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The complete integral closure of monoids and domains II

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Riassunto: *Utilizzando metodi geometrici, vengono costruiti monoidi primari la cui chiusura integrale completa non è completamente integralmente chiusa. Tali monoidi non possono essere ottenuti come monoidi moltiplicativi di domini di integrità con gruppo di divisibilità finitamente generato.*

Abstract: *Using geometrical methods we construct primary monoids whose complete integral closure is not completely integrally closed. Such monoids cannot be realized as multiplicative monoids of integral domains with finitely generated groups of divisibility.*

1 – Introduction

In this note we study the (complete) integral closure of monoids H together with their groups of divisibility $\mathcal{G}(H) = \mathcal{Q}(H)/H^{\times}$. We will show how under certain assumptions these investigations may be reduced to the case where $\mathcal{G}(H) = \mathcal{Q}(H)$ is torsion free (Theorem 1 and Corollary 1); in particular, this works if $\mathcal{G}(H)$ is finitely generated. A monoid with torsion free quotient group may be considered as a submonoid of a real vector space. For such a monoid H we characterize its integral closure $\widetilde{H}$ and the complete integral closure $\widehat{\widetilde{H}}$ of $\widetilde{H}$ in geometrical terms (Theorem 2).

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This allows us to construct primary monoids H whose complete integral closure $\widehat{H}$ is not completely integrally closed (Theorem 3). In [1; Theorem 4] it was proved that $\widehat{\widehat{H}}$ is completely integrally closed for all primary monoids H; hence this latter result is sharp.

In section 5 we characterize integral domains R with finitely generated groups of divisibility whose multiplicative monoids are primary. In particular it will turn out that the complete integral closure of these domains is completely integrally closed.

2 – Preliminaries

Throughout this paper, a monoid means a commutative and cancellative semigroup with unit element. In this section and in the following one we use multiplicative notation.

Let H be a monoid; then $H^\times$ denotes its group of invertible elements and $\mathcal{Q}(H)$ a quotient group of H with $H \subseteq \mathcal{Q}(H)$; H is called reduced if $H^\times = \{1\}$. H is said to be *primary* , if $H \neq H^\times$ and if $a, b \in H$ and $b \notin H^\times$, then $a|b^n$ for some $n \in \mathbb{N}_+$. The integral closure $\widetilde{H} \subseteq \mathcal{Q}(H)$ and the complete integral closure $\widehat{H} \subseteq \mathcal{Q}(H)$ are defined by

$$\widetilde{H} = \{x \in \mathcal{Q}(H) \mid x^n \in H \text{ for some } n \in \mathbb{N}_+\}$$

and

$$\widehat{H} = \{x \in \mathcal{Q}(H) \mid \text{there exists some } c \in H \text{ s.t. } cx^n \in H \text{ for all } n \in \mathbb{N}_+\}.$$

H is called *integrally closed* if $H = \widetilde{H}$, and it is called *completely integrally closed* if $H = \widehat{H}$.

Clearly, we have

$$H \subseteq \widetilde{H} \subseteq \widehat{H} \subseteq \mathcal{Q}(H);$$

furthermore $\widetilde{H}$ and $\widehat{H}$ are integrally closed but in general $\widehat{H}$ is not completely integrally closed. If H is primary then $\widehat{\widehat{H}}$ is completely integrally closed (cf. [1; Theorem 4]).

In analogy to the appropriate notion in ring theory we define

$$\mathcal{G}(H) = \mathcal{Q}(H)/H^\times$$

as the *group of divisibility* of H.

As usual, we say that an abelian group G is *bounded* , if there exists an $n \in \mathbb{N}_+$ such that $g^n = 1$ for all $g \in G$.

PROPOSITION 1. *Let H be a monoid, G an abelian group and $\pi : \mathcal{Q}(H) \to G$ a group epimorphism with kernel K. Then we have:*

1. *If $KH^\times/H^\times$ is a torsion group, then $\pi(\widetilde{H}) = \widetilde{\pi(H)}$ and $\pi(H^\times) = \pi(H)^\times$.*
2. *If $KH^\times/H^\times$ is bounded, then $\pi(\widehat{H}) = \widehat{\pi(H)}$.*
3. *If $K \subseteq H^\times$ and G is a free abelian group, then $H \simeq K \times \pi(H)$.*

PROOF. Clearly, G is a quotient group of $\pi(H)$.

1. Obviously $\pi(\widetilde{H}) \subseteq \widetilde{\pi(H)}$ and $\pi(H^\times) \subseteq \pi(H)^\times$. Conversely, let $y \in \widetilde{\pi(H)} \subseteq G$ be given; then there are $x \in \mathcal{Q}(H), a \in H$ and some $n \in \mathbb{N}_+$ such that $y = \pi(x)$ and $y^n = \pi(a)$. Thus $x^n = as$ for some $s \in \mathrm{Ker}(\pi) = K$. Since $KH^\times/H^\times$ is a torsion group, there exists an $m \in \mathbb{N}_+$ with $s^m \in H^\times$. This yields $x^{nm} \in H, x \in \widetilde{H}$ whence $y = \pi(x) \in \pi(\widetilde{H})$.

Next, let $\pi(a) \in \pi(H)^\times$ with $a \in H$. Then there exist a $b \in H$ and an $s \in K$ such that $ab = s$. Since there is an $m \in \mathbb{N}_+$ for which $s^m \in H^\times$, it follows that $a \in H^\times$.

2. We have $\pi(\widehat{H}) \subseteq \widehat{\pi(H)}$ and in order to verify the opposite inclusion we take an element $y = \pi(x) \in \widehat{\pi(H)}$ with $x = a^{-1}b$ for some $a, b \in H$. Then there exists an element $c \in H$ such that for all $n \in \mathbb{N}_+$ $cx^n = d_n s_n$ for some $d_n \in H$ and $s_n \in K$. Let $\lambda \in \mathbb{N}_+$ be such that $s^\lambda \in H^\times$ for all $s \in K$, whence $c^\lambda x^{\lambda n} \in H$ for all $n \in \mathbb{N}_+$. Setting $c^* = c^\lambda a^{\lambda-1}$ we infer that $c^* x^m \in H$ for all $m \in \mathbb{N}_+$ whence $x \in \widehat{H}$.

3. Since G is free abelian, the exact sequence

$$1 \to K \to \mathcal{Q}(H) \to G \to 1$$

splits; let $\eta\colon G \to \mathcal{Q}(H)$ be the group monomorphism with $\pi \circ \eta = \mathrm{id}_G$. We set $H' = \eta(\pi(H))$; then $H' \simeq \pi(H), H' \subseteq H, \mathcal{Q}(H') = \eta(G)$ and $\mathcal{Q}(H) \simeq K \oplus \mathcal{Q}(H')$. Hence the product $K \times H'$ is a direct one, which obviously is contained in H.

Conversely, let $a \in H$ be given. Then $a = \varepsilon b$ with $\varepsilon \in K$ and $b \in (\eta \circ \pi)(\mathcal{Q}(H))$. Hence $b = \eta(x)$ with $x \in G$,

$$\pi(a) = \pi(b) = (\pi \circ \eta)(x) = x \in \pi(H)$$

and thus $b = \eta(x) \in \eta(\pi(H)) = H'$. □

3 – (Complete) integral closure and groups of divisibility

The aim of this section is to point out a way how to reduce investigations concerning the (complete) integral closure of primary monoids H to reduced primary monoids H^* having torsion free groups of divisibility. The relationship between H and H^* will be most intimate if H is reduced and $\mathcal{G}(H)$ is a direct sum of a bounded group and a free abelian group.

Theorem 1. *Let H be a monoid, $\pi : \mathcal{Q}(H) \to \mathcal{Q}(H)/\widetilde{H}^\times$ the canonical epimorphism and $H^* = \pi(H)$. Then we have*

1. *H^* is reduced, $\mathcal{G}(H^*) = \mathcal{G}(\widetilde{H})$ is torsion free and $\widetilde{H}^\times/H^\times$ is the torsion subgroup of $\mathcal{G}(H)$.*

2. *$\widetilde{H^*} \simeq \widetilde{H}/\widetilde{H}^\times$, and if $\mathcal{G}(\widetilde{H})$ is free abelian then $\widetilde{H} \simeq \widetilde{H}^\times \times \widetilde{H^*}$.*

3. *Suppose $\widetilde{H}^\times/H^\times$ is bounded. Then $\widehat{H^*} \simeq \widehat{H}/\widetilde{H}^\times$, and if $\mathcal{G}(\widetilde{H})$ is free abelian then $\widehat{H} \simeq \widetilde{H}^\times \times \widehat{H^*}$.*

4. *H is primary if and only if H^* is primary.*

Proof.

1. H^* is reduced by Proposition 1; hence $\mathcal{G}(H^*) = \mathcal{Q}(H^*) = \mathcal{Q}(H)/\widetilde{H}^\times = \mathcal{G}(\widetilde{H})$. Clearly $\mathcal{G}(\widetilde{H})$ is torsion free and $\widetilde{H}^\times/H^\times$ is the torsion group of $\mathcal{G}(H)$ because $\mathcal{G}(H)/(\widetilde{H}^\times/H^\times) \simeq \mathcal{G}(\widetilde{H})$.

2. and **3.** are consequences of Proposition 1.

4. This follows from [2; Lemma 2]. □

Corollary 1. *Let H be a reduced monoid and suppose that $\mathcal{G}(H)$ is a direct sum of a bounded group T and a free abelian group. Then $\widetilde{H} \simeq T \times \widetilde{H^*}$ and $\widehat{H} \simeq T \times \widehat{H^*}$.*

Proof. Since $H^\times = \{1\}$, $T = \widetilde{H}^\times$ is the torsion subgroup of $\mathcal{G}(H) = \mathcal{Q}(H)$ and $\mathcal{G}(\widetilde{H}) = \mathcal{G}(H)/T$. Thus the assertion follows immediately from the previous theorem. □

4 – Geometrical methods

In this section we investigate monoids having torsion free quotient groups. A torsion free abelian group G is a flat $\mathbb{Z}$-module and hence the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces an injection $G = G \otimes \mathbb{Z} \hookrightarrow G \otimes \mathbb{Q}$. Using base extension again, the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$ embeds into the real vector space $G \otimes \mathbb{R}$. Throughout this section we use additive notation for the operation of monoids. Therefore we may consider a monoid H with $\mathcal{Q}(H)$ torsion free as a submonoid of the additive group of the real vector space $V = \mathcal{Q}(H) \otimes \mathbb{R}$, and obviously H contains a basis of V. This allows us to study H using geometrical methods in V.

Our first aim is to derive geometrical descriptions of $\widetilde{H}$ and $\widehat{\widehat{H}}$ and to obtain a geometrical characterization of being primary. From this we see that $\widehat{\widehat{H}}$ is completely integrally closed, if the quotient group of H is finitely generated, and it enables us to construct primary monoids H for which $\widehat{H}$ is not completely integrally closed.

We recall some geometrical notations. Let V be a real vector space. For two distinct elements $x, y \in V$

$$[x, y] = \{y + \lambda(x - y) \mid 0 \leq \lambda \leq 1\}$$

denotes the line segment joining x and y; we set $[x, y) = [x, y] \setminus \{y\}$ and $(x, y) = [x, y) \setminus \{x\}$.

Let $M \subseteq V$ be a subset; M is called (algebraically) open if for all $a \in M$ and for all $a \neq x \in V$ there exists some $b \in (a, x)$ for which $[a, b] \subseteq M$. M is called convex if $[a, b] \subseteq M$ for all $a, b \in M$. We denote by

$$\mathcal{C}(M) = \Big\{ \sum_{x \in M} \lambda_x x \Big| \lambda_x \in \mathbb{R}_{\geq 0} \text{ and } \lambda_x = 0 \text{ for all but finitely many } x \in M \Big\}.$$

the convex cone with apex $0 \in V$ which is generated by M; $\mathcal{C}(M) \subseteq V$ is a convex set. Finally

$$\operatorname{lin}(M) = M \cup \{x \in V \mid [a, x) \subseteq M \text{ for some } a \in M \text{ with } a \neq x\}$$

is the set of points which are either in M or linearly accessible from M.

If M is a convex subset with non empty interior in a topological vector space then M is algebraically open if and only if M is topologically open, and lin (M) is just the topological closure $\overline{M}$ of M ([5; §11 A, Lemma]).

THEOREM 2. *Let H be a nontrivial monoid such that $\mathcal{Q}(H)$ is torsion free and suppose that $\mathcal{Q}(H) \subseteq \mathcal{Q}(H) \otimes \mathbb{Q} \subseteq \mathcal{Q}(H) \otimes \mathbb{R} = V$. Then we have*

1. $\widetilde{H} = \mathcal{C}(H) \cap \mathcal{Q}(H)$.

2. $\mathcal{C}(\widetilde{H}) = \mathcal{C}(H)$.

3. $\widehat{\widetilde{H}} = \text{lin}\ (\mathcal{C}(H)) \cap \mathcal{Q}(H)$.

4. *H is primary and reduced if and only if $\mathcal{C}(H) \setminus \{0\}$ is open and $\mathcal{C}(H) \neq V$.*

PROOF.

1. Let $a \in \widetilde{H} \subseteq \mathcal{Q}(H)$; then we have $na \in H \subseteq \mathcal{C}(H)$ for some $n \in \mathbb{N}_+$, which yields $a \in \mathcal{C}(H) \cap \mathcal{Q}(H)$.

Conversely, let $0 \neq a \in \mathcal{C}(H) \cap \mathcal{Q}(H)$ be given. Hence there exist $h_i \in H$ and $\lambda_i \in \mathbb{R}_{>0}$ with $a = \sum_{i=1}^{k} \lambda_i h_i$. Since a and h_i are contained in the rational vector space $\mathcal{Q}(H) \otimes \mathbb{Q}$, $(\lambda_1, \dots, \lambda_k)$ may be interpreted as a solution of a system of linear equations over $\mathbb{Q}$. Since this system has a positive solution, it has a positive rational one, and thus we may assume that all $\lambda_i \in \mathbb{Q}_{>0}$. Let $n \in \mathbb{N}_+$ be such that $n\lambda_i \in \mathbb{N}_+$ for $1 \leq i \leq k$; then $na \in H$ and hence $a \in \widetilde{H}$.

2. We have

$$\mathcal{C}(H) \subseteq \mathcal{C}(\widetilde{H}) \subseteq \mathcal{C}(\mathcal{C}(H)) = \mathcal{C}(H).$$

3. Let $a \in \widehat{\widetilde{H}} \subseteq \mathcal{Q}(H)$ be given; by definition there exists some $c \in \widetilde{H}$ such that $c + a\mathbb{N}_+ \subseteq \widetilde{H} \subseteq \mathcal{C}(H)$. Therefore $\frac{1}{n}c + a \in \mathcal{C}(H)$ for all $n \in \mathbb{N}_+$. Thus

$$[c+a, a) = \bigcup_{n \geq 1} [c+a, \tfrac{1}{n}c + a] \subseteq \mathcal{C}(H)$$

which implies $a \in \text{lin}\ (\mathcal{C}(H))$.

To verify the opposite inclusion, we take an element $a \in \text{lin}\ (\mathcal{C}(H)) \cap \mathcal{Q}(H)$. Hence there is an element $c' \in \mathcal{C}(H)$ such that

$$[c', a) = \{a + \lambda(c' - a) \mid 0 < \lambda \leq 1\} \subseteq \mathcal{C}(H),$$

which implies that $(n+1)(a+\frac{1}{n+1}(c'-a)) = c'+na \in \mathcal{C}(H)$ for all $n \in \mathbb{N}_+$. If $c' = \sum_{\nu=1}^{k} \lambda_\nu c_\nu$ with $c_\nu \in H$ and $\lambda_\nu > 0$, choose $m \in \mathbb{N}_+$ with $0 < \lambda_\nu < m$ for all $1 \le \nu \le k$. Then

$$m(\sum_{\nu=1}^{k} c_\nu) + na \in \mathcal{C}(H) \cap \mathcal{Q}(H) = \widetilde{H}$$

for all $n \in \mathbb{N}_+$, and thus $a \in \widehat{\widetilde{H}}$.

4. Suppose that $\mathcal{C}(H) \setminus \{0\}$ is open and let $a, b \in H \setminus \{0\}$ be given. Then there exists some $c \in [a, a-b) = \{a + \lambda(-b) \mid 0 \le \lambda < 1\}$ such that $[a, c] \subseteq \mathcal{C}(H)$, and therefore $a - \frac{1}{n} b \in \mathcal{C}(H)$ for some $n \in \mathbb{N}_+$. Then $na - b \in \mathcal{C}(H) \cap \mathcal{Q}(H) = \widetilde{H}$ and hence there exists some $m \in \mathbb{N}_+$ such that $m(na-b) \in H$ which yields $mna - b \in H$. Since $a, b \in H \setminus \{0\}$ were arbitrary, we conclude that either H is reduced and primary or $H = H^\times$, which yields the contradiction $\mathcal{C}(H) = V$.

Conversely, suppose that H is a reduced, primary monoid; then obviously $\mathcal{C}(H) \ne V$. Let $0 \ne a \in \mathcal{C}(H)$ and $x \in V$ be given. Since H contains a basis of V, there are $h_1, \ldots, h_n \in H$ such that $a = \sum_{i=1}^{n} \rho_i h_i$, $x - a = \sum_{i=1}^{k} \mu_i h_i - \sum_{i=k+1}^{n} \mu_i h_i$ with all $\rho_i, \mu_i \ge 0$. We may assume that $\rho_1 > 0$.
If $k = n$ everything is clear; otherwise we have

$$\begin{aligned}[a, x] &= \Big\{\rho_1 h_1 + (\sum_{i=2}^{n} \rho_i h_i + \lambda \sum_{i=1}^{k} \mu_i h_i) - \lambda \sum_{i=k+1}^{n} \mu_i h_i \Big| \; 0 \le \lambda \le 1\Big\} \\ &= \Big\{\frac{\rho_1}{n-k} \sum_{i=k+1}^{n} (h_1 - \lambda \frac{\mu_i(n-k)}{\rho_1} h_i) + c_\lambda \Big| 0 \le \lambda \le 1\Big\} \\ &\qquad\qquad\qquad\qquad\qquad \text{with } c_\lambda \in \mathcal{C}(H).\end{aligned}$$

Since H is reduced, $h_1 \notin H^\times$, and since H is primary, there is an $m \in \mathbb{N}_+$ such that $mh_1 - h_i \in H$ for $k+1 \le i \le n$. Thus $h_1 - \lambda \frac{\mu_i(n-k)}{\rho_1} h_i \in \mathcal{C}(H)$ if $\lambda \le \frac{\rho_1}{m\mu_i(n-k)}$ for all $k+1 \le i \le n$ with $\mu_i > 0$, which implies that $[a, b] \subseteq \mathcal{C}(H)$ for some $b \in (a, x)$. □

Remark. Let H be a monoid such that $\mathcal{Q}(H)$ is finitely generated and torsion free. Then $\mathcal{Q}(H) \otimes \mathbb{R}$ has finite dimension, the interior of

$\mathcal{C}(H)$ is non empty and hence

$$\widehat{\widehat{H}} = \overline{\mathcal{C}(H)} \cap \mathcal{Q}(H).$$

This immediately shows that $\widehat{\widehat{H}}$ is completely integrally closed. Since every monoid with finitely generated quotient group is a G-monoid, this also follows from [1; Theorem 4].

This result cannot be enbettered even for primary monoids: the next theorem exhibits primary monoids $H \subseteq \mathbb{Z}^s$ with $2 \le s \le \infty$ for which all inclusions

$$H \subset \widetilde{H} \subset \widehat{H} \subset \widehat{\widehat{H}}$$

are strict; in particular $\widehat{H} \neq \widehat{\widehat{H}}$.

Theorem 3. *For every $d \in \mathbb{N}_+ \cup \{\infty\}$ there exist reduced primary monoids H with $\dim_{\mathbb{R}}(\mathcal{Q}(H) \otimes \mathbb{R}) = d+1$ such that $\widehat{H}$ is not completely integrally closed.*

Proof.

1. Let $d \in \mathbb{N}_+ \cup \{\infty\}$ be given and put $I = \{i \in \mathbb{N} \mid 0 \le i < d+1\}$. For $0 \neq i \in I$ choose $\xi_i \in \mathbb{R}_{>0}$ with $\sum_{i=1}^d \xi_i < \infty$ and define

$$H = \Big\{(x_i)_{i\in I} \in \mathbb{N}_+^I \Big| \sup_{i\in I} |x_i| < \infty \text{ and } \sum_{i=1}^d \xi_i x_i \le x_0^2\Big\}.$$

Thus H consists of all bounded sequences $x \in \mathbb{N}_+^I$ for which x_0 is sufficiently large. If $(x_i)_{i\in I}, (y_i)_{i\in I} \in H$ then $\sum_{i=1}^d \xi_i(x_i + y_i) \le x_0^2 + y_0^2 \le (x_0+y_0)^2$, hence $H \subseteq \mathbb{Z}^I$ is a monoid which obviously is reduced, and we claim that

$$\mathcal{Q}(H) = \Big\{(x_i)_{i\in I} \in \mathbb{Z}^I \Big| \sup_{i\in I} |x_i| < \infty\Big\}. \tag{1}$$

Indeed, a sequence $(x_i)_{i\in I} \in \mathbb{Z}^I$ is bounded if and only if it is the difference of two bounded sequences in $\mathbb{N}_+^I$ with sufficiently large first component.

2. In the second step we will determine $\widetilde{H}$ and $\widehat{\widehat{H}}$. Put

$$H_1 = \Big\{(x_i)_{i\in I} \in \mathbb{N}_+^I \Big| \sup_{i\in I} |x_i| < \infty\Big\} \cup \{0\}$$

and

$$H_2 = \Big\{ (x_i)_{i \in I} \in \mathbb{N}^I \Big| \sup_{i \in I} |x_i| < \infty \Big\}.$$

Let $x = (x_i)_{i \in I} \in H_1$ and $m \in \mathbb{N}_+$ with $m \geq \sum_{i=1}^d \xi_i x_i$. Then we obtain $mx \in H$ and hence $\frac{1}{n} x \in \mathcal{C}(H)$ for all $n \in \mathbb{N}_+$. Thus we have

$$\bigcup_{n \geq 1} \frac{1}{n} H_1 \subseteq \mathcal{C}(H) \subseteq \mathbb{R}^I_{>0} \cup \{0\}, \tag{2}$$

where the second inclusion is obvious. Now Theorem 2.1 and (1) yield

$$\widetilde{H} = \mathcal{C}(H) \cap \mathcal{Q}(H) = H_1.$$

Now let $y = (y_i)_{i \in I} \in H_2$. Then $x = (\max\{y_i, 1\})_{i \in I} \in H_1 \subseteq \mathcal{C}(H)$ and for any $n \in \mathbb{N}_+$ $x^{(n)} = (\max\{y_i, \frac{1}{n}\})_{i \in I} \in \frac{1}{n} H_1 \subseteq \mathcal{C}(H)$, which yields $[x, y) = \bigcup_{n \geq 1} [x, x^{(n)}] \subseteq \mathcal{C}(H)$. Thus we have shown that

$$H_2 \subseteq \text{lin } \big(\mathcal{C}(H)\big) \subseteq \mathbb{R}^I_{\geq 0}.$$

Using Theorem 2.3 and (1) yields

$$\widehat{\widehat{H}} = \text{lin } \big(\mathcal{C}(H)\big) \cap \mathcal{Q}(H) = H_2.$$

3. Now we will show that H is a primary monoid. If d is finite we deduce from (2) that $\mathcal{C}(H) = \mathbb{R}^{d+1}_{>0} \cup \{0\}$, thus by Theorem 2.4 H is primary.

For $d = \infty$ we will give a direct proof which is easier than showing that $\mathcal{C}(H)$ is (algebraically) open. Let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in H \setminus \{0\}$. For sufficiently large $n \in \mathbb{N}$ we have $nx_i > y_i$ for all $i \in I$ and $\sum_{i=1}^d \xi_i (nx_i - y_i) \leq (nx_0 - y_0)^2$, which shows $nx - y \in H$.

4. In the final step we show that

$$\widehat{H} = \Big\{ (x_i)_{i \in I} \in H_2 \Big| x_0 > 0 \Big\} \cup \{0\}.$$

This yields $\widehat{H} \neq \widehat{\widehat{H}}$ and thus $\widehat{H}$ is not completely integrally closed. Since $\widehat{H} \subseteq \widehat{\widehat{H}} = H_2$ we must prove that for $0 \neq x \in H_2$ we have: $x \in \widehat{H}$ if and only if $x_0 \neq 0$.

First consider $x = (x_i)_{i \in I} \in H_2$ with $x_0 = 0$ and $x_j \neq 0$ for some $j \in I$. For any $c = (c_i)_{i \in I} \in H$ we have for sufficiently large $n \in \mathbb{N}_+$: $\sum_{i=1}^{d} \xi_i(c_i + nx_i) \geq \xi_j n x_j > c_0^2$, thus $c + nx \notin H$ and $x \notin \widehat{H}$.

Now let $x = (x_i)_{i \in I} \in H_2$ with $x_0 \geq 1$. Put $m = \max_{i \in I} |x_i|$, choose $c_0 \in \mathbb{N}_+$ with $c_0 \geq m \sum_{i=1}^{d} \xi_i$ and let $c = (c_0, 1, 1, \dots) \in H$. Then one easily checks that $c + nx \in H$ holds for all $n \in \mathbb{N}$ which proves $x \in \widehat{H}$. $\square$

Remark.

1. Considering the finite dimensional case $d \in \mathbb{N}_+$, a quick glance at the above proof exhibits

$$\mathcal{C}(H) = \mathbb{R}_{>0}^{d+1}, \quad \operatorname{lin}\left(\mathcal{C}(H)\right) = \mathbb{R}_{\geq 0}^{d+1}, \quad \widetilde{H} = \mathbb{N}_{+}^{d+1},$$

$$\widehat{H} = \mathbb{N}_+ \times \mathbb{N}^d \text{ and } \widehat{\widehat{H}} = \mathbb{N}^{d+1}\,.$$

2. For every submonoid $H \subseteq \mathbb{Z}$ with $\mathcal{Q}(H) = \mathbb{Z}$ we have $\widetilde{H} \in \{-\mathbb{N}, \mathbb{N}, \mathbb{Z}\}$. Thus $\widetilde{H}$ is completely integrally closed and the assumption $d + 1 \geq 2$ in the previous theorem cannot be improved.

5 – Integral domains

For an integral domain R let $R^\bullet = R \setminus \{0\}$ denote the multiplicative monoid of R, $R^\times = R^{\bullet\times}$ its group of units, $\mathcal{G}(R) = \mathcal{G}(R^\bullet)$ the group of divisibility and $\overline{R}$ the integral closure of R in its quotient field.

We shall make use of the following simple observation: let A, B be two domains with $A \subseteq B$ whose quotient fields coincide; then they give rise to the exact sequence of abelian groups

$$1 \to B^\times/A^\times \to \mathcal{G}(A) \to \mathcal{G}(B) \to 1 \tag{3}$$

Theorem 4. *Let R be an integral domain. Then the following conditions are equivalent:*

1. *$R^\bullet$ is primary and $\mathcal{G}(R)$ is finitely generated.*

2. *R is a one-dimensional local noetherian domain, its integral closure $\overline{R}$ is a finitely generated R-module and $(\overline{R}^\times : R^\times) < \infty$.*

Proof.

1. $\Longrightarrow$ 2. R is one-dimensional and local because $R^\bullet$ is primary ([4; Theorem 4.1]). Since $\mathcal{G}(R)$ is finitely generated, $\overline{R}$ is a finitely generated R-module and $\overline{R}^\times/R^\times$ is finite by [3; Theorem 3.9].

It remains to verify that R is noetherian. $\mathcal{G}(R)$ has finite torsion free rank and therefore, by [3; Theorem 2.1], $\overline{R}$ is the intersection of just finitely many valuation overrings, say $\overline{R} = \bigcap_{i=1}^n V_i$ with $V_i \nsubseteq V_j$ for $i \neq j$. Then, for $1 \leq i \leq n$, we have $V_i = \overline{R}_{P_i}$ for $P_i \triangleleft \overline{R}$ prime ([6; Theorem 12.2]). Since $\overline{R}$ is one-dimensional, all V_i have rank one; setting $A = R$ and $B = V_i$ in (3) we infer that $\mathcal{G}(V_i)$ is finitely generated and hence isomorphic to $\mathbb{Z}$; again from [6; Theorem 12.2] it follows that $\overline{R}$ is a principal ideal domain. So finally R is noetherian by the theorem of Eakin-Nagata.

2. $\Longrightarrow$ 1. $R^\bullet$ is primary by [4; Theorem 4.1]. $\overline{R}$ is a Dedekind domain with finitely many prime ideals, whence it is a principal ideal domain. Thus

$$\overline{R}^\bullet = F \times \overline{R}^\times$$

where F is a free abelian monoid with finite basis, and hence $\mathcal{G}(\overline{R}) = \mathcal{Q}(F)$ is a free abelian group of finite rank. Thus the exact sequence (3) (with $A = R$ and $B = \overline{R}$) splits and

$$\mathcal{G}(R) = \mathcal{G}(\overline{R}) \oplus \overline{R}^\times/R^\times$$

is finitely generated. $\square$

Remarks. Let R be an integral domain satisfying the equivalent conditions of the previous theorem.

1. Since R is noetherian we infer $\overline{R} = \widehat{R}$ and being a Dedekind domain $\widehat{R}$ is completely integrally closed.

2. The number of prime ideals of $\overline{R}$ equals the torsion free rank of $\mathcal{G}(R)$.

3. $R^\bullet$ is a finitely primary monoid (cf. [2; Theorem 2]).

In a final example we discuss a class of domains satisfying the conditions of Theorem 4.

Example. Let $\mathfrak{o}$ be an order in a Dedekind domain R (i. e. $\mathfrak{o} \subseteq R$ is a subring, the quotient fields of $\mathfrak{o}$ and R coincide, and R is a finitely generated $\mathfrak{o}$-module). Then for every $(0) \neq \mathfrak{p} \in \operatorname{spec}(\mathfrak{o})$ $\mathfrak{o}_\mathfrak{p}$ is a one-dimensional local noetherian domain, $\overline{\mathfrak{o}_\mathfrak{p}}$ is a finitely generated $\mathfrak{o}_\mathfrak{p}$-module and $(\overline{\mathfrak{o}_\mathfrak{p}}^\times : \mathfrak{o}_\mathfrak{p}^\times) \leq (R : F)$ where F is the conductor of $\mathfrak{o} \subseteq R$ (cf. [7; Kap. I, Satz 12.11]). Hence if R has the finite norm property, then $\mathfrak{o}_\mathfrak{p}$ satisfies condition 2 of Theorem 4.

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