1)}|^2 du + \frac{\rho_1}{\rho_0} \int\limits_{S_1} \psi^{(1)} \frac{\partial\bar{\psi}^{(1)}}{\partial n} ds\,. \end{aligned}$$

By repeated use of Green's first theorem, and taking into account (3.2), the transmission conditions (2.10), the boundary behaviour on S_c and dividing throughout by $\overline{k_0^2}$, we get from (3.4)

$$\begin{aligned}\frac{1}{\overline{k_0^2}}\int_{r=R}\psi^{(0)}\frac{\partial\bar{\psi}^{(0)}}{\partial n}ds &= -\int_{\Omega_{0,R}}|\psi^{(0)}|^2du+\frac{1}{\overline{k_0^2}}\int_{\Omega_{0,R}}|\operatorname{grad}\psi^{(0)}|^2du+\\ &-\sum_{j=1}^{\infty}\frac{\rho_0}{\rho_j}\frac{\overline{k_j^2}}{\overline{k_0^2}}\int_{\Omega_j}|\psi^{(j)}|^2du+\sum_{j=1}^{\infty}\frac{1}{\overline{k_0^2}}\frac{\rho_0}{\rho_j}\int_{\Omega_j}|\operatorname{grad}\psi^{(j)}|^2du\,. \end{aligned}\tag{3.5}$$

The convergence of the series in (3.5) follows by (3.1), and by noting that

$$\sum_{j=1}^{\infty}\int_{\Omega_j}|\psi^{(j)}|^2du=\|\psi_\Omega\|^2_{L^2(\Omega)}<+\infty \qquad \text{and}$$

$$\sum_{j=1}^{\infty}\int_{\Omega_j}|\operatorname{grad}\psi^{(j)}|^2du=\|\psi_\Omega\|^2_{H^1(\Omega)}<+\infty\,,$$

where $\psi_\Omega(\mathbf{r})=\psi^{(j)}(\mathbf{r})$, in Ω_j, $j=1,2,\dots$.

Taking imaginary parts in (3.5), and taking into account (3.1) we get

$$\begin{aligned}\operatorname{Im}\left(\frac{1}{\overline{k_0^2}}\int_{r=R}\psi^{(0)}\frac{\partial\bar{\psi}^{(0)}}{\partial n}ds\right) &= \left(\operatorname{Im}\frac{1}{\overline{k_0^2}}\right)\int_{\Omega_{0,R}}|\operatorname{grad}\psi^{(0)}|^2du+\\ &+\left(\operatorname{Im}\frac{1}{\overline{k_0^2}}\right)\sum_{j=1}^{\infty}\frac{\rho_0}{\rho_j}\int_{\Omega_j}|\operatorname{grad}\psi^{(j)}|^2du\,.\end{aligned}\tag{3.6}$$

As $R\to\infty$, $\Omega_{0,R}$ tends to Ω_0. Moreover, since $\psi^{(0)}$ satisfies the radiation conditions (2.5) and (2.6), it follows, [12], that the left-hand side of (3.6) tends to zero as $R\to\infty$. Since $\operatorname{Im}\dfrac{1}{\overline{k_0^2}}=\dfrac{\operatorname{Im}k_0^2}{|k_0|^4}$, and $\operatorname{Im}k_0^2=2\operatorname{Re}k_0\operatorname{Im}k_0$, we obtain from (3.6), that

$$\frac{2\operatorname{Re}k_0\operatorname{Im}k_0}{|k_0|^4}\sum_{j=0}^{\infty}\frac{\rho_0}{\rho_j}\int_{\Omega_j}|\operatorname{grad}\psi^{(j)}|^2du=0\,.\tag{3.7}$$

If $\operatorname{Im} k_0 > 0$ then we must have that each $\psi^{(j)}$ is a constant in Ω_j. But by (2.6), $\psi^{(0)}$ must be equal to zero in Ω_0, and then by (2.10) each other $\psi^{(j)}$, $j = 1, 2, \ldots$, must also be equal to zero in Ω_j. Hence in this case, (H) has only the trivial solution.

In the case where $\operatorname{Im} k_0 = 0$, by writing the RHS of (3.6) as in (3.7) we obtain that

$$\operatorname{Im}\left(\int\limits_{r=R} \psi^{(0)} \frac{\partial \bar{\psi}^{(0)}}{\partial n} ds\right) = 0 \,. \tag{3.8}$$

From (2.5) it follows that,

$$k_0 \int\limits_{r=R} |\psi^{(0)}|^2 ds + \operatorname{Im}\left(\int\limits_{r=R} \psi^{(0)} \frac{\partial \bar{\psi}^{(0)}}{\partial n} ds\right) = o(1)\,, \quad \text{as} \quad R \to \infty\,, \tag{3.9}$$

and hence by (3.8) we have

$$\int\limits_{r=R} \left|\psi^{(0)}\right|^2 ds = o(1)\,, \quad \text{as} \quad R \to \infty\,. \tag{3.10}$$

Therefore, using Rellich's Theorem [6], it follows, that $\psi^{(0)} = 0$ in Ω_0. If we prove that $\psi^{(1)} = 0$ in Ω_1, then by the same argument $\psi^{(2)}$ will turn to be zero in Ω_2, etc.

By Holmgren's uniqueness theorem, [13], we have that the solution of the Cauchy problem

$$\begin{aligned} &\Delta\psi^{(1)} + k_1^2 \psi^{(1)} = 0\,, \quad &&\text{in} \quad \Omega_1 \\ &\psi^{(1)} = \frac{\partial \psi^{(1)}}{\partial n} = 0\,, &&\text{on} \quad S_0 \end{aligned}$$

is equal to zero, in $\Omega_1 \cap D$, where D is a neighborhood of any point of S_0.

Since $\psi^{(1)}$ is analytic, [6], it follows by the unique continuation principle that $\psi^{(1)} = 0$ in Ω_1. The proof of Theorem 3.1 is, hence, complete.

REMARK 3.2 As it can be seen by its proof, Theorem 3.1 is valid for any parameters k_j, ρ_j, satisfying (3.1). When, in particular, k_j, ρ_j, denote the wave number, and the mass density in Ω_j, respectively, we are led to the following classes of acoustic transmission problems, that satisfy (3.1):

(i) The class of damped layers with $d_j = d_0$ for each j, i.e.

$$k_j^2 = \frac{\omega}{c_j^2}(\omega + id_0) \in \mathbb{C}\,, \quad \text{and}$$

(ii) The class of undamped layers, i.e.

$$k_j^2 = \frac{\omega^2}{c_j^2} \in \mathbb{R}\,.$$

Consider, now, the nonhomogeneous transmission problem consisting of

$$\Delta\psi^{(j)} + k_j^2\psi^{(j)} = f_j\,, \qquad \text{in} \quad \Omega_j$$

$$\left.\begin{aligned} &\psi^{(j+1)} - \psi^{(j)} = 0 \\ &\frac{1}{\rho_{j+1}}\frac{\partial\psi^{(j+1)}}{\partial n} - \frac{1}{\rho_j}\frac{\partial\psi^{(j)}}{\partial n} = 0 \end{aligned}\right\}, \quad \text{on} \quad S_j$$

for all $j = 0, 1, 2, \ldots$, with the homogeneous Dirichlet condition on S_c.

Suppose also that $\psi^{(0)}$ satisfies Sommerfeld's radiation condition (2.5), and that (3.1) is true. This problem will be denoted by (N).

Let $k(\mathbf{r}) = k_j^2$, $f(\mathbf{r}) = f_j(\mathbf{r})$, $q = \dfrac{1}{\rho_j}$, $\psi(\mathbf{r}) = \psi^{(j)}(\mathbf{r})$ in Ω_j, $j = 0, 1, 2, \ldots$, (i.e. $\psi(\mathbf{r}) = \psi_0(\mathbf{r})$, $\mathbf{r} \in \Omega_0$ and $\psi(\mathbf{r}) = \psi_\Omega(\mathbf{r})$, $\mathbf{r} \in \Omega$) and define $R(\Omega_0) = \Big\{\psi^{(0)} \in H^1_{loc}(\Omega_0) : \psi^{(0)} = O\Big(\dfrac{1}{r}\Big)$ and $\dfrac{\partial\psi^{(0)}}{\partial n} - ik_0\psi^{(0)} = o\Big(\dfrac{1}{r}\Big)$, $r \to \infty\Big\}$.

A function $\psi \in H^1(\Omega) \times R(\Omega_0)$ is called a generalized solution of (N), for $f \in L^2(\mathbb{R}^3)$, iff

$$\int\limits_{\mathbb{R}^3 - \Omega_c} \Big(\sum_{m=1}^{3} q\frac{\partial\psi(\mathbf{r})}{\partial r_m}\frac{\partial\varphi(\mathbf{r})}{\partial r_m} - qk(\mathbf{r})\psi(\mathbf{r})\varphi(\mathbf{r})\Big)du = -\int\limits_{\mathbb{R}^3 - \Omega_c} qf(\mathbf{r})\varphi(\mathbf{r})du \tag{3.11}$$

for every $\varphi \in H^1(\Omega) \times R(\Omega_0)$.

As in the standard theory [14], (N) can be written in the form

$$\psi + A\psi = F \tag{3.12}$$

where, since we work in $H^1(\Omega)\times R(\Omega_0)$, the operator $A : H^1(\Omega)\times R(\Omega_0) \to H^1(\Omega) \times R(\Omega_0)$ is compact, [8].

Before moving on to prove that (N) has a unique classical solution, we will state the following regularity result, [4], [12]:

PROPOSITION 3.1. *Under all our assumptions, the generalized solution of* (N) *is in* $C(\mathbb{R}^3)\cap C^{2,a}(\Omega_j)$, *provided* $f \in C^{0,a}(\overline{\Omega}_j)$, $j = 0, 1, 2, \ldots$, $a \in (0, 1)$.

We then conclude this section with

THEOREM 3.2. (N) *has a unique classical solution.*

PROOF. The homogeneous transmission problem (H), can be written as

$$\psi + A\psi = 0 \tag{3.13}$$

where A is the compact operator appearing in (3.12). Since in our case A is self-adjoint, the adjoint homogeneous transmission problem which can be transformed in the form

$$w + A^* w = 0\,, \tag{3.14}$$

coincides with (3.13). By Fredholm's Alternative, a necessary and sufficient condition for the existence and uniqueness of a generalized solution of (3.12), is

$$(F, \tilde{\psi}_s)_{H^1(\mathbb{R}^3)} = 0\,, \tag{3.15}$$

where $\tilde{\psi}_s$, $s = 1, 2, \ldots, \mu$, are the linearly independent solutions of (3.13).

By Proposition 3.1 the generalized solutions of (3.13) are classical. But by Theorem 3.1, (3.13) has only the trivial solution, whereby (3.15) is automatically satisfied. Therefore (N) has a unique generalized solution, which - again by Proposition 3.1 - is classical.

4 – Near and far field

As it is well known, the integral representations of the scattering problems contain all the information about the transmission, boundary and radiation conditions, and the equation which governs the phenomena. In this Section we shall construct an integral representation for the solution in Ω_0, when the infinitely stratified scatterer has a rigid core, i.e. when the boundary condition on S_c is

$$\frac{\partial \psi(\mathbf{r})}{\partial n} = 0\,, \qquad \mathbf{r} \in S_c\,. \tag{4.1}$$

In the case that the boundary condition on S_c is of Dirichlet or Robin type, we can, similarly derive analogous integral representations.

In what follows by “the scattering problem” we shall refer to the acoustic scattering by an infinitely stratified scatterer with a rigid core, as it was described previously.

For simplicity reasons, in this and in the following Section we shall assume that $k_j^2 \in \mathbb{R}$. In this case, the wavenumber k_j, in the region Ω_j, is expressed in terms of k_0 by the relation

$$k_j^2 = \frac{\gamma_0}{\gamma_j}\frac{\rho_j}{\rho_0}k_0^2\,, \qquad j = 1, 2, \dots\,,$$

where Y_j is the compressibility in Ω_j.

So (3.1) reduces to $\sup \dfrac{\rho_j}{\gamma_j} < +\infty$ and $\sup \dfrac{1}{\rho_j} < +\infty$, conditions that are physically meaningful.

Theorem 4.1. *The total exterior field of the scattering problem has the following integral representation*

$$\begin{aligned} u^{(0)}(\mathbf{r}) = {} & e^{ik_0\hat{\mathbf{k}}\cdot\mathbf{r}} + \\ & + \frac{1}{4\pi}\int_{S_c}\left(\frac{1}{R} - ik_0\right)\frac{e^{ik_0R}}{R}\widehat{\mathbf{R}}\cdot\hat{\mathbf{n}}\psi(\mathbf{r}')ds(\mathbf{r}') + \\ & + \frac{k_0^2}{4\pi}\sum_{j=1}^{\infty}\left(\frac{\gamma_0}{\gamma_j} - 1\right)\int_{\Omega_j}\frac{e^{ik_0R}}{R}\psi(\mathbf{r}')du(\mathbf{r}') + \\ & + \frac{1}{4\pi}\sum_{j=1}^{\infty}\left(1 - \frac{\rho_0}{\rho_j}\right)\int_{\Omega_j}\left(\frac{1}{R} - ik_0\right)\frac{e^{ik_0R}}{R}\widehat{\mathbf{R}}\cdot\nabla\psi(\mathbf{r}')du(\mathbf{r}')\,, \end{aligned} \tag{4.2}$$

where $\mathbf{R} = |\mathbf{r} - \mathbf{r}'|$.

PROOF. The scattered field $\psi^{(0)}$ satisfies in Ω_0 the well known [7] Helmholtz's integral representation

$$\psi^{(0)}(\mathbf{r}) = \frac{1}{4\pi}\int\limits_{S_0}\left[\psi^{(0)}(\mathbf{r}')\frac{\partial}{\partial n}\frac{\mathrm{e}^{ik_0R}}{R} - \frac{\mathrm{e}^{ik_0R}}{R}\frac{\partial}{\partial n}\psi^{(0)}(\mathbf{r}')\right]ds(\mathbf{r}')\,. \tag{4.3}$$

Since the incident wave belongs to the kernel of the differential operator of the problem we have that

$$\int\limits_{S_0}\left[\mathrm{e}^{ik_0\hat{\mathbf{k}}\cdot\mathbf{r}}\frac{\partial}{\partial n}\frac{\mathrm{e}^{ik_0R}}{R} - \frac{\mathrm{e}^{ik_0R}}{R}\frac{\partial}{\partial n}\mathrm{e}^{ik_0\hat{\mathbf{k}}\cdot\mathbf{r}}\right]ds(\mathbf{r}') = 0\,. \tag{4.4}$$

Using (2.4), (4.3), (4.4) and introducing the transmission conditions given by (2.7), we obtain

$$\begin{aligned} u^{(0)}(\mathbf{r}) &= \mathrm{e}^{ik_0\hat{\mathbf{k}}\cdot\mathbf{r}} + \frac{1}{4n}\int\limits_{S_0}\psi^{(1)}(\mathbf{r}')\frac{\partial}{\partial n}\frac{\mathrm{e}^{ik_0R}}{R}ds(\mathbf{r}') = \\ &= \frac{1}{4n}\frac{\rho_0}{\rho_1}\int\limits_{S_0}\frac{\mathrm{e}^{ik_0R}}{R}\frac{\partial}{\partial n}\psi^{(1)}(\mathbf{r}')ds(\mathbf{r}')\,. \end{aligned} \tag{4.5}$$

In order to incorporate the boundary condition which is satisfied on the surface S_c of the core, we work as follows. Applying, successively, Green's first identity on $\psi^{(j)}$, $R^{-1}\exp(ik_0R)$ in Ω_j, using that $\psi^{(j)}$, $R^{-1}\exp(ik_0R)$ are solutions of (3.2) in Ω_j and Ω_0, respectively, and introducing the transmission conditions given by (2.10), we have of each surface integral of (4.5), the relations

$$\begin{aligned} \int\limits_{S_0}\psi^{(1)}(\mathbf{r}')\frac{\partial}{\partial n}\frac{\mathrm{e}^{ik_0R}}{R}ds(\mathbf{r}') &= \int\limits_{S_c}\psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{\mathrm{e}^{ik_0R}}{R}ds(\mathbf{r}') + \\ - k_0^2\sum_{j=1}^{\infty}\int\limits_{\Omega_j}\psi(\mathbf{r}')\frac{\mathrm{e}^{ik_0R}}{R}du(\mathbf{r}') &+ \sum_{j=1}^{\infty}\int\limits_{\Omega_j}\nabla\psi(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}\frac{\mathrm{e}^{ik_0R}}{R}du(\mathbf{r}') \end{aligned} \tag{4.6}$$

and

$$\frac{\rho_0}{\rho_1}\int\limits_{S_0}\frac{\mathrm{e}^{ik_0R}}{R}2\frac{\partial}{\partial n}\psi^{(1)}(\mathbf{r}')ds(\mathbf{r}') = -k_0^2\sum_{j=1}^{\infty}\frac{\gamma_0}{\gamma_j}\int\limits_{\Omega_j}\frac{\mathrm{e}^{ik_0R}}{R}\psi(\mathbf{r}')du(\mathbf{r}')+ \tag{4.7}$$
$$+\sum_{j=1}^{\infty}\frac{\rho_0}{\rho_j}\int\limits_{\Omega_j}\nabla\psi(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}\frac{\mathrm{e}^{ik_0R}}{R}du(\mathbf{r}')\,.$$

Substituting (4.6), (4.7) into (4.5) we conclude the proof.

Remark 4.1 It is clear that the series appearing in (4.2) converge uniformly. The functions ψ (solution of Helmholtz's equation in Ω_j), and $\nabla\psi$ are bounded in every bounded region Ω_j and γ_j, are also supposed to be bounded, something that is physically meaningful. In addition, if $|\Omega_j|$ denotes the volume of the region Ω_j then $\sum\limits_{j=1}^{\infty}|\Omega_j| = |\Omega|$.

The behaviour of the scattered wave in the region of radiation (far field) is described by the normalized (dimensionless) scattering amplitude g, which is defined by the relation

$$\psi^{(0)}(\mathbf{r}) = g(\hat{\mathbf{r}},\hat{\mathbf{k}})h_0^1(k_0r) + 0\left(\frac{1}{r^2}\right), \quad r\to\infty\,, \tag{4.8}$$

where $h_0^1(x) = \exp(ix)/ix$ is the zeroth order spherical Hankel function of the first kind. In order to express g in closed form we use the asymptotic relations

$$R = |\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + 0\left(\frac{1}{r}\right), \quad r\to\infty \tag{4.9}$$

$$\widehat{\mathbf{R}} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \hat{\mathbf{r}} + 0\left(\frac{1}{r}\right), \quad r\to\infty \tag{4.10}$$

$$\frac{1}{R} = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + 0\left(\frac{1}{r^2}\right) \quad r\to\infty \tag{4.11}$$

$$\frac{\mathrm{e}^{ik_0R}}{R}\frac{\mathrm{e}^{ik_0R|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathrm{e}^{ik_0r}}{r}\mathrm{e}^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'} + 0\left(\frac{1}{r^2}\right), \quad r\to\infty\,. \tag{4.12}$$

Substituting (4.9) - (4.12) into (4.2) and using (2.4), (4.8) we prove the following

THEOREM 4.2. *The normalized amplitude for the scattering problem is given by the formula*

$$
\begin{aligned}
g(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \frac{k_0^2}{4\pi}\int_{S_c}\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\psi(\mathbf{r}')\mathrm{e}^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'}ds(\mathbf{r}')+ \\
+\frac{ik_0^3}{4\pi}\sum_{j=1}^{\infty}\left(\frac{\gamma_0}{\gamma_j}-1\right)\int_{\Omega_j}\psi(\mathbf{r}')\mathrm{e}^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'}du(\mathbf{r}')+ \\
+\frac{k_0^2}{4\pi}\sum_{j=1}^{\infty}\left(1-\frac{\rho_0}{\rho_j}\right)\int_{\Omega_j}\hat{\mathbf{r}}\cdot\nabla\psi(\mathbf{r}')\mathrm{e}^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'}du(\mathbf{r}')\,.
\end{aligned}
\tag{4.13}
$$

5 – The low-frequency theory

The incident plane wave $u^{inc}(\mathbf{r}) = \exp(ik_0\hat{\mathbf{k}}\cdot\mathbf{r})$ is analytic at $k_0 = 0$, and we assume that the fields $\psi^{(j)}(\mathbf{r})$ and $u^{(0)}(\mathbf{r})$ are also analytic at $k_0 = 0$.

Therefore, a convergent Taylor series for $\psi(\mathbf{r})$ exists in powers of k_0

$$
\psi(\mathbf{r}) = \sum_{m=0}^{\infty}\frac{(ik_0)^m}{m!}\phi_m(\mathbf{r})\,,\quad r\in\Omega_j\,,\quad j=0,1,2,\dots \tag{5.1}
$$

and for $u^{(0)}(\mathbf{r})$

$$
u^{(0)}(\mathbf{r}) = \sum_{m=0}^{\infty}\frac{(ik_0)^m}{m!}F_m^{(0)}(\mathbf{r})\,,\quad \mathbf{r}\in\Omega_0\,, \tag{5.2}
$$

where the low frequency coefficients $\phi_m(\mathbf{r})$, and $F_m^{(0)}(\mathbf{r})$ are independent of k_0. Substituting (5.1), (5.2) into (2.3), (2.4), (2.7), (2.8) and equating coefficients of equal powers of k_0, the previous scattering problem is reduced to a sequence of potential problems, that can be solved iteratively. Therefore, the coefficients $\phi_m^{(j)}(\mathbf{r}) = \phi_m(\mathbf{r})$ in Ω_j satisfy, for

$m = 0, 1, 2, \ldots$

(5.3) $$\Delta \phi_m^{(j)}(\mathbf{r}) = m(m-1)\frac{\gamma_0 \rho_j}{\gamma_j \rho_0}\phi_{m-2}^{(j)}(\mathbf{r}), \quad \mathbf{r} \in \Omega_j$$

(5.4) $$\phi_m^{(j+1)}(\mathbf{r}) = \phi_m^{(j+2)}(\mathbf{r}), \quad \mathbf{r} \in S_j$$

(5.5) $$\rho_{j+2}\frac{\partial \phi_m^{(j+1)}(\mathbf{r})}{\partial n} = \rho_{j+1}\frac{\partial \phi_m^{(j+2)}(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S_j$$

for $j = 0, 1, 2, \ldots$, and

(5.6) $$F_m^{(0)}(\mathbf{r}) = \phi_m^{(0)}(\mathbf{r}) + (\hat{\mathbf{k}} \cdot \mathbf{r})^m, \quad \mathbf{r} \in \Omega_0$$

(5.7) $$F_m^{(0)}(\mathbf{r}) = \phi_m^{(1)}(\mathbf{r}), \quad \mathbf{r} \in S_0$$

(5.8) $$\rho_1 \frac{\partial F_m^{(0)}(\mathbf{r})}{\partial n} = \rho_0 \frac{\partial \phi_m^{(1)}(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S_0 .$$

Also, ϕ_m satisfies the boundary condition

(5.9) $$\frac{\partial}{\partial n}\phi_m(\mathbf{r}) = 0, \quad \mathbf{r} \in S_c .$$

In order to derive a low-frequency expression for the total exterior field, we substitute (5.1), (5.2) and the expansions

(5.10) $$\mathrm{e}^{ik_0 \hat{\mathbf{k}} \cdot \mathbf{r}} = \sum_{m=0}^{\infty} \frac{(ik_0)^m}{m!}(\hat{\mathbf{k}} \cdot \mathbf{r})^m$$

(5.11) $$\frac{\mathrm{e}^{ik_0 R}}{R} = \sum_{m=0}^{\infty} \frac{(ik_0)^m}{m!} R^{m-1}$$

into (4.2). After tedious calculations we arrive at

THEOREM 5.1. *The low-frequency coefficients of the total exterior* *field for the scattering problem have the following expression*

$$(5.12)\qquad F_m^{(0)}(\mathbf{r}) =$$

$$= (\hat{\mathbf{k}}\cdot\mathbf{r})^m - \frac{1}{4\pi}\sum_{q=0}^{m}\binom{m}{q}(m-q-1)\int_{S_c}(\mathbf{R}\cdot\hat{\mathbf{n}})R^{m-q-3}\phi_q(\mathbf{r}')ds(\mathbf{r}')+$$
$$-\frac{1}{4\pi}\sum_{j=1}^{\infty}\Big(\frac{\gamma_0}{\gamma_j}-1\Big)\sum_{q=0}^{m}\binom{m}{q}(m-q-1)\int_{\Omega_j}R^{m-q-3}\phi_q(\mathbf{r}')du(\mathbf{r}')+$$
$$-\frac{1}{4\pi}\sum_{j=1}^{\infty}\Big(1-\frac{\rho_0}{\rho_j}\Big)\sum_{q=0}^{m}\binom{m}{q}(m-q-1)\int_{\Omega_j}R^{m-q-3}\mathbf{R}\cdot\nabla\phi_q(\mathbf{r}')du(\mathbf{r}')\,.$$

Since it is needed in applications, we construct bounds for the low-frequency coefficients.

THEOREM 5.2. *For the low-frequency coefficients of the scattering problem we have*

$$(5.13)\qquad \big|\phi_m(\mathbf{r})\big| \le B_m \quad \textit{and} \quad \Big|\big|\nabla\phi_m(\mathbf{r})\big|\Big| \le b_m$$

for all $r\in\Omega_j$, $j=1,2,\dots$, $m=0,1,2,\dots$, *with*

$$(5.14)\qquad B_m = \begin{cases} \displaystyle\sum_{n=0}^{\frac{m}{2}}\frac{m!}{(2n)!}\Big[\frac{\gamma_0\rho}{\gamma\delta_0}(\mathrm{e}^{2a}-1)\Big]^{\frac{m}{2}-n}\xi_{2n}\,,\ m:\ \textit{even}\\ \displaystyle\sum_{n=0}^{\frac{m-1}{2}}\frac{m!}{(2n+1)!}\Big[\frac{\gamma_0\rho}{\gamma\delta_0}(\mathrm{e}^{2a}-1)\Big]^{\frac{m-1}{2}-n}\xi_{2n+1}\,,\ m:\ \textit{odd}\end{cases}$$

$$(5.15)\qquad b_m = C_m\Big[\frac{\xi_m}{d_m}+m(m-1)d_mB_m\Big],$$

where

$$\xi_m = \sup\Big\{\sup_{\mathbf{r}\in S_j}|\phi_m(\mathbf{r})|\,,\ j=1,2,\dots\Big\},$$
$$\rho = \sup\{\rho_j\,,\ j=1,2,\dots\}\,,$$
$$\gamma = \inf\{\gamma_j\,,\ j=1,2,\dots\}\,,$$

C_m, d_m *are known constants.*

Proof. As it is well known, [13], for the solutions of Poisson's equation $\Delta u = f$ we have the a priori bounds

$$\sup_{\Omega_j} |u| \leq \sup_{\partial\Omega_j} |u| + C \sup_{\Omega_j} |f| \,, \tag{5.16}$$

where C is less than or equal to $(\mathrm{e}^{2a} - 1)$.

Applying (5.16) repeatedly, with respect to m, in the equations (5.3) and taking into account that $\xi_m < \infty$, a condition that is physically meaningful, we construct the bounds B_m, [3]. Similarly, using the well known gradient estimates for Poisson's equation [13], we derive the bounds b_m.

Remark 5.1 The term $(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})^m$ in (5.12), gives the main contribution of the incident wave to the corresponding approximation.

Remark 5.2 The two series appearing in (5.12) converge uniformly, because the solutions, and their gradients, of Poisson's equation are bounded in each Ω_j. For the first series we have

$$\left| \left(\frac{\gamma_0}{\gamma_j} - 1 \right) \sum_{q=0}^{m} \binom{m}{q} (m-q-1) \int_{\Omega_j} R^{m-q-3} \phi_q(\mathbf{r}') du(\mathbf{r}') \right| \leq$$
$$\leq \left(\frac{\gamma_0}{\gamma} + 1 \right) \sum_{q=1}^{m} \binom{m}{q} (m-q-1) \int_{\Omega_j} (2r)^{m-q-3} B_q du(\mathbf{r}') \leq A |\Omega_j| \,,$$

where A is independent of j and $\sum_{j=1}^{\infty} |\Omega_j| = |\Omega|$. Using the Weierstrass M-test, we conclude that the series converge uniformly.

In order to derive the low-frequency expansions for the scattering amplitude g, the expansions (5.10), (5.11) and

$$\mathrm{e}^{-ik_0 \hat{\mathbf{r}} \cdot \mathbf{r}'} = \sum_{m=0}^{\infty} (-1)^m \frac{(ik_0)^m}{m!} (\hat{\mathbf{r}} \cdot \mathbf{r}')^m$$

are substituted into (4.13), and we obtain the following

Theorem 5.3. *The low-frequency normalized scattering amplitude* *for the scattering problem is given by*

$$
\begin{aligned}
(5.17)\qquad & g(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \\
&= \frac{ik_0}{4\pi}\sum_{m=0}^{\infty}\frac{(ik_0)^m}{m!}\sum_{q=0}^{m}\binom{m}{q}(-1)^q\Big\{-k_0 i\int_{S_c}(\hat{\mathbf{r}}\cdot\mathbf{r}')^q(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}})\phi_{m-q}(\mathbf{r}')ds(\mathbf{r}')+ \\
&+ k_0^2\sum_{j=1}^{\infty}\Big(\frac{\gamma_0}{\gamma_j}-1\Big)\int_{\Omega_j}(\hat{\mathbf{r}}\cdot\mathbf{r}')^q\phi_{m-q}(\mathbf{r}')du(\mathbf{r}')+ \\
&- ik_0\sum_{j=1}^{\infty}\Big(1-\frac{\rho_0}{\rho_j}\Big)\int_{\Omega_j}(\hat{\mathbf{r}}\cdot\mathbf{r}')^q\hat{\mathbf{r}}\cdot\nabla\phi_{m-q}(\mathbf{r}')du(\mathbf{r}')\Big\}.
\end{aligned}
$$

REMARK 5.3 Using estimates (5.13) for the low-frequency coefficients, we can derive from (5.17) the following bound for the normalized scattering amplitude:

$$
\begin{aligned}
(5.18)\qquad |g(\hat{\mathbf{r}},\hat{\mathbf{k}})| \le \frac{k_0}{4\pi}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\frac{k_0^m}{m!}\binom{m}{q}a^q\Big\{k_0B_{m-1}|S_c|+ \\
+ k_0^2\Big(\frac{\gamma_0}{\gamma}+1\Big)B_{m-q}|\Omega| + k_0\Big(1+\frac{\rho_0}{\rho}\Big)b_{m-q}|\Omega|\Big\}.
\end{aligned}
$$

REMARK 5.4 The volume integrals appearing in the second and third terms of (5.17), represent, respectively, the moments of the pressure field, and the velocity field projected on the direction of observation $\hat{\mathbf{r}}$, in each layer of the scatterer.

We conclude this Section with some comments on certain special cases of physical interest.

REMARK 5.5 We consider the condition

$$\rho_j = \rho_{j+1}\,,\quad \gamma_j = \gamma_{j+1}\,,\quad j \ge n\,,\quad n \in \mathbb{N}_0\,.$$

When $n = 0$, then no scattering occurs by the layers of the scatterer, but only by the core i.e. we have scattering by a soft ($\varepsilon = 1$) or rigid ($\varepsilon = 0$), or resistive ($0 < \varepsilon < 1$) simple scatterer.

When $n = 1$, we have scattering by a penetrable body with an impenetrable core, [7].

When $n = N > 1$, we have scattering by a finitely-layered scatterer, [2].

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