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The Theorem of Totten for planar spaces

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Riassunto: *Si dimostra che uno spazio planare irriducibile* $\mathbf{S}$ *con* v *punti e* π *piani, tale che* $n^3 \leq v < (n+1)^3$ *e* $\pi \leq v + n^2 + n$ *per* $n \geq 4$, *esiste se e solo se* n *è una potenza di un numero primo ed inoltre se* $\mathbf{S}$ *è immergibile in* $PG(3, n)$, *oppure se* $\mathbf{S}$ *è lo spazio affine* $AG(3, n)$ *con uno spazio proiettivo generalizzato di dimensione* 3 *all'infinito. Questo risultato è analogo a quello relativo alla classificazione degli spazi lineari ristretti di Totten.*

Abstract: *It is shown that an irreducible planar space* $\mathbf{S}$ *with* v *points and* π *planes such that* $n^3 \leq v < (n+1)^3$ *and* $\pi \leq v + n^2 + n$ *for some integer* $n \geq 4$ *exists iff* n *is a prime power and* $\mathbf{S}$ *either can be embedded in* $PG(3, n)$, *or* $\mathbf{S}$ *is the affine space* $AG(3, n)$ *with a generalized projective* 3*-space at infinity. This result is an analogue to the classification of restricted linear spaces by Totten.*

1 – Introduction

The famous result of de Bruijn [2] states that every linear space has at least as many lines as points with equality if it is a (possibly degenerate) projective plane. After the classification of linear spaces with one, two, or three more lines than points [1, 12, 10], Totten obtained the classification of all *restricted* linear spaces, that is linear spaces satisfying $(b-v)^2 \leq v$ where v is the number of points and b is the number of lines. If n denotes the unique integer satisfying $n^2 \leq v < (n+1)^2$, then the

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condition $(b-v)^2 \leq v$ means that $b \leq v+n$. The result of Totten is the following. If $b < v+n$ and $n \geq 4$, then the linear space can be embedded in a projective plane of order n. If however $b = v+n$, then there is a class of exceptional spaces, called projectively inflated affine planes.

A result analogous to the de Bruijn-Erdös result has been proved for geometric lattices by Greene [4]. It states that a geometric lattice of rank $r \geq 3$ has at least as many hyperplanes as points with equality iff it is modular. One might try to generalize also Totten's Theorem to higher dimensional linear spaces. For planar spaces, this was started by Hafner [5]. Following Hafner, we call a planar space *restricted*, if the numbers v of its points and π of its planes satisfy $n^3 \leq v < (n+1)^3$ and $\pi \leq v+n^2+n$ for some integer $n \geq 2$. There are two reasons for this choice of the bound on π. On the one hand one wants to include affine 3-spaces in the classification. On the other hand, as in the 2-dimensional case, there will be a class of exceptional planar spaces satisfying $\pi = v+n^2+n$, the so called inflated affine spaces. Hafner showed that in a restricted planar space with $n^3 \leq v < (n+1)^3$ every plane has at most n^2+n+1 points and every line has at most $n+1$ points. We give a much shorter proof for these results and moreover complete the classification of restricted planar spaces for $n \geq 4$.

Let us recall some terminology. A (*finite*) *linear space* is a pair $(\mathcal{P}, \mathcal{L})$ consisting of a finite set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that any two distinct points P and Q occur in a unique line, denoted by PQ, every line has at least two points, and there are three non-collinear points. A linear space is called *degenerate*, if it has a line containing all but one of the points. A set of points that contains the line PQ for any of its points P and Q is called a *subspace*.

A *planar space* is a linear space $(\mathcal{P}, \mathcal{L})$ together with a family Π of at least two subspaces, called *planes*, such that any three non-collinear points are in a unique plane, and every plane has at least three non-collinear points. The planar space $\mathbf{S}$ is called *reducible* (and else *irreducible*) if there exists a non-trivial partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ with the properties that every line is contained in one component or has a unique point in each component, and every plane is contained in one component or consists of a point in one component and a line in the other component. Let $\mathbf{S}$ be a reducible planar space with partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ as above. If $\mathbf{S}_i$ is the structure *induced* by $\mathbf{S}$ on $\mathcal{P}_i$, $i = 1, 2$ (that is $\mathbf{S}_i$ consists of the points of $\mathcal{P}_i$ and the lines and planes of $\mathbf{S}$ that have all its points in $\mathcal{P}_i$), then we say that $\mathbf{S}$ is the *direct product* of $\mathbf{S}_1$ and $\mathbf{S}_2$. Notice that $\mathbf{S}_i$ may by a point, a line, a linear space, or a planar space. In this paper, we call a planar space *degenerate*, if it is the direct product of either two lines, or a plane and a point. A *generalized* projective 3-space is a modular geometric lattice of rank 4, that is a 3-dimensional projective space $PG(3, q)$, a direct product of two lines, or a direct product of a point and a projective plane.

The following definition generalizes the notation of inflated affine planes [11]. Consider a projective space $\mathbf{P} = PG(3, n)$ and a plane E_∞ of $\mathbf{P}$. Let $\mathcal{P}_s$ be a set of points of E_∞ with three non-collinear points, and let $\mathcal{P}$ be the set consisting of those points of $\mathbf{P}$ that do not lie in $E_\infty \setminus \mathcal{P}_s$. Then $\mathbf{P}$ induces a linear space $\mathbf{L}$ on $\mathcal{P}$ and a linear space $\mathbf{L}_\infty$ on $\mathcal{P}_s$. Suppose that $\mathbf{L}_\infty$ has a family of (at least two) subspaces that makes it into a planar space $\mathbf{S}_\infty$. Then there are two ways to make $\mathbf{L}$ into a planar space. The first possibility is to adjoin as planes all sets $E \cap \mathcal{P}$ where E is a plane of $\mathbf{P}$. The so-obtained planar space $\mathbf{S}_1$ is the one induced by $\mathbf{P}$ on $\mathcal{P}$. The second possibility is to adjoin the same set of planes, except that the plane $E_\infty \cap \mathcal{P}$ is replaced by the planes of $\mathbf{S}_\infty$. The so-obtained planar space $\mathbf{S}_2$ is called an *inflated affine* 3*-space*; it consists of an affine 3-space and a planar space $\mathbf{S}_\infty$ *at infinity*. Since generalized projective 3-spaces have the same number of points and planes, an affine 3-space of order n with a generalized projective 3-space at infinity has $n^2 + n$ more planes than points.

Now we can state the main result of this paper.

Theorem 1.1. *Let $\mathbf{S}$ be a non-degenerate planar space and denote by n the positive integer such that $n^3 \leq v < (n+1)^3$. If $n \geq 4$ and if $\mathbf{S}$ is restricted, then n is a prime power and $\mathbf{S}$ can be embedded in $PG(3, n)$, or $\mathbf{S}$ is $AG(3, n)$ with a generalized projective 3-space at infinity.*

2 – Preliminary results

In this section we prove some preliminary results. For a finite linear space $\mathbf{L} = (\mathcal{P}, \mathcal{L})$, we call the number r_P of lines on a point P the *degree* of P, and the number k_l of points on a line l the degree of l. By a *weight function* we mean a function w from $\mathcal{L}$ into $\Re$; then $w(l)$ is called the *weight* of the line l. For further terminology and well-known results such as the DE BRUIJN-ERDÖS Theorem [2] and TOTTEN's Theorem [11] we refer to [7].

LEMMA 2.1. *Let* $\mathbf{L}$ *be a non-degenerate linear space with* b *lines. Then* $b \geq 2r_P$ *for every point* P.

PROOF. Let v denote the number of points, let P be any point, and denote by x the number of 2-lines on P. Then $v \geq 2r_P + 1 - x$. Since $\mathbf{L}$ is not degenerate, it induces a linear space $\mathbf{L}'$ on the set of points other than P. It has $v-1$ points, and thus, by the de Bruijn-Erdös Theorem, at least $v-1$ lines. Since the 2-lines on P are lines of $\mathbf{L}$ but not of $\mathbf{L}'$, it follows that $b \geq v - 1 + x$. $\square$

LEMMA 2.2. *Let* $\mathbf{L}$ *be a linear space with* b *lines,* w *a weight function,* $s, \epsilon \in \Re$, *and put* $S := \sum_{l \in \mathcal{L}} w(l)$.

(a) *If* $w(l_1) + w(l_2) \geq 2s$ *for all intersecting lines* l_1 *and* l_2, *then* $S \geq bs$.

(b) *Suppose that* $w(l_1) + w(l_2) + w(l_3) \geq 3s - \epsilon$ *for all distinct non-confluent lines* l_1, l_2 *and* l_3 *of which at least two meet. If* $\epsilon = 0$, *then* $S \geq bs$. *If* s *and the weight of every line is an integer, then* $S \geq bs - 1$ *if* $\epsilon = 1$, *and* $S \geq b\Big(s - \dfrac{1}{2}\Big)$ *if* $\epsilon = 2$.

PROOF. a) Denote by $l_1, \dots, l_u$ the lines whose weight is less than s. By hypothesis, these lines are disjoint. We may assume that $w(l_j) \leq w(l_i)$ for $j < i$. Put $\mathcal{P}_i := \bigcup_{j=1}^{i} l_j$ and let $\mathcal{L}_i$ denote the set consisting of the lines that meet $\mathcal{P}_i$, $i = 1, \dots, u$. Then $|\mathcal{P}_i| \geq 2i$ and thus $|\mathcal{L}_i| \geq 2i$ by the de Bruijn-Erdös Theorem. Hence we can choose distinct lines $g_1, \dots, g_u$ with $g_i \in \mathcal{L}_i$ and $g_i \neq l_1, \dots, l_u$. For each index i there is an index $j \leq i$ such that g_i and l_j meet. By Hypothesis, $w(g_i) + w(l_j) \geq 2s$; hence, $w(g_i) + w(l_i) \geq 2s$. Since the lines l_i are the only lines having weight less than s, it follows that the average weight of the lines is at least s.

b) In view of a), we may assume that there exist intersecting lines l_1 and l_2 satisfying $w(l_1) + w(l_2) \leq 2s - \epsilon$ (use the integrality hypothesis for $\epsilon \in \{1, 2\}$). Put $w_1 := (l_1)$, $w_2 := w(l_2)$, $P = l_1 \cap l_2$ and $r = r_P$. We may assume that $w_1 \leq w_2 \leq w(l)$ for all lines $l \neq l_1$ on P. By hypothesis, $w(l) \geq 3s - \epsilon - w_1 - w_2$ for all lines not containing P. Hence

$$S \geq w_1 + (r-1)w_2 + (b-r)(3s - \epsilon - w_1 - w_2) = \\ = bs - \epsilon + (b+1-2r)(2s - \epsilon - w_1 - w_2) + (r-2)(s - \epsilon - w_1)\,.$$

Since $b \geq 2r$ (Lemma 2.1), it follows that $S \geq bs - \epsilon + (r-2)(s - \epsilon - w_1)$. If $\epsilon \in \{0,1\}$, $w_1 \leq w_2$ and $w_1 + w_2 \leq 2s - \epsilon$ imply that $w_1 \leq s - \epsilon$ and thus $S \geq bs - \epsilon$. If $\epsilon = 2$, then only $w_1 \leq s-1$, so $S \geq bs - 2 - (r-2)$; now $S \geq b\Big(s - \dfrac{1}{2}\Big)$ follows from Lemma 2.1. □

Lemma 2.3. *Suppose that* $\mathbf{S}$ *is a planar space with* v *points, and* E *is a plane of* $\mathbf{S}$.

a) *Let* l_1, l_2, l_3 *be distinct lines of* E *with* $\emptyset \neq l_1 \cap l_2 \notin l_3$. *If* $w_i + 1$ *is the number of planes on* l_i, $i = 1,2,3$, *then* $\dfrac{1}{27}(w_1 + w_2 + w_3)^3 \geq w_1 w_2 w_3 \geq v - |E|$.

b) *If* E *has* b *lines, then* E *meets at least* $b\sqrt[3]{v - |E|}$ *other planes.*

Proof. Part b) follows form a) and Lemma 2.2 b), applied with $\epsilon = 0$ and $s = \sqrt[3]{v - |E|}$. The first inequality in a) holds for all non-negative integers. Every point P not in E determines a triple (E_1, E_2, E_3) where E_i is the plane on P and l_i. Since distinct points yield distinct triples, we obtain $w_1 w_2 w_3 \geq v - |E|$. □

3 – The maximum number of points of planes and lines

For the rest of this paper, $\mathbf{S}$ denotes a non-degenerate restricted planar space with v points and π planes, where $n^3 \leq v < (n+1)^3$ and $\pi \leq v + n^2 + n$ for some integer $n \geq 4$. For a point P, we denote by π_P the number of planes on P. Two distinct lines h and l are *coplanar* if they are in a common plane, which is denoted by hl in this case. For a point P and a line l not on P, the plane on P and l is denoted by Pl. For a point P, we denote the *quotient geometry* at P by $\mathbf{S}/P$; this is the linear space whose 'points' are the lines of $\mathbf{S}$ on P and whose 'lines' are the planes of $\mathbf{S}$ on P, where incidence is inclusion. Notice that $\pi_P \geq r_P$

by the de Bruijn-Erdös Theorem, and that a plane E that does contain P is canonically embedded in $\mathbf{S}/P$ (via the map $x \to Px$ for every point and line x of E).

LEMMA 3.1.
a) *If E is a plane, then at most one line of E is only in two planes.*
b) *The quotient geometries at points are non-degenerate linear spaces.*

PROOF. Assume that some plane E has two distinct lines l_1 and l_2 such that l_i lies only in E and one more plane E_i, $i = 1, 2$. Then every point is in E or in $E_1 \cap E_2$. Since $\mathbf{S}$ is non-degenerate, it follows that $l := E_1 \cap E_2$ is a line. Put $k = k_l$ and $e = |E|$.

Assume that $l \cap E = \emptyset$. Then at most $\frac{e}{2}$ lines of E are coplanar to l. Since E has at least e lines, it follows that at least $\frac{e}{2}$ lines of E lie in at least k planes other than E. It follows that $\pi \geq 1 + \frac{e}{2} + \frac{e}{2} \cdot k$. Since $\pi \leq v + n^2 + n = e + k + n^2 + n$, we obtain $(e-2)(k-1) \leq 2n^2 + 2n$. Since $e + k = v$, $e \geq 3$, $k \geq 2$, $v \geq n^3$, and $n \geq 4$, this is a contradiction. Hence l and E meet in a point P.

Since $\mathbf{S}$ is non-degenerate, we have $e \leq v - 2$ and the plane E is a non-degenerate linear space. Put $r := r_P$, and denote by b the number of lines of E. Every line of E that does not contain P, lies in k planes, which meet l in a unique point. Every other plane contains P. Since P is on r planes, and since E has $b - (r-1)$ lines that do not contain P, we obtain $\pi = r + (b + 1 - r)(k-1) = b + k - 1 + (b-r)(k-2)$. Since $v = e + k - 1 \leq b + k - 1$, we obtain $(b-r)(k-2) \leq \pi - v \leq n^2 + n$. Since E is non-degenerate, we have $b \geq 2(r-1)$ by Lemma 2.1. Hence $2(b-r) \geq b - 2 \geq e - 2$. Together it follows $(e-2)(k-2) \leq 2(n^2+n)$. But $e \leq v - 2$, and thus $(e-2)(k-2) = (e-2)(v-e-1) \geq v - 4 \geq n^3 - 4 > 2(n^2+n)$, a contradiction. This proves a) and b) is an immediate consequence of a). $\square$

LEMMA 3.2. *If E is a non-degenerate plane with at least b lines, then $b\sqrt[3]{v-b} \leq v + n^2 + n$.*

Proof. Let b' be the number of lines of E. Then E has at most b' points, so $\pi \geq b'\sqrt[3]{v-b'}$ by Lemma 2.3 b). Lemma 3.1 says that at most one line of E lies in only two planes. Hence $\pi \geq 2b'$ and thus $4b' \leq 2\pi \leq 2(v+n^2+n) \leq 3v$. It follows that the function $b' \to {b'}^3(v-b')$ is monotone increasing for the possible values of b' (since the first derivation is $3{b'}^2v - 4{b'}^3$). The assertion follows from $b' \geq b$ and $\pi \leq v+n^2+n$. □

Lemma 3.3. *A plane with the maximum number of points is non-degenerate.*

Proof. Consider a plane E with the maximum number e of points, and assume that E is degenerate, that is E has a line l with $k := e-1$ points. Then every plane on l has e points, so l is in $v-k$ planes. Let P be the point of E not on l. Then P lies on k lines of E. Lemma 3.1 a) implies that P lies on at least $2k-1$ planes other than E. Hence $\pi \geq v-k+2k-1 = v+k-1$, so $k \leq \pi - v + 1 \leq n^2+n+1$.

In E the point P lies on k lines, all of which have degree 2. Let α be the minimum number of planes on one of these lines. Since $k+1$ is the maximum number of points in a plane, we must have $v-2 \leq \alpha(k-1)$. Furthermore, P lies on at least $k(\alpha-1)$ planes other than E, so $\pi \geq (v-k)+k(\alpha-1)$. Hence $n^2+n \geq \pi - v \geq v-2k+\alpha-2$. Using $k \leq n^2+n+1$, $\alpha \geq 2$, $n \geq 4$ and $v \geq n^3$ gives a contradiction. □

Lemma 3.4. *Every plane has at most n^2+n+1 points.*

Proof. Let E be a plane with the maximum number of points, put $e := |E|$, and denote by b_E the number of lines of E. By Lemma 3.3, E is non-degenerate. Lemma 3.1 implies $2b_E \leq \pi \leq v+n^2+n$. Since $n^3 \leq v \leq (n+1)^3-1$, Lemma 3.2 implies that $b_E < (n+2)^2$. Now, the main result of [3] gives $e \leq n^2+3n+3 = (n+1)^2+(n+1)+1$.

Assume that $e \geq n^2+n+2$. Since E is non-degenerate, the main result of [3], shows that $b_E \geq b := n^2+3n+1$. Using Lemma 3.2 and $v \geq n^3$, we conclude that $v > n^3+2n^2 > (n-1)e$. Hence $v-|E| > n^3$, and every line is in at least n planes.

Let $\mathcal{L}(E)$ be the set of lines of E. For $l \in \mathcal{L}(E)$, we denote by $w(l)+1$ the number of planes through l. Then $w(l) \geq n-1$ for $l \in \mathcal{L}(E)$. Also

$\pi \geq 1 + \sum_{l \in \mathcal{L}(E)} w(l)$. Consider three non-confluent lines l_1, l_2, l_3 of E at least two of which meet, and put $w_i := w(l_i)$. Then $w_1 w_2 w_3 \geq v - |E| > n^3$ by 2.3. Using $w_i \geq n-1$, we obtain $w_1 + w_2 + w_3 \geq 3n+1$. Lemma 2.2 (used with $s = n+1$, $\epsilon = 2$) shows that $\pi \geq b_E\Big(n + \dfrac{1}{2}\Big)$. Hence $\pi - |E| \geq b_E\Big(n - \dfrac{1}{2}\Big) \geq (n^2+3n+1)\Big(n - \dfrac{1}{2}\Big)$. Using $v \geq \pi - n^2 - n$, we obtain $v - |E| > n^2(n+1)$. Hence $w_1 w_2 w_3 > n^2(n+1)$ by Lemma 2.3 a). Using $w_i \geq n-1$, we obtain $w_1 + w_2 + w_3 \geq 3n+2$. Lemma 2.2 (used with $s = n+1$, $\epsilon = 1$) gives $\pi \geq b_E(n+1) - 1 \geq (n^2+3n+1)(n+1)$. But $\pi \leq v + n^2 + n$ and $v \leq (n+1)^3 - 1$, a contradiction. ∎

LEMMA 3.5.

a) *Every lines lies in at least n planes.*

b) *Every line has less than $3n$ points.*

c) *Suppose that E is a non-degenerate plane E with b lines. Then E meets at least $bn - 1$ planes $\neq E$ in a line. In particular $\pi \geq bn$.*

PROOF. a) Since $v \geq n^3 > (n-1)(n^2+n+1)$, this follows from Lemma 3.4.

b) Assume that l is a line with $k \geq 3n$ points. By a), each of the $v - k$ points not on l is on at least $k(n-1)$ planes that meet l in precisely one point. Since each such plane has at most $n^2 + n$ points not on l (Lemma 3.4), it follows that $(v-k)k(n-1) \leq \pi(n^2+n)$.

From Lemma 3.4, we get $k \leq n^2 + n$. Hence $3n \leq k \leq v - 3n$, so $(v-k)k \geq 3n(v-3n)$ and thus $3(v-3n)(n-1) \leq \pi(n+1)$. But $\pi \leq v + n^2 + n$ and $v \geq n^3$, a contradiction.

c) Consider three non-confluent lines $l_1, l_2, l_3 \in \mathcal{M}$ at least two of which meet, and denote by $1 + w_i$ the number of planes on l_i. Lemma 2.3 shows that $w_1 w_2 w_3 \geq v - |E| \geq n^3 - (n^2+n+1) > (n-1)^2 n$. Since, by a), $w_i \geq n-1$, it follows that $w_1 + w_2 + w_3 \geq 3n - 1$. The assertion follows from Lemma 2.2 b). ∎

LEMMA 3.6. *Every point lies on at least $n^2 + n$ planes.*

Proof. Let P be a point lying on the minimum number of planes, and assume that $\pi_P < n^2+n$. Put $r := r_P$. Since, by Lemma 3.1 b), $\mathbf{S}/P$ is a non-degenerate linear space, it follows from the main result of [3] that $r \le n^2-n+2$.

Choose lines l_1, l_2 on P such that $k_l \le k_{l_2} \le k_{l_1}$ for all lines l on P other than l_1. Put $k_{l_1} = k_1+1$ and $k_{l_2} = k_2+1$. The plane $E := l_1 l_2$ has at least $2+k_1k_2$ lines. We have $n^3-1 \le v-1 \le k_1+(r-1)k_2$. Since $k_1 < 3n-1$ (Lemma 3.5) and $r \le n^2-n+2$, it follows that $k_2 > n$. Hence $k_1k_2 \ge k_2^2 \ge (n+1)^2$, and thus E has at least $b := n^2+2n+3$ lines. It follows that every point not on E lies on at least b planes.

Recall that every plane has at most $e := n^2+n+1$ points. By the choice of P, every point of E lies on at least π_P planes. Thus, if t is the number of incident point-plane pairs, then $t \ge (v-|E|)b + |E|\pi_P \ge (v-e)b + e\pi_P$. On the other hand, since the planes not on P have at most r points, we have $t \le (\pi - \pi_P)r + \pi_P \cdot e$. Using $\pi \le v + n^2 + n$, we obtain $(v-e)b \le (v+n^2+n-\pi_P)r$. Since $r \le n^2-n+2$, $v \ge n^3$ and $\pi_P \ge r$, this is a contradiction. $\square$

Lemma 3.7. *If $E_1, \dots, E_v$ are distinct planes, then there exists an enumeration $P_1, \dots, P_v$ of the points such that $P_i \notin E_i$, $i = 1, \dots, v$.*

Proof. Let F_i be the complement of the set of points of E_i. We have to find an enumeration $P_1, \dots, P_v$ of the points such that $P_i \in F_i$. By P. Hall's marriage theorem [6], it suffices to show that $f := |\bigcup_{F\in\mathcal{F}} F| \le |\mathcal{F}|$ for every subset $\mathcal{F}$ of $\{F_1, \dots, F_v\}$. This is trivial, if $|\mathcal{F}| \le 1$ or $f = v$. If $|\mathcal{F}| \ge 2$ and $f \le v-2$, let P and Q be two points not in $\bigcup_{F\in\mathcal{F}} F$; then the planes corresponding to the elements of $\mathcal{F}$ contain the line $l := PQ$, so $|\mathcal{F}| \le v - k_l$. On the other hand, since $|\mathcal{F}| \ge 2$, the elements of $\mathcal{F}$ cover all points not on l, so $f = v-k_l$. Finally consider the case that $f = v-1$, so there is a unique point P contained in all planes corresponding to the elements of $\mathcal{F}$. We have to show that $|\mathcal{F}| \le v-1$, that is that P is in at most $v-1$ planes.

Assume that $\pi_P \ge v$. Then at most n^2+n planes do not contain P. If b is the number of lines of a plane E not containing P, then Lemma 3.5 implies that E meets $b(n-1)$ planes not on P. Hence $b \le n+2$ and thus $|E| \le n+2$. Now, if l is a line not on P, then the plane Pl has at most

n^2+n+1 points, and every other plane on l has at most $n+2$ points. Thus, if l is on $c+1$ planes, then $n^3 \leq v \leq n^2+n+1+c(n-k_l) \leq n^2+n+1+cn$ giving $c \geq c_0 := n^2-n-1$. If l_1, l_2 are distinct lines not on P, then at least $2c_0-1$ planes contain l_1 or l_2 but not P. But $2c_0-1 > n^2+n$, a contradiction. ∎

LEMMA 3.8. *There exists a non-incident point-plane pair* (P, E) *satisfying* $\pi_P \leq |E|+n+1$ *and* $\pi_P \leq r_P+n$.

PROOF. Choose enumerations $E_1,\dots,E_\pi$ of the planes and $P_1,\dots,P_v$ of the points such that $|E_i| \geq |E_j|$ for $i<j$, and $P_i \notin E_i$ for $i \leq v$. Put $m := \min\{\pi_{P_i} - |E_i| | i=1,\dots,v\}$ and $e := |E_v|$. Then $m \geq 0$ and $e \leq n^2+n+1$. Since $\sum \pi_{P_i} = \sum |E_i|$, and $|E_i| \leq e$ for $i>v$, we have $vm \leq (\pi - v)e \leq n(n+1)e$.

Assume that $m \geq n+2$. Since $e \leq n^2+n+1$ and $v \geq n^3$, we get $v < n^3+2n$ and $e \geq n^2+n$. Hence $\pi < n^3+n^2+3n$. Let b be the number of lines of E_v. Then $b \leq n^2+n+2$ by 3.5. Since E_v is non-degenerate (Lemma 3.5 b), Totten's Theorem implies that E_v can be embedded in a projective plane of order n. Hence $b = n^2+n+1$. If $\mathcal{A}$ is the set consisting of the planes that do not contain a line of E_v, then $|\mathcal{A}| \leq \pi - bn \leq 2n$ by 3.5. Each point P_i not in E_v is on $\pi_{P_i} - b \geq |E_i|+m-b \geq |E_v|+m-b \geq n+1$ planes of $\mathcal{A}$. Counting incident pairs (P_i, A) with $P_i \notin E_v$ and $A \in \mathcal{A}$, we obtain thus $(v-e)(n+1) \leq |\mathcal{A}|(n^2+n+1)$. But $v \geq n^3$, $|\mathcal{A}| \leq 2n$ and $e \leq n^2+n+1$, a contradiction.

Let i be an index with $\pi_{P_i} = |E_i|+m \leq r_{P_i}+m$. If $m \leq n$ or $r_{P_i} \geq |E_i|+1$, we are done. Suppose therefore that $m=n+1$ and $e_i := |E_i| = r_{P_i}$. Then E_i and $\mathbf{S}/P_i$ are isomorphic linear spaces. Hence E_i has $b := \pi_{P_i} = e_i+n+1$ lines. From $vm \leq n(n+1)e$, we get $v \leq en$. Since $\pi \geq bn$ (by 3.5), $e_i \geq e$ and $\pi \leq v+n^2+n$, we obtain $e_i = e$, $\pi = bn$, $v = en$, and $b = e+n+1$. Since $\pi = bn$, some line l of E_i lies in at most n planes. Hence l is in a plane E with $|E| \geq k_l + (v-k_l)/n > v/n = e$. Let P be a point not in E_i or E. Using 3.5 and $\pi = bn$, we see that every plane on P meets E in a line, so $\pi_P = b = e_i+n+1 \leq |E|+n$. Also $|E| \leq r_P$ and thus $\pi_P \leq r_P+n$, since $P \notin E$. ∎

THEOREM 3.9. *Every line has at most* $n+1$ *points.*

PROOF. By Lemma 3.8, there exist a point P and a plane E satisfying $P \notin E$, $\pi_P \leq |E|+n+1$, and $\pi_P \leq r_P+n$. Then $\pi_P \geq n^2+n$ and thus

$r_P \geq n^2$ and $|E| \geq n^2 - 1$ by Lemma 3.6. By Totten's Theorem either $\mathbf{S}/P$ can be embedded in a projective plane of order n, or $\pi_P = r_P + n$ and $\mathbf{S}/P$ is an inflated affine plane of order n. Since E is embedded in $\mathbf{S}/P$, it follows that every line of E has at most $n+1$ points.

We call a point of E *good*, if it lies on at most $n+1$ lines of E. Then E has at least $n^2 - 1$ good points (if $\mathbf{S}/P$ and hence E can be embedded, then even every point of E is good; if $\mathbf{S}/P$ is an inflated affine plane, then $\mathbf{S}/P$ has precisely n^2 'points' of degree $n+1$, and since $|E| \geq \pi_P - n - 1 = r_P - 1$, it follows that E has at least $n^2 - 1$ points that lie on at most $n+1$ lines of E).

The main result of [3] shows that E has at least $n^2 + n$ lines. Thus, if $\mathcal{A}$ is the set consisting of the planes that have no line in common with E, then Lemma 3.5 shows that $|\mathcal{A}| \leq \pi - (n^2+n)n \leq v + n - n^3$.

Assume that there exists a line h of degree $k = n+1+d > n+1$. By Lemma 3.5, we have $d < 2n-1$. Put $t := n^2 - n - 2$. Consider a point $Q \notin E \cup h$. We claim that Q lies on at least dt planes of $\mathcal{A}$. In fact, let F be the plane on Q and h. Since lines of E have at most $n+1$ points, at least $n^2 - 1 - (n+1) = t$ good points X of E do not lie in F. Consider such a good point X. Then the line QX is not coplanar to h and thus lies in at least k planes. Since X is good, at least $k - (n+1) = d$ planes on QX meet E only in the point X. Since there are t choice for X, we conclude that Q lies on at least dt planes of $\mathcal{A}$.

Counting the number s of incident point-plane pairs with points $Q \notin E \cup h$ and planes of $\mathcal{A}$, we conclude that $s \geq (v - |E| - k)dt$. Since every plane has at most $e := n^2 + n + 1$ points, it follows that $s \geq (v - n^2 - 2n - 2 - d)dt$. Since $1 \leq d \leq 2n-1$, it follows that $s \geq (v - n^2 - 2n - 3)t$. On the other hand $s \leq |\mathcal{A}|e \leq (v + n - n^3)e$. Comparing both bounds for s using $v \leq (n+1)^3 - 1$, gives a contradiction in the case that $n \geq 5$.

Hence $n = 4$ and now the bounds for s yield only that $v \geq 90$. We count the number z of all incident point-plane pairs to obtain a contradiction. Since, by Lemma 3.6, every point lies on at least $b := n^2 + n$ planes and since the points outside $E \cup h$ lie on at least $b + dt \geq b + t$ planes, we have $z \geq vb(v - e - k)t$. On the other hand, since planes have at most e points, we have $z \leq \pi e \leq (v + n^2 + n)e$. Compare both bounds for z using $n = 4$ to obtain $v \leq 70 + 10k/9$. But $k \leq 3n - 1 = 11$ (Lemma 3.5) and $v \geq 90$, a contradiction. $\square$

4 – Embedding the planes

We call a line *long* (and otherwise *short*), if it has degree $n+1$. A line is called *ordinary* (and otherwise *special*), if it lies in at most $n+1$ planes. A point P is called *ordinary* (and otherwise *special*), if the quotient geometry $\mathbf{S}/P$ can be embedded in a projective plane of order n.

LEMMA 4.1.

a) *If P is an ordinary point, then $r_P \geq n^2$ and $\pi_P \geq n^2+n$. If $r_P \geq n^2+2$, then $\pi_P = n^2+n+1$.*

b) *If P is special, then $r_P \geq n^2$ and $\pi_P \geq n^2+n+2$. If $r_P \geq n^2+1$, then $\pi_P \geq n^2+n+3$. Also $\pi_P \geq r_P + n$ with equality only if $\mathbf{S}/P$ is an inflated affine plane of order n.*

PROOF. Let P a point. Since every line has at most $n+1$ points and since $v \geq n^3$, we have $r_P \geq (v-1)/n > n^2-1$. This implies a). Now suppose that P is special. By Lemma 3.1 b), $\mathbf{S}/P$ is non-degenerate. Lemma 5.1 in [7] yields $\pi_P \geq n^2+n+2$, and the Theorem of Totten shows that $\pi_P \geq r_P + n$ with equality only if $\mathbf{S}/P$ is an inflated affine plane of order n. If $\pi_P = n^2+n+2$, then a result of STINSON [9] shows that $r_P = n^2$. □

LEMMA 4.2. *Suppose that P is a point outside a long line l. If every plane on P meets l, then P is an ordinary point.*

PROOF. Suppose that $\mathbf{L}$ is a linear space with $r \geq n^2$ points that has a line l' of degree $n+1$ meeting every other lines. We claim that $\mathbf{L}$ can be embedded in a projective plane of order n (then the lemma follows for $\mathbf{L} = \mathbf{S}/P$ and $l' = Pl$). Since l' meets every line, every point not on l' has degree $n+1$. This implies that every line has degree at most $n+1$. Hence $r \leq n^2+n+1$. If every point of l' has degree at most $n+1$, then $\mathbf{L}$ has at most n^2+n+1 lines, and Lemma 5.1 of [7] shows that $\mathbf{L}$ can be embedded.

Assume a point $L \in l'$ has degree $d \geq n+2$. Then some line h on L has degree $x \leq 1 + (r - |l'|)/(d-1) \leq r/(n+1)$. Choose a point $H \in h \setminus \{L\}$. Then some line $g \neq h$ on H has degree at least $1 + (r-x)/n > n$. Hence g has degree $n+1$, so L is on a line x missing g. Then every point of x is on at least $|g|+1 = n+2$ lines, a contradiction. □

Lemma 4.3. *Suppose that some ordinary long line l has at least n special points. Then $v \geq n^3 + 2n$, $r_P \geq n^2 + 2$ for every point P, and every long line lies in at least $n + 1$ planes.*

Proof. Denote by π' number of planes meeting l. It follows from Lemma 4.1 that $\pi' \geq n^2+n+n(n^2+1)$. Hence $v \geq \pi - n^2 - n \geq n(n^2+1)$. Theorem 3.9 implies that $r_P \geq n^2+1$ for every point P. Now Lemma 4.1 shows that $\pi' \geq n^2+n+n(n^2+2)$ giving $v \geq n^3+2n$. Hence every point is on at least n^2+2 lines. Using Lemma 3.4, it follows every long line is on at least $n+1$ planes. $\square$

Lemma 4.4. *Suppose that l is an ordinary long line with at least n special points. Then the ordinary points do not lie in a common plane.*

Proof. Let d be the number of planes that miss l, let r denote the minimum point degree, let s denote the maximum number of short lines on a point of degree r, and let t be the number of ordinary points of l. Then $v \leq 1 + rn - s$. It follows from Lemma 4.1 and 4.3 that $r \geq n^2+2$, that l lies on $n+1$ planes, that ordinary points lie on n^2+n+1 planes, and special points lie on at least $r+n$ planes. Let m be the number of special points of l that lie on more than $r+n$ planes. Then l meets at least $w := n + 1 + tn^2 + (n + 1 - t)(r - 1) + m$ planes, so $\pi \geq w + d$. Use $\pi \leq v + n^2 + n$ to obtain $d + s + m \leq n + (n^2 + 1 - r)(1 - t)$. We may assume that $d \geq n - 1$ (otherwise Lemma 4.2 implies that there exist at least $v - |l| - d(n^2 + n + 1) > n^2 + n + 1$ ordinary points). Then $s + m \leq 1 + (n^2 + 1 - r)(1 - t)$. It follows that $t \neq 0$ (otherwise $s + m \leq n^2 + 2 - r$, so $m = 0$ and $r = n^2 + 2$; in this case, every point of l is special and on r lines and $r + n = n^2 + n + 2$ planes, contradicting Lemma 4.1). Hence $t = 1$ and $s + m \leq n - d \leq 1$. Since $m \leq 1$, we can find two special points $P_1, P_2 \in l$ that lie on $r + n$ planes. Lemma 4.1 b) implies that P_i is on r lines and that $\mathbf{S}/P_i$ is an inflated affine plane of order n, $i = 1, 2$. It follows that P_1 lies on precisely n^2 ordinary lines. Since $s \leq 1$, there exists a plane E on l such that every line of E on P_1 or P_2 is long. Consequently E has n^2+n+1 points. From $s \leq 1$, we also deduce that P_1 lies on an ordinary long line l' that is not contained in E. As for l, we can show that l' contains an ordinary point P'. Then P' is not in E, and since E has $n^2 + n + 1$ points, it follows that $\mathbf{S}/P'$ and E are projective planes of order n.

Since $d > 0$ there exists a plane F missing l. Since E is a projective plane, we have $|E \cap F| \leq 1$. Hence F contains a line h with $h \cap E = \emptyset$. Distinct planes on h meet E in disjoint sets of points. Since E is a projective plane, this implies that at most on plane on h meets E in a line. Since each point of E spans a plane with h, it follows at least $n^2 - n$ planes on h meet E in a point not on l. Hence $d \geq n^2 - n$. But $m + d \leq n - s$, a contradiction. □

Lemma 4.5. *Suppose there exists an ordinary long line. Then every plane can be embedded in a projective plane of order n.*

Proof. In view of Lemma 4.4 we may assume that every ordinary long line has at least two ordinary points. Suppose that there exists an ordinary long line l. Since it has two ordinary points, every plane not on l can be embedded. Assume that l is contained in a plane E that can not be embedded. Then every ordinary point is in E, and thus every ordinary long line is a line of E. Consider an ordinary point P on l. Then P lies on at most n^2 lines which are not in E and these lines are short. Hence $v \leq |E| + n^2(n-1)$. It follows that $|E| \geq n^2$ and $\pi \leq n^3 + n + |E|$. If b is the number of lines in E, then $\pi \geq bn$ by Lemma 3.5. Hence $bn \leq n^3 + n + |E|$. Since $|E| \leq n^2 + n + 1$, it follows that $b \leq n^2 + n + 2$. Since E can not be embedded, we get $|E| \leq n^2+2$ from Totten's Theorem. Hence $bn \leq n^3+b+|E| \leq n^3+n^2+n+2$, so $b \leq n^2+n+1$. Since $|E| \geq n^2$, Lemma 5.1 of [7] shows that E can be embedded, a contradiction. □

Theorem 4.6. *Every plane can be embedded in a projective plane of order n. There exists an ordinary point.*

Proof. The preceding two lemmas allow us to assume that every ordinary line is short. By Lemma 3.8, there is non-incident point-plane pair (P, E) satisfying $\pi_P \leq |E| + n + 1$ and $\pi_P \leq r_P + n$.

First consider the case that P is ordinary. Then every line on P is ordinary and thus short. Hence $v = n^3$, $r_P = n^2 + n + 1$, and every line on P has n points. Consequently, $\mathbf{S}/P$ is a projective plane of order n. It follows that every plane on P has n^2 points. Since $\pi \leq v + n^2 + n$, Lemma 3.5 shows that every plane has at most n^2+n+1 lines. Lemma 5.1

of [7] implies that the planes on P can be embedded. The planes not containing P can be embedded, since P is ordinary.

Now consider the case that P is special. Lemma 4.1 implies $\pi_P = r_P + n$ and that $\mathbf{S}/P$ is an inflated affine plane. Hence $r_P \leq n^2+n+1$, and P is on n^2 ordinary lines, which are short. This implies that $v \leq 1 + r_P n - n^2$ and $\pi \leq 1 + r_P n + n$. From $v \geq n^3$, we get $r_P \geq n^2+n$. Let b the number of lines of E. From Lemma 3.5 we get $b \leq \pi/n$. Hence $b \leq r_P + 1$. Since $|E| \geq \pi_P - n - 1 = r_P - 1$, Totten's Theorem shows that $b = n^2 + n + 1$ and that E can be embedded in a projective plane of order n.

Denote by c the number of planes that do not contain P and do not meet E in a line. Since P lies on $\pi_P - b = r_P + n - b$ planes that do not meet E in a line, Lemma 3.5 shows that $\pi \geq bn + r_P + n - b + c$. Compare this with $\pi \leq 1 + r_P n + n$ to obtain $r_P = n^2 + n + 1$ and $c \leq 1$.

Since P lies on n^2 ordinary lines, and since $|E| \geq r_P - 1$, we can find an ordinary line l on P that meets E in a point R and contains a third point P'. Since E has $n^2 + n + 1$ lines and is embedded, R lies on $n + 1$ lines of E. Hence every plane on l meets E in a line. Since $c \leq 1$, it follows that P' lies on at most one plane that does not meet E in a line. Hence $\pi_{P'} \leq b + 1 \leq |E| + 3$ and thus $\pi_P \leq r_{P'} + 3 < r_{P'} + n$, so P' is an ordinary point. Now we can apply the first case to P' to complete the proof. ∎

5 – Classification of S

LEMMA 5.1.

a) *A long line meets every line that is coplanar to it.*

b) *Suppose that a plane E misses a long line l. Then every point and line of E is special. If l is an ordinary line, then $|E| \leq n + 1$.*

c) *If a line l lies in exactly c planes, then $r_P \leq 1 + cn$ for every point $P \in l$.*

d) *If P is a point, then $r_P \leq n^2 + n + 1$. If P does not lie on a long line, then $r_P = n^2 + n + 1$, $v = n^3$, and every line on P has precisely n points.*

e) *If $v = n^3$ and if there exists a special line, then every line lies in at least $n + 1$ planes.*

Proof. a) Let E be a plane containing the long line l. By Theorem 4.6, E can be embedded in a projective plane of order n, so l meets every line of E.

b) It follows from a) that no line of E is coplanar to l. Hence every line of E lies in at least $|l|+1 = n+2$ planes. Consequently, the lines and therefore also the points of E are special. Part a) also implies that l spans distinct planes with distinct points of E, so l is in at least $|E|$ planes. Thus, if l is ordinary, then $|E| \leq n+1$.

c) This is immediate, since all planes are embedded in projective planes of order n.

d) By Theorem 4.6 there exists an ordinary point P. Every line on P is ordinary. Now apply part c) to a line l on P and Q to conclude that $r_P \leq n^2+n+1$. Since $v \geq n^3$, the second assertion is an immediate consequence.

e) Suppose that $v = n^3$ and that the line h lies in n planes (cf. Lemma 3.5); we shall show that every line is ordinary. Then h lies in a plane E with at least $k_h + (v-k_h)/n$ points. Hence $|E| > n^2+1$. Since E can be embedded in a projective plane of order n, it follows that E has $b := n^2+n+1$ lines. Let P be any point of E not on h. Then every line on P that is not in E has at most n points, since h is in n planes. Hence P is on at most $n+1$ long lines, so $v-1 \leq r_P(n-1)+n+1$, which implies that $r_P > n^2+1$. Part c) shows that every line on P lies in at least $n+1$ planes. Hence every line of E other than h lies in at least $n+1$ planes. Thus E meets at least $bn-1$ other planes in a line. Since $\pi \leq v+n^2+n = bn$, it follows that E meets every plane in a line, and that every line of E other than h is in precisely $n+1$ planes. Hence every line of E is ordinary. Since E meets every plane in a line, we see that every point not on E is on b lines and thus an ordinary point. This implies that also all lines not in E are ordinary. ∎

Lemma 5.2. *If s is a special line, then there exists an ordinary long line missing s.*

Proof. Assume that every ordinary long line meets s. First we consider the case that there exists an ordinary long line l that meets s in a point H. Then each point $P \in l \setminus \{H\}$ is on at least n^2+n+1 planes (otherwise Lemma 4.1 shows that $r_P \leq n^2+1$ and P is ordinary;

since $v \geq n^3$, it follows that some line on P is long and misses s, a contradiction). Since l lies in at most $n+1$ planes, it follows that there are at least n^3 planes that do not contain l but meet l in a point other than H. Hence l meets at least $n^3 + \pi_H$ planes. Lemma 4.2 implies that every point of $s \setminus \{H\}$ lies on a plane missing l. Hence there are at least $k_s - 1$ planes missing l, so $\pi \geq n^3 + \pi_H + k_s - 1$. Since $\pi_H \geq n^2 + n + 2$ (Lemma 4.1), it follows that $v \geq n^3 + 1 + k_s$. In view of Lemma 4.6, there exists an ordinary point P. Since s meets every ordinary long line, P lies on at most k_s long lines. Hence $v \leq 1 + r_P(n-1) + k_s$. Since $r_P \leq n^2 + n + 1$, we obtain $v \leq n^3 + k_s$, a contradiction.

Now consider the case that there does not exist an ordinary long line. Let P be an ordinary point, put $E := Ps$, denote by b the number of lines in E, and by d the number of planes that do not meet E in a line. Lemma 5.1 d) shows that $r_P = n^2 + n + 1$, $v = n^3$, and that every line on P has length n. Hence $\pi_P = n^2 + n + 1$, $\mathbf{S}/P$ is a projective plane of order n, and E has n^2 points. By Lemma 5.1 e), every line of E is in at least $n+1$ planes. Since s lies in at least $n+2$ planes, it follows that $\pi \geq bn + 2 + d$. Since $\pi \leq v + n^2 + n$, it follows that $b \leq n^2 + n$. Since $|E| = n^2$ and E is embedded in a projective plane, we have $b = n^2 + n$. Since every point X not on E is on at least $n^2 + n + 1$ planes (use Lemma 4.1 if X is special, and the argument we used for P if X is ordinary), it follows that $d(n^2+n+1) \geq v - |E|$. Hence $d \geq n - 1$ and thus $\pi \geq bn+2+d \geq n^3+n^2+n+1 > v+n^2+n$, a contradiction. ∎

Lemma 5.3. a) *Every long line is ordinary.*

b) *Suppose that the special line s lies on $n+1+\delta$ planes. Then $\delta+1$ of the planes on s have at most $n+1$ points and have only special lines and points.*

c) *A special point lies on a special line.*

Proof. a) Assume that s is a special long line. By Lemma 5.2, there exists an ordinary long line l missing s. Since s is special, it lies on a plane E missing l. Clearly, $|E| > k_s = n+1$. However, $|E| \leq n+1$ by Lemma 5.1 b), a contradiction.

b) Lemmas 5.2 shows that there exists an ordinary long line l missing s. Lemma 5.1 a) shows that l and s are not coplanar. Hence δ planes on s miss l. Let E be a plane on s missing l, and P a point of E not on s. Then $|E| \le n+1$ by Lemma 5.1 b). Hence not every line of E on P has n points, so Lemma 5.1 d) shows that P lies on a long line l'. Since $|E| \le n+1$, the line l' is not a line of E. Hence l' misses s, which implies that s lies on δ planes that miss l'. Since E meets l' and misses l, it follows that s lies on $\delta+1$ planes that miss either l or l'. Since, by a), l and l' are ordinary, the assertion follows from Lemma 5.1 b).

c) In view of Lemmas 3.1 b) and 4.1, it suffices to show that any non-degenerate linear space with $r \ge n^2$ points in which every point is on at most $n+1$ lines has at most n^2+n+1 lines. Let $\mathbf{L}$ be a linear space with these properties. Then every line of $\mathbf{L}$ has at most $n+1$ points. If every line of $\mathbf{L}$ has at most n points, then $r \ge n^2$ implies that all lines of $\mathbf{L}$ have n points and every point of $\mathbf{L}$ is on $n+1$ lines; in this case $\mathbf{L}$ is an affine plane of order n and has thus n^2+n lines. If some l of $\mathbf{L}$ has $n+1$ points, then l must meet every line, and therefore $\mathbf{L}$ has at most n^2+n+1 lines. ∎

Lemma 5.4. *A long line has at most one special point.*

Proof. Let l be a long line and suppose that l has two special points P and P'. Let α be the number of planes on l, and denote them by $E_1, \dots, E_\alpha$. If P has degree at most n in E_i, then put $g_i := 0$. If P has degree $n+1$ in E_i, then let g_i+1 denote the minimum degree of the lines of E_i on P. Similarly define g_i', $i = 1, \dots, \alpha$, for P'. Put $G := \sum_{i=1}^{\alpha} g_i$ and $G' := \sum_{i=1}^{\alpha} g_i'$. Since E_i can be embedded in a projective plane of order n, we have $|E_i| \le n+1+(n-1)^2+g_i+g_i'$. Hence,

$$(1)\quad n^3 \le v = n+1+\sum_{i=1}^{\alpha}\big(|E_i|-n-1\big) \le n+1+\alpha(n-1)^2+G+G'\,.$$

Consider distinct planes E_i and E_j on l. We claim that $g_i+g_j \le n$. To see this, we may assume that P has degree $n+1$ in E_i and E_j. By Lemma 5.3 c), P lies on a special line x. We may assume that $x \not\subseteq E_i$. Since P has degree $n+1$ in E_i, Lemma 5.3 b) implies that x lies on a plane that meets E_i in a special line x_i. The same Lemma shows that x_i lies in a plane E that has at most $n+1$ points and meets E_j in a line x_j.

Hence $|x_i \cup x_j| \leq |E| \leq n+1$. Since $1+g_i$ is minimum degree of a line of E_i on P, it follows that $1+g_i+g_j \leq |x_i \cup x_j| \leq n+1$.

Hence $g_i + g_j \leq n$ for distinct indices $i, j \in \{1, \ldots, \alpha\}$. It follows easily that $G \leq \frac{1}{2}\alpha n$. Also $G \leq \frac{1}{2}(\alpha-1)n$, if $g_i = 0$ for at least one index x. Similarly $G' \leq \frac{1}{2}\alpha n$. Using inequality (1), it follows that $\alpha > n$, and thus $\alpha = n+1$, since l is ordinary.

Assume that some point of $X \in l$ lies on at most n^2+n planes. Then $r_X \leq n^2+1$ by the main result of [3]. Hence, if $n+1-d_i$ is the degree of X in E_i, then $d := \sum_{i=1}^{n+1} d_i \geq n$. We can improve the above bound for $|E_i|$ now to $|E_i| \leq n+1+(n-1)^2+g_i+g_i'-d_i(n-2)$ and then inequality (1) gives $n^3 \leq v \leq n+1+\alpha(n-1)^2+G+G'-d(n-2)$. But $d \geq n$, $\alpha = n+1$, and $G, G' \leq \frac{1}{2}n(n+1)$, a contradiction.

Hence l lies in $n+1$ planes and every point of l lies on at least n^2+n+1 planes. If P lies on $n^2+n+1+p$ planes and if P' lies on $n^2+n+1+p'$ planes, then it follows that l meets at least $n^3+n^2+n+1+p+p'$ planes. Hence $v \geq \pi - n^2 \geq n^3+1+p+p'$. Using $v \leq n+1+(n+1)(n-1)^2+G+G'$, we obtain $G+G' \geq n^2-1+p+p'$. Since $p, p' \geq 1$ (Lemma 4.1), we have $G+G' > n^2$.

We may assume that $G \geq G'$, which implies that $G > \frac{1}{2}n^2$. This implies that $g_i > 0$ for $i = 1, \ldots, n+1$, which means that P has degree $n+1$ in every plane E_i. Hence $r_P = n^2+n+1$. Since P is a special point, it follows that $\pi_P \geq r_P + n \geq n^2+2n+1$.

Hence $p \geq n$. It follows that $G+G' \geq n^2-1+p+p' \geq n^2+n$. Since $G, G' \leq \frac{1}{2}n(n+1)$ we obtain equality. Hence $G = G' = \frac{1}{2}n(n+1)$, $p = n$ and $p' = 1$. However $G' > \frac{1}{2}n^2$ implies as before that $p' \geq n$, a contradiction. □

Lemma 5.5.

a) *Every line that has two special points is special and has only special points.*

b) *Every plane with three non-collinear special points has only special points.*

Proof. a) Since lines on ordinary points are ordinary, it suffices to show that an ordinary line has at most one special point. Assume on the contrary that there exists an ordinary line g with two special points S and S'. The preceding lemma says that g is short. Let s be a special line on S and let s' be a special line on S' (Lemma 5.3 c). Lemma 5.3 b) shows that s' lies in a plane T that has at most $n+1$ points. This implies that some line of T on S' has less than n points. Lemma 5.1 d) implies therefore that S' lies on a long line l. Since s has only special points, Lemma 5.4 shows that l misses s, and then Lemmas 5.1 a) shows that l and s are not coplanar. Thus s lies in $n+1$ distinct planes that meet l. Hence Lemma 5.3 b) yields that s lies in a plane F that has only special points and lines and that meets l in a special point X. Since g is ordinary, we have $F \neq E$. Hence $X \neq S'$ and thus l contains two special points, contradicting Lemma 5.4.

b) Suppose that E is a plane with three non-collinear special points Q, R, S, and denote by χ the set consisting of all special points of E. By a), any line that has two points in χ is contained in χ. Assume that E has an ordinary point P. Then all lines PX, $X \in \chi$, are distinct and hence $|\chi| \leq n+1$. Thus the lines SQ and SR have together at most $n+1$ points. Lemma 5.1 d) implies that S lies on a long line l. Lemma 5.4 shows that S is the only special point of l. Hence l misses the line $s := QR$. Lemma 5.1 a) shows that l is not a line of E. Hence distinct points of l span distinct planes with s. Since E has an ordinary point, it follows that each of the $n+1$ planes on s that meets l has an ordinary point. This contradicts Lemma 5.3 b). ∎

Now we are in position to complete the proof of the main theorem. If every line is ordinary (so also every point is ordinary by Lemma 5.3 c), then Theorem 1.3 and the final remark in [8] show that n is a prime power and $\mathbf{S}$ can be embedded in $PG(3,n)$.

From now on we shall assume that there exists at least one special point. We denote the set of special points by $\mathcal{P}_\infty$. It follows from Lemma 5.3 c) and b) that there are two planes all of whose points lie in $\mathcal{P}_\infty$. Thus Lemma 5.5 implies that $\mathbf{S}$ induces a planar space $\mathbf{S}_\infty$ on the set $\mathcal{P}_\infty$. We define a new planar space $\mathbf{S}'$ as follows. It has the same structure as $\mathbf{S}$ except that we replace all planes that are contained in $\mathcal{P}_\infty$ be a new plane E_∞ whose point set is $\mathcal{P}_\infty$. Lemma 5.5 garanties that $\mathbf{S}'$

is in fact a planar space. It follows from Lemma 5.3 b) that every special line of $\mathbf{S}$ lies in $\mathbf{S}'$ on at most $n+1$ planes. As before, it follows from [8] that n is a prime power and that $\mathbf{S}'$ can be embedded in $PG(3,n)$. Let $\mathbf{A}$ denote planar structure obtained by removing the plane E_∞ and all points and lines of E_∞ from $\mathbf{S}'$ (we do not call $\mathbf{A}$ a planar space, since it might have lines with one point or planes without three non-collinear points). Since $\mathbf{S}'$ is embedded in $PG(3,n)$, it follows that $\mathbf{A}$ is embedded in $AG(3,n)$.

Let π_∞ be the number of planes contained in $\mathcal{P}_\infty$, that is π_∞ is the number of planes of $\mathbf{S}_\infty$, and put $v_\infty := |\mathcal{P}_\infty|$. From the result of Greene mentioned in the introduction, we get that $\pi_\infty \geq v_\infty$ with equality iff $\mathbf{S}_\infty$ is a generalized projective space. Let v_0 be the number of points and π_0 be the number of planes of $\mathbf{A}$. Then $v = v_0 + v_\infty$ and $\pi = \pi_0 + \pi_\infty$. The planar space $\mathbf{S}'$ has $\pi_0 + 1$ planes.

Since the points of $\mathcal{P}_\infty$ are all contained in the plane E_∞ of $\mathbf{S}'$, we have $v_\infty \leq n^2+n+1$. Hence $v_0 = v - v_\infty \geq n^3 - n^2 - n - 1$. Since $\mathbf{A}$ is embedded in $AG(3,n)$, it follows that $\pi_0 \geq v_0 + n^2 + n$ with equality iff $\mathbf{A} = AG(3,n)$. Since

$$v + n^2 + n \geq \pi = \pi_0 + \pi_\infty \geq \pi_0 + v_\infty \geq v_0 + n^2 + n + v_\infty = v + n^2 + n\,,$$

it follows that $\mathbf{A} = AG(3,n)$ and $\pi_\infty = v_\infty$. Hence $\mathbf{S}_\infty$ is a generalized projective space, and $\mathbf{S}$ is $AG(3,n)$ with the generalized projective space $\mathbf{S}_\infty$ at infinity. This completes the proof of Theorem 1.1.

In order to complete the classification, it remains to investigate the structure of the generalized projective 3-space $\mathbf{S}_\infty$. We know that $\mathbf{S}_\infty$, considered as a linear space, is embedded in $PG(2,n)$. Thus, we have to determine which linear spaces that are embedded in $PG(2,n)$ can be endowed with a set of planes to obtain a generalized projective 3-space. It is not too difficult to see that there are only the following three possibilities.

$\mathbf{S}_\infty$ is a projective 3-space $PG(3,m)$ with $n = m^e$ for an integer $e \geq 4$.

$\mathbf{S}_\infty$ is the direct product of a desarguesian projective plane $PG(2,m)$ and a point which $n = m^e$ for some integer $e \geq 3$.

$\mathbf{S}_\infty$ is contained in the union of two lines of $PG(2,n)$.

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