Indefinite Extrinsic Spheres

著者	DRAGOMIR Sorin, DUGGAL Krishan L.
journal or publication title	Tsukuba journal of mathematics
volume	32
number	2
page range	335-348
year	2008-10
URL	http://hdl.handle.net/2241/00144079

TSUKUBA J. MATH.
Vol. 32 No. 2 (2008), 335–348

INDEFINITE EXTRINSIC SPHERES

By

Sorin Dragomir[1] and Krishan L. Duggal[2]

Abstract. We study the lightlike geometry of the second fundamental form of the intersection between an algebraic hypersurface in $\mathbf{C}_s^n$ and a pseudosphere $S_{2s}^{2n-1}(2/\sqrt{k})$ including a class of extrinsic spheres which are not homotopy spheres.

Let $\mathbf{C}_s^n$ denote $\mathbf{C}^n$ with the quadratic form $-|z_1|^2 - \cdots - |z_s|^2 + |z_{s+1}|^2 + \cdots + |z_n|^2$. Let $F \in \mathbf{C}[z]$ be a homogenous polynomial and $S = \{z \in \mathbf{C}_s^n : F_{z_j}(z) = 0, 1 \le j \le n\}$ where $F_{z_j} = \partial F/\partial z_j$. Then $\tilde{M} = \tilde{M}(F) = \{z \in \mathbf{C}_s^n : F(z) = 0\} \backslash S$ is a complex hypersurface. Let $S_{2s}^{2n-1}(r) = \{z \in \mathbf{C}_s^n : \sum_{j=1}^n \varepsilon_j |z_j|^2 = r^2\}$ be the pseudosphere of radius $r > 0$ (where $\varepsilon_j = -1$ for any $1 \le j \le s$ and $\varepsilon_{s+a} = 1$ for any $1 \le a \le n-s$). We set

$$M = M(F) = \tilde{M} \cap S_{2s}^{2n-1}(2/\sqrt{k})$$

($k > 0$). We wish to study the geometry of the second fundamental form of M in both $\tilde{M}$ and $\mathbf{C}_s^n$. In the positive definite case ($s = 0$) M is an example of an extrinsic sphere (cf. B-Y. Chen, [5]) which is not even homeomorphic to a sphere (cf. B-Y. Chen, [6]). Let $\Lambda_0 = \{z \in \mathbf{C}_s^n : \sum_{j=1}^n \varepsilon_j |z_j|^2 = 0\}$ be the null cone and $C = \{z \in \mathbf{C}_s^n : (F_{z_1}(z), \ldots, F_{z_n}(z)) \in \Lambda_0\}$. Our result is

Theorem A. *Assume that $\tilde{M}(F) \subset C$. Let $\tilde{\nabla}$ be the induced connection on $\tilde{M} = \tilde{M}(F)$ as a 2-lightlike submanifold of $\mathbf{C}_s^n$. Then* i) *$M = M(F)$ is an extrinsic sphere in $(\tilde{M}, \tilde{g}, \tilde{\nabla})$.* ii) *There is a free action of S^1 on M such that M/S^1 is a complex manifold and $S^1 \to M \to M/S^1$ a principal circle bundle.* iii) *If M/S^1 is 1-connected and* $\dim_{\mathbf{C}} M/S^1 > 2$ *then either $\pi_1(M) = \pi_2(M) = 0$ or*

[1] Università della Basilicata, Dipartimento di Matematica e Informatica, Via dell'Ateneo Lucano 10, Contrada Macchia Romana, 85100 Potenza, Italy, e-mail: sorin.dragomir@unibas.it
[2] University of Windsor, Department of Mathematics and Statistics, Windsor, Ontario N9B3P4, Canada, e-mail: yq8@uwindsor.ca
Received January 24, 2008.

$\pi_1(M) = \pi_2(M) = \mathbf{Z}$ so that in general M is not homeomorphic to a sphere. iv) *The immersion $M \hookrightarrow \mathbf{C}^n_s$ is totally umbilical of mean curvature $-(k/4)\xi$ if and only if $\tilde{M}$ is a complex hyperplane $a_1 z_1 + \cdots + a_n z_n = 0$ with $a = (a_1, \dots, a_n) \in \Lambda_0$. Here $\xi = z^j \partial/\partial z^j + \bar{z}^j \partial/\partial \bar{z}^j$. Moreover* v) *$M$ is an indefinite CR submanifold of $\mathbf{C}^n_s$ and hence a CR manifold of CR dimension $n-2$ whose extrinsic Levi form is given by $\tilde{L}_x(w) = -\varepsilon_j |w^j|^2 \xi_x$ for any $w = w^j(\partial/\partial z^j)_x \in T_{1,0}(M)_x$ and any $x \in M$.*

Here $\tilde{g}$ is the first fundamental form of $\tilde{M} \hookrightarrow \mathbf{C}^n_s$. Also $\pi_k(M)$ is the k-th homotopy group of M. An appealing question is whether CR functions on $M(F)$ extend (at least locally) holomorphically to $\mathbf{C}^n$. The convex hull of the image of $\tilde{L}_x$ has empty interior in the transversal space $\mathrm{tr}(M)_x$ thus killing a hope to generalize Theorem 1 in [4], p. 200–201, to the case of the extrinsic sphere $M(F)$. Throughout this paper we emphasize on the geometric features of $M(F)$ and relegate all analytic considerations to further work.

1. A Reminder of Lightlike Geometry

We adopt the notations and conventions in [8]. Let (M^{2n+2}, J, G) be an indefinite Kähler manifold, of complex dimension $n+1$ $(n \geq 1)$ where J denotes the complex structure and G the indefinite Riemannian metric of index $2s$ $(0 < s < n+1)$, cf. e.g. [1], p. 55. Then $G(JX, JY) = G(X, Y)$ for any $X, Y \in T(M^{2n+2})$ and $DJ = 0$ where D is the Levi-Civita connection of (M^{2n+2}, G). Let M be a real m-dimensional *lightlike* submanifold of (M^{2n+2}, G) i.e. G is degenerate on $T(M)$. For each point $x \in M$ we set

$$(\mathrm{Rad}\, TM)_x = \{\xi \in T_x(M) : g_x(\xi, X) = 0, X \in T_x(M)\}.$$

Here $g = j^* G$ is the induced metric on M and $j : M \hookrightarrow M^{2n+2}$ is the inclusion. A fundamental assumption in lightlike geometry is that $\mathrm{Rad}\, TM : x \in M \mapsto (\mathrm{Rad}\, TM)_x$ is a smooth distribution on M of rank $r \geq 1$. Let $T(M)^\perp \to M$ be the normal bundle of j. Clearly $(\mathrm{Rad}\, TM)_x \subseteq T(M)^\perp_x$ for any $x \in M$ hence $r \leq \min\{m, k\}$ where $k = 2(n+1) - m$ is the codimension of M in M^{2n+2}. $\mathrm{Rad}\, TM$ is referred to as the *radical distribution* of M. We shall also use the terminology ([8], p. 141–150) in Table 1 below. Cf. also R. Roşca, [9]. One of the main techniques used in this paper is that of *screen distributions*. Precisely we consider smooth distributions $S(TM)$ and $S(TM^\perp)$ on M such that

$$T(M) = S(TM) \oplus_{\mathrm{orth}} \mathrm{Rad}\, TM, \tag{1}$$

$$T(M)^\perp = S(TM^\perp) \oplus_{\mathrm{orth}} \mathrm{Rad}\, TM. \tag{2}$$

Table 1. Classification of lightlike submanifolds according to the rank of their radical distribution.

M	r
(I) r-lightlike submanifold	$1 \le r < \min\{m, k\}$
(II) co-isotropic	$1 \le r = k < m$
(III) isotropic	$1 \le r = m < k$
(IV) totally lightlike	$1 \le r = m = k$

We write $V \oplus_{\text{orth}} W$ whenever the sum $V + W$ is direct and the spaces V, W are orthogonal. *A posteriori* both $S(TM)$ and $S(TM^{\perp})$ are nondegenerate (cf. e.g. Proposition 2.1 in [8], p. 5). $S(TM)$ (respectively $S(TM^{\perp})$) is referred to as a *tangential* (respectively *normal*) *screen distribution*. Such a choice of screen distributions on M leads to the construction of a vector bundle $\text{tr}(TM) \to M$ which is complementary to $T(M)$ in $T(M^{2n+2})$. Although $\text{tr}(TM) \to M$ will prove to contain a lightlike vector bundle it may be used (as a lightlike analog to the normal bundle of a nondegnerate submanifold of M^{2n+2}) to build a theory similar to that of the second fundamental form in Riemannian geometry (cf. [8]).

Let M be a real m-dimensional r-lightlike submanifold of the semi-Riemannian manifold (M^{2n+2}, G). As $S(TM)$ is nondegenerate

$$T(M^{2n+2}) = S(TM) \oplus_{\text{orth}} S(TM)^{\perp}.$$

It is immediate that $S(TM^{\perp}) \subset S(TM)^{\perp}$. Indeed, if $X \in S(TM^{\perp}) \subset T(M)^{\perp}$ then X is orthogonal to $T(M) \supset S(TM)$ hence X is orthogonal to $S(TM)$ i.e. $X \in S(TM)^{\perp}$. In particular

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{\text{orth}} S(TM^{\perp})^{\perp}. \tag{3}$$

Note that $\text{Rad}\, TM \subset S(TM^{\perp})^{\perp}$. Indeed if $X \in \text{Rad}\, TM$ then $X \in T(M)$ and X is perpendicular to $T(M)^{\perp} \supset S(TM^{\perp})$ hence $X \in S(TM^{\perp})^{\perp}$. Next we need transversal vector bundles and the corresponding Gauss formula.

Let $\{\xi_1, \dots, \xi_r\} \subset \Gamma^{\infty}(U, \text{Rad}\, TM)$ be a local frame. Let $F \to M$ be a vector bundle such that

$$S(TM^{\perp})^{\perp} = (\text{Rad}\, TM) \oplus F$$

(so that F has rank r). Let $\{V_1, \dots, V_r\} \subset \Gamma^{\infty}(U, F)$ be a local frame. Let us set

$$g_{jk} = G(\xi_j, V_k), \quad 1 \le j, k \le r.$$

Then $\det[g_{jk}] \neq 0$ everywhere on U. Let $[g^{jk}] := [g_{jk}]^{-1}$ and let us set

$$N_i = -\frac{1}{2} g^{ki} g^{\ell j} G(V_k, V_\ell) \xi_j + g^{ji} V_j. \tag{4}$$

Then $G(N_i, \xi_j) = \delta_{ij}$ and $G(N_i, N_j) = 0$. In particular it follows that $\{\xi_1, \dots, \xi_r, N_1, \dots, N_r\}$ is a local frame of $S(TM^\perp)^\perp$ on U. Moreover we set

$$\mathrm{ltr}(TM)_x = \sum_{i=1}^{r} \mathbf{R} N_{i,x}, \quad x \in U. \tag{5}$$

By a result in [8] (cf. Theorem 1.2, p. 144) $\mathrm{ltr}(TM)_x$ is well defined i.e. its definition doesn't depend upon the local frames $\{\xi_j : 1 \le j \le r\}$ of Rad TM and $\{V_j : 1 \le j \le r\}$ of F at x. Also $\mathrm{ltr}(TM) = \bigcup_{x \in M} \mathrm{ltr}(TM)_x$ is a vector bundle over M and

$$S(TM^\perp)^\perp = (\mathrm{Rad}\ TM) \oplus \mathrm{ltr}(TM). \tag{6}$$

We call $\mathrm{ltr}(TM) \to M$ a *lightlike transversal* vector bundle with respect to the screen distributions $S(TM)$ and $S(TM^\perp)$. Note that the construction of a lightlike transversal vector bundle does depend upon the choice of $F \to M$. A *transversal* vector bundle $\mathrm{tr}(TM) \to M$ is given by

$$\mathrm{tr}(TM) = \mathrm{ltr}(TM) \oplus_{\mathrm{orth}} S(TM^\perp).$$

We emphasize that

$$\begin{aligned} T(M^{2n+2}) &= S(TM) \oplus S(TM)^\perp \\ &= S(TM) \oplus [S(TM^\perp) \oplus S(TM^\perp)^\perp] \\ &= S(TM) \oplus S(TM^\perp) \oplus (\mathrm{Rad}\ TM) \oplus \mathrm{ltr}(TM) \end{aligned}$$

hence

$$T(M^{2n+2}) = T(M) \oplus \mathrm{tr}(TM). \tag{7}$$

Let $\tan_x : T_x(M^{2n+2}) \to T_x(M)$ and $\mathrm{tra}_x : T_x(M^{2n+2}) \to \mathrm{tr}(TM)_x$ be the projections associated with the direct sum decomposition (7). We set

$$\nabla_X Y = \tan(D_X Y), \quad h(X, Y) = \mathrm{tra}(D_X Y),$$

for any $X, Y \in T(M)$. Then ∇ is a torsion-free linear connection on M and h is $C^\infty(M)$-bilinear symmetric (as D is torsion-free). In particular

$$D_X Y = \nabla_X Y + h(X, Y) \tag{8}$$

In general however ∇ is not a metric connection but rather

$$(\nabla_X g)(Y, Z) = G(h(X, Y), Z_{\mathrm{Rad}\, TM}) + G(h(X, Z), Y_{\mathrm{Rad}\, TM})$$

for any $X, Y, Z \in T(M)$. Here $X_{\mathrm{Rad}\, TM}$ denotes the Rad TM-component of $X \in T(M)$ with respect to the decomposition (1). Equation (8) is referred to as the *Gauss formula* while ∇ and h are respectively the *induced connection* and the *second fundamental form* (of the given immersion $M \hookrightarrow M^{2n+2}$) associated to the transversal bundle $\mathrm{tr}(TM) \to M$.

2. Indefinite CR Submanifolds

The complex structure J on M^{2n+2} induces a "tangential" complex structure on M (cf. e.g. (1.12) in [7], p. 5)

$$T_{1,0}(M)_x = T^{1,0}(M^{2n+2})_x \cap [T_x(M) \otimes_{\mathbf{R}} \mathbf{C}], \quad x \in M.$$

If the spaces $T_{1,0}(M)_x$ have the same dimension for any $x \in M$ then $(M, T_{1,0}(M))$ is a CR manifold and $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ is its Levi, or maximally complex, distribution (cf. [7], p. 4). When the ambient space is endowed with a Kähler metric G it is a meaningful problem to study the extrinsic geometry of M in (M^{2n+2}, G). To this end A. Bejancu, [2], examined the class of the *CR submanifolds* where one additionally requires the anti-invariance condition $JH(M)^\perp \subseteq T(M)^\perp$. Though smaller than the class of real submanifolds possessing a well defined (i.e. of constant rank) induced CR structure, A. Bejancu's CR submanifolds do include the generic, totally real, and invariant (i.e. complex) submanifolds as particular cases (and lead to an unifying treatment of the geometry of their second fundamental forms, cf. e.g. [12]). When G is an indefinite Kähler metric, a lightlike analog to A. Bejancu's class was proposed by B. Sahin et al., [10]. Let us adopt the following definition. The synthetic object $(M, S(TM), S(TM^\perp), \mathscr{D})$ is called an *indefinite CR submanifold* if 1) Rad TM is J-invariant (that is $J_x(\mathrm{Rad}\, TM)_x = (\mathrm{Rad}\, TM)_x$ for any $x \in M$) and $\mathscr{D} : x \in M \mapsto \mathscr{D}_x$ is a C^∞ distribution on M such that 2) $\mathscr{D}_x \subseteq S(TM)_x$ and the distribution $\mathscr{D} \oplus \mathrm{Rad}\, TM$ is J-invariant, 3) the perp distribution $\mathscr{D}^\perp \subseteq S(TM)$ satisfies $J_x \mathscr{D}_x^\perp \subseteq S(TM^\perp)_x$ and 4) $S(TM)_x = \mathscr{D}_x \oplus \mathscr{D}_x^\perp$, for any $x \in M$.

This slightly generalizes the concept in [10], p. 141, where one requests that $\mathscr{D}$ (rather than $\mathscr{D} \oplus \mathrm{Rad}\, TM$) be J-invariant and the resulting notion is referred to as a *screen CR submanifold*. Clearly both $\mathscr{D}$ and $\mathscr{D}^\perp$ are nondegenerate. Note that no lightlike real hypersurface may be organized as an indefinite CR submanifold. Indeed if M is an indefinite CR submanifold and $m = 2n + 1$ (and $k = 1$) then $r = 1$, a contradiction (r must be even). Also one checks easily

Proposition 1. *Let M be an indefinite CR manifold. If M is co-isotropic, isotropic or totally lightlike then M is a complex submanifold of M^{2n+2}.*

Cf. also Proposition 3.2 in [10], p. 144. In the positive definite case ($s = 0$) the distribution Rad TM is trivial and $(M, \mathscr{D})$ is an ordinary CR submanifold of M^{2n+2} (in the sense of [2]). On the other hand, by a result of D. E. Blair & B-Y. Chen, [3], any proper (i.e. $\mathscr{D} \neq (0)$ and $\mathscr{D}^{\perp} \neq (0)$) CR submanifold is a CR manifold (in the sense of [7], p. 4). It will be shortly seen that an indefinite CR submanifold of an indefinite Kähler manifold is a CR manifold, as well.

Let $(M, \mathscr{D})$ be an indefinite CR submanifold of $\mathbf{C}_s^{n+1}$. Let $x \in M$ and let $\mathrm{tra}_x : T(\mathbf{C}_s^{n+1}) \to \mathrm{tr}(TM)_x$ be the projection associated to the decomposition (7) (with $M^{2n+2} = \mathbf{C}_s^{n+1}$). Exploiting the analogy between the transversal bundle of a lightlike submanifold and the normal bundle of a nondegenerate submanifold we introduce the following notion. The *extrinsic Levi form* of $(M, \mathscr{D})$ is given by

$$\tilde{L}_x(w) = \frac{i}{2}\,\mathrm{tra}_x(J[W, \overline{W}])_x, \quad w \in T_{1,0}(M)_x,$$

where W is a C^{∞} section in $T_{1,0}(M) = \{X - iJX : X \in \mathscr{D} \oplus \mathrm{Rad}\, TM\}$ such that $W_x = w$. By (7) and by the formal integrability property of $\mathscr{D} \oplus \mathrm{Rad}\, TM$ (cf. Lemma 5 in Section 4) the definition of $\tilde{L}_x(w)$ doesn't depend upon the choice of the section W extending w. An explicit description of $\tilde{L}_x(w)$ (in terms of defining functions) is given in Lemma 6.

3. The Geometry of $M(F)$

Let $F \in \mathbf{C}[z]$ be a homogeneous polynomial of degree d. A real vector field $X = Z^j \partial/\partial z^j + \overline{Z}^j \partial/\partial \bar{z}^j$ is tangent to $\tilde{M}$ if and only if $Z^j F_{z^j} = 0$. Let us set

$$V = \varepsilon^j \left(\overline{F}_{\bar{z}_j} \frac{\partial}{\partial z^j} + F_{z^j} \frac{\partial}{\partial \bar{z}_j} \right), \quad W = JV.$$

Throughout $\varepsilon^j = \varepsilon_j$ and $z^j = z_j$. Then $V, W \in T(\tilde{M})^{\perp}$. On the other hand $\{V, W\}$ are linearly independent at each point of $\mathbf{C}_s^n \backslash S$ hence $\{V, W\}$ is a global frame of $T(\tilde{M})^{\perp}$. Note that $X = AV + BW \in \mathrm{Rad}\, T\tilde{M}$ if and only if $(A + iB)\varepsilon^j |F_{z^j}|^2 = 0$ along $\tilde{M}$, hence

$$(\mathrm{Rad}\, T\tilde{M})|_{\tilde{M}\backslash C} = (0), \quad (\mathrm{Rad}\, T\tilde{M})|_{\tilde{M}\cap C} = T(\tilde{M})^{\perp}|_{\tilde{M}\cap C},$$

so that $\tilde{M}$ is a 2-lightlike complex submanifold provided that $\tilde{M}\backslash C = \emptyset$. Moreover $X \in T(M)$ if and only if $Z^j F_{z^j} = 0$ and $\varepsilon_j(Z^j \bar{z}_j + \overline{Z}^j z_j) = 0$ along M. Let us set $\xi = z^j \partial/\partial z^j + \bar{z}^j \partial/\partial \bar{z}^j$. Then

$$G(\xi, X) = \varepsilon_j(z_j \bar{Z}^j + \bar{z}_j Z^j) = 0$$

for any $X \in T(S^{2n-1}_{2s}(2/\sqrt{k}))$ hence $\xi \in T(S^{2n-1}_{2s}(2/\sqrt{k}))^\perp$. Also (by complex homogeneity) $\xi(F) = z^j F_{z^j} = dF = 0$ along $\tilde{M}$ hence $\xi \in T(\tilde{M})$. Note that $\{\xi, V, W\}$ are linearly independent at each point of M. Indeed $\alpha\xi + AV + BW = 0$ implies $\alpha z^j + (A + iB)\varepsilon_j \bar{F}_{\bar{z}_j} = 0$ hence (by contraction with $\varepsilon_j \bar{z}_j$) $0 = \alpha\varepsilon_j |z^j|^2 + (A + iB)\bar{z}^j \bar{F}_{\bar{z}^j} = 4\alpha/k$ i.e. $\alpha(z) = 0$ for any $z \in M$, etc. Hence $\{\xi, V, W\}$ is a global frame of the normal bundle $T(M)^\perp$ of the immersion $M \hookrightarrow \mathbf{C}^n_s$. A calculation shows that

Lemma 1. *The radical distribution of $M = M(F)$ is* Rad $TM = (\text{Rad } T\tilde{M})|_M$ *and $S(TM^\perp) := \mathbf{R}\xi|_M$ is a normal screen distribution. If $M \backslash C = \varnothing$ then* Rad TM *is smooth so that M is a* 2*-lightlike submanifold of $\mathbf{C}^n_s$.*

It should be observed that the assumptions in Theorem A do not allow for quadrics $\tilde{M} = Q_{n-1}$. Indeed if $F = z_1^d + \cdots + z_n^d$ then $C = \{z \in \mathbf{C}^n : \varepsilon_j |z^j|^{2(d-1)} = 0\}$. Next if $d = 2$ then $C = \Lambda_0$ and $M = Q_{n-1} \cap S^{2n-1}_{2s}(2/\sqrt{k})$ doesn't intersect the null cone.

We assume from now on that $M \backslash C = \varnothing$. Then

Lemma 2. *$J\xi = i(z^j \partial/\partial z^j - \bar{z}_j \partial/\partial \bar{z}_j)$ is tangent to M. If $E \to M$ is a vector bundle such that $T(M) = E \oplus (\text{Rad } TM) \oplus \mathbf{R}J\xi|_M$ then $S(TM) := E \oplus \mathbf{R}J\xi|_M$ is a tangential screen distribution for the immersion $M \hookrightarrow \mathbf{C}^n_s$.*

An extrinsic sphere is a totally umbilical submanifold of a Riemannian manifold whose mean curvature vector is everywhere nonzero and parallel in the normal bundle (cf. e.g. [5]). The notion admits the following generalization to the semi-Riemannian context. Let M be a sumanifold of the semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Let $\tilde{\nabla}$ be a linear connection on $\tilde{M}$. Let us assume that M is r-lightlike with $r \geq 0$ (if $r = 0$ then M is a semi-Riemannian submanifold). We say M is $\tilde{\nabla}$-*totally umbilical* in $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ if there is a transversal vector bundle $\text{tr}(TM) \to M$ (if $r = 0$ then $\text{tr}(TM) \to M$ is the normal bundle of the immersion $M \hookrightarrow \tilde{M}$) such that $\text{tra}(\tilde{\nabla}_X Y) = g(X, Y)H$ for some $H \in \text{tr}(TM)$ and any $X, Y \in T(M)$. If additionally H (the $\tilde{\nabla}$-*mean curvature*) can be chosen such that $H_x \neq 0$ for any $x \in M$ and $\text{tra}(\tilde{\nabla}_X H) = 0$ for any $X \in T(M)$ then M is a $\tilde{\nabla}$-*extrinsic sphere* in $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ (if $r = 0$ then $\text{tra} : T(\tilde{M}) \to \text{tr}(TM)$ is the orthogonal projection).

PROOF OF THEOREM A. Let $\iota : M \hookrightarrow \tilde{M}$ be the inclusion and $N(\iota)$, respectively $\mathrm{Rad}(d\iota)$, the normal bundle and radical distribution of ι. As ξ is tangent to $\tilde{M}$ and orthogonal to M it follows that $N(\iota) = \mathbf{R}\xi|_M$. Next

$$\mathrm{Rad}(d\iota) = T(M) \cap \mathbf{R}\xi|_M = (0)$$

because ξ is space-like. Hence $T(M)$ is nondegenerate in $T(\tilde{M}, \tilde{g})$ and

$$T_x(\tilde{M}) = T_x(M) \oplus_{\mathrm{orth}} \mathbf{R}\xi_x, \quad x \in M. \tag{9}$$

Under the assumptions in Theorem A one has $T(\tilde{M})^\perp = \mathrm{Rad}\, T\tilde{M}$ hence we may take $S(T\tilde{M}^\perp) = (0)$ and $\mathrm{tr}(T\tilde{M}) = \mathrm{ltr}(T\tilde{M})$ while the lightlike transversal vector bundle $\mathrm{ltr}(T\tilde{M}) \to \tilde{M}$ is built as follows. Let $S(T\tilde{M})$ be a tangential screen distribution for the immersion $\tilde{M} \hookrightarrow \mathbf{C}^n_s$ such that $\xi \in S(T\tilde{M})$. Then we may decompose as $T(\mathbf{C}^n_s) = S(T\tilde{M}) \oplus S(T\tilde{M})^\perp$ and $\mathrm{Rad}\, T\tilde{M} \subset S(T\tilde{M})^\perp$. Therefore we may choose a complement $\tilde{F}$ to $\mathrm{Rad}\, T\tilde{M}$ in $S(T\tilde{M})^\perp$ and build $\{\tilde{N}_1, \tilde{N}_2\}$ such that $\{\tilde{\xi}_i, \tilde{N}_j : i, j \in \{1, 2\}\}$ is a local frame of $S(T\tilde{M})^\perp$ and

$$G(\tilde{N}_i, \tilde{\xi}_j) = \delta_{ij}, \quad G(\tilde{N}_i, \tilde{N}_j) = 0,$$

where $\tilde{\xi}_1 = V|_{\tilde{M}}$ and $\tilde{\xi}_2 = W|_{\tilde{M}}$. Finally one sets $\mathrm{ltr}(T\tilde{M}) = \mathbf{R}\tilde{N}_1 \oplus \mathbf{R}\tilde{N}_2$. We shall need the Gauss formula (for the immersion $\tilde{M} \hookrightarrow \mathbf{C}^n_s$)

$$D_X Y = \tilde{\nabla}_X Y + \tilde{h}(X, Y), \quad X, Y \in T(\tilde{M}).$$

In general $\tilde{\nabla}\tilde{g} \neq 0$ (so that $\tilde{\nabla}$ is not the Levi-Civita connection of $(\tilde{M}, \tilde{g})$). Let us observe that

$$D_X \xi = X, \quad X \in T(S^{2n-1}_{2s}(2/\sqrt{k})). \tag{10}$$

As $\xi \in S(T\tilde{M})$ it follows that $S(T\tilde{M})^\perp$ is orthogonal to ξ hence $G(\tilde{N}_i, \xi) = 0$. Hence (by $DG = 0$ and by (10))

$$G(\tilde{\nabla}_X Y, \xi) = G(D_X Y, \xi) = X(G(Y, \xi)) - G(Y, D_X \xi) = -G(Y, X)$$

for any $X, Y \in T(M)$ so that the normal component (with respect to (9)) of $\tilde{\nabla}_X Y$ is $-(k/4)g(X, Y)\xi$ and we may set $H = -(k/4)\xi$. Moreover

$$\mathrm{tra}(\tilde{\nabla}_X H) = -\frac{k^2}{16} G(D_X \xi, \xi) = 0.$$

The first statement in Theorem A is proved. To prove the next assertion we need the indefinite Hopf S^1-fibration $\Pi : S^{2n-1}_{2s}(2/\sqrt{k}) \to \mathbf{C}P^{n-1}_s(k)$ (cf. e.g. [1]). If we set $\Lambda_+ = \{z \in \mathbf{C}^n_s : \sum_{j=1}^n \varepsilon_j |z_j|^2 \geq 0\}$ then the base complex manifold is the open subset of the complex projective space $\mathbf{C}P^{n-1}$ given by $\mathbf{C}P^{n-1}_s(k) =$

$(\Lambda_+ \backslash \Lambda_0)/(\mathbf{C}\backslash\{0\})$. As F is homogeneous the S^1-action descends to an action on $\tilde{M}$, and then on M, such that $M/S^1 = \Pi(M)$. As $S^1 \to M \to \Pi(M)$ is a principal circle bundle the proof of Proposition 2 in [6], p. 205, implies $\pi_k(M) \approx \pi_k(M/S^1)$ for and $k \geq 3$, $\pi_2(M/S^1) = \mathbf{Z}$, and the statement in Theorem A about the homotopy groups of M.

Next we need to compute the second fundamental form h of the immersion $M \hookrightarrow \mathbf{C}^n_s$.

Lemma 3. *For any* $X, Y \in T(M)$ *if* $X = Z^j \partial/\partial z^j + \bar{Z}^j \partial/\partial/\partial \bar{z}^j$ *and* $Y = W^j \partial/\partial z^j + \overline{W}^j \partial/\partial \bar{z}^j$ *then*

$$h(X, Y) = -\alpha^i(X, Y) N_i - \frac{k}{4} g(X, Y)\xi$$

where

$$\alpha^1(X, Y) = F_{z^j z^k} Z^j W^k + \bar{F}_{\bar{z}^j \bar{z}^k} \bar{Z}^j \overline{W}^k, \tag{11}$$

$$\alpha^2(X, Y) = i(F_{z^j z^k} Z^j W^k - \bar{F}_{\bar{z}^j \bar{z}^k} \bar{Z}^j \overline{W}^k), \tag{12}$$

and $\{N_1, N_2\}$ *are given by* (4) *with* $\xi_1 = V|_M$ *and* $\xi_2 = W|_M$.

Proof. As $G(\xi, N_i) = G(\xi, \xi_i) = 0$ then (by (10))

$$G(h(X, Y), \xi) = G(\mathrm{tra}(D_X Y), \xi) = -G(Y, X).$$

Similarly $\alpha^i(X, Y) = -G(D_X \xi_i, Y)$. Moreover $D_{\partial/\partial z^j} V = \varepsilon^k F_{z_k z_j} \partial/\partial \bar{z}^k$ yields (11)–(12). Q.e.d.

If $F(z) = a_1 z_1 + \cdots + a_n z_n$ and $a \in \Lambda_0$ then $C = \mathbf{C}^n$ and (by Lemma 3) $h = -(k/4) g \otimes \xi$. Viceversa (11)–(12) yield $F_{z^j z^k}(z) = 0$ along M hence (by the homogeneity of F) along $\tilde{M}$. Then (by the proof of Theorem 4 in [6], p. 206–207) F must be linear (cf. also [11]).

Let us fix a bundle $E \to M$ as in Lemma 2 and the consider the corresponding tangential screen distribution $S(TM)$. Moreover let $\mathscr{D}(F)$ be the orthogonal complement of $\mathbf{R}J\xi$ in $S(TM)$. We need

Lemma 4 (Cf. [10]). *Let* M *be a* m*-dimensional* 2ρ*-lightlike submanifold of the complex* $(n+1)$*-dimensional indefinite Kähler manifold* (M^{2n+2}, J, G)*. Let us assume that* 1) Rad TM *is* J*-invariant*, 2) $r = 2\rho < m$, *and* 3) *the codimension* k *of* M *in* M^{2n+2} *is* $k = 2\rho + 1$*. Then* M *may be organized as an indefinite CR submanifold.*

Any orientable real hypersurface in a Hermitian manifold is a CR submanifold (in the sense of [2]) in a natural way. The lightlike analog to this situation is provided by Lemma 4. No proof is provided in [10] and the proof of Proposition 3.4 in [8], p. 203, doesn't apply (as claimed in [10], p. 143). Therefore we give a complete proof of Lemma 4 as follows. Let us choose any normal screen distribution $S(TM^{\perp})$ so that the decomposition (2) holds. The assumption $k = r + 1$ implies that $\dim_{\mathbf{R}} S(TM^{\perp})_x = 1$ for any $x \in M$. We claim that

$$JS(TM^{\perp}) \subseteq T(M). \tag{13}$$

Indeed let $x \in M$ and let us set $\langle\, , \rangle = G_x$. As $S(TM^{\perp})_x$ is 1-dimensional we may choose $v \in S(TM^{\perp})_x$ such that $v \neq 0$. Now on one hand $\langle v, J_x v\rangle = 0$ (as G and J are compatible) so that

$$J_x v \perp S(TM^{\perp})_x. \tag{14}$$

On the other hand if $\xi \in (\mathrm{Rad}\, TM)_x$ then (by the J-invariance of Rad TM) one has $\langle \xi, J_x v\rangle = -\langle J_x \xi, v\rangle = 0$ hence

$$J_x v \perp (\mathrm{Rad}\, TM)_x. \tag{15}$$

Then (2) and (14)–(15) imply that $J_x v \in (T_x(M)^{\perp})^{\perp} = T_x(M)$ and (13) is proved. Next we claim that the sum $(\mathrm{Rad}\, TM) + JS(TM^{\perp})$ is direct. Indeed if $X \in (\mathrm{Rad}\, TM) \cap JS(TM^{\perp})$ then $X = JV$ for some $V \in S(TM^{\perp})$ and $X \perp T(M) \supset JS(TM^{\perp})$ implies that

$$0 = G(X, JV) = G(JV, JV) = G(V, V)$$

hence $V = 0$ because $S(TM^{\perp})$ is a nondegenerate distribution of rank 1. Thus $X = 0$ i.e.

$$(\mathrm{Rad}\, TM) \cap JS(TM^{\perp}) = (0). \tag{16}$$

At this point we choose a tangential screen distribution $S(TM)$ such that $S(TM) \supset JS(TM^{\perp})$. For instance, let us choose a complement E to $(\mathrm{Rad}\, TM) \oplus JS(TM^{\perp})$ in $T(M)$ i.e.

$$T(M) = E \oplus (\mathrm{Rad}\, TM) \oplus JS(TM^{\perp}),$$

and let us set by definition

$$S(TM) := E \oplus JS(TM^{\perp}).$$

Finally let $\mathscr{D}$ be the perp of $JS(TM^{\perp})$ in $S(TM)$. As $JS(TM^{\perp})$ is nondegenerate

$$S(TM) = \mathscr{D} \oplus JS(TM^{\perp})$$

and $\mathscr{D}^{\perp} = JS(TM^{\perp})$. To complete the proof of Lemma 4 we need to check that $\mathscr{D} \oplus \mathrm{Rad}\, TM$ is J-invariant. As $\mathrm{Rad}\, TM$ is J-invariant it suffices to show that $J\mathscr{D} \subset \mathscr{D} \oplus \mathrm{Rad}\, TM$. To this end let $X \in \mathscr{D}$ and $W \in T(M)^{\perp}$. Then (by (2)) $W = Y + \xi$ for some $Y \in S(TM^{\perp})$ and $\xi \in \mathrm{Rad}\, TM$. We have

$$G(JX, W) = -G(X, JY) - G(X, J\xi) = 0.$$

Indeed $G(X, JY) = 0$ because X belongs to $\mathscr{D}$ while JY belongs to $JS(TM^{\perp})$ and these are orthogonal spaces. Also $G(X, J\xi) = 0$ because of the invariance of the radical distribution. Summing up $JX \in (T(M)^{\perp})^{\perp} = T(M)$ i.e. JX is tangential. Also if $V \in S(TM^{\perp})$ then $G(JX, JV) = G(X, V) = 0$ as X is tangential and V is normal. Hence

$$JX \perp JS(TM^{\perp}) \tag{17}$$

and we may conclude that

$$JX \in \mathscr{D} \oplus \mathrm{Rad}\, TM. \tag{18}$$

To prove (18) we let $JX = Y + JV + \xi$ for some $Y \in \mathscr{D}$, $V \in S(TM^{\perp})$ and $\xi \in \mathrm{Rad}\, TM$ then taking the inner product with JV gives (by (17)) $G(V, V) = 0$ i.e. $V = 0$. Q.e.d.

Let us go back to the proof of Theorem A. By Lemma 4 the pair $(M, \mathscr{D}(F))$ is an indefinite CR manifold hence a CR manifold of hypersurface type (of CR dimension $n-2$). Indeed any indefinite CR submanifold admits a natural CR structure described by the following

LEMMA 5. *Let $(M, S(TM), S(TM^{\perp}), \mathscr{D})$ be a m-dimensional indefinite CR submanifold of the complex $(n+1)$-dimensional indefinite Kähler manifold (M^{2n+2}, J, G) such that M is a r-lightlike submanifold of (M^{2n+2}, G), $0 < r < \min\{m, 2n+2-m\}$. Let $H(M) := \mathscr{D} \oplus \mathrm{Rad}\, TM$ and $T_{1,0}(M) := \{X - iJX : X \in H(M)\}$. Then $(M, T_{1,0}(M))$ is a CR manifold of type $(p+\rho, q)$ where $r = 2\rho$ and*

$$2p = \dim_{\mathbf{R}} \mathscr{D}_x, \quad q = \dim_{\mathbf{R}} \mathscr{D}_x^{\perp}, \quad x \in M,$$

and $H(M)$ its Levi distribution. Also the CR structure $T_{1,0}(M)$ and the induced CR structure from M^{2n+2} coincide. In particular the inclusion $j : M \to M^{2n+2}$ is a CR immersion i.e. $(d_x j)T_{1,0}(M)_x \subset T^{1,0}(M^{2n+2})_x$ for any $x \in M$. The CR manifold $(M, T_{1,0}(M))$ is generically embedded in M^{2n+2} if $n+1 = m - p - \rho$.

PROOF OF LEMMA 5. Let us consider the distribution $H(M) = \mathscr{D} \oplus \mathrm{Rad}\ TM$. Then

$$T(M) = S(TM) \oplus \mathrm{Rad}\ TM = \mathscr{D} \oplus \mathscr{D}^{\perp} \oplus \mathrm{Rad}\ TM$$

that is

$$T(M) = H(M) \oplus \mathscr{D}^{\perp}. \tag{19}$$

Let $X, Y \in H(M)$ and $Z \in \mathscr{D}^{\perp}$. Then (as D is symmetric and $DJ = 0$)

$$\begin{aligned} G([JX, JY], Z) &= G(D_{JX}JY - D_{JY}JX, Z) \\ &= -G(D_{JX}Y - D_{JY}X, JZ) = G(h(JY, X) - h(JX, Y), JZ). \end{aligned}$$

The last equality holds due to $JZ \in J\mathscr{D}^{\perp} \subseteq S(TM^{\perp}) \subset T(M)^{\perp}$. Next (as h is symmetric)

$$\begin{aligned} G([JX, JY], Z) &= G(h(X, JY) - h(Y, JX), JZ) \\ &= G(D_X JY - D_Y JX, JZ) = G(D_X Y - D_Y X, Z) = G([X, Y], Z) \end{aligned}$$

hence $[JX, JY] - [X, Y]$ is orthogonal to $\mathscr{D}^{\perp}$ so that

$$[JX, JY] - [X, Y] \in H(M), \quad X, Y \in H(M). \tag{20}$$

Next

$$[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i\{[X, JY] + [JX, Y]\}$$

(by the integrability of J)

$$= [X, Y] - [JX, JY] - i\{[X, Y] - [JX, JY]\}$$

hence (by (20)) $[X - iJX, Y - iJY] \in T_{1,0}(M)$ for any $X, Y \in H(M)$.

To prove that $T_{1,0}(M) = T^{1,0}(M^{2n+2}) \cap [T(M) \otimes \mathbf{C}]$ we only need to check the inclusion $\supseteq$. Let $\tilde{X} - iJ\tilde{X} \in T^{1,0}(M^{2n+2})$ be tangent to M. Then $\tilde{X} = X + Y$ for some $X \in H(M)$ and $Y \in \mathscr{D}^{\perp}$. Then $X - iJX + Y - iJY \in T(M) \otimes \mathbf{C}$ yet $JY \in J\mathscr{D}^{\perp} \subseteq S(TM^{\perp}) \subset T(M)^{\perp}$ hence $JY \in \mathrm{Rad}\ TM$. Thus (by the J-invariance of $\mathrm{Rad}\ TM$) $Y \in (\mathrm{Rad}\ TM) \cap \mathscr{D}^{\perp} \subset H(M) \cap \mathscr{D}^{\perp} = (0)$. Q.e.d.

To end the proof of Theorem A we apply

LEMMA 6. *Let M be an indefinite CR submanifold of $\mathbf{C}_s^n$ and $x \in M$. Let us assume that $M = \{\zeta \in \mathbf{C}_s^{n+1} : \rho_1(\zeta) = 0, \dots, \rho_k(\zeta) = 0\}$ and that $\{D\rho_1(x), \dots, D\rho_r(x)\}$ is a linear basis in $(\mathrm{Rad}\ TM)_x$ and $\{D\rho_{r+1}(x), \dots, D\rho_k(x)\}$ is a g_x-orthonormal basis in $S(TM^{\perp})_x$. Let $\{\xi_1, \dots, \xi_r\}$ be a local frame of* Rad *TM* *such that* $\xi_{i,x} = (D\rho_i)(x)$. *Let* $\mathrm{ltr}(TM) \to M$ *be a lightlike transversal vector bundle and* $\{N_1, \dots, N_r\}$ *a local frame of* $\mathrm{ltr}(TM) \to M$ *defined on a neighborhood of* x *such that* $G(\xi_i, N_j) = \delta_{ij}$ *and* $G(N_i, N_j) = 0$. *Then the extrinsic Levi form at* x *is given by*

$$\tilde{L}_x(w) = -\sum_{i=1}^{r}\left(\frac{\partial^2\rho_i}{\partial\zeta^A\partial\bar{\zeta}^B}(x)w^A\overline{w}^B\right)N_{i,x} - \sum_{\alpha=1}^{k-r}\left(\frac{\partial^2\rho_{r+\alpha}}{\partial\zeta^A\partial\bar{\zeta}^B}(x)w^A\overline{w}^B\right)D\rho_{r+\alpha}(x), \tag{21}$$

for any $w = w^A(\partial/\partial\zeta^A)_x \in T_{1,0}(M)_x$.

Note that M is given by $\rho_i(z) = 0$, $i \in \{1,2,3\}$ where $\rho_1(z) = F(z) + \overline{F(z)}$, $\rho_2(z) = i(\overline{F(z)} - F(z))$, and $\rho_3(z) = \varepsilon_j|z^j|^2 - 4/k$. We set

$$\tilde{\rho}_i(z) = \rho_i(z), \quad \tilde{\rho}_3(z) = \frac{\sqrt{k}}{2}\rho_3(z), \quad i \in \{1,2\},$$

so that $D\tilde{\rho}_3$ is an orthonormal vector field spanning $S(TM^\perp)$ while $\{D\tilde{\rho}_1, D\tilde{\rho}_2\}$ is a frame of Rad TM. Finally (by Lemma 6) $\tilde{L}_x(w) = -\varepsilon_A\delta_{AB}w^A\overline{w}^B\xi_x$ for any $w \in T_{1,0}(M)_x$, $x \in M$. The proof of Theorem A is complete.

The gradient $D\rho_\ell$ (in Lemma 6) is meant with respect to the flat indefinite Kähler metric G on $\mathbf{C}^n_s$. One may compare $\tilde{L}_x$ as given by (21) and the expression of the extrinsic Levi form in Theorem 1 of [4], p. 160. It is likely that the use of $\mathrm{tr}(TM) \to M$ (rather than $T(M)^\perp$) may lead to new CR extension results for CR functions on indefinite CR submanifolds. Indeed one may consider the convex hull $\Gamma_x \subseteq \mathrm{tr}(TM)_x$ of the image of $\tilde{L}_x : T_{1,0}(M)_x \to \mathrm{tr}(TM)_x$. It is known (cf. e.g. [4], p. 200) that in the positive definite case Γ_x determines the geometry of the open set to which CR functions extend holomorphically.

Proof of Lemma 6. Let $J^* : T^*(\mathbf{C}^n_s) \to T^*(\mathbf{C}^n_s)$ be the dual complex structure i.e. $(J^*\alpha)(v) = \alpha(Jv)$. Then

$$\begin{aligned} 2i\tilde{L}_x(w) &= -\sum_{i=1}^{r} G(\xi_i, J[W, \overline{W}])_x N_{i,x} \\ &\quad - \sum_{\alpha=1}^{k-r} G(D\rho_{r+\alpha}, J[W, \overline{W}])_x D\rho_{r+\alpha}(x) \\ &= -\sum_i (J^*d\rho_i)([W, \overline{W}])_x N_{i,x} - \sum_\alpha (J^*d\rho_{r+\alpha})([W, \overline{W}])_x D\rho_{r+\alpha}(x). \end{aligned}$$

As $d = \partial + \bar{\partial}$ and $J^* \circ \partial = i\partial$ (respectively $J^* \circ \bar{\partial} = -i\bar{\partial}$) the coefficients in the linear combination of $\{N_{i,x}, D\rho_{r+\alpha}(x) : 1 \le i \le r, 1 \le \alpha \le k - r\}$ above are

$$i((\bar{\partial} - \partial)\rho_\ell)[W, \overline{W}] = -2i(d(\bar{\partial} - \partial)\rho_\ell)(W, \overline{W})$$

at the point x hence (again by $d = \partial + \bar{\partial}$)

$$-\frac{1}{2}\tilde{L}_x(w) = \sum_i (\partial\bar{\partial}\rho_i)(W, \overline{W})_x N_{i,x} + \sum_\alpha (\partial\bar{\partial}\rho_{r+\alpha})(W, \overline{W}) D\rho_{r+\alpha}(x)$$

leading to (21). Q.e.d.

References

[1] M. Barros & A. Romero, Indefinite Kähler manifolds, Math. Ann., **261** (1982), 55–62.
[2] A. Bejancu, CR submanifolds of a Kaehler manifold, I–II, Proc. A.M.S., **250** (1979), 335–345.
[3] D. E. Blair & B-Y. Chen, On CR submanifolds of Hermitian manifolds, Israel J. Math., **34** (1979), 353–363.
[4] A. Boggess, CR manifolds and the tangential Cauchy-Riemann complex, Studies in Advanced Math., CRC Press, Inc., Boca Raton-Ann Arbor-Boston-London, 1991.
[5] B-Y. Chen, Extrinsic spheres in Kähler manifolds, Michigan Math. J., **23** (1976), 327–330.
[6] B-Y. Chen, Odd-dimensional extrisic spheres in Kähler manifolds, Rendiconti di Matematica, (1979), 201–207.
[7] S. Dragomir & G. Tomassini, Differential Geometry and Analysis in CR Manifolds, Progress in Mathematics, Vol. 246, Birkhäuser, Boston-Basel-Berlin, 2006.
[8] K. L. Duggal & A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic, vol. 364, 1996.
[9] R. Roşca, Varietăţi izotrope şi pseudoizotrope incluse într-o varietate relativistă, Editura Academiei Republicii Socialiste România, Bucureşti, 1972.
[10] B. Sahin & K. L. Duggal, Screen Cauchy-Riemann lightlike submanifolds, Acta Math. Hungar., (1–2) **106** (2005), 137–165; Generalized Cauchy-Riemann lightlike submanifolds of Kähler manifolds, ibidem, (1–2) **112** (2006), 113–136.
[11] A. Vitter, On the curvature of complex hypersurfaces, Indiana Univ. Math. J., **23** (1974), 813–826.
[12] K. Yano & M. Kon, CR submanifolds of Kaehlerian and Sasakian manifolds, Progress in Math., vol. 30, Ed. by J. Coates & S. Helgason, Birkhäuser, Boston-Basel-Stuttgart, 1983.

Corresponding Author:
Sorin DRAGOMIR
Università degli Studi della Basilicata
Dipartimento di Matematica e Informatica
Via dell'Ateneo Luano 10
85100 Potenza, ITALY