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Local behavior of solutions of quasilinear elliptic equations with coefficients in Morrey spaces

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RIASSUNTO: *In questa nota proviamo una disuguaglianza di Harnack per equazioni quasilineari di tipo ellittico, estendendo i risultati contenuti in [6] e [5].*

ABSTRACT: *In this paper we prove a Harnack inequality for some quasilinear elliptic equations, extending the results in [6] and [5].*

1 – Introduction

In his work [6] J. SERRIN extended the Harnack inequality to quasilinear equations of the form

$$\operatorname{div} A\big(x, u(x), \nabla u(x)\big) + B\big(x, u(x), \nabla u(x)\big) = 0\,. \tag{1.1}$$

In his paper Serrin assumes that the coefficients in the structure conditions belong to appropriate L^p spaces. Precisely his assumptions are

$$\begin{cases} a = \text{ constant} \\ b(x), e(x) \in L^{n/(p-1)} \\ c(x) \in L^{n/(1-\varepsilon)} \\ d(x), f(x), g(x) \in L^{n/(p-\varepsilon)} \end{cases} \tag{1.2}$$

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where a, b, c, d, e, f and g are the coefficients in the structure conditions given in (2.1) of the following section 2 and ε is a positive small number.

Recently Rakotoson and Ziemer in [5] somewhat improved the above mentioned result by Serrin. Indeed in Theorem 3.12 of [5] a Harnack inequality is proved for equation

$$\operatorname{div} A(x, u(x), \nabla u(x)) + B(x, u(x), \nabla u(x)) = \mu$$

under the following assumptions

$$\begin{cases} a \cong \text{ constant}, \ c(x) \in L^{\infty}_{loc} \\ b(x) = d(x) = 0 \\ e(x) \in L^{n/(p-1+\varepsilon)} \\ f(x) \in L^{n/(p+\varepsilon)} + L^{1,\lambda} \\ g(x) \in L^{n/(p+\varepsilon)}. \end{cases} \tag{1.3}$$

We wish to point out that, because of the form of the L^{∞} estimate they get in Theorem 3.4, their constant is blowing up as the radius of the ball, on which the inequality is considered, approaches zero. This in particular makes it impossible to deduce Hölder continuity of the local solution from Harnack inequality.

Our aim in this note is to extend the validity of the Harnack inequality in both papers [6] and [5] weakening the assumptions in [6] and [5] and obtaining the very same conclusion as in Serrin, i.e. an inequality with constant *independent* of the radius of the ball. This is done in Theorem 2 in the following section 4.

As an obvious consequence we obtain the local Hölder continuity of solutions improving Theorem 3.7 in [5], Corollary 1 in [4] and our own Theorem 3 in [7].

Essentially our hypotheses are done using everywhere the Morrey space scale instead of the L^p spaces and it can be shown by examples that in this scale they are optimal in order to have the boundedness of the local solutions.

A comparison between our assumptions and the above recalled ones can be found in Remark 1 at the end of this paper.

About the technique we tried to follow as closely as possible the by now classical work [6], i.e. we will use the Moser's iteration technique.

Clearly, because of the lack of high integrability of our coefficients, many modifications are needed, which are pointed in the proof of the theorems below. In particular we have to use on a regular basis Adams inequality (see Lemma 1 below) instead of Hölder plus Sobolev to estimate products of coefficients times test functions. This we learnt from the paper [5] and the later [4].

We did not give the details of the proof whenever it is essentially the same as in Serrin only stressing the differences in treatment which lead to the basic inequalities.

After this work was completed we heard by Professor Ziemer, whose kindness we gratefully aknowledge, that G. Lieberman has recently obtained a result similar to ours. In his work [3] G. Lieberman proves L^∞ estimate and Harnack principle for positive solutions of a quasilinear second order equation. While the structure assumptions seem to be more general, in some cases, than ours, the class of solutions he considers is different. This because he is forced by the structure to assume a priori the boundedness of the solutions. This, on the contrary, we prove in Theorem 1 in this paper.

2 – Structure hypotheses and preliminary results

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let

$$A(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and

$$B(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

two continuous functions satisfying inequalities of the form

$$\tag{2.1} \begin{cases} \big|A(x,u,\xi)\big| \le a|\xi|^{p-1} + b|u|^{p-1} + e \\ \big|B(x,u,\xi)\big| \le c|\xi|^{p-1} + d|u|^{p-1} + f \\ \xi \cdot A(x,u,\xi) \ge |\xi|^p - d|u|^p - g \end{cases}$$

for a.e. $x \in \Omega$, $\forall\, u \in \mathbb{R}$, $\forall\, \xi \in \mathbb{R}^n$. Here p is a fixed number in $]1,n[$, a is a positive constant and the functions $b(x)$, $c(x)$, $d(x)$, $e(x)$, $f(x)$ and

$g(x)$ are such that

$$
\left\{
\begin{array}{l}
b(x), e(x) \in L^{\frac{p}{p-1},\lambda} \\
c(x) \in L^{p,\lambda} \\
d(x), f(x), g(x) \in L^{1,\lambda}
\end{array}
\right.
\tag{2.2}
$$

where $\lambda = n - p + \alpha$, $\alpha > 0$.

We recall that for $q \in [1, +\infty[$, $\eta \in]0, n[$, $L^{q,\eta} = L^{q,\eta}(\Omega)$ denotes the classical Morrey space of the functions $f \in L^q(\Omega)$ such that

$$
\sup_{\substack{x\in\Omega \\ \rho>0}} \frac{1}{\rho^\eta} \int\limits_{B(x,\rho)\cap\Omega} |f(y)|^q dy = \|f\|^q_{q,\eta;\Omega} < +\infty
$$

where $B(x,\rho)$ is the ball centered at x with radius ρ (whenever x is not relevant we will write B_ρ).

In the following two lemmas are vital

LEMMA 1 (ADAMS [1]). *Let μ a nonnegative measure in $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and $0 < r < +\infty$, there is a constant M with the property that $\mu[B(x,r)] \leq M r^\lambda$, where $\lambda = \frac{s}{p}(n-p)$, $1 < p < s < +\infty$ and $p < n$. If $u \in C_0^\infty(\mathbb{R}^n)$ then*

$$
\Big(\int\limits_{\mathbb{R}^n} |u|^s d\mu\Big)^{\frac{1}{s}} \leq C\, M^{\frac{1}{s}} \Big(\int\limits_{\mathbb{R}^n} |\nabla u|^p dx\Big)^{\frac{1}{p}}
$$

where $C = C(p, \lambda, n)$.

LEMMA 2 (Poincaré, JOHN-NIRENBERG [2]). *Let $B_{\tilde r} \subset R^n$, $u(x) \in H^{1,p}(B_{\tilde r})$ and suppose that for all $B_r \subset B_{\tilde r}$ there exists a constant K such that*

$$
\Big(\int\limits_{B_r} |\nabla u|^p dx\Big)^{1/p} \leq K r^{(n-p)/p}\,.
$$

Then there exist two positive constants p_0 and C depending on K, p, n such that

$$
\Big(\int\limits_{B_{\tilde r}} \mathrm{e}^{p_0 u} dx\Big) \Big(\int\limits_{B_{\tilde r}} \mathrm{e}^{-p_0 u} dx\Big) \leq C |B_{\tilde r}|^2\,.
$$

A function $u(x)$ is said to be a local solution of (1.1) in Ω if $u \in H^{1,p}_{loc}(\Omega)$ and

$$\int_\Omega \Big\{ A(x,u(x),\nabla u(x))\nabla\varphi(x) + B(x,u(x),\nabla u(x))\varphi(x) \Big\} dx = 0 \qquad \forall\,\varphi \in C_0^\infty(\Omega)\,. \tag{2.3}$$

Using Young's inequality and Lemma 1 it is immediately seen that (2.3) is meaningful.

3 – Local boundedness of solutions

The purpose of this section is to show that weak solutions of equation (1.1) are locally bounded.

THEOREM 1. *There exist a positive number r_0, independent of u, such that, if we assume that $u(x)$ is a local solution of equation* (1.1) *in Ω, that conditions* (2.1) *and* (2.2) *hold and that B_r and B_{2r} are balls with the same center with $B_{2r} \subset\subset \Omega$, then for $r \le r_0$ we have*

$$\|u\|_{L^\infty(B_r)} \le C r^{-n/p} \Big\{ \Big(\int_{B_{2r}} |u|^p dx \Big)^{1/p} + h r^{n/p} \Big\}$$

where

$$h = (r^{\frac{p-1}{p}\alpha} \|e\|_{\frac{p}{p-1},\lambda;B_{2r}} + r^\alpha \|f\|_{1,\lambda;B_{2r}})^{\frac{1}{p-1}} + (r^\alpha \|g\|_{1,\lambda;B_{2r}})^{\frac{1}{p}}\,.$$

PROOF. First we prove the theorem for the special case $r = 1$.

Set $v = |u| + h$, where h is a positive constant. From (2.1) we deduce

$$\begin{cases} |A(x,u,\xi)| \le a|\xi|^{p-1} + b_1|v|^{p-1} \\ |B(x,u,\xi)| \le c|\xi|^{p-1} + d_1|v|^{p-1} \\ \xi \cdot A(x,u,\xi) \ge |\xi|^p - d_1|v|^p \end{cases} \tag{3.1}$$

where $b_1 = b + h^{1-p}e$ and $d_1 = d + h^{1-p}f + h^{-p}g$.

Thus, setting

$$h=\left(\|e\|_{\frac{p}{p-1},\lambda;B_2}+\|f\|_{1,\lambda;B_2}\right)^{\frac{1}{p-1}}+\left(\|g\|_{1,\lambda;B_2}\right)^{\frac{1}{p}}$$

we have that the norms of b_1 and d_1 are bounded,

$$\tag{3.2}\begin{cases}\|b_1\|_{\frac{p}{p-1},\lambda;B_2}\le\|b\|_{\frac{p}{p-1},\lambda;B_2}+1\\ \|d_1\|_{1,\lambda;B_2}\le\|d\|_{1,\lambda;B_2}+2\,.\end{cases}$$

For fixed numbers $q\ge 1$ and $\ell>h$ we consider the functions

$$F(v)=\begin{cases}v^q & \text{if}\quad h\le v\le\ell\\ q\ell^{q-1}v-(q-1)\ell^q & \text{if}\quad \ell\le v\end{cases}$$

and

$$G(u)=\operatorname{sign}u\big[F(v)F'(v)^{p-1}-q^{p-1}h^{\beta}\big]\,,\quad -\infty<u<+\infty$$

where β is such that $pq=p+\beta-1$.

As a test function in (2.3) we take

$$\varphi(x)=\eta^p(x)G(u)$$

where $\eta(x)$ is a non negative smooth function with support in B_2.

Substituting $\varphi(x)$ in (2.3) and using the assumptions (3.1) we obtain, as in [6],

$$\tag{3.3}\begin{aligned}\int_{B_2}\eta^p|\nabla w|^p dx\le{}& ap\int_{B_2}\big|(\nabla\eta)w\big|\big|\eta(\nabla w)\big|^{p-1}dx+\\ &+q^{p-1}p\int_{B_2}b_1\big|(\nabla\eta)w\big||\eta w|^{p-1}dx+\\ &+\int_{B_2}c|\eta w|\big|\eta(\nabla w)\big|^{p-1}dx+(1+p)q^{p-1}\int_{B_2}d_1|\eta w|^p dx\,,\end{aligned}$$

where $w=w(x)=F(v)$.

We simplify (3.3) using Young's inequality

$$ab^{p-1} \le \frac{1}{p}\varepsilon^{(1-p)}a^p + \Big(1 - \frac{1}{p}\Big)\varepsilon b^p\,, \quad \forall\, \varepsilon > 0$$

obtaining

$$\int\limits_{B_2} \eta^p |\nabla w|^p dx \le C_1(p+1)q^p \Big\{ \int\limits_{B_2} |w(\nabla\eta)|^p dx + \int\limits_{B_2} |\eta w|^p d\mu \Big\} \tag{3.5}$$

where C_1 is a positive constant depending only on p and a and μ denotes the measure defined by

$$d\mu = (b_1^{\frac{p}{p-1}} + c^p + d_1)dx\,.$$

Noting that for $0 < \rho < +\infty$ we have

$$\mu[B_\rho] \le M\rho^\lambda$$

where

$$M = \|b_1\|^{p/(p-1)}_{p/(p-1),\lambda;B_2} + \|c\|^p_{p,\lambda,B_2} + \|d_1\|_{1,\lambda;B_2}$$

using Lemma 1 we obtain

$$\begin{aligned} \int\limits_{B_2} |\eta w|^p d\mu &\le \Big(\int\limits_{B_2} |\eta w|^s d\mu \Big)^{p/s} \mu(B_2)^{1-p/s} \le \\ &\le C_2(p,\lambda,n) M \Big\{ \int\limits_{B_2} |(\nabla\eta)w|^p dx + \int\limits_{B_2} |(\nabla w)\eta|^p dx \Big\} \end{aligned} \tag{3.6}$$

where s is defined by $(n-p)s/p = \lambda$.

Substituting (3.6) in (3.5) and assuming that

$$C_1(p+1)C_2 M < 1 \tag{3.7}$$

we obtain

$$\|\eta(\nabla w)\|_{L^p(B_2)} \le C_3(p,a)q\|w(\nabla\eta)\|_{L^p(B_2)}$$

and by Sobolev inequality we have

$$\|\eta w\|_{L^{p^*}(B_2)} \le C_4 q \|w(\nabla\eta)\|_{L^p(B_2)} \tag{3.8}$$

where C_4 is a constant depending only on p, a and n.

Now proceeding exactly as in [6] pp. 258-259, by the familiar iteration procedure, we obtain

$$\|u\|_{L^\infty(B_1)} \le C\{\|u\|_{L^p(B_2)} + h\}\,.$$

This proves the theorem in the special case $r = 1$. The general case $r \ne 1$ is obtained by dilation.

We note that, in this case, condition (3.7) yields the number r_0 in the statement of the theorem, this is easily seen having in mind the way in which the $L^{p,\lambda}$ norms in the "constant" M vary under dilation.

4 – Harnack's inequality and the Hölder continuity of solutions

THEOREM 2. *There exists a number $r_0 > 0$, independent of u, such that if assume that $u(x)$ is a non negative local solution of equation* (1.1) *and that conditions* (2.1) *and* (2.2) *hold, then for $r \le r_0$ and $B_{3r} \subset\subset \Omega$ we have*

$$\max_{B_r} u(x) \le C\{\min_{B_r} u(x) + h\}$$

where

$$C = C\big(p, n, a, \lambda, r_0, \|b\|_{p/(p-1),\lambda;B_{3r}}, \|c\|_{p,\lambda;B_{3r}}, \|d\|_{1,\lambda;B_{3r}}\big)$$
$$h = \big(r^{\frac{p-1}{p}\alpha}\|e\|_{\frac{p}{p-1},\lambda;B_{3r}} + r^\alpha\|f\|_{1,\lambda;B_{3r}}\big)^{\frac{1}{p-1}} + \big(r^\alpha\|g\|_{1,\lambda;B_{3r}}\big)^{\frac{1}{p}}\,.$$

PROOF. Also in this case we prove the theorem assuming $r = 1$.

Proceeding as in Theorem 1, setting $v = |u| + h$, we deduce the new conditions (3.1) and (3.2) and then, taking as a test function in (2.3) $\varphi(x) = \eta^p v^\beta$, where $\eta(x)$ is a non negative smooth function with support in B_3 and $\beta \in \mathbb{R}$, we obtain

$$\begin{aligned}\int\limits_{B_3} |\nabla v|^p \eta^p v^{\beta-1} dx \le C_1(p,a)\big(1+|\beta|^{-1}\big)\Big\{ \int\limits_{B_3} |\nabla \eta|^p v^{p+\beta-1} dx + \\ + \int\limits_{B_3} \eta^p v^{p+\beta-1} d\mu \Big\}.\end{aligned} \tag{4.1}$$

Here we may assume that $v(x) \ge \varepsilon > 0$. Otherwise we may replace v by $v+\varepsilon$ and let $\varepsilon \to 0$ in the final result.

Setting

$$w(x) = \begin{cases} v^q(x) \quad \text{where} \quad pq = p+\beta-1 & \text{if} \quad \beta \ne 1-p \\ \log v(x) & \text{if} \quad \beta = 1-p \end{cases}$$

(4.1) yields

$$\begin{aligned}\int\limits_{B_3} \eta^p |\nabla w|^p dx \le C_1 |q|^p \big(1+|\beta|^{-1}\big)^p \Big\{ \int\limits_{B_3} |\nabla \eta|^p w^p dx + \\ + \int\limits_{B_3} \eta^p w^p d\mu \Big\} \quad \text{if} \quad \beta \ne 1-p\end{aligned} \tag{4.2}$$

$$\int\limits_{B_3} \eta^p |\nabla w|^p dx \le C_1 \Big\{ \int\limits_{B_3} |\nabla \eta|^p dx + \int\limits_{B_3} \eta^p d\mu \Big\} \quad \text{if} \quad \beta = 1-p\,. \tag{4.2$'$}$$

We consider first the (4.2)$'$. Using Lemma 1 we have

$$\int\limits_{B_3} \eta^p |\nabla w|^p dx \le C_2 \int\limits_{B_3} |\nabla \eta|^p dx \tag{4.3}$$

where C_2 is a positive constant depending on $p, n, a, \lambda, \|b\|_{p/(p-1),\lambda;B_3}$, $\|c\|_{p,\lambda;B_3}$ and $\|d\|_{1,\lambda;B_3}$.

Let B_h be an arbitrary open ball contained in B_2 with the same center. We choose $\eta(x)$ so that $\eta(x) = 1$ in B_h, $0 \le \eta \le 1$ in $B_3 \setminus B_h$ and $|\nabla \eta| \le 3/h$, $h > 1$.

From (4.3) we have

$$\|\nabla w\|_{L^p(B_h)} \le C_3 h^{(n-p)/p}$$

where C_3 is a positive constant depending on the same arguments of C_2.

Thus, from Lemma 2, we have that there exist two positive constants p_0 and k depending on p, n and C_3 such that

$$\Big(\int_{B_2} \mathrm{e}^{p_0 w} dx\Big) \Big(\int_{B_2} \mathrm{e}^{-p_0 w} dx\Big) \le K\,.$$

Since $w = \log v$ we have

$$\phi(p_0, 2) \le K\phi(-p_0, 2) \tag{4.4}$$

where, for any real number $p \ne 0$, we have

$$\phi(p, h) = \Big(\int_{B_h} |v|^p dx\Big)^{1/p}\,.$$

We consider now the (4.2). Using Lemma 1 and proceeding as in Theorem 1 we have

$$\int_{B_3} \eta^p |\nabla w|^p dx \le C_1 |q|^p (1 + |\beta|^{-1})^p \int_{B_3} |\nabla \eta|^p w^p dx \tag{4.5}$$

provided

$$C_1 |q|^p (1 + |\beta|^{-1})^p C(p, \lambda, n)\{\|b_1\|^{p/(p-1)}_{p/(p-1),\lambda;B_3} + \|c\|^p_{p,\lambda,B_3} + \|d_1\|_{1,\lambda;B_3}\} < 1$$

and using Sobolev inequality, (4.5) yields

$$\|\eta w\|_{L^{p^*}(B_3)} \le C_4 |q| (1 + |\beta|) \|(\nabla\eta) w\|_{L^p(B_3)} \tag{4.6}$$

where C_4 is a positive constant depending only on p, a and n.

Now following the pattern of [6] pp. 267-268 we have

$$\phi(\infty, 1) \le C'\phi(p_0', 2)$$

and

$$\phi(-\infty, 1) \ge C''\phi(p_0, 2)$$

where C' and C'' are two positive constants depending only on p, a and n, and $p'_0 \leq p_0$. Recalling (4.4) we obtain

$$\phi(\infty, 1) \leq C\phi(-\infty, 1)$$

where $C = C\big(p, n, a, \lambda, \|b\|_{p/(p-1),\lambda;B_3}, \|c\|_{p,\lambda;B_3}, \|d\|_{1,\lambda;B_3}\big)$ that is

$$\max_{B_1} u(x) \leq C\{ \min_{B_1} u(x) + h\} .$$

The general case $r \neq 1$ follows again by dilation.

As in [6], using Harnack's inequality, we deduce the following result

Corollary 1. *Let $u(x)$ be a weak solution of equation* (1.1). *If we assume that* (2.1) *and* (2.2) *hold, then u is locally Hölder continuous on Ω.*

Remark 1 We wish to point out that it can be seen by simple examples that our assumptions (2.2) on the coefficients of the structure conditions are sharp in the scale of Morrey spaces (for more details see Remark 2 in [7]).

Furthermore, we wish to show that our hypotheses are more general than those in [6] and, at least in some instances, than those in [5] and [4].

Indeed comparing (2.2) with (1.2) we have

$$L^{n/(p-\varepsilon)} \subsetneqq L^{1,\lambda} \quad ; \quad L^{n/(1-\varepsilon)} \subsetneqq L^{p,\lambda} \quad ; \quad L^{n/(p-1)} \subsetneqq L^{p/(p-1),\lambda} .$$

While, with respect to [5] where it is assumed $b = d = 0$ we have $b \in L^{p/(p-1),\lambda}$ $d \in L^{1,\lambda}$ and about e, f and g the comparison goes similarly to the above with [6].

Similar considerations can be seen to hold with respect to [4].

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