Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(7): 16208–16227.
DOI: 10.3934/math.2023829
Received: 09 February 2023
Revised: 06 April 2023
Accepted: 17 April 2023
Published: 06 May 2023

Research article

Time-dependent asymptotic behavior of the solution for evolution equation with linear memory

Tingting Liu[1], **Tasneem Mustafa Hussain Sharfi**[2] **and Qiaozhen Ma**[1,*]

[1] College of Mathematics and Statistics, Northwest Normal University, Gansu, 730070, Lanzhou, China

[2] Faculty of Mathematical science, University of Khartoum, Sudan

* **Correspondence:** Email: maqzh@nwnu.edu.cn.

Abstract: In this article, by using the operator decomposition technique, we discuss the existence of a time-dependent global attractor for a nonlinear evolution equation with linear memory within the theory of time-dependent space. Furthermore, the regularity and asymptotic structure of the time-dependent attractor are proved, which means that the time-dependent attractor of the evolution equation converges to the attractor of the limit wave equation when the coefficient $\varepsilon(t) \to 0$ as $t \to \infty$.

Keywords: linear memory; time-dependent global attractor; regularity; asymptotic structure
Mathematics Subject Classification: 35B40, 35B41, 37L30

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\tau \in \mathbb{R}$, we consider the following equation:

$$\begin{cases} u_{tt} - k(0)\triangle u - \triangle u_t - \varepsilon(t)\triangle u_{tt} - \int_0^\infty k'(s)\triangle u(t-s)ds + f(u) = g(x), & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = u_0(x,t), \quad u_t(x,t) = \partial_t u_0(x,t), \quad x \in \Omega, \quad t \le \tau, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u = u(x,t) : \Omega \times (\tau, \infty) \to \mathbb{R}$ is an unknown function, and $u_0 : \Omega \times (-\infty, \tau] \to \mathbb{R}$ is a given past history of u, $k(0), k(\infty) > 0$ and $k'(s) \le 0$ for every $s \in \mathbb{R}^+$, $g(x) \in L^2(\Omega)$. $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function with

$$\lim_{t\to+\infty} \varepsilon(t) = 0; \tag{1.2}$$

especially, there exists a positive constant L such that

$$\sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)|] \leq L. \tag{1.3}$$

The function $f \in C^1(\mathbb{R})$, $f(0) = 0$, satisfies the conditions

$$|f'(s)| \leq C(1 + |s|^2), \quad \forall s \in \mathbb{R}, \tag{1.4}$$

and

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\lambda_1, \quad \forall s \in \mathbb{R}, \tag{1.5}$$

where C is a positive constant, and λ_1 is the first eigenvalue of $A = -\triangle$ with Dirichlet boundary value condition.

Nonlinear evolution equations of this type arise as models of a vibration of a nonlinear elastic rod, which are used to represent the propagation of lengthwise-waves in nonlinear elastic rods and ion-sonic of space transformation by weak nonlinear effect; see for details [1–3].

Equation (1.1) becomes a strongly damped wave equation with a linear memory term when the coefficient function $\varepsilon(t) \equiv 0$, and it was discussed clearly in [4] and the references therein. When $\varepsilon(t) \equiv \varepsilon$, Eq (1.1) becomes an autonomous evolution equation, and the long-time behavior of the solutions can be well characterized by using the concept of global attractors under the framework of semigroups. In this case, when $\mu(s) = -k'(s)$ vanishes, Eq (1.1) reduces to the damped wave equation, which has been extensively discussed by many authors. For instance, Xie and Zhong [5, 6] systematically investigated the existence of global attractors for (1.1) on weak and strong Hilbert spaces, respectively. Based on the global well-posedness results given in [7], Sun, Yang and Duan [8] constructed the uniformly asymptotic regularity of solution with respect to $\varepsilon \in [0, 1]$ for (1.1) when $g \in L^2(\Omega)$ and $g \in H^{-1}(\Omega)$, respectively, and they also obtained the existence of exponential attractors as well as the upper-semicontinuity of global attractors.

If $\varepsilon(t)$ is dependent on t, then Eq (1.1) becomes more complex and interesting. In this case, the long-time behavior of the solutions for (1.1) can be well characterized by the concept of time-dependent global attractors under the framework of processes, which have been discussed in [9–13]. Recently, Ma, Wang and Liu [14] investigated the existence and regularity of the time-dependent attractors for wave equations by using the operator decomposition technique along with compactness of translation theorem, also they proved the asymptotic structure as in [13]. In [15], they verified the asymptotic compactness of wave equations with nonlinear damping and linear memory by using the contractive functions method which was introduced in [10].

For our problem, by using the method of contractive functions [10], Liu and Ma [16] have obtained the existence of time-dependent global attractors of a nonlinear evolution equation with nonlinear damping and $\mu(s) = -k'(s) = 0$ in (1.1). For problem (1.1), we first introduce a new variable that is used to construct a relatively complicated triple solution space. Second, in order to prove compactness and regularity we use the decomposition technique as in [14]. Finally, we also prove the asymptotic structure of time-dependent global attractor as $\varepsilon(t) \to 0$ when $t \to \infty$.

The rest of this article is organized as follows: In the next section, we define some function set, and we recall some basic definitions and abstract results. In Section 3, the existence and regularity of time-dependent global attractor are obtained. Finally, in Section 4 we prove the asymptotic structure of time-dependent global attractor.

2. Preliminaries

As in [4], we introduce the new variable

$$\eta^t(x,s) = u(x,t) - u(x,t-s),$$

and differentiating the above equation, we get

$$\eta_t^t(s) = -\eta_s^t(s) + u_t(t), \tag{2.1}$$

with

$$\eta_t = \frac{\partial}{\partial t}\eta, \eta_s = \frac{\partial}{\partial s}\eta.$$

For simplicity, we set $\mu(s) = -k'(s)$ and $k(\infty) = 1$, where the memory component μ satisfies the following conditions:

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \int_0^\infty \mu(s)ds = m_0 < +\infty, \quad \forall s \in \mathbb{R}^+, \tag{2.2}$$

$$\mu'(s) \leq -\rho\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+ \text{ and some } \rho > 0. \tag{2.3}$$

Then, we can reformulate (1.1) as the following dynamical system:

$$\begin{cases} u_{tt} - \triangle u - \triangle u_t - \varepsilon(t)\triangle u_{tt} - \int_0^\infty \mu(s)\triangle\eta^t(s)ds + f(u) = g(x), \\ \eta_t^t + \eta_s^t = u_t, \end{cases} \tag{2.4}$$

with initial boundary conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq \tau, \\ \eta^t(x,s) = 0, \quad (x,s) \in \partial\Omega \times \mathbb{R}^+, \quad t \geq \tau, \\ u(x,\tau) = u_0(x),\ u_t(x,\tau) = u_1(x),\ \eta^t(x,0) = 0, \quad x \in \Omega, \\ \eta^\tau(x,s) = \eta^0(x,s), \quad (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{2.5}$$

where

$$u_0(x) = u_0(x,\tau), u_1(x) = \partial_t u_0(x,t)|_{t=\tau},$$

and

$$\eta_0 = \eta^0(x,s) = u_0(x,\tau) - u_0(x,\tau-s).$$

Without loss of generality, set $H = L^2(\Omega)$ with inner product$\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For $s \in \mathbb{R}^+$ we define the hierarchy of (compactly) nested Hilbert spaces

$$H^s = D(A^{\frac{s}{2}}), \quad \langle w,v\rangle_s = \langle A^{\frac{s}{2}}w, A^{\frac{s}{2}}v\rangle, \quad \|w\|_s = \|A^{\frac{s}{2}}w\|.$$

Especially, we have the embedding $H^{s+1} \hookrightarrow H^s$. Also, we denote $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

For $s \in \mathbb{R}^+$, let $L^2_\mu(\mathbb{R}^+; H^s)$ be the family of Hilbert spaces of functions $\varphi : \mathbb{R}^+ \to H^s$, endowed with the inner product and norm, respectively,

$$\langle \varphi_1, \varphi_2 \rangle_{\mu,s} = \langle \varphi_1, \varphi_2 \rangle_{\mu,H^s} = \int_0^\infty \mu(s)\langle \varphi_1(s), \varphi_2(s)\rangle_{H^s} ds,$$

$$\|\varphi\|^2_{\mu,s} = \|\varphi\|^2_{\mu,H^s} = \int_0^\infty \mu(s)\|\varphi(s)\|^2_s ds.$$

We also need the spaces

$$H^1_\mu(\mathbb{R}^+; H^s) = \{\varphi : \varphi(r), \partial_r \varphi(r) \in L^2_\mu(\mathbb{R}^+; H^s)\}.$$

Now, for $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$, we introduce the following time-dependent spaces

$$\mathcal{H}^s_t = H^{s+1} \times H^{s+1}_t \times L^2_\mu(\mathbb{R}^+; H^{s+1}),$$

with norms

$$\|z\|^2_{\mathcal{H}^s_t} = \|\{u, v, \eta^t\}\|^2_{\mathcal{H}^s_t} = \|u\|^2_{s+1} + \|v\|^2_s + \varepsilon(t)\|v\|^2_{s+1} + \|\eta^t\|^2_{\mu,s+1},$$

where the space H^{s+1}_t is endowed with the time-dependent norm $\|v\|^2_s + \varepsilon(t)\|v\|^2_{s+1}$.

The symbol is always omitted whenever zero. In particular, the time-dependent phase space where we settle the problem is

$$\mathcal{H}_t = H^1 \times H^1_t \times L^2_\mu(\mathbb{R}^+; H^1),$$

endowed with the time-dependent product norms

$$\|z\|^2_{\mathcal{H}_t} = \|\{u, v, \eta^t\}\|^2_{\mathcal{H}_t} = \|u\|^2_1 + \|v\|^2 + \varepsilon(t)\|v\|^2_1 + \|\eta^t\|^2_{\mu,1}.$$

Now we recall some basic definitions and abstract results that will help us to get our main results.

Definition 2.1. *[9, 12] Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two parameter family of mappings $U(t, \tau) : X_\tau \to X_t, t \geq \tau, t, \tau \in \mathbb{R}$ with properties*

(i) $U(\tau, \tau) = Id$ is the identity operator on X_τ , $\tau \in \mathbb{R}$;
(ii) $U(t, s)U(s, \tau) = U(t, \tau)$, $\forall t \geq s \geq \tau, \tau \in \mathbb{R}$.

For every $t \in \mathbb{R}$, let X_t be a family of normed spaces, and we define the $R-$ball of X_t as follows:

$$\mathbb{B}_t(R) = \{z \in X_t : \|z\|_{X_t} \leq R\}.$$

We denote the Hausdorff semi-distance of two nonempty sets $A, B \subset X_t$ by

$$\text{dist}_{X_t}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_{X_t}.$$

Definition 2.2. *[9, 12] A family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded if there exists $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$, $\forall t \in \mathbb{R}$.*

Definition 2.3. *[9, 12] We say $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for the process $U(t,\tau)$, if $B_t \subset \mathbb{B}_t(R)$ is uniformly bounded and there exist $t_0 = t_0(\mathfrak{C}) \geq 0$ such that*

$$U(t,\tau)C_\tau \subset B_t, \quad \tau \leq t - t_0,$$

for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$.

Definition 2.4. *[9, 12] A family $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}$ is called pullback attracting if it is uniformly bounded and*

$$\lim_{\tau\to-\infty} \text{dist}_{X_t}(U(t,\tau)C_\tau, K_t) = 0,$$

for every uniformly bounded family $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$.

Definition 2.5. *[9, 12] The time-dependent global attractor is the smallest family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}} \in \mathbb{K}$, where $\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} : K_t \subset X_t$ compact, $\mathfrak{K}$ pullback attracting$\}$, i.e. $A_t \subset K_t, \forall t \in \mathbb{R}$, for any element $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} \in \mathbb{K}$.*

Definition 2.6. *[12] The process $U(t,\tau)$ is called*

- *closed if $U(t,\tau)$ is a closed map for any pair of fixed times $t \geq \tau$;*
- *T-closed for some $T > 0$ if $U(t, t-T)$ is a closed map for all t.*

Definition 2.7. *[12] We say that $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant if*

$$U(t,\tau)A_\tau = A_t, \forall t \geq \tau.$$

Remark 2.1. *[12] If the time-dependent global attractor $\mathfrak{A}$ exists, and the process $U(t,\tau)$ is strongly continuous (or norm-to-weak continuous, or closed, or T-closed), then $\mathfrak{A}$ is invariant.*

Theorem 2.1. *[12] If $U(t,\tau)$ is asymptotically compact, then there exists a unique time-dependent attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$.*

If $U(t,\tau)$ is a T-closed process for some $T > 0$ and possesses a time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$, then $\mathfrak{A}$ is invariant.

In order to prove the asymptotic structure of the time-dependent global attractors for the process $U(t,\tau)$, we recall some results from [13, 14].

Here, we will focus on the case of a process $U(t,\tau)$ acting on a family of spaces $\{Z_t\}_{t\in\mathbb{R}}$ of the form

$$Z_t = \mathcal{X} \times \mathcal{Y}_t,$$

where $\mathcal{X}$ is a normed space, and $\{\mathcal{Y}_t\}_{t\in\mathbb{R}}$ is a family of normed space, endowed with the product norm

$$\|(x,y)\|^2_{Z_t} = \|x\|^2_{\mathcal{X}} + \|y\|^2_{\mathcal{Y}_t}.$$

Let $\Pi_t : Z_t \to \mathcal{X}$ be the projection on the first component of Z_t , that is, $\Pi_t(x,y) = x$. Accordingly, if $C_t \subset Z_t$, then $\Pi_t C_t = \{x \in \mathcal{X} : (x,y) \in C_t\}$. If $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$, then $\Pi\mathfrak{C} = \{\Pi_t C_t\}_{t\in\mathbb{R}}$.

Definition 2.8. *[13, 14] Let $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ be the time-dependent global attractor of $U(t,\tau)$. If $\mathfrak{A}$ is invariant, then $A_t = \{z(t) \in Z_t : z \text{ CBT } of \ U(t,\tau)\}$. Accordingly, we can write*

$$\mathfrak{A} = \{z : t \to z(t) \in Z_t \text{ with } z \text{ a CBT of } U(t,\tau)\},$$

where $z : t \mapsto z(t) \in Z_t$ is a complete bounded trajectory CBT *of $U(t,\tau)$ if*

$$\sup_{t\in\mathbb{R}} \|z(t)\|_{Z_t} \leq \infty \text{ and } z(t) = U(t,\tau)z(\tau), \forall t \geq \tau, \tau \in \mathbb{R}.$$

Lemma 2.1. *[13] Assume that, for any sequence $z_n = (x_n, y_n)$ of a complete bounded trajectory (CBT) of the process $U(t,\tau)$ and any $t_n \to \infty$, there exists a complete bounded trajectory (CBT) w of a semigroup $S(t)$ and $s \in \mathbb{R}$ for which*

$$\|x_n(s + t_n) - w(s)\|_{X} \to 0,$$

as $n \to \infty$ up to a subsequence. Then,

$$\lim_{t\to\infty} \mathrm{dist}_{X}(\Pi_t A_t, A_\infty) = 0,$$

where A_∞ is the global attractor in the phase space X for the autonomous system corresponding to the non-autonomous system with the coefficient $\varepsilon(t) \equiv 0$.

Let $F(u) = \int_0^u f(s)ds$, and we can obtain the following lemma:

Lemma 2.2. *[14] From dissipation condition (1.5), there exist two positive constants k_1 and k_2 and for some $0 < \nu < 1$ such that*

$$\langle f(u), u\rangle \geq -(1-\nu)\|u\|_1^2 - k_1, \quad \forall u \in H_0^1(\Omega), \tag{2.6}$$

$$2\langle F(u), 1\rangle \geq -(1-\nu)\|u\|_1^2 - k_2, \quad \forall u \in H_0^1(\Omega). \tag{2.7}$$

Lemma 2.3. *[14] Let $Y(t) : [\tau,\infty) \to \mathbb{R}^+$ be an absolutely continuous function satisfying the inequality*

$$\frac{d}{dt}Y(t) + 2\epsilon Y(t) \leq h(t)Y(t) + k,$$

for some $\epsilon > 0, k \geq 0$ and where $h : [\tau,\infty) \to \mathbb{R}^+$ fulfills

$$\int_\tau^\infty h(s)ds \leq m,$$

with $m \geq 0$. Then,

$$Y(t) \leq Y(\tau)e^m e^{-\epsilon(t-\tau)} + k\epsilon^{-1}e^m.$$

Within this article, we often use Hölder and Young inequalities and denote positive constants by C, which will change in different lines or even in the same line.

3. Existence of time-dependent global attractors

3.1. Well-posedness and time-dependent absorbing set

In order to obtain the well-posedness of the solution associated with (2.4)–(2.5), we first make a priori estimates as follows:

Lemma 3.1. *Assume that (1.2)–(1.5) and (2.2)–(2.3) hold, and then for any initial data $z_\tau = z(\tau) = (u_0, u_1, \eta_0) \in \mathbb{B}_\tau(R_0) \subset \mathcal{H}_\tau$, there exists a constant $R > 0$, such that*

$$\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t} \leq R, \quad \forall \tau \leq t.$$

Proof. Multiplying $(2.4)_1$ with $2u_t + 2\delta u$ and integrating on Ω, we find that

$$
\begin{aligned}
&\frac{d}{dt}(\|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + (1+\delta)\|u\|_1^2 + 2\delta\langle u_t, u\rangle + 2\delta\varepsilon(t)\langle \nabla u_t, \nabla u\rangle + \|\eta^t(s)\|_{\mu,1}^2 + 2\langle F(u), 1\rangle - 2\langle g, u\rangle)\\
&+ 2\|u_t\|_1^2 + 2\delta\|u\|_1^2 - 2\delta\|u_t\|^2 - (2\delta\varepsilon(t) + \varepsilon'(t))\|u_t\|_1^2 + 2\langle \eta^t, \eta_s^t\rangle_{\mu,1} - 2\delta\int_0^\infty \mu(s)\langle \triangle\eta^t(s), u(t)\rangle ds\\
&+ 2\delta\langle f, u\rangle - 2\delta\langle g, u\rangle = 2\delta\langle \varepsilon'(t)\nabla u_t, \nabla u\rangle. \qquad (3.1)
\end{aligned}
$$

First, from condition (1.3), and by the Hölder, Young and Poincaré inequalities, there holds

$$2\delta\langle \varepsilon'(t)\nabla u_t, \nabla u\rangle \leq 2\delta L\|u_t\|_1\|u\|_1 \leq \frac{1}{2}\|u_t\|_1^2 + 2\delta^2 L^2\|u\|_1^2,$$

where $\|u\|_1^2 \geq \lambda_1\|u\|^2,\ \forall u \in H^2(\Omega)$.

Let

$$
\begin{aligned}
E(t) =& \|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + (1+\delta)\|u\|_1^2 + 2\delta\langle u_t, u\rangle + 2\delta\varepsilon(t)\langle \nabla u_t, \nabla u\rangle + \|\eta^t(s)\|_{\mu,1}^2\\
&+ 2\langle F(u), 1\rangle - 2\langle g, u\rangle, \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
I(t) =& (\frac{3}{2} - 2\delta\varepsilon(t) - \varepsilon'(t))\|u_t\|_1^2 + (2\delta - 2\delta^2 L^2)\|u\|_1^2 - 2\delta\|u_t\|^2 + 2\langle \eta^t, \eta_s^t\rangle_{\mu,1}\\
&- 2\delta\int_0^\infty \mu(s)\langle \triangle\eta^t(s), u(t)\rangle ds + 2\delta\langle f, u\rangle - 2\delta\langle g, u\rangle. \qquad (3.3)
\end{aligned}
$$

Then,

$$\frac{d}{dt}E(t) + I(t) \leq 0. \qquad (3.4)$$

Integrating (3.4) from τ to t, we have

$$E(t) \leq -\int_\tau^t I(s)ds + E(\tau). \qquad (3.5)$$

Next, we estimate (3.2) and (3.3), respectively. By using (1.3), (2.7) and the Hölder, Young, Poincaré inequalities, it follows that

$$2\delta|\langle u_t, u\rangle| \leq 2\delta\|u_t\|\|u\| \leq \delta\|u_t\|^2 + \frac{\delta}{\lambda_1}\|u\|_1^2,$$

$$2\delta\varepsilon(t)|\langle \nabla u_t, \nabla u\rangle| \leq \delta\varepsilon(t)\|u_t\|_1^2 + \delta L\|u\|_1^2,$$

$$2|\langle g, u\rangle| \leq \frac{\nu}{2}\|u\|_1^2 + \frac{2}{\lambda_1\nu}\|g\|^2.$$

Then,

$$E(t) \geq (1-\delta)\|u_t\|^2 + (\frac{\nu}{2} - \frac{\delta}{\lambda_1} - L\delta)\|u\|_1^2 + \varepsilon(t)(1-\delta)\|u_t\|_1^2 + \|\eta^t(s)\|_{\mu,1}^2 - (\frac{2}{\lambda_1\nu}\|g\|^2 + k_2). \qquad (3.6)$$

Using (2.2), (2.3) there holds

$$2\langle \eta^t, \eta^t_s \rangle_{\mu,1} \geq \frac{2\rho}{2}\|\eta^t(s)\|^2_{\mu,1} = \rho\|\eta^t(s)\|^2_{\mu,1},$$

and

$$2\delta|\langle \int_0^\infty \mu(s)\triangle\eta^t(s)ds, u(t)\rangle| \leq \frac{\rho}{2}\|\eta^t(s)\|^2_{\mu,1} + \frac{2\delta^2 m_0}{\rho}\|u\|^2_1.$$

Hence, from (2.6) and the condition (1.3), we get

$$\begin{aligned}
I(t) \geq &(\frac{3}{2} - 2\delta\varepsilon(t) - \varepsilon'(t))\|u_t\|^2_1 + (2\delta - 2\delta^2 L^2 - \frac{2\delta^2 m_0}{\rho})\|u\|^2_1 - 2\delta\|u_t\|^2 + \frac{\rho}{2}\|\eta^t(s)\|^2_{\mu,1} \\
&+ 2\delta\langle f, u\rangle - 2\delta\langle g, u\rangle \\
\geq &(\frac{1}{2} - 2\delta\varepsilon(t) - \varepsilon'(t))\|u_t\|^2_1 + 2\delta(\nu - \delta - \delta L^2 - \frac{\delta m_0}{\rho})\|u\|^2_1 + (\lambda_1 - 2\delta)\|u_t\|^2 + \frac{\rho}{2}\|\eta^t(s)\|^2_{\mu,1} \\
&- 2\delta k_1 - \frac{1}{2\lambda_1}\|g\|^2 \\
\geq &\delta\varepsilon(t)\|u_t\|^2_1 + \delta\nu\|u\|^2_1 + \delta\|u_t\|^2 + \frac{\rho}{2}\|\eta^t(s)\|^2_{\mu,1} - (\frac{1}{2\lambda_1}\|g\|^2 + 2\delta k_1), \qquad (3.7)
\end{aligned}$$

where we have chosen $0 < \delta$ small enough such that

$$1 - \delta \geq \delta,\ \frac{\nu}{2} - \frac{\delta}{\lambda_1} - L\delta \geq \frac{\nu}{4},\ \frac{1}{2} - 2\delta\varepsilon(t) - \varepsilon'(t) > \delta\varepsilon(t),\ \nu - \delta - \delta L^2 - \frac{\delta m_0}{\rho} > \frac{\nu}{2},\ \ \lambda_1 - 2\delta > \delta.$$

Let $M_1 = \min\{\frac{\nu}{4}, \delta\}$, $M_2 = \min\{\delta, \nu\delta, \frac{\rho}{2}\}$, $m_1 = \frac{2}{\lambda_1\nu}\|g\|^2 + k_2$, $m_2 = \frac{1}{2\lambda_1}\|g\|^2 + 2\delta k_1$, and then from (3.5) we arrive at

$$\begin{aligned}
&M_1[\|u_t\|^2 + \varepsilon(t)\|u_t\|^2_1 + \|u\|^2_1 + \|\eta^t(s)\|^2_{\mu,1}] - m_1 \\
&\leq -\int_\tau^t (M_2[\|u_t(r)\|^2 + \varepsilon(r)\|u_t(r)\|^2_1 + \|u(r)\|^2_1 + \|\eta^r(s)\|^2_{\mu,1}] - m_2)dr + E(\tau).
\end{aligned}$$

Therefore, taking $K_0 > \frac{m_2}{M_2}$, we have

$$\|u_t(t)\|^2 + \varepsilon(t)\|u_t(t)\|^2_1 + \|u(t)\|^2_1 + \|\eta^t(s)\|^2_{\mu,1} \leq K_0,\ \forall\, t \geq t_0.$$

As a result, if (u, u_t, η) is the solution of the system, let $\mathcal{B}_t = \bigcup_{t\geq\tau} U(t,\tau)B_\tau$, where

$$\mathcal{B}_\tau = \{(u_0, u_1, \eta^0) \in \mathcal{H}_\tau : \|u_1\|^2 + \varepsilon(\tau)\|u_1\|^2_1 + \|u_0\|^2_1 + \|\eta^0(s)\|^2_{\mu,1} \leq K_0\}.$$

Then, $\mathcal{B}_t$ is a bounded absorbing set for process $\{U(t,\tau)\}_{t\geq\tau}$.

On the other hand, from the above discussion, there exists a positive constant $R(R_0) > 0$ such that

$$\|u\|^2_1 + \|u_t\|^2 + \varepsilon(t)\|u_t\|^2_1 + \|\eta^t\|^2_{\mu,1} \leq R,\ \forall\, t \geq t_0 \geq \tau.$$

The proof is completed. □

Lemma 3.2. *Let the assumptions (1.2)–(1.5) and (2.2)–(2.3) hold, and then for any initial data* $z_\tau = z(\tau) = (u_0, u_1, \eta_0) \in \mathcal{H}_\tau$ *, on any interval* $[\tau, t]$ *with* $t > \tau$*, there exists a unique solution* $(u(t), u_t(t), \eta^t(s))$ *of the system (2.4)–(2.5) satisfying*

$$u \in C([\tau, t]; H_0^1(\Omega)), u_t \in C([\tau, t]; H_0^1(\Omega)), \eta^t \in C([\tau, t]; L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))).$$

Furthermore, let $z_i(\tau) \in \mathcal{H}_\tau$ *be the initial data such that* $\|z_i(\tau)\|_{\mathcal{H}_\tau} \le R_0, (i = 1, 2)$*, and* $z_i(t)$ *be the solution of problem (2.4)–(2.5). Then, there exists* $\widetilde{C} = \widetilde{C}(R_0) > 0$*, such that*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_t} \le e^{\widetilde{C}(t-\tau)}\|z_1(\tau) - z_2(\tau)\|_{\mathcal{H}_\tau}, \forall t \ge \tau. \tag{3.8}$$

Thus, the system (2.4)–(2.5) generates a strongly continuous process $U(t, \tau)$*, where* $U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ *acting as* $U(t, \tau)z(\tau) = \{u(t), u_t(t), \eta^t(s)\}$*, with the initial data* $z_\tau = \{u_0, u_1, \eta_0\} \in \mathcal{H}_\tau$*.*

Proof. Based on Lemma 3.1, we can obtain the existence of a solution for problem (2.4)–(2.5) by using the Faedo-Galerkin approximation method, and the degenerate coefficient function $\varepsilon(t)$ in (2.4) is not causing a new difficult. See for details [5, 12, 17].

Consequently, we only need to verify the estimate (3.8). For this purpose, we assume that $z_i(t) = \{u_i(t), u_{it}(t), \eta_i^t(s)\}(i = 1, 2)$ are the solutions of (2.4)–(2.5) with the corresponding initial data $z_i(\tau) = \{u_i^0(\tau), u_i^1(\tau), \eta_i^0(s)\}(i = 1, 2)$, and there exists $R_0 > 0$ such that $\|z_i(\tau)\|_{\mathcal{H}_\tau} \le R_0, i = 1, 2$. □

According to Lemma 3.1 we ensure that

$$\|U(t, \tau)z_i(\tau)\|_{\mathcal{H}_t} \le R,\ i = 1, 2. \tag{3.9}$$

Let $\overline{z}(t) = \{\overline{u}(t), \overline{u}_t(t), \overline{\eta}^t(s)\} = U(t, \tau)z_1(\tau) - U(t, \tau)z_2(\tau)$, and then $\overline{z}(t)$ satisfies the following equation:

$$\overline{u}_{tt} - \triangle\overline{u} - \triangle\overline{u}_t - \varepsilon(t)\triangle\overline{u}_{tt} - \int_0^\infty \mu(s)\triangle\overline{\eta}^t(s)ds + f(u_1) - f(u_2) = 0. \tag{3.10}$$

Taking the inner product of (3.10) with $2\overline{u}_t$ in $L^2(\Omega)$, we get

$$\begin{aligned}&\frac{d}{dt}[\|\overline{u}_t\|^2 + \|\overline{u}\|_1^2 + \varepsilon(t)\|\overline{u}_t\|_1^2 + \|\overline{\eta}^t\|_{\mu,1}^2] - \varepsilon'(t)\|\overline{u}_t\|_1^2 + 2\|\overline{u}_t\|_1^2 + 2\langle\overline{\eta}^t, \overline{\eta}_s^t\rangle_{\mu,1}\\ =& - 2\langle f(u_1) - f(u_2), \overline{u}_t\rangle.\end{aligned} \tag{3.11}$$

In line with (1.4), (3.9), Hölder inequality, Young inequality and embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, it follows that

$$\begin{aligned}-2\langle f(u_1) - f(u_2), \overline{u}_t\rangle \le& C\int_\Omega (1 + |u_1|^2 + |u_2|^2)|\overline{u}||\overline{u}_t|dx\\ \le& C(1 + \|u_1\|_{L^6}^2 + \|u_2\|_{L^6}^2)\|\overline{u}\|_{L^6}\|\overline{u}_t\|\\ \le& C(1 + \|u_1\|_1^2 + \|u_2\|_1^2)\|\overline{u}\|_1\|\overline{u}_t\|\\ \le& C\|\overline{u}\|_1\|\overline{u}_t\|\\ \le& C_R(\|\overline{u}\|_1^2 + \|\overline{u}_t\|^2);\end{aligned} \tag{3.12}$$

meanwhile, (2.2) and (2.3) mean

$$\langle\overline{\eta}^t(s), \overline{\eta}_s^t(s)\rangle_{\mu,1} \ge \frac{\rho}{2}\|\overline{\eta}^t(s)\|_{\mu,1}^2. \tag{3.13}$$

Together with (3.12) and (3.13), from (3.11) we deduce

$$\frac{d}{dt}[\|\overline{u}_t\|^2 + \|\overline{u}\|_1^2 + \varepsilon(t)\|\overline{u}_t\|_1^2 + \|\overline{\eta}^t\|_{\mu,1}^2] \leq C_R(\|\overline{u}\|_1^2 + \|\overline{u}_t\|^2) + \rho\|\overline{\eta}^t(s)\|_{\mu,1}^2.$$

So, according to the norm of (2.5), we can claim

$$\frac{d}{dt}\|\overline{z}(t)\|_{\mathcal{H}_t}^2 \leq \widetilde{C}\|\overline{z}(t)\|_{\mathcal{H}_t}^2, \tag{3.14}$$

where $\widetilde{C} = \max\{C_R, \rho\}$. Thus, by using the Gronwall lemma with (3.14), we conclude the result (3.8).

Remark 3.1. *Based on the argument, there exists R such that $\mathfrak{B} = \{\mathbb{B}_t(R)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\geq\tau}$ associated with (2.4) and (2.5), and for $M_0(R_0) > 0$ there holds*

$$\sup_{z_\tau\in\mathbb{B}_\tau(\mathrm{R}_0)}\{\|U(t,\tau)z_\tau\|_{\mathcal{H}_t}^2 + \int_\tau^\infty \|u_t(y)\|_1^2 dy\} \leq M_0, \ \forall\tau\in\mathbb{R}. \tag{3.15}$$

Proof. Let $\delta \equiv 0$ in equality (3.4), and we get that

$$\frac{d}{dt}[\|u_t\|^2 + \varepsilon(t)\|u_t\|_1^2 + \|u\|_1^2 + \|\eta^t(s)\|_{\mu,1}^2 + 2\langle F(u), 1\rangle - 2\langle g, u\rangle] + \|u_t(y)\|_1^2 \leq 0.$$

Integrating on $[\tau, t]$ and using inequality (3.6), we have $\int_\tau^\infty \|u_t(y)\|_1^2 dy \leq M_0(> 0)$. Then, together with Lemma 3.1, we conclude that (3.15) is true. □

3.2. Asymptotic compactness and time-dependent global attractor

In this section, we do as in [14]. We find a suitable decomposition of the process, which is the sum of a decaying part and compact part. By a direct application of the abstract Theorem 2.1, we do this strategy to show that the process is asymptotically compact, and then the existence of the time-dependent global attractor is obtained.

For decomposition we write $f = f_0 + f_1$, where $f_0, f_1 \in C^2(\mathbb{R})$ satisfy

$$|f_1'(u)| \leq C(1 + |u|^{\gamma-1}), \quad 1 < \gamma < 3, \ \forall u \in \mathbb{R}, \tag{3.16}$$

$$|f_0''(u)| \leq C(1 + |u|), \quad \forall u \in \mathbb{R}, \tag{3.17}$$

$$\liminf_{|u|\to\infty} \frac{f_1(u)}{u} > -\lambda_1, \ \forall u \in \mathbb{R}, \tag{3.18}$$

$$f_0(0) = f_0'(0) = 0, \quad f_0(u)u \geq 0, \ \forall u \in \mathbb{R}. \tag{3.19}$$

Let $\mathfrak{B} = \{\mathbb{B}_t(\mathrm{M}_0)\}_{t\in\mathbb{R}}$ be a time-dependent absorbing set. Then, for any $z \in \mathbb{B}_\tau(\mathrm{M}_0)$ and fixed $\tau \in \mathbb{R}$, we decompose the process $U(t,\tau)$ as follows:

$$U(t,\tau)z = \{u(t), u_t(t), \eta^t(s)\} = U_0(t,\tau)z + U_1(t,\tau)z,$$

where

$$U_0(t,\tau)z = \{v(t), v_t(t), \zeta^t(s)\} \text{ and } U_1(t,\tau)z = \{w(t), w_t(t), \xi^t(s)\},$$

solve respectively the systems

$$\begin{cases} v_{tt} + Av + Av_t + \varepsilon(t)Av_{tt} + \int_0^\infty \mu(s)A\zeta^t(s)ds + f_0(v) = 0, \\ \zeta_t^t(s) = -\zeta_s^t(s) + v_t(t), \\ v|_{\partial\Omega} = 0, v(x,\tau) = u_0(x), v_t(x,\tau) = u_1(x), \\ \zeta^t|_{\partial\Omega} = 0, \zeta^0(x,s) = u_0(x) - u_0(x,\tau - s), \end{cases} \tag{3.20}$$

and

$$\begin{cases} w_{tt} + Aw + Aw_t + \varepsilon(t)Aw_{tt} + \int_0^\infty \mu(s)A\xi^t(s)ds + f(u) - f_0(v) = g(x), \\ \xi_t^t(s) = -\xi_s^t(s) + w_t(t), \\ w|_{\partial\Omega} = 0, w(x,\tau) = 0, w_t(x,\tau) = 0, \\ \xi^t|_{\partial\Omega} = 0, \xi^0(x,s) = 0. \end{cases} \tag{3.21}$$

In the following lemma, the constant $C > 0$ depends only on $\mathfrak{B}$.

Lemma 3.3. *If (1.2)–(1.5), (2.2)–(2.3) and (3.16)–(3.19) hold, then there exists $\delta = \delta(\mathfrak{B}) > 0$ such that*

$$\|U_0(t,\tau)z(\tau)\|_{\mathcal{H}_t} \le Ce^{-\delta(t-\tau)}. \tag{3.22}$$

Proof. Repeating word by word the proof of Lemma 3.1 in the case of $U_0(t,\tau)$, we can get the bound

$$\|U_0(t,\tau)z(\tau)\|_{\mathcal{H}_t} \le C. \tag{3.23}$$

Multiplying Eq $(3.20)_1$ by $2v_t + 2\delta v$ and integrating on Ω, we find that

$$\begin{aligned} &\frac{d}{dt}(\|v_t\|^2 + \varepsilon(t)\|v_t\|_1^2 + (1+\delta)\|v\|_1^2 + 2\delta\langle v_t, v\rangle + 2\delta\varepsilon(t)\langle\nabla v_t, \nabla v\rangle + \|\zeta^t(s)\|_{\mu,1}^2 + 2\langle F_0(v), 1\rangle) \\ &+2\|v_t\|_1^2 + 2\delta\|v\|_1^2 - 2\delta\|v_t\|^2 - (2\delta\varepsilon(t) + \varepsilon'(t))\|v_t\|_1^2 + 2\langle\zeta^t, \zeta_s^t\rangle_{\mu,1} + 2\delta\int_0^\infty \mu(s)\langle A\zeta^t(s), v(t)\rangle ds \\ &+2\delta\langle f_0, v\rangle = 2\delta\langle\varepsilon'(t)Av_t, Av\rangle. \end{aligned} \tag{3.24}$$

Define

$$E_0(t) = \|v_t\|^2 + \varepsilon(t)\|v_t\|_1^2 + (1+\delta)\|v\|_1^2 + 2\delta\langle v_t, v\rangle + 2\delta\varepsilon(t)\langle\nabla v_t, \nabla v\rangle + \|\zeta^t(s)\|_{\mu,1}^2 + 2\langle F_0(v), 1\rangle,$$

where

$$F_0(s) = \int_0^s f_0(y)dy.$$

Then, we get

$$\begin{aligned} &\frac{d}{dt}E_0(t) + 2\|v_t\|_1^2 + 2\delta\|v\|_1^2 - 2\delta\|v\|^2 - (2\delta\varepsilon(t) + \varepsilon'(t))\|v_t\|_1^2 + 2\langle\zeta^t, \zeta_s^t\rangle_{\mu,1} \\ &+2\delta\int_0^\infty \mu(s)\langle A\zeta^t(s), v(t)\rangle ds + 2\delta\langle f_0, v\rangle = 2\delta\langle\varepsilon'(t)Av_t, Av\rangle. \end{aligned} \tag{3.25}$$

From (3.17) and (3.23), we have

$$\frac{1}{2}\|U_0(t,\tau)z(\tau)\|_{\mathcal{H}_t}^2 \le E_0(t) \le C\|U_0(t,\tau)z(\tau)\|_{\mathcal{H}_t}^2. \tag{3.26}$$

Therefore, by the same steps of the proof of Lemma 3.1, we deduce

$$\frac{d}{dt}E_0(t) + \delta\|U_0(t,\tau)z(\tau)\|^2_{\mathcal{H}_t} \le 0.$$

Thus, combining with (3.26) and using the Gronwall lemma with the above, we complete the proof. □

Remark 3.2. *Under the assumptions of Lemma 3.3, the following uniformly bounded holds:*

$$\sup_{t\ge\tau}[\|U(t,\tau)z(\tau)\|_{\mathcal{H}_t} + \|U_0(t,\tau)z(\tau)\|_{\mathcal{H}_t} + \|U_1(t,\tau)z(\tau)\|_{\mathcal{H}_t}] \le C. \tag{3.27}$$

Lemma 3.4. *If (1.2)–(1.5),(2.2)–(2.3) and (3.16)–(3.19) hold, then there exists $M = M(\mathfrak{B}) > 0$ such that*

$$\|U_1(t,\tau)z(\tau)\|_{\mathcal{H}_t^\sigma} \le M,\ \forall t \ge \tau,$$

where

$$0 < \sigma \le \min\{\frac{1}{2}, \frac{3-\gamma}{2}\}. \tag{3.28}$$

Proof. Multiplying Eq $(3.21)_1$ by $2A^\sigma w_t + 2\delta A^\sigma w$ and integrating it over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}E_1(t) + 2\|w_t\|^2_{\sigma+1} + 2\delta\|w\|^2_{\sigma+1} - 2\delta\|w_t\|^2_\sigma - (2\delta\varepsilon(t) + \varepsilon'(t))\|w_t\|^2_{\sigma+1} + 2\langle\xi^t, \xi^t_s\rangle_{\mu,\sigma+1} \\
&+2\delta\int_0^\infty \mu(s)\langle A\xi^t(s), A^\sigma w(t)\rangle ds + 2\delta\langle f(u) - f_0(v) - g, A^\sigma w\rangle \\
&=2\delta\varepsilon'(t)\langle Aw_t, A^\sigma w\rangle + I_1 + I_2 + I_3,
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
E_1(t) =& \|U_1(t,\tau)z\|^2_{\mathcal{H}_t^\sigma} + \delta\|w\|^2_{\sigma+1} + 2\delta\langle w_t, A^\sigma w\rangle + 2\delta\varepsilon(t)\langle Aw_t, A^\sigma w\rangle \\
&+ 2\langle f(u) - f_0(v) - g, A^\sigma w\rangle + C, \\
I_1 =& 2\langle[f_0'(u) - f_0'(v)]u_t, A^\sigma w\rangle, \\
I_2 =& 2\langle f_0'(v)w_t, A^\sigma w\rangle, \\
I_3 =& 2\langle f_1'(u)u_t, A^\sigma w\rangle.
\end{aligned}$$

Now, by using (1.3), (3.17), (3.27) and the embedding inequality ($\sigma < \frac{\sigma+1}{2}$), we have

$$\begin{aligned}
2\langle f(u) - f_0(v), A^\sigma w\rangle &\le 2\|f(u) - f_0(v)\|\|A^\sigma w\| \\
&\le C\|A^\sigma w\| \\
&\le C\|A^{\frac{\sigma+1}{2}} w\| \\
&\le \frac{1}{4}\|w\|^2_{\sigma+1} + C,
\end{aligned}$$

$$2\langle g, A^\sigma w\rangle \le 2\|g\|\|A^\sigma w\| \le C\|g\|^2 + \frac{1}{4}\|w\|^2_{\sigma+1}.$$

Then, using the Hölder, Young inequalities, we get

$$2\delta\langle w_t, A^\sigma w\rangle \le 2\delta\|w_t\|_\sigma\|w\|_\sigma \le 2\delta\|w_t\|_\sigma^2 + \frac{\delta}{2}\|w\|_\sigma^2;$$
$$2\delta\varepsilon(t)\langle Aw_t, A^\sigma w\rangle \le 2\delta\varepsilon(t)\|w_t\|_{\sigma+1}\|w\|_\sigma \le \frac{\varepsilon(t)}{2}\|w_t\|_{\sigma+1}^2 + 2L\delta^2\|w\|_\sigma^2.$$

Choose δ small enough and $C > 0$ large enough, and we can obtain

$$\frac{1}{2}\|U_1(t,\tau)z(\tau)\|_{\mathcal{H}_t^\sigma}^2 \le E_1(t) \le 2\|U_1(t,\tau)z(\tau)\|_{\mathcal{H}_t^\sigma}^2 + 2C. \tag{3.30}$$

Hence, exploiting (3.17), (3.27) and some Sobolev embeddings $H^{1+\sigma} \hookrightarrow L^{\frac{6}{1-2\sigma}}$, $H^{1-\sigma} \hookrightarrow L^{\frac{6}{1+2\sigma}}$, and the continuous embedding $H^{\frac{(3p-6)}{2p}} \hookrightarrow L^p(\Omega)$ $(p > 2)$, we have

$$\begin{aligned}
I_1 &\le C\int_\Omega (1 + |u| + |v|)\cdot|w|\cdot|u_t|\cdot|A^\sigma w|dx\\
&\le C(1 + \|u\|_{L^6} + \|v\|_{L^6})\cdot\|w\|_{L^{\frac{6}{1-2\sigma}}}\cdot\|u_t\|\cdot\|A^\sigma w\|_{L^{\frac{6}{1+2\sigma}}}\\
&\le C(1 + \|u\|_1 + \|v\|_1)\cdot\|w\|_{\sigma+1}^2\cdot\|u_t\|\\
&\le C\|u_t\|\|w\|_{\sigma+1}^2 \le \frac{\delta}{4}\|w\|_{\sigma+1}^2 + \frac{C^2}{\delta}\|u_t\|^2\|w\|_{\sigma+1}^2\\
&\le \frac{\delta}{4}E_1(t) + C\|u_t\|^2\|w\|_{\sigma+1}^2;\\
I_2 &\le C(\|v\|_{L^6} + \|v\|_{L^6}^2)\cdot\|w_t\|_{L^{\frac{6}{3-2\sigma}}}\cdot\|A^\sigma w\|_{L^{\frac{6}{1+2\sigma}}}\\
&\le C(\|v\|_1 + \|v\|_1^2)\cdot\|w_t\|_\sigma\cdot\|A^\sigma w\|_{1-\sigma}\\
&\le C\|v\|_1\cdot\|w_t\|_\sigma\cdot\|w\|_{\sigma+1} + C\|v\|_1^2\cdot\|w_t\|_\sigma\cdot\|w\|_{\sigma+1}\\
&\le \frac{\delta}{2}\|w_t\|_\sigma^2 + C(\|v\|_1^2 + \|v\|_1^4)\cdot\|w\|_{\sigma+1}^2.
\end{aligned}$$

Also, by using (3.16), we have

$$\begin{aligned}
I_3 &\le C\int_\Omega (1 + |u|^{\gamma-1}).|u_t|.|A^\sigma w|dx\\
&\le C\|u\|_{L^{\frac{6(\gamma-1)}{2(1-\sigma)}}}^{\gamma-1}.\|u_t\|.\|A^\sigma w\|_{L^{\frac{6}{1+2\sigma}}} + C\|u_t\|\cdot\|A^\sigma w\|\\
&\le \|u_t\|^2\cdot\|w\|_{\sigma+1}^2 + C.
\end{aligned}$$

In addition, (2.2) and (2.3) mean

$$2\langle\xi^t, \xi_s^t\rangle_{\mu,\sigma+1} \ge \rho\|\xi^t(s)\|_{\mu,\sigma+1}^2,$$

and

$$2\delta\int_0^\infty \mu(s)\langle\triangle\xi^t(s), A^\sigma w(t)\rangle ds \le \frac{\rho}{2}\|\xi^t(s)\|_{\mu,\sigma+1}^2 + \frac{2m_0\delta^2}{\rho}\|w\|_{\sigma+1}^2.$$

As a consequence, we can write (3.29) as

$$\frac{d}{dt}E_1(t) + \delta E_1(t) + \Gamma \le \frac{\delta}{2}E_1(t) + C\|u_t\|^2\|w\|_{\sigma+1}^2 + C(\|v\|_1^2 + \|v\|_1^4)\|w\|_{\sigma+1}^2 + C.$$

We can see that for $0 < \delta$ small enough,

$$\Gamma =(1 - \varepsilon'(t) - 3\delta\varepsilon(t))\|w_t\|^2_{\sigma+1} + (\frac{\lambda_1}{2} - 3\delta - \frac{\delta}{2})\|w_t\|^2_{\sigma} + (\frac{\rho}{2} - \delta)\|\xi^t\|^2_{\mu,\sigma+1} + (\delta - \delta^2 - \delta^2 L^2 - \frac{2m_0\delta^2}{\rho})\|w\|^2_{\sigma+1} - 2\delta^2\langle w_t, A^\sigma w\rangle - 2\delta^2\varepsilon(t)\langle Aw_t, A^\sigma w\rangle > 0.$$

According to (3.24) and taking δ small enough, we get

$$\frac{d}{dt}E_1(t) + \frac{\delta}{2}E_1(t) \le q(t)E_1(t) + C,$$

where $q(t) = C(\|u_t\|^2 + \|v\|_1^2 + \|v\|_1^4)$. Remark 3.1 and Lemma 3.3 imply that

$$\int_\tau^\infty q(y)dy \le C.$$

Now, applying Lemma 2.3, we get

$$E_1(t) \le CE_1(\tau)e^{-\frac{\delta}{4}(t-\tau)} + C \le C.$$

Together with (3.22), the proof is completed. □

Especially, taking $\sigma = \frac{1}{3}$, we directly get

$$\|U_1(t,\tau)z(\tau)\|_{\mathcal{H}_t^{\frac{1}{3}}} \le C. \tag{3.31}$$

The proof is similar the above estimation, here we omit it.

Remark 3.3. *In order to obtain a compact subset of $\mathcal{H}_t$, we also need the compactness of the memory term which is verified and proved in Lemma 3.6 in [14].*

Theorem 3.1. *Assume that (1.2)–(1.5), (2.2)–(2.3), (3.16)–(3.19) hold. The process $U(t,\tau) : \mathcal{H}_\tau \to \mathcal{H}_t$ generated by problem (2.4)–(2.5) has an invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$.*

Proof. Denote the closure of C_t in $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ by $\overline{C}_t$. According to Lemma 3.4 and Remark 3.3, we consider the family $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}$, where

$$K_t = \{(u, u_t) \in \mathrm{H}^{\sigma+1} \times \mathrm{H}_t^{\sigma+1} : \|u\|_{\sigma+1} + \varepsilon(t)\|u_t\|_{\sigma+1} + \|u_t\|_\sigma \le M\} \times \bar{C}_t \subset \mathcal{H}_t.$$

Applying the compact embedding $H^{\sigma+1} \times H_t^{\sigma+1} \hookrightarrow H^1_0(\Omega) \times H^1_0(\Omega)$, together with the compactness of C_t in $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$, we know that K_t is compact in $\mathcal{H}_t$; since the injection constant M is independent of t, the set K is uniformly bounded. Finally, by Theorem 2.1 and Lemmas 3.1, 3.3 and 3.4, we conclude that there exists a unique time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$, Furthermore, from the strong continuity of the process state in Lemma 3.2 and from Remark 2.1, the $\mathfrak{A}$ is invariant. □

3.3. Regularity of time-dependent global attractor

The main result of this subsection is to prove A_t is bounded in $\mathcal{H}_t^1$. Fix $\tau \in \mathbb{R}$, and for $z \in A_\tau$ we decompose again the process $U(t,\tau)z$ into the sum $U_2(t,\tau)z + U_3(t,\tau)z$, where

$$U_2(t,\tau)z = \{v(t), v_t(t), \zeta^t(s)\} \text{ and } U_3(t,\tau)z = \{w(t), w_t(t), \xi^t(s)\},$$

solve respectively the systems

$$\begin{cases} v_{tt} + Av + Av_t + \varepsilon(t)Av_{tt} + \int_0^\infty \mu(s)A\zeta^t(s)ds = 0, \\ \zeta_t^t(s) = -\zeta_s^t(s) + v_t(t), \\ U_2(t,\tau)z(\tau) = (u_0, u_1, \zeta^0), \end{cases} \tag{3.32}$$

and

$$\begin{cases} w_{tt} + Aw + Aw_t + \varepsilon(t)Aw_{tt} + \int_0^\infty \mu(s)A\xi^t(s)ds + f(u) = g(x), \\ \xi_t^t(s) = -\xi_s^t(s) + w_t(t), \\ U_3(t,\tau)z(\tau) = 0. \end{cases} \tag{3.33}$$

As a particular case of Lemma 3.3, we learn that

$$\|U_2(t,\tau)z(\tau)\|_{\mathcal{H}_t} \le Ce^{-\delta(t-\tau)}, \ \forall t \ge \tau. \tag{3.34}$$

Lemma 3.5. *If (1.2)–(1.5), (2.2)–(2.3), (3.16)–(3.19) hold, then there exists $M_1 = M_1(\mathfrak{A}) > 0$ such that*

$$\|U_3(t,\tau)z\|_{\mathcal{H}_t^1} \le M_1, \ \forall t \ge \tau.$$

Proof. Multiplying equation of $(3.33)_1$ by $2Aw_t + 2\delta Aw$ and integrating it over Ω, using $(3.33)_2$ we get

$$\begin{aligned} &\frac{d}{dt}E_3(t) + 2\|w_t\|_2^2 - 2\delta\|w_t\|_1^2 + 2\delta\|w\|_2^2 - (\varepsilon'(t) + 2\delta\varepsilon(t))\|w_t\|_2^2 + 2\langle \xi^t, \xi_s^t\rangle_{\mu,2} \\ &+ 2\delta \int_0^\infty \mu(s)\langle A\xi^t(s), Aw(t)\rangle ds - 2\delta\langle g, Aw\rangle \\ &= 2\delta\varepsilon'(t)\langle Aw_t, Aw\rangle - 2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle, \end{aligned} \tag{3.35}$$

where

$$E_3(t) = \|U_3(t,\tau)z\|_{\mathcal{H}_t^1}^2 + \delta\|w\|_2^2 + 2\delta\langle w_t, Aw\rangle + 2\delta\varepsilon(t)\langle Aw_t, Aw\rangle - 2\langle g, Aw\rangle + C,$$

and

$$2\langle \xi^t, \xi_s^t\rangle_{\mu,2} \ge \rho\|\xi^t(s)\|_{\mu,2}^2;$$

$$2\delta \int_0^\infty \mu(s)\langle \triangle\xi^t(s), Aw(t)\rangle ds \le \frac{\rho}{2}\|\xi^t(s)\|_{\mu,2}^2 + \frac{2m_0\delta^2}{\rho}\|w\|_2^2.$$

Choose $\delta > 0$ small enough and $C > 0$ large enough, and then we can obtain

$$\frac{1}{4}\|U_3(t,\tau)z\|_{\mathcal{H}_t^1}^2 \le E_3(t) \le 2\|U_3(t,\tau)z\|_{\mathcal{H}_t^1}^2 + 2C. \tag{3.36}$$

By some calculation as in Lemma 3.4 and taking δ small enough, we deduce

$$\frac{d}{dt}E_3(t) + \delta E_3(t) \le -2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle + \delta C.$$

We know from (3.31) that A_t is bounded in $\mathcal{H}_t^{\frac{1}{3}}$. Consequently, exploiting some Sobolev embeddings $H^{\frac{(3p-6)}{2p}} \hookrightarrow L^p (p \geq 2)$, $H^{\frac{1}{3}} \hookrightarrow L^{\frac{18}{7}}(\Omega)$, $H^{\frac{4}{3}} \hookrightarrow L^{18}(\Omega)$, there holds

$$\|f(u)\|_1 = \|f'(u)A^{\frac{1}{2}}u\| \leq \|f'(u)\|_{L^9}\|A^{\frac{1}{2}}u\|_{L^{\frac{18}{7}}} \leq C(1 + \|u\|_{L^{18}}^2) \leq C,$$

so

$$\begin{aligned} -2\langle f(u), Aw_t\rangle - 2\delta\langle f(u), Aw\rangle &\leq 2\|f(u)\|_1(\|w_t\|_1 + \|w\|_1) \\ &\leq \frac{\delta}{2}E_3(t) + C, \end{aligned}$$

where C depends on δ, L. We finally get

$$\frac{d}{dt}E_3(t) + \frac{\delta}{2}E_3(t) \leq C,$$

and then applying the Gronwall lemma and calling (3.36) we get the result. □

Theorem 3.2. *Assume that (1.2)–(1.5), (2.2)–(2.3), (3.16)–(3.19) hold. Then A_t is bounded in $\mathcal{H}_t^1$, with a bound independent of t.*

Proof. From (3.34) and Lemma 3.5, for all $t \in \mathbb{R}$, it yields

$$\lim_{\tau\to-\infty} \text{dist}_t(U(t,\tau)A_\tau, K_t^1) = 0,$$

where

$$K_t^1 = \{z \in \mathcal{H}_t^1 : \|z\|_{\mathcal{H}_t^1} \leq M_1\}.$$

Since $\mathfrak{A}$ is invariant, this means

$$\text{dist}_t(A_t, K_t^1) = 0.$$

Hence, $A_t \subset \overline{K_t^1} = K_t^1$, and we get that A_t is bounded in $\mathcal{H}_t^1$ with a bound independent of $t \in \mathbb{R}$. □

Lemma 3.6. *For any $\tau \in \mathbb{R}, z = (u, u_t, \eta^t) \in A_t$, there exists a positive constant C, such that*

$$\sup_{t\geq\tau}\{\|u_t\|_1^2 + \|u\|_2^2 + \varepsilon(t)\|u_t\|_2^2 + \|\eta^t\|_{\mu,2}^2 + \int_\tau^\infty \|u_t(y)\|_2^2 dy\} \leq C. \tag{3.37}$$

Proof. Similar to the proof of Remark 3.1, we can easily get the result. □

4. Asymptotic structure of the time-dependent attractor

In this section we investigate the relationship between the time-dependent global attractor of $U(t,\tau)$ for problem (2.4) and the global attractor of the limit equation formally corresponding to (2.4) when $t \to +\infty$. If $\varepsilon(t) = 0$ in (2.4), we can obtain the following wave equation:

$$\begin{cases} u_{tt} - \triangle u - \triangle u_t - \int_0^\infty \mu(s)\triangle\eta^t(s)ds + f(u) = g(x), & x \in \Omega, \quad t > 0, \\ \eta_t^t + \eta_s^t = u_t, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad x \in \partial\Omega. \end{cases} \tag{4.1}$$

Within our assumptions on Sections 1 and 2, it is well known that Eq (4.1) generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ acting on the space $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega))$ associated with the problem (4.1), such that $S(t)\{u_0,u_1,\eta_0\}=\{u(t),u_t(t),\eta^t(s)\}$ is the solution of (4.1), where $\{u_0,u_1,\eta_0\}$ is the initial data of (4.1). Furthermore, $\{S(t)\}_{t\geq 0}$ admits the (classical) global attractor A_∞ in the space of $H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega))$. See [4, 17] for details.

Also, we know that, for any fixed $s\in\mathbb{R}$,

$$A_\infty=\{\omega(s):\omega\ \text{ CBT }\ \text{of }\ S(t)\},$$

where $\omega:\mathbb{R}\to H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega))$ is called a CBT of $S(t)$.

Next, we establish the asymptotic closeness of the time-dependent global attractor $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ of the process generated by (2.4) and the global attractor A_∞ of the semigroup $\{S(t)\}_{t\geq 0}$ generated by (4.1).

That is, we can obtain the following result.

Theorem 4.1. *Under the assumptions (1.2)–(1.5), (2.2)–(2.3), (3.16)–(3.19), the following limits holds*

$$\lim_{t\to\infty}\operatorname{dist}_{H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega))}(\Pi_t A_t,A_\infty)=0. \tag{4.2}$$

To prove (4.2), we need to prove the following Lemma which is based on Lemma 4.1.

Lemma 4.1. *For any sequence $z_n=(u_n,\partial_t u_n,\eta_n^t)$ of CBT for the process $U(t,\tau)$ and any $t_n\to\infty$, there exists a CBT $y=(w,w_t,\xi^t)$ of the semigroup $S(t)$ such that, for every $T>0$,*

$$\sup_{t\in[-T,T]}\|u_n(t+t_n)-w(t)\|_1\to 0, \tag{4.3}$$

$$\sup_{t\in[-T,T]}\|\partial_t u_n(t+t_n)-w_t(t)\|\to 0, \tag{4.4}$$

and

$$\sup_{t\in[-T,T]}\|\eta_n^{t+t_n}(s)-\xi^t(s)\|_{\mu,1}\to 0, \tag{4.5}$$

as $n\to\infty$, up to a subsequence.

Proof. From (3.37), for every $T>0$, $u_n(\cdot+t_n)$ is bounded in $L^\infty(-T,T,H^2)\cap W^{1,2}(-T,T,H_0^1(\Omega))$, and $\partial_t u_n(\cdot+t_n)$ is bounded in $L^\infty(-T,T,H^1)\cap L^2(-T,T,H_0^2(\Omega))\cap W^{1,2}(-T,T,H(\Omega))$. Then by direct application of Corollary 5 in [18] show that $(u_n(\cdot+t_n),\partial_t u_n(\cdot+t_n))$ is relatively compact in $C([-T,T],H_0^1(\Omega)\times L^2(\Omega))$. □

In addition, by Remark 3.3 and together with (3.37), we know that the sequence $\eta_t^{\cdot+t_n}(s)$ is bounded in the space $L^\infty(-T,T;L_\mu^2(\mathbb{R}^+,H^2)\cap H_\mu^1(\mathbb{R}^+,H_0^1(\Omega)))$, so $\eta_t^{\cdot+t_n}(s)$ is relatively compact in $C([-T,T],$ $L_\mu^2(\mathbb{R}^+,H_0^1(\Omega)))$. Hence there exists a function

$$(w(\cdot),w_t(\cdot),\xi^{\cdot}(s))=y:\mathbb{R}\times\mathbb{R}\times(\mathbb{R},\mathbb{R}^+)\to H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega)),$$

such that

$$u_n(\cdot+t_n)\to w(\cdot),\quad \partial_t u_n(\cdot+t_n)\to w_t(\cdot),\quad \eta_n^{\cdot+t_n}(s)\to\xi^{\cdot}(s),$$

hold.

In particular, $y = (w, w_t, \xi^t(s)) \in C(\mathbb{R}\times\mathbb{R}\times(\mathbb{R},\mathbb{R}^+), H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega)))$. Also, recalling (3.36), we have

$$\sup_{t\in\mathbb{R}} \|y(t)\|_{H_0^1(\Omega)\times L^2(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega))} \le C. \tag{4.6}$$

We are left to show that y solves (4.1). Define

$$v_n(t) = u_n(t + t_n), \varepsilon_n(t) = \varepsilon(t + t_n), \xi_n^t(s) = \eta_n^{t+t_n}(s);$$

then, we write Eq (2.4) of $(v_n(t), \partial_t v_n, \xi_n^t(s))$ in the form

$$\begin{cases} \partial_{tt} v_n - \triangle v_n - \triangle\partial_t v_n - \varepsilon_n(t)\triangle\partial_{tt} v_n - \int_0^\infty \mu(s)\triangle\xi_n^t(s)ds + f(v_n) = g, \\ \partial_t\xi_n^t + \partial_s\xi_n^t = \partial_t v_n. \end{cases}$$

We first prove that the sequence $\varepsilon_n(t)\triangle\partial_{tt}v_n$ converges to zero in the distributional sense. Indeed, for every fixed $T > 0$ and every smooth H-valued function φ supported on $(-T, T)$, we have

$$\int_{-T}^{T} \varepsilon_n(t)\langle\Delta\partial_{tt}v_n, \varphi(t)\rangle dt = -\int_{-T}^{T} \varepsilon_n(t)\langle\Delta\partial_t v_n, \varphi(t)\rangle dt - \int_{-T}^{T} \varepsilon_n'(t)\langle\Delta\partial_t v_n, \varphi(t)\rangle dt.$$

Then, exploiting (3.37) again, we get

$$\begin{aligned} |\int_{-T}^{T} \varepsilon_n(t)\langle\Delta\partial_{tt}v_n, \varphi(t)\rangle dt| &\le c\int_{-T}^{T} \sqrt{\varepsilon_n(t)}\sqrt{\varepsilon_n(t)}\|\Delta\partial_t v_n\|dt \\ &\quad + C\int_{-T}^{T} \frac{|\varepsilon_n'(t)|}{\sqrt{\varepsilon_n(t)}}\sqrt{\varepsilon_n(t)}\|\Delta\partial_t v_n\|dt \\ &\le C\int_{-T}^{T} \sqrt{\varepsilon_n(t)}\sqrt{\varepsilon_n(t)}\|\partial_t v_n\|_2 dt \\ &\quad + C\int_{-T}^{T} \frac{|\varepsilon_n'(t)|}{\sqrt{\varepsilon_n(t)}}\sqrt{\varepsilon_n(t)}\|\partial_t v_n\|_2 dt \\ &\le C\int_{-T}^{T} \sqrt{\varepsilon_n(t)}dt + C\int_{-T}^{T} \frac{|\varepsilon_n'(t)|}{\sqrt{\varepsilon_n(t)}}dt \\ &\le CT\sup_{t\in[-T,T]} \sqrt{\varepsilon_n(t)} + C(\sqrt{\varepsilon_n(-T)} - \sqrt{\varepsilon_n(T)}), \end{aligned}$$

where the constant $C > 0$ also depends on φ. Since

$$\lim_{n\to\infty}[\sup_{t\in[-T,T]} \varepsilon_n(t)] = 0,$$

we reach the desired conclusion

$$\lim_{n\to\infty}\int_{-T}^{T} \varepsilon_n(t)\langle\triangle\partial_{tt}v_n, \varphi(t)\rangle dt = 0.$$

Now, taking into account (1.4), for every $T > 0$, we have the convergence

$$\triangle v_n + f(v_n) \to \triangle w + f(w),$$

in the topology of $L^\infty(-T,T;H^{-1})$ for every $T>0$. At the same time, the convergences

$$\partial_{tt}v_n(t)-\partial_t\triangle v_n(t)\to w_{tt}(t)-\triangle w_t(t),\ \partial_t\xi_n^t(s)\to\partial_t\xi^t(s),$$

hold (up to subsequence) in the distributional sense. Therefore, we end up with the equality

$$w_{tt}-\triangle w-\triangle w_t-\int_0^\infty\mu(s)\triangle\xi^t(s)ds+f(w)=g(x),$$

which together with (4.6), proves that $y(t)$ is a CBT of the semigroup $S(t)$.

Proof. Proof of Theorem 4.1. According to Lemma 4.1, for our problem, we can apply Lemma 2.1 with $\mathcal{X}=H_0^1(\Omega)\times L_\mu^2(\mathbb{R}^+,H_0^1(\Omega)),\mathcal{Y}_t=H_0^1(\Omega)$, the latter space endowed with the norm $\|\cdot\|_{\mathcal{Y}_t}=\sqrt{\varepsilon(t)}\|\cdot\|_1+\|\cdot\|$. Combining with Lemma 2.1, the result here should include the convergence of u_t in the space $L^2(\Omega)$, namely, (4.4). Consequently, we complete the proof of Theorem 4.1. □

5. Conclusions

Based on the theory of time-dependent attractor in time-dependent space, we discussed the asymptotic compactness for the nonlinear evolution equation with linear memory. By the method of operator decomposition, which overcoming the difficulty caused by the degenerate coefficient and memory term, and then the regularity and asymptotic structure of the time-dependent attractor are also proved, that means the combination for time-dependent attractor with the global attractor of the limit wave equation when the coefficient $\varepsilon(t)\to 0$ as $t\to\infty$.

Acknowledgments

This work was supported partially by the National Natural Science Foundation of China (No. 11961059, 12101502).

Conflict of interest

The authors declare no conflict of interest.

References

1. I. L. Bogolubskey, Some examples of inelastic soliton interaction, *Comput. phys. Commun.*, **13** (1977), 149–155. https://doi.org/10.1016/0010-4655(77)90009-1
2. W. G. Zhu, Nonlinear waves in elastic rods, *Acta Solid Mech. Sin.*, **1** (1980), 247–253.
3. P. A. Clarkson, R. J. Levegue, R. Saxton, Solitary-wave interaction in elastic rods, *Stud. Appl. Math.*, **75** (1986), 95–121. https://doi.org/10.1002/sapm198675295
4. V. Pata, A. Zucchi, Attractor for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.*, **11** (2001), 505–529.
5. Y. Q. Xie, C. K. Zhong, The existence of global attractors for a class nonlinear evolution equation, *J. Math. Anal. Appl.*, **336** (2007), 54–69. https://doi.org/10.1016/j.jmaa.2006.03.086

6. Y. Q. Xie, C. K. Zhong, Asymptotic behavior of a class of nonlinear evolution equations, *Nonlinear Anal.*, **71** (2009), 5095–5105. https://doi.org/10.1016/j.na.2009.03.086
7. A. N. Carvalho, J. W. Cholewa, Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equations of the second order in time, *Trans. Amer. Math. Soc.*, **361** (2009), 2567–2588.
8. C. Y. Sun, L. Yang, J. Q. Duan, Asymptotic behavior for a semilinear second order evolution equation, *Trans. Amer. Math. Soc.*, **363** (2011), 6085–6109.
9. F. Di Plinio, G. S. Duane, R. Temam, Time dependent attractor for the oscillon equation, *Discrete Contin. Dyn. Syst.*, **29** (2011), 141–167. https://doi.org/10.3934/dcds.2011.29.141
10. F. J. Meng, M. H. Yang, C. K. Zhong, Attractors for wave equations with nonlinear damping on time-dependent space, *Discrete Contin. Dyn. Syst. Ser. B*, **21** (2016), 205–225. http://doi.org/10.3934/dcdsb.2016.21.205
11. F. J. Meng, C. C. Liu, Necessary and sufficient conditions for the existence of time-dependent global attractor and application, *J. Math. Phys.*, **58** (2017), 032702. https://doi.org/10.1063/1.4978329
12. M. Conti, V. Pata, R. Temam, Attractors for processes on time-dependent spaces. Applications to wave equations, *J. Differ. Equ.*, **255** (2013), 1254–1277. https://doi.org/10.1016/j.jde.2013.05.013
13. M. Conti, V. Pata, Asymptotic structure of the attractor for processes on time-dependent spaces, *Nonlinear Anal.-Real*, **19** (2014), 1–10. https://doi.org/10.1016/j.nonrwa.2014.02.002
14. Q. Z. Ma, J. Wang, T. T. Liu, Time-dependent asymptotic behavior of the solution for wave equations with linear memory, *Comput. Math. Appl.*, **76** (2018), 1372–1387. https://doi.org/10.1016/j.camwa.2018.06.031
15. Q. Z. Ma, J. Wang, T. T. Liu, Time-dependent attractor of wave equations with nonlinear damping and linear memory, *Open Math.* **17** (2019), 89–103. https://doi.org/10.1515/math-2019-0008
16. T. T. Liu, Q. Z. Ma, Time-dependent global attractor of nonlinear evolution equation with nonlinear damping, *J. Sichuan Univ.*, **54** (2017), 917–924.
17. S. Borini, V. Pata, Uniform attractors for a strongly damped wave equation with linear memory, *Asymptot. Anal.*, **20** (1999), 263–277.
18. J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pur. Appl.*, **146** (1987), 65–96.

AIMS Press