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Research article

A coupled system of p-Laplacian implicit fractional differential equations depending on boundary conditions of integral type

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Abstract: The objective of this article is to investigate a coupled implicit Caputo fractional p-Laplacian system, depending on boundary conditions of integral type, by the substitution method. The Avery-Peterson fixed point theorem is utilized for finding at least three solutions of the proposed coupled system. Furthermore, different types of Ulam stability, i.e., Hyers-Ulam stability, generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability, are achieved. Finally, an example is provided to authenticate the theoretical result.

Keywords: p-Laplacian operator; coupled system of fractional differential equations; positive solutions; stability in the form of Ulam
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1. Introduction

Derivatives of integer orders are the particular form of fractional order derivatives. When the notation $\frac{d^n}{dt^n}$ was presented by Leibniz at the end of 17^{th} century for n^{th} order derivative, L'Hospital asked from him can we take $n = \frac{1}{2}$? Leibniz replied "this is an apparent Paradox, from which one day useful consequences will be drawn", and this was the origin of fractional derivatives Riemann-Liouville derivative was originated after the great contributions of mathematicians Fourier and Laplace, and due to this notion of fractional derivative the fractional calculus was developed. After that, researchers showed interest in fractional calculus [4–9, 13, 17–20, 28]. Fractional derivatives are universal operators, which cover different physical phenomena and areas of mathematical modeling such as: control theory [26], dynamical process [24], electro-chemistry [16], image and signal processing [22], mathematical biology [21], etc.

The existence of solutions is the basic subject for the investigation of the fractional differential equations. Numerous mathematicians worked on the existence of solutions for different Boundary Value Problems, Using different fixed point theorems (see [23, 29, 32]). In [2] Ahmad *et al.* studied the existence of coupled fractional differential equations involving a p-Laplacian operator by applying fixed point theorems. In [11, 12], the authors got theoretical results related to fractional differential equations with p-Laplacian operator (p-LO).

It can be easily observed that obtaining the exact solution of non linear models is a tough and challenging task, to overcome the difficulty, a lot of approximation methods were established. Hyers-Ulam Stability (HUS) can predict the gap between exactness and approximations of the solutions. This notion was initiated (in 1940) by Ulam [25] and further extended by Hyers to abstract spaces, after one year. Recent results shows that different models with different boundary conditions are investigated for HUS [3, 15, 30, 31].

Mohammed et al. [14], explored the existence and uniqueness of solution of a model which involves the Caputo-Katugampola fractional derivative:

$$\begin{cases} {}^cD^{\beta}(\phi_p[{}^cD_1^{\alpha,\rho}\varpi(\varrho)]) = A(\phi_p[{}^cD_1^{\alpha,\rho}\varpi(\varrho)]) + f(\varrho, \varpi(\varrho), {}^cD_1^{\alpha,\rho}\varpi(\varrho)) = 0, \ \varrho \in [\varrho_0, T], \\ \varpi(\varrho_0) + B_1\varpi(T) = C_1 \int_{\varrho_0}^{T} g_1(s, \varpi(s), {}^cD_1^{\gamma,\rho}\varpi(s))ds, \\ \varpi'(\varrho_0) + B_2\varpi'(T) = C_2 \int_{\varrho_0}^{T} g_1(s, \varpi(s), {}^cD_1^{\gamma,\rho}\varpi(s))ds, \\ \phi_p({}^cD_1^{\alpha,\rho}\varpi(\varrho_0)) = \varpi_0, \quad (\phi_p({}^cD_1^{\alpha,\rho}\varpi(\varrho_0)))' = \varpi_1, \end{cases}$$

where ${}^cD^{\beta}$ is a Caputo fractional derivative of order $\beta \in (1, 2)$, ${}^cD_1^{\gamma,\rho}$, $\gamma \in (0, 1)$ and ${}^cD^{\alpha,\rho}$, $\alpha \in (1, 2)$, $\rho > 0$ are Caputo-Katugampola fractional derivative, ϕ_p, $p > 1$ is a p-LO.

Hira et al. [27], investigated the existence, uniqueness and HU stability of solutions to nonlinear coupled FDEs:

$$\begin{cases} {}^cD_{0^+}^{\beta_1}(L_p({}^cD_{0^+}^{\alpha_1}\varpi(\varrho))) = A_1(\varrho)\varpi(\varrho) + \Phi(\varrho, \varpi(\varrho), {}^cD_{0^+}^{\beta_1}(L_p({}^cD_{0^+}^{\alpha_1}\omega(\varrho)))), \\ {}^cD_{0^+}^{\beta_2}(L_p({}^cD_{0^+}^{\alpha_2}y(\varrho))) = A_2(\varrho)\omega(\varrho) + \Psi(\varrho, {}^cD_0^{\beta_2}(L_p({}^cD_{0^+}^{\alpha_2}\varpi(\varrho))), \omega(\varrho)), \ \varrho \in \mathcal{T}, \\ {}^cD_{0^+}^{\alpha_1}\varpi(0)) = \varpi(0) = \varpi''(0) = 0, \\ \varpi'(T) = C_1 \int_0^T g_1(s, \varpi(s), {}^cD_{0^+}^{\beta_1}(L_p({}^cD_{0^+}^{\alpha_1}\omega(s)))), \\ {}^cD_{0^+}^{\alpha_2}\omega(0)) = \omega(0) = \omega''(0) = 0, \\ \omega'(T) = C_2 \int_0^T g_2(s, {}^cD_{0^+}^{\beta_1}(L_p({}^cD_{0^+}^{\alpha_1}\varpi(s))), \omega(s)), \end{cases} \tag{1.1}$$

where $2 < \alpha_i \le 3$, $0 < \beta_1 \le 1$, η_i, $\gamma_i > 0$, $\psi_i \in L[0, T]$, and ${}^cD_{0^+}^{\alpha_i}$ and ${}^cD_{0^+}^{\beta_i}$ are the Caputo derivatives of order α_i and β_i, $i = 1, 2$, respectively. $L_p(s) = |s|^{p-2}s$ is a p-LO, where $\frac{1}{p} + \frac{1}{q} = 1$, and L_q denotes inverse of p-Laplacian. $A_i : \mathcal{T} \to \Re$ are closed bounded linear operators for any $\rho \in \mathcal{T} = [0, T]$, and Φ, Ψ, $g_k : \mathcal{T} \times \Re \times \Re \to \Re$, $(k = 1, 2)$ are continuous functions $i = 1, 2$. $C_k \in \Re^{n\times n}$, $(k = 1, 2)$.

In [12], Lu et al. investigated the nonlinear fractional BVP with p-Laplacian operator

$$\begin{cases} D_{0+}^{\mathcal{B}}[\vartheta_p D_{0+}^{\mathcal{Z}}X(\alpha)] = F(\alpha, X(\alpha)), \ \ 0 < \alpha < 1, \\ X(0) = X'(0) = X'(1) = 0, \\ D_{0+}^{\mathcal{Z}}X(0) = D_{0+}^{\mathcal{Z}}X(1)) = 0, \end{cases}$$

where $\mathcal{Z} \in (2, 3]$, and $\mathcal{B} \in (1, 2]$. $D^{\mathcal{Z}}$, $D^{\mathcal{B}}$ represents the standard Riemann-Liouville fractional derivatives.

In [2], Ahmad et al. investigated the system:

$$\begin{cases} {}^cD^{\mathcal{Z}_1}[\vartheta_p D^{\mathcal{B}} X(\alpha)] + K_1(\alpha)F_1(\alpha, X(\alpha), {}^c D^{\mathcal{Z}_2}[\vartheta_p(D^{\mathcal{B}} Y(\alpha)]) = 0, \quad \alpha \in]0,1[, \\ {}^cD^{\mathcal{Z}_2}[\vartheta_p D^{\mathcal{B}} Y(\alpha)] + K_2(\alpha)F_2(\alpha, {}^c D^{\mathcal{Z}_1}[\vartheta_p D^{\mathcal{B}} X(\alpha)], Y(\alpha)) = 0, \quad \alpha \in]0,1[, \\ \Big([\vartheta_p D^{\mathcal{B}} X(\alpha)]\Big)^i = \Big([\vartheta_p (D^{\mathcal{B}} Y(\alpha)]\Big)^i = 0, \quad i = \overline{0, m-1}, \\ I^{k-\mathcal{B}} X(0) = I^{k-\mathcal{B}} Y(0) = 0, \quad k = \overline{2, m}, \\ D^{\delta}(X(1)) = D^{\delta}(Y(1)) = 0, \end{cases}$$

where ${}^cD^{\mathcal{Z}}$ and $D^{\mathcal{B}}$ respectively denotes the Caputo and Riemann-Liouville FD of order $\mathcal{Z}$ and $\mathcal{B}$, $m-1 < \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{B} \leq m$, $m \in \{4, 5, \dots\}$, and $1 < \delta \leq 2$, $K_1(\cdot)$, $K_2(\cdot)$ are linear and bounded operators on R.

Zhang et al. [33], studied the existence of

$$\begin{cases} {}^cD^{\mathcal{B}}\vartheta_p({}^cD^{\mathcal{Z}}X(\alpha)) + F(\alpha, {}^c D^{\mathcal{B}} X(\alpha)) = 0, \quad \alpha \in (0,1), \\ \Big(\vartheta_p({}^cD^{\mathcal{Z}}X(0))\Big)^i = \vartheta_p({}^cD^{\mathcal{Z}}X(1)) = 0, \quad i = 1, 2, \dots, m-1 =: \overline{1, m-1}, \\ X(0) + X'(0) = \int_0^1 S_1(\varpi)X(\varpi)d\varpi + a, \quad X(1) + X'(1) = \int_0^1 S_2(\varpi)X(\varpi)d\varpi + b, \\ X^j(0) = 0, \quad j = \overline{2, n-1}, \end{cases}$$

where $\mathcal{Z}$ and $\mathcal{B}$ are the orders of Caputo fractional derivatives ${}^cD^{\mathcal{Z}}$ and ${}^cD^{\mathcal{B}}$ respectively.

Following [33], in this article, we present existence and stability analysis of the model of the form

$$\begin{cases} {}^cD^{\mathcal{B}_1}\vartheta_p({}^cD^{\mathcal{Z}_1}X(\alpha)) - F_1(\alpha, {}^cD^{\gamma_1}X(\alpha), {}^cD^{\gamma_2}Y(\alpha)) = 0, \quad \alpha \in (0,1), \\ {}^cD^{\mathcal{B}_2}\vartheta_p({}^cD^{\mathcal{Z}_2})Y(\alpha) - F_2(\alpha, {}^cD^{\gamma_1}X(\alpha), {}^cD^{\gamma_2}Y(\alpha)) = 0, \quad \alpha \in (0,1), \\ \Big(\vartheta_p({}^cD^{\mathcal{Z}_1}X(0))\Big)^i = \vartheta_p({}^cD^{\mathcal{Z}_1}X(1)) = 0, \quad i = \overline{1, m-1}, \\ \Big(\vartheta_p({}^cD^{\mathcal{Z}_2}Y(0))\Big)^i = \vartheta_p({}^cD^{\mathcal{Z}_2}Y(1)) = 0, \quad i = \overline{1, m-1}, \\ X(0) + X'(0) = \int_0^1 S_1(\varpi)X(\varpi)d\varpi + \int_0^1 \mathtt{H}_1(\varpi)d\varpi, \\ X(1) + X'(1) = \int_0^1 S_2(\varpi)X(\varpi)d\varpi + \int_0^1 \mathtt{H}_2(\varpi)d\varpi, \\ Y(0) + Y'(0) = \int_0^1 S_3(\varpi)Y(\varpi)d\varpi + \int_0^1 \mathtt{H}_3(\varpi)d\varpi, \\ Y(1) + Y'(1) = \int_0^1 S_4(\varpi)Y(\varpi)d\varpi + \int_0^1 \mathtt{H}_4(\varpi)d\varpi, \\ X^j(0) = 0, \quad Y^j(0) = 0, \quad j = \overline{2, n-1}, \end{cases} \tag{1.2}$$

where $1 < m-1 < \mathcal{B}_1; \mathcal{B}_2 < m$; $1 < n-1 < \mathcal{Z}_1; \mathcal{Z}_2 < n$; $\mathcal{Z}_1 - \mathcal{B}_1; \mathcal{Z}_2 - \mathcal{B}_2 > 1$, $\mathcal{Z}_1; \mathcal{Z}_2; \mathcal{B}_1$, and $\mathcal{B}_2$ be the orders of Caputo fractional derivatives ${}^cD^{\mathcal{Z}_1}$, ${}^cD^{\mathcal{Z}_2}$, ${}^cD^{\mathcal{B}_1}$ and ${}^cD^{\mathcal{B}_2}$ respectively, and $0 < \gamma_1, \gamma_2 \leq 1$. The

p-Laplacian operator is represented by ϑ_p and is defined as $\vartheta_p(\varpi) = |\varpi|^{p-2}\varpi$, $p > 1$, $\vartheta_p^{-1} = \vartheta_q$, $\frac{1}{p}+\frac{1}{q}$=1. $S_1, S_2, S_3, S_4 \in C([0,1], R^+)$, F_1, and F_2 are appropriate functions. $\mathtt{H}_1, \mathtt{H}_2, \mathtt{H}_3, \mathtt{H}_4 \in C([0,1], R^+)$ are perturbation functions.

($\mathfrak{C}_1$) $0 < \int_0^1 \mathtt{H}_1(\varpi)d\varpi < \int_0^1 \mathtt{H}_2(\varpi)d\varpi < 2\int_0^1 \mathtt{H}_1(\varpi)d\varpi < +\infty$, $0 \le S_1(\alpha) \le S_2(\alpha) \le 2S_1(\alpha)$, $0 \le \int_0^1 S_1(\varpi)d\varpi$, $\int_0^1 S_2(\varpi)d\varpi < 1$, $0 \le S_3(\alpha) \le S_4(\alpha) \le 2S_3(\alpha)$, $0 \le \int_0^1 S_3(\varpi)d\varpi$, $\int_0^1 S_4(\varpi)d\varpi < 1$.

The remaining manuscript is as follows: Section 2 contains basic definitions, auxiliary lemmas and related theorems. In Section 3, we present the existence theory for the problem (1.2) and gives some related properties of Green function. In Section 4, we obtain at least three positive solutions of coupled system (1.2) by using the Avery-Peterson fixed point theorem. Section 5 contains HU type stability results, and Section 6 provide an example to authenticate the theoretical result.

2. Preliminaries

Here, we are presenting an important literature concerning the Caputo fractional derivatives and integral, which gives us help throughout this article, for the details, reader should study [1, 10].

Definition 2.1. *Let $X \in L^1([0,T], R^+)$ be a function. Then the $\mathcal{Z}$ order fractional integral is defined by*

$$I^{\mathcal{Z}}X(\varpi) = \frac{1}{\Gamma(\mathcal{Z})}\int_0^{\alpha}(\alpha-\varpi)^{\mathcal{Z}-1}X(\varpi)d\varpi \quad \alpha > 0, \quad \mathcal{Z} > 0,$$

on the condition that the integral on right side exists, where Γ is the Euler Gamma function, defined as

$$\Gamma(\mathcal{Z}) = \int_0^{\infty}\alpha^{\mathcal{Z}-1}e^{-\alpha}d\alpha.$$

Definition 2.2. *Let $X : [0,T] \to R$ be a function. Then the $\mathcal{Z} >$ order fractional derivative is defined as*

$${}^cD^{\mathcal{Z}}X(\alpha) = \frac{1}{\Gamma(n-\mathcal{Z})}\int_0^{\alpha}(\alpha-\varpi)^{n-\mathcal{Z}-1}X^n(\varpi)d\varpi,$$

on the condition that the integral on the right side exists, and $[\mathcal{Z}]$ denote the integer part of a real number $\mathcal{Z}$, where $n = 1 + [\mathcal{Z}]$.

Definition 2.3. *Let $X : [0,T] \to R$ be a function. Then the $\mathcal{Z}$ order sequential fractional derivative is defined as:*

$$D^{\mathcal{Z}}X(\alpha) = D^{\mathcal{Z}_1}D^{\mathcal{Z}_2}D^{\mathcal{Z}_3}\dots D^{\mathcal{Z}_m}X(\alpha),$$

where $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \dots, \mathcal{Z}_m)$ is any multi-index, and the operator $D^{\mathcal{Z}}$ can either be Riemann-Liouville or Caputo or any other kind of integro-differential operator.

Lemma 2.1. *For any $\mathcal{Z} > 0$, the Caputo FDE ${}^cD^{\mathcal{Z}}X(\alpha) = 0$ has a solution of the form*

$$X(\alpha) = e_0 + e_1\alpha + e_2\alpha^2 + \cdots + e_{n-1}\alpha^{n-1},$$

where $e_i \in R$, $i = \overline{0, n-1}$, and $n = 1 + [\mathcal{Z}]$.

Lemma 2.2. *For any $\mathcal{Z} > 0$, we have*

$$I^{\mathcal{Z}}({}^cD^{\mathcal{Z}}X(\alpha)) = X(\alpha) + e_0 + e_1\alpha + e_2\alpha^2 + \cdots + e_{n-1}\alpha^{n-1},$$

where $e_i \in R, i = \overline{0, n-1}$, and $n = 1 + [\mathcal{Z}]$.

Lemma 2.3. *[33] Let $F_1 \in C([0,1], R)$, and $1 < m-1 < \mathcal{B}_1 < m$. Then*

$$\begin{cases} D^{\mathcal{B}_1}\omega(\alpha) = -F_1(\alpha), & 0 < \alpha < 1, \\ \omega(1) = \omega^i(0) = 0, & i = \overline{1, m-1}, \end{cases} \tag{2.1}$$

has a solution (unique)

$$\omega(\alpha) = \int_0^1 K_{\mathcal{B}_1}(\alpha, \varpi)F_1(\varpi)d\varpi,$$

where

$$K_{\mathcal{B}_1}(\alpha, \varpi) = \frac{1}{\Gamma(\mathcal{B}_1)}\begin{cases} (1-\varpi)^{\mathcal{B}_1-1} + (\alpha-\varpi)^{\mathcal{B}_1-1}, & 0 \le \varpi \le \alpha \le 1, \\ (1-\varpi)^{\mathcal{B}_1-1}, & 0 \le \alpha \le \varpi \le 1. \end{cases} \tag{2.2}$$

Let

$$M_{S_1} = \int_0^1 S_1(\varpi)d\varpi, \quad M'_{S_1} = \int_0^1 \varpi S_1(\varpi)d\varpi, \quad M_{S_2} = \int_0^1 S_2(\varpi)d\varpi, \quad M'_{S_2} = \int_0^1 \varpi S_2(\varpi)d\varpi,$$

$$M_{S_3} = \int_0^1 S_3(\varpi)d\varpi, \quad M'_{S_3} = \int_0^1 \varpi S_3(\varpi)d\varpi, \quad M_{S_4} = \int_0^1 S_4(\varpi)d\varpi, \quad M'_{S_4} = \int_0^1 \varpi S_4(\varpi)d\varpi,$$

$$\eta_1(\alpha) = \frac{(2-\alpha)\delta_{S_1} + (\alpha-1)\delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})}{1 - \delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})(M'_{S_1} - M_{S_1})}, \tag{2.3}$$

$$\eta_2(\alpha) = \frac{(2-\alpha)\delta_{S_1}\delta_{S_2}(M'_{S_1} - M_{S_1}) + (\alpha-1)\delta_{S_2}}{1 - \delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})(M'_{S_1} - M_{S_1})}, \tag{2.4}$$

$$\begin{aligned} \psi_1(\alpha) = &(2-\alpha)\int_0^1 \mathrm{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathrm{H}_2(\varpi)d\varpi \\ &+ \int_0^1 \Big[\eta_1(\alpha)S_1(\varpi) + \eta_2(\alpha)S_2(\varpi)\Big] \times \Big[(2-\varpi)\int_0^1 \mathrm{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathrm{H}_2(\theta)d\theta\Big]d\varpi, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \psi_2(\alpha) = &(2-\alpha)\int_0^1 \mathrm{H}_3(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathrm{H}_4(\varpi)d\varpi \\ &+ \int_0^1 \Big[\eta_3(\alpha)S_3(\varpi) + \eta_4(\alpha)S_4(\varpi)\Big] \times \Big[(2-\varpi)\int_0^1 \mathrm{H}_3(\theta)d\theta + (\varpi-1)\int_0^1 \mathrm{H}_4(\theta)d\theta\Big]d\varpi. \end{aligned} \tag{2.6}$$

From $(\mathfrak{C}_1)$, we know for $\alpha \in (0,1)$,

$$\begin{aligned} &2S_1(\alpha) > S_2(\alpha) > \alpha S_2(\alpha) > \alpha S_1(\alpha), \\ &2S_3(\alpha) > S_4(\alpha) > \alpha S_4(\alpha) > \alpha S_3(\alpha), \end{aligned}$$

and

$$1 > \int_0^1 S_2(\varpi)d\varpi > \int_0^1 \varpi S_2(\varpi)d\varpi > \int_0^1 \varpi S_1(\varpi)d\varpi > 0,$$

$$1 > \int_0^1 S_2(\varpi)d\varpi > \int_0^1 S_1(\varpi)d\varpi > \int_0^1 \varpi S_1(\varpi)d\varpi > 0,$$
$$1 > \int_0^1 S_4(\varpi)d\varpi > \int_0^1 \varpi S_4(\varpi)d\varpi > \int_0^1 \varpi S_3(\varpi)d\varpi > 0,$$
$$1 > \int_0^1 S_4(\varpi)d\varpi > \int_0^1 S_3(\varpi)d\varpi > \int_0^1 \varpi S_3(\varpi)d\varpi > 0,$$
$$\int_0^1 2S_1(\varpi)d\varpi > \int_0^1 S_2(\varpi)d\varpi, \quad \int_0^1 2\varpi S_1(\varpi)d\varpi > \int_0^1 \varpi S_2(\varpi)d\varpi,$$
$$\int_0^1 S_3(\varpi)d\varpi > \int_0^1 S_4(\varpi)d\varpi, \quad \int_0^1 2\varpi S_3(\varpi)d\varpi > \int_0^1 \varpi S_4(\varpi)d\varpi.$$

Implies

$$1 > M_{S_2} > M'_{S_2} > M'_{S_1} > 0, \;\; 1 > M_{S_2} > M_{S_1} > M'_{S_1} > 0, \;\; 2M_{S_1} > M_{S_2}, \;\; 2M'_{S_1} > M'_{S_2}, \tag{2.7}$$
$$1 > M_{S_4} > M'_{S_4} > M'_{S_3} > 0, \;\; 1 > M_{S_4} > M_{S_3} > M'_{S_3} > 0, \;\; 2M_{S_3} > M_{S_4}, \;\; 2M'_{S_3} > M'_{S_4}. \tag{2.8}$$

3. Existence of solutions

For our results we need an assumption and the lemma.

$(\mathfrak{C}_2)$ $\delta_{S_1}^{-1} = 1 - 2M_{S_1} + M'_{S_1} \neq 0$, $\quad \delta_{S_2}^{-1} = 1 + M_{S_2} - M'_{S_2} \neq 0$ and $\delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})(M'_{S_1} - M_{S_1}) \neq 1$.

Lemma 3.1. *Let $(\mathfrak{C}_2)$ hold, $n-1 < \mathcal{Z}_1 \leq n$ and $h_1 : \mathcal{J} \to R$ are is an appropriate function. The BVP:*

$$\begin{cases} {}^cD^{\mathcal{Z}_1}X(\alpha) - h_1(\alpha) = 0, \quad \alpha \in (0,1), \\ X(0) + X'(0) = \int_0^1 S_1(\varpi)X(\varpi)d\varpi + \int_0^1 \mathtt{H}_1(\varpi)d\varpi, \\ X(1) + X'(1) = \int_0^1 S_2(\varpi)X(\varpi)d\varpi + \int_0^1 \mathtt{H}_2(\varpi)d\varpi, \\ X^j(0) = 0, \quad j = \overline{2, n-1}, \end{cases} \tag{3.1}$$

has solution(unique) of the form

$$X(\alpha) = \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)h_1(\varpi)d\varpi + \psi_1(\alpha), \tag{3.2}$$

where

$$G_{\mathcal{Z}_1}(\alpha,\varpi) = G_1(\alpha,\varpi) + G_2(\alpha,\varpi), \tag{3.3}$$

$$G_1(\alpha,\varpi) = \frac{1}{\Gamma(\mathcal{Z}_1)} \begin{cases} (\mathcal{Z}_1 - \varpi)(1-\alpha)(1-\varpi)^{\mathcal{Z}_1-2} + (\alpha-\varpi)^{\mathcal{Z}_1-1}, & 0 \leq \varpi \leq \alpha \leq 1, \\ (\mathcal{Z}_1 - \varpi)(1-\alpha)(1-\varpi)^{\mathcal{Z}_1-2}, & 0 \leq \alpha \leq \varpi \leq 1, \end{cases} \tag{3.4}$$

and

$$G_2(\alpha,\varpi) = \eta_1(\alpha)\Big(\int_0^1 S_1(\theta)G_1(\varpi,\theta)d\theta\Big) + \eta_2(\alpha)\Big(\int_0^1 S_2(\theta)G_1(\varpi,\theta)d\theta\Big). \tag{3.5}$$

Proof. Consider

$$^{c}D^{\mathcal{Z}_1}X(\alpha) = h_1(\alpha). \tag{3.6}$$

Using Lemma 2.2, we get

$$X(\alpha) = \frac{1}{\Gamma(\mathcal{Z}_1)} \int_0^{\alpha} (\alpha - \varpi)^{\mathcal{Z}_1 - 1} h_1(\varpi) d\varpi + e_0 + e_1\alpha + \cdots + e_{n-1}\alpha^{n-1}. \tag{3.7}$$

Using boundary conditions of (3.1), we get

$$\begin{aligned} e_0 =& 2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi + 2\int_0^1 S_1(\varpi)X(\varpi)d\varpi \\ &- \int_0^1 S_2(\varpi)X(\varpi)d\varpi + \frac{1}{\Gamma(\mathcal{Z}_1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-1}h_1(\varpi)d\varpi + \frac{1}{\Gamma(\mathcal{Z}_1-1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-2}h_1(\varpi)d\varpi, \\ e_1 =& -\int_0^1 \mathtt{H}_1(\varpi)d\varpi + \int_0^1 \mathtt{H}_2(\varpi)d\varpi - \int_0^1 S_1(\varpi)X(\varpi)d\varpi + \int_0^1 S_2(\varpi)X(\varpi)d\varpi \\ &- \frac{1}{\Gamma(\mathcal{Z}_1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-1}h_1(\varpi)d\varpi - \frac{1}{\Gamma(\mathcal{Z}_1-1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-2}h_1(\varpi)d\varpi, \end{aligned}$$

and $e_2 = e_3 = \cdots = e_{n-1} = 0$. Putting e's in (3.7) and

$$\begin{aligned} X(\alpha) =& (2-\alpha)\int_0^1 S_1(\varpi)X(\varpi)d\varpi + (\alpha-1)\int_0^1 S_2(\varpi)X(\varpi)d\varpi + (2-\alpha)\int_0^1 \mathtt{H}_1(\varpi)d\varpi \\ &+ (\alpha-1)\int_0^1 \mathtt{H}_2(\varpi)d\varpi + \frac{1-\alpha}{\Gamma(\mathcal{Z}_1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-1}h_1(\varpi)d\varpi + \frac{1-\alpha}{\Gamma(\mathcal{Z}_1-1)}\int_0^1 (1-\varpi)^{\mathcal{Z}_1-2}h_1(\varpi)d\varpi \\ &+ \frac{1}{\Gamma(\mathcal{Z}_1)}\int_0^t (\alpha-\varpi)^{\mathcal{Z}_1-1}h_1(\varpi)d\varpi \\ =& \int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + (2-\alpha)\int_0^1 S_1(\varpi)X(\varpi)d\varpi + (\alpha-1)\int_0^1 S_2(\varpi)X(\varpi)d\varpi \\ &+ (2-\alpha)\int_0^1 \mathtt{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathtt{H}_2(\varpi)d\varpi \\ =& \int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + (2-\alpha)\mathcal{A}_{S_1} + (\alpha-1)\mathcal{A}_{S_2} + (2-\alpha)\int_0^1 \mathtt{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathtt{H}_2(\varpi)d\varpi, \end{aligned} \tag{3.8}$$

where $G_1(\alpha,\varpi)$ is given in (3.4), $\mathcal{A}_{S_1} = \int_0^1 S_1(\varpi)X(\varpi)d\varpi$ and $\mathcal{A}_{S_2} = \int_0^1 S_2(\varpi)X(\varpi)d\varpi$.

In view of (3.8), we get

$$\begin{aligned} S_1(\alpha)X(\alpha) =& S_1(\alpha)\int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + (2-\alpha)\mathcal{A}_{S_1}S_1(\alpha) + (\alpha-1)\mathcal{A}_{S_2}S_1(\alpha) \\ &+ (2-\alpha)S_1(\alpha)\int_0^1 \mathtt{H}_1(\varpi)d\varpi + (\alpha-1)S_1(\alpha)\int_0^1 \mathtt{H}_2(\varpi)d\varpi. \end{aligned} \tag{3.9}$$

Integrating (3.9) from 0 to 1, we obtain

$$\mathcal{A}_{S_1} = \int_0^1 S_1(\varpi)X(\varpi)d\varpi$$

$$
\begin{aligned}
&= \int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h(\theta)d\theta\Big)d\varpi + (2\mathcal{A}_{S_1} - \mathcal{A}_{S_2})\int_0^1 S_1(\varpi)d\varpi + (\mathcal{A}_{S_2} \\
&\quad - \mathcal{A}_{S_1})\int_0^1 \varpi S_1(\varpi)d\varpi + 2\int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi \\
&\quad + \int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi \\
&= \int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi + (2\mathcal{A}_{S_1} - \mathcal{A}_{S_2})M_{S_1} + (\mathcal{A}_{S_2} - \mathcal{A}_{S_1})M'_{S_1} \\
&\quad + 2\int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi \\
&\quad + \int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi.
\end{aligned}
$$

Implies that

$$
\begin{aligned}
\mathcal{A}_{S_1} =& \delta_{S_1}\int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi + \delta_{S_1}\mathcal{A}_{S_2}(M'_{S_1} - M_{S_1}) \\
&+ 2\delta_{S_1}\int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \delta_{S_1}\int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi \\
&+ \delta_{S_1}\int_0^1 \varpi S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \delta_{S_1}\int_0^1 S_1(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi.
\end{aligned}
\tag{3.10}
$$

Similarly, we obtain

$$
\begin{aligned}
\mathcal{A}_{S_2} &= \int_0^1 S_2(\varpi)X(\varpi)d\varpi \\
&= \int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi + (2\mathcal{A}_{S_1} - \mathcal{A}_{S_2})\int_0^1 S_2(\varpi)d\varpi + (\mathcal{A}_{S_2} - \mathcal{A}_{S_1})\int_0^1 \varpi S_2(\varpi)d\varpi \\
&\quad + 2\int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi \\
&\quad + \int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi \\
&= \int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi + (2\mathcal{A}_{S_1} - \mathcal{A}_{S_2})M_{S_2} + (\mathcal{A}_{S_2} - \mathcal{A}_{S_1})M'_{S_2} \\
&\quad + 2\int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi \\
&\quad + \int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi.
\end{aligned}
$$

Implies that

$$
\mathcal{A}_{S_2} = \delta_{S_2}\int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi + \delta_{S_2}\mathcal{A}_{S_1}(2M_{S_2} - M'_{S_2})
$$

$$+ 2\delta_{S_2}\int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi - \delta_{S_2}\int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta d\varpi$$
$$+ \delta_{S_2}\int_0^1 \varpi S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi - \delta_{S_2}\int_0^1 S_2(\varpi)\int_0^1 \mathtt{H}_2(\theta)d\theta d\varpi. \tag{3.11}$$

From (3.10) and (3.12), we get

$$\begin{aligned}\mathcal{A}_{S_1} =& \frac{\delta_{S_1}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{\delta_{S_1}\delta_{S_2}(M'_{S_1}-M_{S_1})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{\delta_{S_1}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta\\ &+ (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_1(\varpi)d\varpi + \frac{\delta_{S_1}\delta_{S_2}(M'_{S_1}-M_{S_1})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\\ &\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_2(\varpi)d\varpi\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathcal{A}_{S_2} =& \frac{\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{\delta_{S_2}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_1(\varpi)d\varpi\\ &+ \frac{\delta_{S_2}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_2(\varpi)d\varpi.\end{aligned} \tag{3.13}$$

Putting (3.12) and (3.13) in (3.8), we get

$$\begin{aligned}X(\alpha) =& \int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + (2-\alpha)\int_0^1 \mathtt{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathtt{H}_2(\varpi)d\varpi \qquad (3.14)\\ &+ \frac{(2-\alpha)\delta_{S_1} + (\alpha-1)\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_1(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{(2-\alpha)\delta_{S_1}\delta_{S_2}(M'_{S_1}-M_{S_1}) + (\alpha-1)\delta_{S_2}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 S_2(\varpi)\Big(\int_0^1 G_1(\varpi,\theta)h_1(\theta)d\theta\Big)d\varpi\\ &+ \frac{(2-\alpha)\delta_{S_1} + (\alpha-1)\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_1(\varpi)d\varpi\\ &+ \frac{(2-\alpha)\delta_{S_1}\delta_{S_2}(M'_{S_1}-M_{S_1}) + (\alpha-1)\delta_{S_2}}{1-\delta_{S_1}\delta_{S_2}(2M_{S_2}-M'_{S_2})(M'_{S_1}-M_{S_1})}\int_0^1 [(2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta]S_2(\varpi)d\varpi.\end{aligned}$$

From (2.3) and (2.4), we can write (3.14) as

$$
\begin{aligned}
X(\alpha) = & \int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + (2-\alpha)\int_0^1 \mathrm{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathrm{H}_2(\varpi)d\varpi \\
& + \int_0^1 \eta_1(\alpha)\Big(\int_0^1 S_1(\theta)G_1(\theta,\varpi)d\theta\Big)h_1(\varpi)d\varpi + \int_0^1 \eta_2(\alpha)\Big(\int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta\Big)h_1(\varpi)d\varpi \\
& + \int_0^1 \eta_1(\alpha)[(2-\varpi)\int_0^1 \mathrm{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathrm{H}_2(\theta)d\theta]S_1(\varpi)d\varpi \\
& + \int_0^1 \eta_2(\alpha)[(2-\varpi)\int_0^1 \mathrm{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathrm{H}_2(\theta)d\theta]S_2(\varpi)d\varpi \\
= & \int_0^1 G_1(\alpha,\varpi)h(\varpi)d\varpi + (2-\alpha)\int_0^1 \mathrm{H}_1(\varpi)d\varpi + (\alpha-1)\int_0^1 \mathrm{H}_2(\varpi)d\varpi \\
& + \int_0^1 \Bigg[\eta_1(\alpha)\Big(\int_0^1 S_1(\theta)G_1(\theta,\varpi)d\theta\Big)h_1(\varpi) + \eta_2(\alpha)\Big(\int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta\Big)h_1(\varpi) \\
& + \Big(\eta_1(\alpha)S_1(\varpi) + \eta_2(\alpha)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathrm{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathrm{H}_2(\theta)d\theta\Big)\Bigg]d\varpi \\
= & \int_0^1 G_1(\alpha,\varpi)h_1(\varpi)d\varpi + \int_0^1 G_2(\alpha,\varpi)h_1(\varpi)d\varpi + \psi_1(\alpha) \\
= & \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)h_1(\varpi)d\varpi + \psi_1(\alpha).
\end{aligned}
$$

□

Lemma 3.2. *The coupled BVP* (1.2) *is equivalent to the following system of integral equations*

$$
\begin{cases}
X(\alpha) = \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha), \\
Y(\alpha) = \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_2(\alpha),
\end{cases} \tag{3.15}
$$

where $G_{\cdot}(\alpha,\varpi)$ *and* $K_{\cdot}(\alpha,\varpi)$ *are given by* (3.3) *and* (2.2)*.*

Proof. Using Lemmas 2.3 and 3.1, set $\omega(\alpha) = \vartheta_p(D^{\mathcal{Z}_1}X(\alpha))$ and $h_1(\alpha) = F_1(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))$, and we have

$$X(\alpha) = \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha).$$

Similarly, we can set $\omega(\alpha) = \vartheta_p(D^{\mathcal{Z}_2}Y(\alpha))$ and $h_2(\alpha) = F_2(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))$, we obtain

$$Y(\alpha) = \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_2(\alpha).$$

On the other hand, if (X, Y) satisfy (3.15), we can easily prove that (X, Y) satisfy the pair of boundary value problem (1.2). Hence, the proof is completed. □

Lemma 3.3. *We use* $G_{\mathcal{Z}}(\alpha,\varpi) = (G_{\mathcal{Z}_1}(\alpha,\varpi), G_{\mathcal{Z}_2}(\alpha,\varpi))$ *and* $K_{\mathcal{B}}(\alpha,\varpi) = (K_{\mathcal{B}_1}(\alpha,\varpi), K_{\mathcal{B}_2}(\alpha,\varpi))$ *as the Green's functions of the proposed system* (1.2) *having the following properties:*

(I) $K_{\mathcal{B}}(\alpha,\varpi) \geq 0$ *is continuous for all* $\alpha,\varpi \in [0,1]$*;*

(II) $K_{\mathcal{B}}(\alpha,\varpi) \leq K_{\mathcal{B}}(s,\varpi)$ *for all* $\alpha,\varpi \in [0,1]$*;*

(III) $\int_0^1 K_{\mathcal{B}_1}(\alpha,\varpi)d\varpi = \frac{1-\alpha^{\mathcal{B}_1}}{\Gamma(\mathcal{B}_1+1)} \leq \frac{1}{\Gamma(\mathcal{B}_1+1)}$ *for all* $\alpha,\varpi \in [0,1]$*;*
$\int_0^1 K_{\mathcal{B}_2}(\alpha,\varpi)d\varpi = \frac{1-\alpha^{\mathcal{B}_2}}{\Gamma(\mathcal{B}_2+1)} \leq \frac{1}{\Gamma(\mathcal{B}_2+1)}$ *for all* $\alpha,\varpi \in [0,1]$*;*

(IV) $G_{\mathcal{Z}}(\alpha,\varpi) \geq 0$ *is continuous for all* $\alpha,\varpi \in [0,1]$*;*

(V) $\psi(\alpha) = (\psi_1(\alpha), \psi_2(\alpha)) \geq 0$ *for all* $\alpha \in [0,1]$*;*

Proof. The proofs of (I) and (II) can be seen in [33].

(III) For $\alpha \in [0,1]$, we have

$$\int_0^1 K_{\mathcal{B}_1}(\alpha,\varpi)d\varpi = \int_0^y \Big((1-\varpi)^{\mathcal{B}_1-1} + (\alpha-\varpi)^{\mathcal{B}_1-1}\Big)d\varpi + \int_y^1 (1-\varpi)^{\mathcal{B}_1-1}d\varpi = \frac{1-\alpha^{\mathcal{B}_1}}{\Gamma(\mathcal{B}_1+1)} \leq \frac{1}{\Gamma(\mathcal{B}_1+1)}.$$

Also,

$$\int_0^1 K_{\mathcal{B}_2}(\alpha,\varpi)d\varpi = \int_0^y \Big((1-\varpi)^{\mathcal{B}_2-1} + (\alpha-\varpi)^{\mathcal{B}_2-1}\Big)d\varpi + \int_y^1 (1-\varpi)^{\mathcal{B}_2-1}d\varpi = \frac{1-\alpha^{\mathcal{B}_2}}{\Gamma(\mathcal{B}_2+1)} \leq \frac{1}{\Gamma(\mathcal{B}_2+1)}.$$

(IV) Firstly, from (3.4), one get $G_1(\alpha,\varpi) \geq 0$, $\alpha,\varpi \in [0,1]$, and from ($\mathfrak{C}_1$), $\delta > 0$ and $G_1(\alpha,\varpi) \geq 0$, we have

$$\begin{aligned}
\frac{\partial G_2(\alpha,\varpi)}{\partial t} =& \frac{-\delta_{S_1} + \delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})}{1 - \delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})(M'_{S_1} - M_{S_1})} \int_0^1 S_1(\theta)G_1(\theta,\varpi)d\theta \\
&+ \frac{-\delta_{S_1}\delta_{S_2}(M'_{S_1} - M_{S_1}) + \delta_{S_2}}{1 - \delta_{S_1}\delta_{S_2}(2M_{S_2} - M'_{S_2})(M'_{S_1} - M_{S_1})} \int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta \\
=& \delta \int_0^1 \delta_{S_1}\delta_{S_2}[(2M_{S_2} - M'_{S_2} - \delta_{S_2}^{-1})S_1(\theta) - (M'_{S_1} - M_{S_1} + \delta_{S_1}^{-1})S_2(\theta)]G_1(\theta,\varpi)d\theta > 0.
\end{aligned}$$

Thus $G_2(\alpha,\varpi)$ is a increasing on $\alpha \in [0,1]$.

Utilizing (3.5), we get

$$\begin{aligned}
G_2(\alpha,\varpi) \geq& G_2(0,\varpi) = \eta_1(0)\Big(\int_0^1 S_1(\theta)G_1(\theta,\varpi)d\theta\Big) + \eta_2(0)\Big(\int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta\Big) \\
&\geq \int_0^1 (\eta_1(0)S_1(\theta) + \eta_2(0)\frac{1}{2}S_1(\theta))G_1(\theta,\varpi)d\theta \\
&= \int_0^1 (\eta_1(0) + \frac{\eta_2(0)}{2})S_1(\theta)G_1(\theta,\varpi)d\theta \geq 0.
\end{aligned}$$

So that $G_{\mathcal{Z}_1}(\alpha,\varpi) \geq 0$. Similarly $G_{\mathcal{Z}_2}(\alpha,\varpi) \geq 0$. Hence, $G_{\mathcal{Z}}(\alpha,\varpi) \geq 0$.

(V) From $(\mathfrak{C}_1)$ and (2.5), we know

$$\psi_1'(\alpha) = -\int_0^1 \mathtt{H}_1(\varpi)d\varpi + \int_0^1 \mathtt{H}_2(\varpi)d\varpi + \eta_1'(\alpha)\int_0^1 S_1(\varpi)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi$$
$$+ \eta_2'(\alpha)\int_0^1 S_2(\varpi)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi > 0, \quad \alpha \in [0,1].$$

Thus $\psi_1(\alpha)$, is increasing on $\alpha \in [0,1]$.

From (2.5), we have

$$\psi_1(\alpha) \geq \psi_1(0) = 2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi$$
$$+ \int_0^1 \Big(\eta_1(0)S_1(\varpi) + \eta_2(0)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi > 0, \quad \alpha \in [0,1].$$

Similarly $\psi_1(\alpha) \geq 0$, and hence $\psi(\alpha) \geq 0$. □

Lemma 3.4. *For $\kappa \in (0, \frac{1}{2})$, let*

$$\max_{\alpha\in[0,1]} G_{\mathcal{Z}_1}(\alpha,\varpi) \leq G_1(0,\varpi) + G_2(1,\varpi),$$
$$\max_{\alpha\in[0,1]} G_{\mathcal{Z}_2}(\alpha,\varpi) \leq G_3(0,\varpi) + G_4(1,\varpi),$$
$$\min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_1}(\alpha,\varpi) \geq \rho_{\mathcal{Z}_1}(G_1(0,\varpi) + G_2(1,\varpi)),$$
$$\min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_2}(\alpha,\varpi) \geq \rho_{\mathcal{Z}_2}(G_3(0,\varpi) + G_4(1,\varpi)),$$

where $\rho_{\mathcal{Z}_1} = \frac{2M'_{S_1} - M'_{S_2}}{1 + M_{S_2} + M'_{S_1} - M_{S_1}}$ *and* $\rho_{\mathcal{Z}_2} = \frac{2M'_{S_3} - M'_{S_4}}{1 + M_{S_4} + M'_{S_3} - M_{S_3}}$.

Proof. First Step. We need to get

$$\min_{\alpha\in[0,\kappa]} G_1(\alpha,\varpi) \geq (1-\kappa)G_1(0,\varpi) > \frac{1}{2}\max_{\alpha\in[0,1]} G_1(\alpha,\varpi). \tag{3.16}$$

For $\varpi < \alpha$, where $\varpi \in [0,1)$ and $\alpha \in [0,\kappa]$, we have

$$\Gamma(\mathcal{Z}_1)\frac{\partial G_1(\alpha,\varpi)}{\partial\alpha} = -(\mathcal{Z}_1 - \varpi)(1-\varpi)^{\mathcal{Z}_1-2} + (\mathcal{Z}_1 - 1)(\alpha-\varpi)^{\mathcal{Z}_1-2} < 0,$$

which implies that $G_1(\alpha,\varpi)$ is decreasing function monotonically with respect to $\alpha \in [\varpi,\kappa]$, so that

$$G_1(\alpha,\varpi) \leq G_1(s,\varpi) = (\mathcal{Z}_1 - \varpi)(1-\varpi)^{\mathcal{Z}_1-1} < (\mathcal{Z}_1 - \varpi)(1-\varpi)^{\mathcal{Z}_1-2} = G_1(0,\varpi)$$

and

$$\frac{G_1(\alpha,\varpi)}{G_1(0,\varpi)} = \frac{(\mathcal{Z}_1-\varpi)(1-\alpha)(1-\varpi)^{\mathcal{Z}_1-2} + (\alpha-\varpi)^{\mathcal{Z}_1-1}}{(\mathcal{Z}_1-\varpi)(1-\varpi)^{\mathcal{Z}_1-2}}$$
$$= 1 - \alpha + \frac{(\alpha-\varpi)^{\mathcal{Z}_1-1}}{(\mathcal{Z}_1-\varpi)(1-\varpi)^{\mathcal{Z}_1-2}}$$

$$\geq 1-\alpha$$
$$\geq 1-\kappa > \frac{1}{2}.$$

Now if $\varpi \geq \alpha$ and $\alpha \in [0,\kappa]$,

$$\Gamma(\mathcal{Z}_1)\frac{\partial G_1(\alpha,\varpi)}{\partial \alpha} = -(\mathcal{Z}_1-\varpi)(1-\varpi)^{\mathcal{Z}_1-2} < 0,$$

which implies that $G_1(\alpha,\varpi)$ is a monotone decreasing function with respect to $\alpha \in [0,\varpi]$ and $\alpha \in [0,\kappa]$, so that

$$G_1(\alpha,\varpi) \leq G_1(0,\varpi) = (\mathcal{Z}_1-\varpi)(1-\varpi)^{\mathcal{Z}_1-2}$$

and

$$\begin{aligned}\frac{G_1(\alpha,\varpi)}{G_1(0,\varpi)} &= \frac{(\mathcal{Z}_1-\varpi)(1-\alpha)(1-\varpi)^{\mathcal{Z}_1-2}}{(\mathcal{Z}_1-\varpi)(1-\varpi)^{\mathcal{Z}_1-2}}\\ &= 1-\alpha \geq 1-\kappa > \frac{1}{2}.\end{aligned}$$

Thus, (3.16) is satisfied.

Similarly, we get

$$\min_{\alpha\in[0,\kappa]} G_3(\alpha,\varpi) \geq (1-\kappa)G_3(0,\varpi) > \frac{1}{2}\max_{\alpha\in[0,1]} G_3(\alpha,\varpi). \tag{3.17}$$

Step 2. We prove

$$\min_{\alpha\in[0,\kappa]} G_2(\alpha,\varpi) \geq \rho_{\mathcal{Z}_1} G_2(1,\varpi) = \rho_{\mathcal{Z}_1}\max_{\alpha\in[0,1]} G_2(\alpha,\varpi). \tag{3.18}$$

From Lemma 3.3, we know that $G_2(\alpha,\varpi)$ is increasing for $\alpha \in [0,1]$, in such a way that

$$\min_{\alpha\in[0,\kappa]} G_2(\alpha,\varpi) = G_2(0,\varpi), \quad \max_{\alpha\in[0,1]} G_2(\alpha,\varpi) = G_2(1,\varpi).$$

By $(\mathfrak{C}_1)$ and (2.7), we have

$$\begin{aligned}\frac{G_2(0,\varpi)}{G_2(0,\varpi)} &= \frac{\int_0^1 (\eta_1(0)S_1(\theta)+\eta_2(0)S_2(\theta))G_1(\theta,\varpi)d\theta}{\int_0^1 (\eta_1(1)S_1(\theta)+\eta_2(1)S_2(\theta))G_1(\theta,\varpi)d\theta}\\ &\geq \frac{\int_0^1 (\frac{1}{2}\eta_1(0)S_2(\theta)+\eta_2(0)S_2(\theta))G_1(\theta,\varpi)d\theta}{\int_0^1 (\frac{1}{2}\eta_1(1)S_2(\theta)+\eta_2(1)S_2(\theta))G_1(\theta,\varpi)d\theta}\\ &= \frac{(\frac{1}{2}\eta_1(0)+\eta_2(0))\int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta}{(\frac{1}{2}\eta_1(1)+\eta_2(1))\int_0^1 S_2(\theta)G_1(\theta,\varpi)d\theta}\\ &= \frac{2M'_{S_1}-M'_{S_2}}{1+M_{S_2}+M'_{S_1}-M_{S_1}}\end{aligned}$$

$$= \rho_{\mathcal{Z}_1}.$$

Obviously, $\rho_{\mathcal{Z}_1} > 0$, and

$$\rho_{\mathcal{Z}_1} - \frac{1}{2} = \frac{2M'_{S_1} - M'_{S_2}}{1 + M_{S_2} + M'_{S_1} - M_{S_1}} - \frac{1}{2} = \frac{M_{S_1} + 3M'_{S_1} - M_{S_2} - 2M'_{S_2} - 1}{2(1 + M_{S_2} + M'_{S_1} - M_{S_1})} < 0.$$

Thus $0 < \rho_{\mathcal{Z}_1} < \frac{1}{2}$ and (3.18) is satisfied.

By the same procedure,

$$\min_{\alpha\in[0,\kappa]} G_4(\alpha,\varpi) \geq \rho_{\mathcal{Z}_2} G_4(1,\varpi) = \rho_{\mathcal{Z}_2} \max_{\alpha\in[0,1]} G_4(\alpha,\varpi). \tag{3.19}$$

Finally, from (3.16) and (3.18), we can easily show that the following results hold:

$$\begin{aligned}\max_{\alpha\in[0,1]} G_{\mathcal{Z}_1}(\alpha,\varpi) &\leq \max_{\alpha\in[0,1]} G_1(\alpha,\varpi) + \max_{\alpha\in[0,1]} G_2(\alpha,\varpi)\\ &= G_1(0,\varpi) + G_2(1,\varpi),\end{aligned}$$

and

$$\begin{aligned}\min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_1}(\alpha,\varpi) &\geq \min_{\alpha\in[0,\kappa]} G_1(\alpha,\varpi) + \min_{\alpha\in[0,\kappa]} G_2(\alpha,\varpi)\\ &\geq \frac{1}{2} G_1(0,\varpi) + \rho_{\mathcal{Z}_1} G_2(1,\varpi)\\ &> \rho_{\mathcal{Z}_1}(G_1(0,\varpi) + G_2(1,\varpi))\\ &\geq \rho_{\mathcal{Z}_1} \max_{\alpha\in[0,1]} G_{\mathcal{Z}_1}(\alpha,\varpi).\end{aligned}$$

Similarly, from (3.17) and (3.19), we can also easily show that

$$\max_{\alpha\in[0,1]} G_{\mathcal{Z}_2}(\alpha,\varpi) \leq G_3(0,\varpi) + G_4(1,\varpi),$$

and

$$\min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_2}(\alpha,\varpi) > \rho_{\mathcal{Z}_2} \max_{\alpha\in[0,1]} G_{\mathcal{Z}_2}(\alpha,\varpi).$$

□

Lemma 3.5. *If* $(\mathfrak{C}_1)$ *is satisfied, then* $\psi(\alpha)$ *holds the following properties:*

(I) $\psi_1(\alpha) \leq \psi_1(1) = \max_{\alpha\in[0,1]} \psi_1(\alpha)$ *and* $\psi_2(\alpha) \leq \psi_2(1) = \max_{\alpha\in[0,1]} \psi_2(\alpha)$;

(II) $\min_{\alpha\in[0,\kappa]} \psi_1(\alpha) \geq \rho_{\mathcal{Z}_1} \max_{\alpha\in[0,1]} \psi_1(\alpha)$ *and* $\min_{\alpha\in[0,\kappa]} \psi_2(\alpha) \geq \rho_{\mathcal{Z}_2} \max_{\alpha\in[0,1]} \psi_2(\alpha)$.

Proof. Using Lemma 3.3 and (2.5), implies $\psi_1(\alpha)$ is increasing on $\alpha \in [0,1]$, and thus

$$\begin{aligned}\min_{\alpha\in[0,\kappa]} \psi_1(\alpha) = \psi_1(0) =& 2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi\\ &+ \int_0^1 \Big(\eta_1(0)S_1(\varpi) + \eta_2(0)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi,\end{aligned}$$

$$\max_{\alpha\in[0,1]} \psi_1(\alpha) = \psi_1(1)$$
$$= \int_0^1 \mathtt{H}_1(\varpi)d\varpi + \int_0^1 \Big(\eta_1(1)S_1(\varpi) + \eta_2(1)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi$$

and

$$\frac{\psi_1(0)}{\psi_1(1)}$$
$$=\frac{2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi + \int_0^1 \Big(\eta_1(0)S_1(\varpi) + \eta_2(0)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi}{\int_0^1 \mathtt{H}_1(\varpi)d\varpi + \int_0^1 \Big(\eta_1(1)S_1(\varpi) + \eta_2(1)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi} \tag{3.20}$$

$$\geq\frac{2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi + \int_0^1 \Big(\frac{1}{2}\eta_1(0)S_2(\varpi) + \eta_2(0)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi}{\int_0^1 \mathtt{H}_1(\varpi)d\varpi + \int_0^1 \Big(\frac{1}{2}\eta_1(1)S_2(\varpi) + \eta_2(1)S_2(\varpi)\Big)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi}$$
$$=\frac{2\int_0^1 \mathtt{H}_1(\varpi)d\varpi - \int_0^1 \mathtt{H}_2(\varpi)d\varpi + \Big(\frac{1}{2}\eta_1(0) + \eta_2(0)\Big)\int_0^1 S_2(\varpi)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi}{\int_0^1 \mathtt{H}_1(\varpi)d\varpi + \Big(\frac{1}{2}\eta_1(1) + \eta_2(1)\Big)\int_0^1 S_2(\varpi)\Big((2-\varpi)\int_0^1 \mathtt{H}_1(\theta)d\theta + (\varpi-1)\int_0^1 \mathtt{H}_2(\theta)d\theta\Big)d\varpi}. \tag{3.21}$$

Setting $\mathtt{H}_2(0) = 2\mathtt{H}_1(0)$ and $\mathtt{H}_1(1) = 0$, then (3.20) implies

$$\frac{\psi_1(0)}{\psi_1(1)} \geq \frac{2M'_{S_1} - M'_{S_2}}{1 + M_{S_2} + M'_{S_1} - M_{S_1}}$$
$$= \rho_{\mathcal{Z}_1}.$$

Also,

$$\frac{\psi_2(0)}{\psi_2(1)} \geq \rho_{\mathcal{Z}_2}.$$

Thus,

$$\psi_1(\alpha) \leq \psi_1(1) = \max_{\alpha\in[0,1]} \psi_1(\alpha), \quad \min_{\alpha\in[0,\kappa]} \psi_1(\alpha) \geq \rho_{\mathcal{Z}_1} \max_{\alpha\in[0,1]} \psi_1(\alpha),$$
$$\psi_2(\alpha) \leq \psi_2(1) = \max_{\alpha\in[0,1]} \psi_2(\alpha), \quad \min_{\alpha\in[0,\kappa]} \psi_2(\alpha) \geq \rho_{\mathcal{Z}_2} \max_{\alpha\in[0,1]} \psi_2(\alpha).$$

□

4. Major findings

Suppose that μ_1 and μ_2 be nonnegative convex and continuous functionals on Θ, μ_3 be nonnegative concave and continuous functional on Θ and μ_4 be a nonnegative continuous functional on Θ. For $M_1, M_2, M_3, M_4 > 0$, let us introduce the convex sets:

$$\Theta(\mu_1, M_1) = \{(X,Y) \in \Theta : \mu_1(X,Y) < M_1\},$$

$$\Theta(\mu_1,\mu_3;M_3,M_1)=\{(X,Y)\in\Theta: M_3\le\mu_3(X,Y),\ \mu_1(X,Y)\le M_1\},$$
$$\Theta(\mu_1,\mu_2,\mu_3;M_3,M_2,M_1)=\{(X,Y)\in\Theta: M_3\le\mu_3(X,Y),\ \mu_2(X,Y)\le M_2,\ \mu_1(X,Y)\le M_1\},$$

and a closed set

$$\Theta(\mu_1,\mu_4;M_4,M_1)=\{(X,Y)\in\Theta: M_4\le\mu_4(X,Y),\ \mu_1(X,Y)\le M_1\}.$$

Lemma 4.1. *Let Θ be a cone in a real Banach space χ. Let μ_1 and μ_2 be nonnegative continuous convex functionals on Θ, μ_3 be nonnegative continuous concave functional on Θ and μ_4 be a nonnegative continuous functionals on Θ satisfying $\mu_4(\lambda(X,Y))\le\lambda\mu_4(X,Y)$ for $0\le\lambda\le 1$, such that for some positive numbers $\mathcal{N}$ and M_1,*

$$\mu_3(X,Y)\le\mu_4(X,Y),\ \|(X,Y)\|\le\mathcal{N}\mu_1(X,Y), \tag{4.1}$$

for all $(X,Y)\in\overline{\Theta(\mu_1,M_1)}$. Suppose

$$\mathcal{T}:\overline{\Theta(\mu_1,M_1)}\to\overline{\Theta(\mu_1,M_1)}$$

is completely continuous and there exist positive numbers M_2, M_3 and M_4 with $M_4<M_3$ such that

$(\mathfrak{C}_3)$: $\{(X,Y)\in\Theta(\mu_1,\mu_2,\mu_3;\ M_3,M_2,M_1):\mu_3(X,Y)>M_3\}\neq\emptyset$ and $\mu_3(X,Y)>M_3$ for $(X,Y)\in\Theta(\mu_1,\mu_2,\mu_3;\ M_3,M_2,M_1)$;

$(\mathfrak{C}_4)$: $\mu_3(\mathcal{T}(X,Y))>M_3$ for $(X,Y)\in\Theta(\mu_1,\mu_3;M_3,M_1)$ with $\mu_2(\mathcal{T}(X,Y))>M_4$;

$(\mathfrak{C}_5)$: $(0,0)\notin\Theta(\mu_1,\mu_4;M_4,M_1)$ and $\mu_4(\mathcal{T}(X,Y))<M_4$ for $(X,Y)\in\Theta(\mu_1,\mu_4;M_4,M_1)$ with $\mu_4(X,Y)=M_4$.

Then, $\mathcal{T}$ has at least three fixed points $(X_1,Y_1),(X_2,Y_2),(X_3,Y_3)\in\overline{\Theta(\mu_1,M_1)}$ such that

$$\mu_1(X_j,Y_j)\le M_1,\ j=1,2,3;\ \mu_3(X_1,Y_1)>M_3,\ M_4<\mu_3(X_2,Y_2),\ \mu_4(X_2,Y_2)<M_3,\ \mu_3(X_2,Y_2)<M_4.$$

Let $\chi=\{(X,Y)\in C([0,1],R)\times C([0,1],R)\}$ be endowed with the norm $\|(X,Y)\|=\max_{\alpha\in[0,1]}|(X,Y)|$, then χ is a Banach space.

We define a set $\Theta\subset\chi$ by

$$\Theta=\{(X,Y)\in\chi:\ (X(\alpha),Y(\alpha))\ge 0,\ \ \min_{\alpha\in[0,1]}(X,Y)(\alpha)\ge\rho\max_{\alpha\in[0,1]}(X,Y)(\alpha)\}.$$

For $(X,Y),(\widetilde{X},\widetilde{Y})\in\Theta$ and $\nu_1,\nu_2\ge 0$, it is easy to obtain that $\nu_1(X,Y)(\alpha)+\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\ge 0$, and

$$\begin{aligned}\min_{\alpha\in[0,\kappa]}\{\nu_1(X,Y)(\alpha)+\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\}&\ge\min_{\alpha\in[0,\kappa]}\{\nu_1(X,Y)(\alpha)\}+\min_{\alpha\in[0,\kappa]}\{\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\}\\&\ge\rho\max_{\alpha\in[0,1]}\{\nu_1(X,Y)(\alpha)\}+\rho\max_{\alpha\in[0,1]}\{\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\}\\&=\rho(\max_{\alpha\in[0,1]}\{\nu_1(X,Y)(\alpha)\}+\max_{\alpha\in[0,1]}\{\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\})\\&\ge\rho\max_{\alpha\in[0,1]}\{\nu_1(X,Y)(\alpha)+\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\}.\end{aligned}$$

Thus, for $(X,Y),(\widetilde{X},\widetilde{Y})\in\Theta$ and $\nu_1,\nu_2\ge 0$, $\nu_1(X,Y)(\alpha)+\nu_2(\widetilde{X},\widetilde{Y})(\alpha)\in\Theta$. And if $(X,Y)\in\Theta$, $(X,Y)\neq 0$, it can be seen that $-(X,Y)\notin\Theta$. Thus, Θ is a cone in χ.

Let $\mathcal{T} : \Theta \to \chi$ be an operator, i.e., $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$, defined by

$$\mathcal{T}_1(X,Y)(\alpha) = \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha),$$
$$\mathcal{T}_2(X,Y)(\alpha) = \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_2(\alpha).$$

Lemma 4.2. *If* $(\mathfrak{C}_1)$ *is true, then* $\mathcal{T} : \Theta \to \Theta$ *is a completely continuous operator.*

Proof. For $(X,Y) \in \Theta$, it is easy to know that $\mathcal{T}$ is a continuous operator, and $\mathcal{T}(X,Y)(\alpha) \geq 0$. By (3.15), we have

$$\begin{aligned}
&\min_{\alpha\in[0,\kappa]} \mathcal{T}_1(X,Y)(\alpha)\\
&= \min_{\alpha\in[0,\kappa]} \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \min_{\alpha\in[0,\kappa]} \psi_1(\alpha)\\
&= \int_0^1 \min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \min_{\alpha\in[0,\kappa]} \psi_1(\alpha)\\
&\geq \int_0^1 \rho_{\mathcal{Z}_1} \min_{\alpha\in[0,\kappa]} G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \rho_{\mathcal{Z}_1} \min_{\alpha\in[0,\kappa]} \psi_1(\alpha)\\
&\geq \rho_{\mathcal{Z}_1} \max_{\alpha\in[0,1]} \mathcal{T}_1(X,Y)(\alpha).
\end{aligned}$$

Similarly, for $\min_{\alpha\in[0,\kappa]} \mathcal{T}_2(X,Y)(\alpha) \geq \rho_{\mathcal{Z}_2} \max_{\alpha\in[0,1]} \mathcal{T}_2(X,Y)(\alpha)$. So, $\mathcal{T}(\Theta) \subseteq \Theta$.

Set $\Theta_\tau \subset \Theta$ be bounded, i.e., $\exists$ a constant $\tau > 0$ in such a way that $\|(X,Y)\| \leq \tau$, for every $(X,Y) \in \Theta_\tau$. Let $M_0 = \max_{\alpha\in[0,1],(X,Y)} |F_1(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))| > 0$
and $N_0 = \max_{\alpha\in[0,1],(X,Y)} |F_2(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))| > 0$. For $(X,Y) \in \Theta_\tau$, from Lemmas 3.3 and 3.4, we have

$$\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta) \leq \vartheta_q(M_0 \int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)d\theta) \leq \vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})$$

and

$$\begin{aligned}
|\mathcal{T}_1(X,Y)(\alpha)| &= |\int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha)|\\
&\leq \int_0^1 |G_{\mathcal{Z}_1}(\alpha,\varpi)|\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})d\varpi + |\psi_1(\alpha)|\\
&\leq \vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)}) \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)d\varpi + \psi_1(1)\\
&\leq \vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)}) \int_0^1 (G_1(\alpha,\varpi) + G_2(\alpha,\varpi))d\varpi + \psi_1(1) := M_{01}.
\end{aligned} \tag{4.2}$$

Similarly,

$$\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta) \leq \vartheta_q(N_0 \int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)d\theta) \leq \vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})$$

and

$$
\begin{aligned}
|\mathcal{T}_2(X,Y)(\alpha)| =& |\int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_2(\alpha)| \\
\leq & \int_0^1 |G_{\mathcal{Z}_2}(\alpha,\varpi)|\vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})d\varpi + |\psi_2(\alpha)| \\
\leq & \vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})\int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)d\varpi + \psi_2(1) \\
\leq & \vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})\int_0^1 (G_3(\alpha,\varpi)+G_4(\alpha,\varpi))d\varpi + \psi_2(1) := N_{01}.
\end{aligned} \tag{4.3}
$$

So that $\mathcal{T}(X,Y) \leq \max\{M_{01}, N_{01}\}$, which implies $\mathcal{T}(\Theta_\tau)$ is uniformly bounded. Next, we will show that $\mathcal{T}(\Theta_\tau)$ is equicontinuous. Since $G_{\mathcal{Z}}(\alpha,\varpi)$, $\psi(\alpha)$ are continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for any $\epsilon > 0$, there exists a constant $\varsigma_1 > 0$, such that

$$
|G_{\mathcal{Z}_1}(\alpha_1,\varpi) - G_{\mathcal{Z}_1}(\alpha_2,\varpi)| < \frac{\epsilon}{4\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})},
$$
$$
|\psi_1(\alpha_1) - \psi_1(\alpha_2)| < \frac{\epsilon}{4},
$$

for $\alpha_1, \alpha_2 \in [0,1]$ with $|\alpha_1 - \alpha_2| < \varsigma_1$. Therefore,

$$
\begin{aligned}
& |\mathcal{T}_1(X,Y)(\alpha_1) - \mathcal{T}_1(X,Y)(\alpha_2)| \\
=& |\int_0^1 G_{\mathcal{Z}_1}(\alpha_1,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha_1) \qquad (4.4) \\
& - \int_0^1 G_{\mathcal{Z}_1}(\alpha_2,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi + \psi_1(\alpha_2)| \\
\leq & \int_0^1 |G_{\mathcal{Z}_1}(\alpha_1,\varpi) - G_{\mathcal{Z}_1}(\alpha_2,\varpi)|\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi \\
& + |\psi_1(\alpha_1) - \psi_1(\alpha_2)| \\
\leq & \frac{\epsilon}{4\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})}\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)}) + \frac{\epsilon}{4} \\
=& \frac{\epsilon}{2}. \qquad (4.5)
\end{aligned}
$$

Similarly

$$
|\mathcal{T}_2(X,Y)(\alpha_1) - \mathcal{T}_2(X,Y)(\alpha_2)| \leq \frac{\epsilon}{2}. \tag{4.6}
$$

Combining (4.4) and (4.6), we obtain

$$
|\mathcal{T}(X,Y)(\alpha_1) - \mathcal{T}(X,Y)(\alpha_2)| \leq \epsilon.
$$

Implies, $\mathcal{T}(\Theta_\tau)$ is equicontinuous.

Thus Arzel*à*-Ascoli Theorem implies that $\mathcal{T}$ is a completely continuous operator.

□

Define the functionals

$$\mu_1(X,Y)=\|(X,Y)\|,\ \mu_2(X,Y)=\mu_4(X,Y)=\max_{\alpha\in[0,1]}|(X,Y)(\alpha)|,\ \mu_3(X,Y)=\min_{\alpha\in[0,\kappa]}|(X,Y)(\alpha)|,$$

then μ_1 and μ_2 are continuous nonnegative convex functionals, μ_3 is a continuous nonnegative concave functional, μ_4 is a continuous nonnegative functional, and

$$\rho\mu_2(X,Y)\le\mu_3(X,Y)\le\mu_2(X,Y)=\mu_4(X,Y),\ \|(X,Y)\|\le\mathcal{N}\mu_1(X,Y).$$

Thus, (4.1) in Lemma 4.1 is fulfilled.

Let

$$\begin{aligned} l_1&=\int_0^1 G_{\mathcal{Z}_1}(0,\varpi)-G_{\mathcal{Z}_1}(1,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)d\theta)d\varpi,\\ l_2&=\int_0^1 G_{\mathcal{Z}_2}(0,\varpi)-G_{\mathcal{Z}_2}(1,\varpi)\vartheta_q(\int_0^\kappa K_{\mathcal{B}_2}(\varpi,\theta)d\theta)d\varpi,\\ l_3&=\int_0^1 G_{\mathcal{Z}_1}(0,\varpi)-G_{\mathcal{Z}_1}(1,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)d\theta)d\varpi,\\ l_4&=\int_0^1 G_{\mathcal{Z}_2}(0,\varpi)-G_{\mathcal{Z}_2}(1,\varpi)\vartheta_q(\int_0^\kappa K_{\mathcal{B}_2}(\varpi,\theta)d\theta)d\varpi. \end{aligned}$$

Theorem 4.1. *Let* $(\mathfrak{C}_1)$ *is true, and* $\exists$ *constants* $M_1,M_3,M_4\ge\psi(1)$ *with* $M_4<M_3<\rho M_1\min\frac{l_2l_4}{l_1l_3}$ *and* $M_2=\frac{M_3}{\rho}$*, in such a way that*

$(\mathfrak{C}_6)$ $F_1(\alpha,X,Y)\le\vartheta_p(\frac{M_1-\psi_1(1)}{l_1})$ *and* $F_2(\alpha,X,Y)\le\vartheta_p(\frac{M_1-\psi_2(1)}{l_2})$, $(\alpha,X,Y)\in[0,1]\times[0,M_1]\times[0,M_1]$;

$(\mathfrak{C}_7)$ $F_1(\alpha,X,Y)>\vartheta_p(\frac{M_3-\rho_{\mathcal{Z}_1}\psi_1(1)}{\rho_{\mathcal{Z}_1}l_3})$ *and* $F_2(\alpha,X,Y)>\vartheta_p(\frac{M_3-\rho_{\mathcal{Z}_2}\psi_2(1)}{\rho_{\mathcal{Z}_2}l_4})$, $(\alpha,X,Y)\in[0,\kappa]\times[M_3,\frac{M_3}{\rho_{\mathcal{Z}_1}}]\times[0,M_1]$;

$(\mathfrak{C}_8)$ $F_1(\alpha,X,Y)<\vartheta_p(\frac{M_4-\psi_1(1)}{l_1})$ *and* $F_2(\alpha,X,Y)<\vartheta_p(\frac{M_4-\psi_2(1)}{l_2})$, $(\alpha,X,Y)\in[0,1]\times[0,M_4]\times[0,M_1]$.

Then, (1.2) *has at least three positive solutions* (X_1,Y_1), (X_2,Y_2), (X_3,Y_3) *satisfying*

$$\|(X_i,Y_i)\|\le M_1\ \ (i=1,2,3), \tag{4.7}$$

$$\begin{aligned} &\min_{\alpha\in[0,\kappa]}\|(X_1,Y_1)(\alpha)\|\le M_3,\ M_4<\min_{\alpha\in[0,\kappa]}\|(X_2,Y_2)(\alpha)\|,\\ &\max_{\alpha\in[0,1]}\|(X_2,Y_2)(\alpha)\|<M_3,\ \ \max_{\alpha\in[0,1]}\|(X_3,Y_3)(\alpha)\|<M_4. \end{aligned}\tag{4.8}$$

Proof. Obviously, the fixed points of $\mathcal{T}$ are equivalent to the solutions of (1.2). For $(X,Y)\in\overline{\Theta(\mu_1,M_1)}$, we get

$$\mu_1(X,Y)=\|(X,Y)\|\le M_1,$$

thus

$$\max_{\alpha\in[0,1]}|(X,Y)|\le M_1,$$

and then

$$0 \le (X, Y) \le M_1.$$

By $(\mathfrak{C}_6)$, we have

$$\begin{aligned}
\max_{\alpha\in[0,1]} |\mathcal{T}_1(X, Y)(\alpha)| &\le \max_{\alpha\in[0,1]} \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta\Big)d\varpi + \max_{\alpha\in[0,1]} \psi_1(\alpha)\\
&\le \int_0^1 (G_1(0, \varpi) - G_2(1, \varpi))\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta\Big)d\varpi + \psi_1(1)\\
&\le \int_0^1 (G_1(0, \varpi) - G_2(1, \varpi))\vartheta_q\Big(\vartheta_p\Big(\frac{M_1}{l_1}\Big) \int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)d\theta\Big)d\varpi + \psi_1(1)\\
&= \frac{M_1 - \psi_1(1)}{l_1} \int_0^1 (G_1(0, \varpi) - G_2(1, \varpi))\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)d\theta\Big)d\varpi + \psi_1(1)\\
&= M_1.
\end{aligned}$$

Similarly,

$$\max_{\alpha\in[0,1]} |\mathcal{T}_2(X, Y)(\alpha)| \le M_1.$$

So,

$$\mu_1(\mathcal{T}(X, Y)) = \|\mathcal{T}(X, Y)\| = \max_{\alpha\in[0,1]} |\mathcal{T}(X, Y)(\alpha)| \le M_1.$$

Thus, $\mathcal{T} : \overline{\Theta(\mu_1, M_1)} \to \overline{\Theta(\mu_1, M_1)}$.

For $(X, Y) = \frac{M_3}{\rho}$, $(X, Y)(\alpha) \in \Theta(\mu_1, \mu_2, \mu_3; M_3, M_2, M_1)$, $\mu_3(\frac{M_3}{\rho}) > M_3$, we get

$$\{(X, Y) \in \Theta(\mu_1, \mu_2, \mu_3; M_3, M_2, M_1) : \mu_3(X, Y) > M_3\} \ne \emptyset.$$

For $(X, Y)(\alpha) \in \Theta(\mu_1, \mu_2, \mu_3; M_3, M_2, M_1)$, we know that $M_3 \le (X, Y)(\alpha) \le M_2 = \frac{M_3}{\rho}$ for $\alpha \in [0, \kappa]$.

In view of $(\mathfrak{C}_7)$,

$$\begin{aligned}
\mu_3(\mathcal{T}_1(X, Y)) &= \min_{\alpha\in[0,\kappa]} |\mathcal{T}_1(X, Y)(\alpha)|\\
&\ge \min_{\alpha\in[0,\kappa]} \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta\Big)d\varpi + \min_{\alpha\in[0,\kappa]} \psi_1(\alpha)\\
&\ge \int_0^1 \rho_{\mathcal{Z}_1}(G_1(0, \varpi) - G_2(1, \varpi))\vartheta_q\Big(\int_0^\kappa K_{\mathcal{B}_1}(\varpi, \theta)\vartheta_p\Big(\frac{M_3 - \rho_{\mathcal{Z}_1}\psi_1(1)}{\rho_{\mathcal{Z}_1} l_3}\Big)d\theta\Big)d\varpi + \rho_{\mathcal{Z}_1}\psi_1(1)\\
&= \rho_{\mathcal{Z}_1}\vartheta_p\Big(\frac{M_3 - \rho_{\mathcal{Z}_1}\psi_1(1)}{\rho_{\mathcal{Z}_1} l_3}\Big) \int_0^\kappa (G_1(0, \varpi) - G_2(1, \varpi))\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)d\theta\Big)d\varpi + \rho_{\mathcal{Z}_1}\psi_1(1)\\
&= M_3.
\end{aligned}$$

Similarly,

$$\mu_3(\mathcal{T}_2(X, Y)) \ge M_3.$$

So, $\mu_3(\mathcal{T}(X,Y)) > M_3$ for all $(X,Y)(\alpha) \in \Theta(\mu_1,\mu_2,\mu_3;M_3,M_2,M_1)$. Hence, $(\mathfrak{C}_3)$ of Lemma 4.1 is fulfilled.

By (4.7), for all $(X,Y) \in \Theta(\mu_1,\mu_3;M_3,M_1)$ with $\mu_2(\mathcal{T}(X,Y)) > M_2 = \frac{M_3}{\rho}$, one get

$$\mu_2(\mathcal{T}(X,Y)) > \rho M_2 = \rho\frac{M_3}{\rho} = M_3.$$

Implies $(\mathfrak{C}_4)$ of Lemma 4.1 is true.

As $\mu_4(0,0) = 0 < M_4$, thus $0 \notin \Theta(\mu_1,\mu_4;M_4,M_1)$. If $(X,Y) \in \Theta(\mu_1,\mu_4;M_4,M_1)$ with $\mu_4(X,Y) = M_4$, we deduced $\mu_1(X,Y) \le M_1$. Thus, $\max_{\alpha\in[0,1]}(X,Y)(\alpha) = M_4$ and $0 \le (X,Y)(\alpha) \le M_1$.

Using $(\mathfrak{C}_8)$, we get

$$\begin{aligned}
\mu_4(\mathcal{T}_1(X,Y)) &= \max_{\alpha\in[0,1]} |\mathcal{T}_1(X,Y)(\alpha)| \\
&\le \min_{\alpha\in[0,1]} \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta,D^{\gamma_1}X(\theta),D^{\gamma_2}Y(\theta))d\theta\Big)d\varpi + \max_{\alpha\in[0,1]}\psi_1(\alpha) \\
&< \int_0^1 \rho_{\mathcal{Z}_1}(G_1(0,\varpi)-G_2(1,\varpi))\vartheta_q\Big(\int_0^\kappa K_{\mathcal{B}_1}(\varpi,\theta)\vartheta_p\Big(\frac{M_4-\psi_1(1)}{l_1}\Big)d\theta\Big)d\varpi + \psi_1(1) \\
&= \frac{M_4-\psi_1(1)}{l_1}\int_0^\kappa (G_1(0,\varpi)-G_2(1,\varpi))\vartheta_q\Big(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)d\theta\Big)d\varpi + \psi_1(1) \\
&= M_4.
\end{aligned}$$

Similarly,

$$\mu_4(\mathcal{T}_2(X,Y)) \le M_4.$$

So, $\mu_3(\mathcal{T}(X,Y)) < M_3$ for all $(X,Y)(\alpha) \in \Theta(\mu_1,\mu_4;M_4,M_1)$. Therefore, the condition $(\mathfrak{C}_4)$ of Lemma 4.1 hold.

To sum up, all the conditions of Lemma 4.1 are verified, and it was noticed that $(X_i,Y_i)(\alpha) \ge \psi(0) > 0$. Hence, the coupled system (1.2) has at least three positive solutions (X_1,Y_1), (X_2,Y_2), (X_3,Y_3) satisfying (4.7) and (4.8). □

5. Stability results

Definition 5.1. *The system* (1.2) *is HU stable if there exist* $\pi_{\mathcal{Z}_1,\mathcal{Z}_2} = \max\{\pi_{\mathcal{Z}_1},\pi_{\mathcal{Z}_2}\} > 0$ *in such a way that, for* $\varepsilon = \max\{\varepsilon_{\mathcal{Z}_1},\varepsilon_{\mathcal{Z}_2}\} > 0$ *and for any* $(X,Y) \in \chi$ *satisfying*

$$\begin{cases} |D^{\mathcal{B}_1}\vartheta_p(D^{\mathcal{Z}_1}X(\alpha)) + F_1(\alpha,D^{\gamma_1}X(\alpha),D^{\gamma_2}Y(\alpha))| \le \varepsilon_{\mathcal{Z}_1}, & \alpha\in[0,1], \\ |D^{\mathcal{B}_2}\vartheta_p(D^{\mathcal{Z}_2})Y(\alpha) + F_2(\alpha,D^{\gamma_1}X(\alpha),D^{\gamma_2}Y(\alpha))| \le \varepsilon_{\mathcal{Z}_2}, & \alpha\in[0,1], \end{cases} \tag{5.1}$$

$\exists\, (\widehat{X},\widehat{Y}) \in \chi$ *satisfying* (1.2) *such that*

$$\|(X,Y)-(\widehat{X},\widehat{Y})\| \le \pi_{\mathcal{Z}_1,\mathcal{Z}_2}\varepsilon, \;\; \alpha\in[0,1].$$

Definition 5.2. *The coupled implicit FDEs* (1.2) *is* $\mathcal{GHU}$ *stable if there exist* $\Phi \in C(R^+,R^+)$ *with* $\Phi(0) = 0$ *such that, for any solution* $(X,Y) \in \chi$ *of inequality* (5.1)*, there exists a solution* $(\widehat{X},\widehat{Y}) \in \chi$ *of* (1.2) *fulfilling*

$$\|(X,Y)-(\widehat{X},\widehat{Y})\| \le \Phi(\varepsilon), \;\; \alpha\in[0,1].$$

Let $\Psi_{\mathcal{Z}_1,\mathcal{Z}_2} = \max\{\Psi_{\mathcal{Z}_1}, \Psi_{\mathcal{Z}_2}\} \in C([0,1], R)$, and $\pi_{\Psi_{\mathcal{Z}_1},\Psi_{\mathcal{Z}_2}} = \max\{\pi_{\Psi_{\mathcal{Z}_1}}, \pi_{\Psi_{\mathcal{Z}_2}}\} > 0$.

Definition 5.3. *The coupled system of implicit FDEs* (1.2) *is said to be HUR stable with respect to* $\Psi_{\mathcal{Z}_1,\mathcal{Z}_2}$ *if there exists constants* $\pi_{\Psi_{\mathcal{Z}_1},\Psi_{\mathcal{Z}_2}}$ *such that, for some* $\varepsilon > 0$ *and for any solution* $(X, Y) \in \chi$ *of the inequality*

$$\begin{cases} |D^{\mathcal{B}_1}\vartheta_p(D^{\mathcal{Z}_1}X(\alpha)) + F_1(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))| \leq \Psi_{\mathcal{Z}_1}(\alpha)\varepsilon_{\mathcal{Z}_1}, & \alpha \in [0,1], \\ |D^{\mathcal{B}_2}\vartheta_p(D^{\mathcal{Z}_2})Y(\alpha) + F_2(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha))| \leq \Psi_{\mathcal{Z}_2}(\alpha)\varepsilon_{\mathcal{Z}_2}, & \alpha \in [0,1], \end{cases} \tag{5.2}$$

there exists $(\widehat{X}, \widehat{Y}) \in \chi$ *satisfying* (1.2) *in such a way that*

$$\|(X, Y) - (\widehat{X}, \widehat{Y})\| \leq \pi_{\Psi_{\mathcal{Z}_1},\Psi_{\mathcal{Z}_2}} \Psi_{\mathcal{Z}_1,\mathcal{Z}_2}\varepsilon, \quad \alpha \in [0,1]. \tag{5.3}$$

Definition 5.4. *The system* (1.2) *is said to be HUR stable with respect to* $\Psi_{\mathcal{Z}_1,\mathcal{Z}_2}$ *if there exists constants* $\pi_{\Psi_{\mathcal{Z}_1},\Psi_{\mathcal{Z}_2}}$ *such that, for any approximate solution* $(X, Y) \in \chi$ *of inequality* (5.2)*, there exists a solution* $(\widehat{X}, \widehat{Y}) \in \chi$ *of* (1.2) *fulfilling*

$$\|(X, Y) - (\widehat{X}, \widehat{Y})\| \leq \pi_{\Psi_{\mathcal{Z}_1},\Psi_{\mathcal{Z}_2}} \Psi_{\mathcal{Z}_1,\mathcal{Z}_2}(\alpha), \quad \alpha \in [0,1]. \tag{5.4}$$

Remark 5.1. *We say that* $(X, Y) \in \chi$ *is a solution of the system of inequalities* (5.1) *if there exist functions* Λ_f*,* $\Lambda_g \in C([0,1], R)$ *depending upon* X, Y*, respectively, such that*

(I)

$$|\Lambda_f(\alpha)| \leq \varepsilon_{\mathcal{Z}_1}, |\Lambda_g(\alpha)| \leq \varepsilon_{\mathcal{Z}_2}, \alpha \in [0,1];$$

(II)

$$\begin{cases} D^{\mathcal{B}_1}\vartheta_p(D^{\mathcal{Z}_1}X(\alpha)) + F_1(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha)) = \Lambda_f(\alpha), \\ D^{\mathcal{B}_2}\vartheta_p(D^{\mathcal{Z}_2})Y(\alpha) + F_2(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha)) = \Lambda_g(\alpha). \end{cases}$$

Lemma 5.1. *Under the assumptions given in Remark 5.1, the solution* $(X, Y) \in \chi$ *of coupled system*

$$\begin{cases} D^{\mathcal{B}_1}\vartheta_p(D^{\mathcal{Z}_1}X(\alpha)) + F_1(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha)) = \Lambda_f(\alpha), \quad \alpha \in [0,1], \\ D^{\mathcal{B}_2}\vartheta_p(D^{\mathcal{Z}_2})Y(\alpha) + F_2(\alpha, D^{\gamma_1}X(\alpha), D^{\gamma_2}Y(\alpha)) = \Lambda_g(\alpha), \quad \alpha \in (0,1), \\ \left(\vartheta_p(D^{\mathcal{Z}_1}X(0))\right)^i = \vartheta_p(D^{\mathcal{Z}_1}X(1)) = 0, \quad i = \overline{1, m-1}, \\ \left(\vartheta_p(D^{\mathcal{Z}_2}Y(0))\right)^i = \vartheta_p(D^{\mathcal{Z}_2}Y(1)) = 0, \quad i = \overline{1, m-1}, \\ X(0) + X'(0) = \int_0^1 S_1(\varpi)X(\varpi)d\varpi + \int_0^1 \mathrm{H}_1(\varpi)d\varpi, \quad X(1) + X'(1) \\ = \int_0^1 S_2(\varpi)X(\varpi)d\varpi + \int_0^1 \mathrm{H}_2(\varpi)d\varpi, \\ Y(0) + Y'(0) = \int_0^1 S_3(\varpi)Y(\varpi)d\varpi + \int_0^1 \mathrm{H}_3(\varpi)d\varpi, \quad Y(1) + Y'(1) \\ = \int_0^1 S_4(\varpi)Y(\varpi)d\varpi + \int_0^1 \mathrm{H}_4(\varpi)d\varpi, \\ X^j(0) = 0, \quad Y^j(0) = 0, \quad j = \overline{2, n-1}, \end{cases} \tag{5.5}$$

satisfies the system of inequalities

$$|X(\alpha) - \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi - \psi_1(\alpha)| \le \vartheta_q(\varepsilon_{\mathcal{Z}_1})\frac{M_{01}-\psi_1(1)}{\vartheta_q(M_0)},$$
$$|Y(\alpha) - \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi - \psi_2(\alpha)| \le \vartheta_q(\varepsilon_{\mathcal{Z}_2})\frac{N_{01}-\psi_2(1)}{\vartheta_q(N_0)}.$$

Proof. In light of Lemma 3.2, a solution of the coupled system (5.1) is

$$\begin{cases} X(\alpha) = & \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi \\ & -\int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)\Lambda_f(\theta)d\theta)d\varpi + \psi_1(\alpha), \\ Y(\alpha) = & \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi \\ & -\int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)\Lambda_g(\theta)d\theta)d\varpi + \psi_2(\alpha), \end{cases} \tag{5.6}$$

Using (4.2) in (5.6), we have

$$\begin{aligned} |X(\alpha) - & \int_0^1 G_{\mathcal{Z}_1}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi,\theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi - \psi_1(\alpha)| \\ & \le \int_0^1 |G_{\mathcal{Z}_1}(\alpha,\varpi)|\vartheta_q(\varepsilon_{\mathcal{Z}_1}\int_0^1 |K_{\mathcal{B}_1}(\varpi,\theta)|d\theta)d\varpi \le \vartheta_q(\varepsilon_{\mathcal{Z}_1})\frac{M_{01}-\psi_1(1)}{\vartheta_q(M_0)}. \end{aligned}$$

Similarly, using (4.3) in (5.6), we get

$$|Y(\alpha) - \int_0^1 G_{\mathcal{Z}_2}(\alpha,\varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_2}(\varpi,\theta)F_2(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi - \psi_2(\alpha)| \le \vartheta_q(\varepsilon_{\mathcal{Z}_2})\frac{N_{01}-\psi_2(1)}{\vartheta_q(N_0)}.$$

□

For the next result, we suppose the following condition.

($\mathfrak{C}_9$) Let $N_1, N_2, S_1, S_2 \in C([0,1], R)$, and there exists $L_1, L_2, L_3, L_4 > 0$ such that

$$|F_1(\alpha, N_1, S_1) - F_1(\alpha, N_2, S_2)| \le L_1|N_1 - N_2| + L_2|S_1 - S_2|,$$
$$|F_2(\alpha, N_1, S_1) - F_2(\alpha, N_2, S_2)| \le L_3|N_1 - N_2| + L_4|S_1 - S_2|.$$

Theorem 5.1. *Suppose the condition* ($\mathfrak{C}_9$) *and Lemma 5.1 satisfies, then* (1.2) *is HU stable, if* $\upsilon_1\upsilon_4 - \upsilon_2\upsilon_3 = \upsilon_4 - \upsilon_2 > 0$, *where*

$$\begin{aligned} \upsilon_1 &= 1 - (M_{01} - \psi_1(1))\frac{\vartheta_q(\frac{L_1}{\Gamma(\mathcal{B}_1-\gamma_1)\Gamma(1-\gamma_1)})}{\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})}, \\ \upsilon_2 &= (M_{01} - \psi_1(1))\frac{\vartheta_q(\frac{L_2}{\Gamma(\mathcal{B}_1-\gamma_2)\Gamma(1-\gamma_2)})}{\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})}, \\ \upsilon_3 &= 1 - (N_{01} - \psi_2(1))\frac{\vartheta_q(\frac{L_3}{\Gamma(\mathcal{B}_2-\gamma_1)\Gamma(1-\gamma_1)})}{\vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})}, \\ \upsilon_4 &= (N_{01} - \psi_2(1))\frac{\vartheta_q(\frac{L_4}{\Gamma(\mathcal{B}_2-\gamma_2)\Gamma(1-\gamma_2)})}{\vartheta_q(\frac{N_0}{\Gamma(\mathcal{B}_2+1)})}. \end{aligned}$$

Proof. Consider $(X, Y) \in \chi$ to be any solution of (5.5), and $(\widehat{X}, \widehat{Y}) \in \chi$ is a solution of the coupled system (1.2), then we take

$$
\begin{aligned}
&|(X(\alpha) - \widehat{X}(\alpha))| \\
\leq&|X(\alpha) - \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi - \psi_1(\alpha) \\
&+ \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta))d\theta)d\varpi \\
&- \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)F_1(\theta, D^{\gamma_1}\widehat{X}(\theta), D^{\gamma_2}\widehat{Y}(\theta))d\theta)d\varpi| \\
\leq&\vartheta_q(\varepsilon_{\mathcal{Z}_1})\frac{M_{01} - \psi_1(1)}{\vartheta_q(M_0)} + \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)|F_1(\theta, D^{\gamma_1}X(\theta), D^{\gamma_2}Y(\theta)) \\
&- F_1(\theta, D^{\gamma_1}\widehat{X}(\theta), D^{\gamma_2}\widehat{Y}(\theta))|d\theta)d\varpi \\
\leq&\vartheta_q(\varepsilon_{\mathcal{Z}_1})\frac{M_{01} - \psi_1(1)}{\vartheta_q(M_0)} + \int_0^1 G_{\mathcal{Z}_1}(\alpha, \varpi)\vartheta_q(\int_0^1 K_{\mathcal{B}_1}(\varpi, \theta)(L_1 D^{\gamma_1}|X(\theta) - \widehat{X}(\theta)| + L_2 D^{\gamma_2}|Y(\theta) - \widehat{Y}(\theta)|)d\theta)d\varpi \\
\leq&(M_{01} - \psi_1(1))\Big(\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} + \frac{\vartheta_q(\frac{L_1}{\Gamma(\mathcal{B}_1-\gamma_1)\Gamma(1-\gamma_1)})}{\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})}|X(\alpha) - \widehat{X}(\alpha)| + \frac{\vartheta_q(\frac{L_2}{\Gamma(\mathcal{B}_1-\gamma_2)\Gamma(1-\gamma_2)})}{\vartheta_q(\frac{M_0}{\Gamma(\mathcal{B}_1+1)})}|Y(\alpha) - \widehat{Y}(\alpha)|\Big)
\end{aligned}
$$

implies that

$$
\upsilon_1\|u - \widehat{X}\| - \upsilon_2\|v - \widehat{Y}\| \leq (M_{01} - \psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)}. \tag{5.7}
$$

Similarly, we have

$$
\upsilon_4\|v - \widehat{Y}\| - \upsilon_3\|u - \widehat{X}\| \leq (N_{01} - \psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)}. \tag{5.8}
$$

From (5.7) and (5.8), we get

$$
\begin{bmatrix} \upsilon_1 & -\upsilon_2 \\ -\upsilon_3 & \upsilon_4 \end{bmatrix}
\begin{bmatrix} \|X - \widehat{X}\| \\ \|Y - \widehat{Y}\| \end{bmatrix}
\leq
\begin{bmatrix} (M_{01} - \psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} \\ (N_{01} - \psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)} \end{bmatrix}
$$

implies

$$
\begin{bmatrix} \|X - \widehat{X}\| \\ \|Y - \widehat{Y}\| \end{bmatrix}
\leq
\begin{bmatrix} \frac{\upsilon_4}{\upsilon_4-\upsilon_2} & \frac{\upsilon_2}{\upsilon_4-\upsilon_2} \\ \frac{\upsilon_3}{\upsilon_4-\upsilon_2} & \frac{\upsilon_1}{\upsilon_4-\upsilon_2} \end{bmatrix}
\begin{bmatrix} (M_{01} - \psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} \\ (N_{01} - \psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)} \end{bmatrix}. \tag{5.9}
$$

From system (5.9), we have

$$
\|X - \widehat{X}\| \leq \frac{\upsilon_4}{\upsilon_4 - \upsilon_2}(M_{01} - \psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} + \frac{\upsilon_2}{\upsilon_4 - \upsilon_2}(N_{01} - \psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)},
$$

$$\|Y-\widehat{Y}\| \le \frac{\upsilon_3}{\upsilon_4-\upsilon_2}(M_{01}-\psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} + \frac{\upsilon_1}{\upsilon_4-\upsilon_2}(N_{01}-\psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)},$$

which implies that

$$\begin{aligned}\|X-\widehat{X}\|+\|Y-\widehat{Y}\| \le & \frac{\upsilon_1}{\upsilon_4-\upsilon_2}(N_{01}-\psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)} + \frac{\upsilon_2}{\upsilon_4-\upsilon_2}(N_{01}-\psi_2(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_2})}{\vartheta_q(N_0)} \\ & + \frac{\upsilon_3}{\upsilon_4-\upsilon_2}(M_{01}-\psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)} + \frac{\upsilon_4}{\upsilon_4-\upsilon_2}(M_{01}-\psi_1(1))\frac{\vartheta_q(\varepsilon_{\mathcal{Z}_1})}{\vartheta_q(M_0)}.\end{aligned}$$

If $\max\{\vartheta_q(\varepsilon_{\mathcal{Z}_1}), \vartheta_q(\varepsilon_{\mathcal{Z}_2})\} = \vartheta_q(\varepsilon_{\mathcal{Z}})$ and $\frac{\upsilon_1}{\upsilon_4-\upsilon_2}\frac{N_{01}-\psi_2(1)}{\vartheta_q(N_0)} + \frac{\upsilon_2}{\upsilon_4-\upsilon_2}\frac{N_{01}-\psi_2(1)}{\vartheta_q(N_0)} + \frac{\upsilon_3}{\upsilon_4-\upsilon_2}\frac{M_{01}-\psi_1(1)}{\vartheta_q(M_0)} + \frac{\upsilon_4}{\upsilon_4-\upsilon_2}\frac{M_{01}-\psi_1(1)}{\vartheta_q(M_0)} = \upsilon_{\mathcal{Z}_1,\mathcal{Z}_2}$, then

$$\|(X,Y)-(\widehat{X},\widehat{Y})\| \le \upsilon_{\mathcal{Z}_1,\mathcal{Z}_2}\vartheta_q(\varepsilon_{\mathcal{Z}}).$$

Hence, system (1.2) is *HU* stable. Also, if

$$\|(X,Y)-(\widehat{X},\widehat{Y})\| \le \upsilon_{\mathcal{Z}_1,\mathcal{Z}_2}\Phi(\vartheta_q(\varepsilon_{\mathcal{Z}})),$$

with $\Phi(0) = 0$, then the solution of system (1.2) is *GHU* stable. □

Theorem 5.2. *Suppose the conditions* $(\mathfrak{C}_9)$ *and Lemma 5.1 holds, then* (1.2) *is GHU stable, if* $\upsilon_1\upsilon_4 - \upsilon_2\upsilon_3 = \upsilon_4 - \upsilon_2 > 0$*, where* υ_1, υ_2, υ_3 *and* υ_4 *is defined in Theorem 5.1.*

Proof. By applying steps of Theorem 5.1, we can easily show that system (1.2) is *GHU* stable, by using Definition 5.4. □

For the next theorem, we assume that

$(\mathfrak{C}_{10})$ $\exists$ non-decreasing functions $\bar{w}_{\mathcal{Z}}, \bar{w}_{\mathcal{B}} \in C([0,1], R^+)$ in such a way that

$$\mathfrak{I}^{\mathcal{Z}}\bar{w}_{\mathcal{Z}}(\alpha) \le \mathcal{L}_{\mathcal{Z}}\bar{w}_{\mathcal{Z}}(\alpha)$$

and

$$\mathfrak{I}^{\mathcal{B}}\bar{w}_{\mathcal{B}}(\alpha) \le \mathcal{L}_{\mathcal{B}}\bar{w}_{\mathcal{B}}(\alpha),$$

where $\mathcal{L}_{\mathcal{Z}}, \mathcal{L}_{\mathcal{B}} > 0$.

Theorem 5.3. *Suppose the conditions* $(\mathfrak{C}_9)$, $(\mathfrak{C}_{10})$ *and Lemma 5.1 satisfies, then system* (1.2) *is HUR stable if* $\upsilon_1\upsilon_4 - \upsilon_2\upsilon_3 = \upsilon_4 - \upsilon_2 > 0$.

Proof. By applying steps of Theorem 5.1, we can easily show that system (1.2) is *HUR* stable, by using Definition 5.3. □

Theorem 5.4. *Suppose the conditions* $(\mathfrak{C}_9)$, $(\mathfrak{C}_{10})$ *and Lemma 5.1 holds, then system* (1.2) *is GHUR stable if* $\upsilon_1\upsilon_4 - \upsilon_2\upsilon_3 = \upsilon_4 - \upsilon_2 > 0$.

Proof. By applying steps of Theorem 5.1, we can easily show that system (1.2) is *GHUR* stable, by using Definition 5.4. □

6. Example

In this section, we demonstrate an example to illustrate the main results.

Example 6.1. *Consider the following system of implicit FDEs:*

$$\begin{cases} D^{\frac{9}{4}}\vartheta_{\frac{3}{2}}(D^{\frac{7}{2}}X(\alpha))+F_1(\alpha,D^{\frac{1}{2}}X(\alpha),D^{\frac{1}{3}}Y(\alpha))=0,\ \ \alpha\in(0,1),\\ D^{\frac{11}{5}}\vartheta_{\frac{3}{2}}(D^{\frac{10}{3}})Y(\alpha)+F_2(\alpha,D^{\frac{1}{2}}X(\alpha),D^{\frac{1}{3}}Y(\alpha))=0,\ \ \alpha\in(0,1),\\ \left(\vartheta_{\frac{3}{2}}(D^{\frac{7}{2}}X(0))\right)'=\vartheta_{\frac{3}{2}}(D^{\frac{7}{2}}X(1))=0,\ \left(\vartheta_{\frac{3}{2}}(D^{\frac{10}{3}}Y(0))\right)'=\vartheta_{\frac{3}{2}}(D^{\frac{10}{3}}Y(1))=0,\\ X(0)+X'(0)=\int_0^1(\varpi+1)X(\varpi)d\varpi+\int_0^1\frac{\varpi d\varpi}{4},\\ X(1)+X'(1)=\int_0^1(3\varpi^2+\varpi)X(\varpi)d\varpi+\int_0^1\frac{\varpi d\varpi}{3},\\ Y(0)+Y'(0)=\int_0^1(2\varpi+1)Y(\varpi)d\varpi+\int_0^1\frac{\varpi d\varpi}{2},\\ Y(1)+Y'(1)=\int_0^1(5\varpi^2+2\varpi)Y(\varpi)d\varpi+\int_0^1\varpi d\varpi,\\ X^j=Y^j,\ \ where\ \ j=2,3. \end{cases} \tag{6.1}$$

Choosing $M_1=25000$, $M_3=65$, $M_4=3$ and $\kappa=\frac{1}{3}$. Reminding ψ_1, ψ_2 from (2.5), (2.6) *and $\rho_{\mathcal{Z}_1}$ and $\rho_{\mathcal{Z}_2}$ from Lemma 3.4, we get*

$$\psi_1(1)=1.87144,\ \ \psi_2(1)=2.32681,\ \ \rho_{\mathcal{Z}_1}=0.046572\ \ and\ \ \rho_{\mathcal{Z}_2}=0.0097325.$$

In (6.1)*, we see that*

$$F_1(\alpha,D^{\frac{1}{2}}X(\alpha),D^{\frac{1}{3}}Y(\alpha))=\tan(\frac{\alpha}{150})+D^{\frac{1}{2}}X(\alpha)+\cos(D^{\frac{1}{3}}Y(\alpha)),$$
$$F_2(\alpha,D^{\frac{1}{2}}X(\alpha),D^{\frac{1}{3}}Y(\alpha))=\frac{\sin(D^{\frac{1}{2}}X(\alpha))-D^{\frac{1}{3}}Y(\alpha)}{160},$$

which satisfies the following conditions:

($\mathfrak{C}_6$) $\sup F_1(\alpha,X,Y)\approx 246.534\le\min\vartheta_p(\frac{M_1-\psi_1(1)}{l_1})\approx 291.732$,
$\sup F_2(\alpha,X,Y)\approx 241.492\le\min\vartheta_p(\frac{M_1-\psi_2(1)}{l_2})\approx 284.618$, $(\alpha,X,Y)\in[0,1]\times[0,25000]\times[0,25000]$;

($\mathfrak{C}_7$) $\inf F_1(\alpha,X,Y)\approx 240.631>\vartheta_p(\frac{M_3-\rho_{\mathcal{Z}_1}\psi_1(1)}{\rho_{\mathcal{Z}_1}l_3})\approx 201.981$ $(\alpha,X,Y)\in[0,\frac{1}{3}]\times[65,\frac{M_3}{\rho_{\mathcal{Z}_1}}]\times[0,25000]$;
$\inf F_2(\alpha,X,Y)\approx 271>\vartheta_p(\frac{M_3-\rho_{\mathcal{Z}_2}\psi_2(1)}{\rho_{\mathcal{Z}_2}l_4})\approx 201.843$, $(\alpha,X,Y)\in[0,\frac{1}{3}]\times[65,\frac{M_3}{\rho_{\mathcal{Z}_2}}]\times[0,25000]$;

($\mathfrak{C}_8$) $\sup F_2(\alpha,X,Y)\approx 198.938<\vartheta_p(\frac{M_4-\psi_1(1)}{l_1})\approx 243.861$,
$\sup F_2(\alpha,X,Y)\approx 183.861<\vartheta_p(\frac{M_4-\psi_2(1)}{l_2})\approx 202.991$, $(\alpha,X,Y)\in[0,1]\times[0,3]\times[0,25000]$.

Then, all assumptions of Theorem 4.1 are satisfied. Thus, BVP (6.1) *has at least three positive solutions (X_1,Y_1), (X_2,Y_2), (X_3,Y_3) satisfying*

$$\|(X_i,Y_i)\|\le 25000\ (i=1,2,3),$$

$$\min_{\alpha\in[0,\kappa]}|(X_1, Y_1)| > 65,\ 3 < \min_{\alpha\in[0,\kappa]}|(X_2, Y_2)|,$$
$$\max_{\alpha\in[0,1]}|(X_2, Y_2)| < 65, \max_{\alpha\in[0,1]}|(X_3, Y_3)| < 3.$$

For HUS, we found $L_1 = L_2 = \frac{1}{150}$, $L_3 = L_4 = \frac{1}{160}$, $\upsilon_1 \approx 0.02961$, $\upsilon_2 \approx 0.9721$, $\upsilon_3 \approx 0.03291$ *and* $\upsilon_4 \approx 0.9948$. *Hence, by Theorem 5.1, we have* $\upsilon_4 - \upsilon_2 \approx 0.9948 - 0.0227 > 0$.

7. Conclusions

In this paper, we considered an implicit coupled *BVP* of p-Laplacian *FDEs* (1.2), which involved disturbing functions. The relating fractional order derivative is taken in Caputo sense. By using the Avery-Peterson fixed point theorem for the proposed problem, we found at least three solutions, under sufficient conditions. In addition, we presented four types of Ulam's stability, i.e., Hyers-Ulam stability, generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability, for the coupled implicit fractional p-Laplacian system, and an example is provided to authenticate the theoretical results.

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Conflict of interest

The authors declare no conflict of interest.

References

1. R. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.*, **59** (2010), 1095–1100. http://dx.doi.org/10.1016/j.camwa.2009.05.010
2. M. Ahmad, A. Zada, J. Alzabut, Stability analysis of a nonlinear coupled implicit switched singular fractional differential equations with p-Laplacian, *Adv. Differ. Equ.*, **2019** (2019), 436. http://dx.doi.org/10.1186/s13662-019-2367-y
3. M. Ahmad, A. Zada, J. Alzabut, Hyers-Ulam stability of a coupled system of fractional differential equations of Hilfer-Hadamard type, *Demonstr. Math.*, **52** (2019), 283–295. http://dx.doi.org/10.1515/dema-2019-0024
4. Asma, J. Gomez-Aguilar, G. Rahman, M. Javed, Stability analysis for fractional order implicit ψ-Hilfer differential equations, *Math. Method. Appl. Sci.*, **45** (2022), 2701–2712. http://dx.doi.org/10.1002/mma.7948
5. R. Dhayal, J. Gomez-Aguilar, J. Torres-Jimenez, Stability analysis of Atangana-Baleanu fractional stochastic differential systems with impulses, *Int. J. Syst. Sci.*, **53** (2022), 3481–3495. http://dx.doi.org/10.1080/00207721.2022.2090638

6. A. Gonzalez-Calderon, L. Vivas-Cruz, M. Taneco-Hernandez, J. Gomez-Aguilar, Assessment of the performance of the hyperbolic-NILT method to solve fractional differential equations, *Math. Comput. Simulat.*, **206** (2023), 375–390. http://dx.doi.org/10.1016/j.matcom.2022.11.022

7. L. Guo, U. Riaz, A. Zada, M. Alam, On implicit coupled Hadamard fractional differential equations with generalized Hadamard fractional integro-differential boundary conditions, *Fractal Fract.*, **7** (2023), 13. http://dx.doi.org/10.3390/fractalfract7010013

8. M. Iswarya, R. Raja, G. Rajchakit, J. Cao, J. Alzabut, C. Huang, Existence, uniqueness and exponential stability of periodic solution for discrete-time delayed BAM neural networks based on coincidence degree theory and graph theoretic method, *Mathematics*, **7** (2019), 1055. http://dx.doi.org/10.3390/math7111055

9. H. Khan, T. Abdeljawad, J. Gomez-Aguilar, H. Tajadodi, A. Khan, Fractional order Volterra integro-differential equation with Mittag-Leffler kernel, *Fractals*, **29** (2021), 2150154. http://dx.doi.org/10.1142/S0218348X21501541

10. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and application of fractional differential equation*, Amsterdam: Elsevier, 2006.

11. X. Liu, M. Jia, W. Ge, Multiple solutions of a p-Laplacian model involving fractional derivatives, *Adv. Differ. Equ.*, **2013** (2013), 126. http://dx.doi.org/10.1186/1687-1847-2013-126

12. H. Lu, Z. Han, S. Sun, J. Liu, Existence on positive solutions for boundary value problems of nonlinear fractional differential equations with p-Laplacian, *Adv. Differ. Equ.*, **2013** (2013), 30. http://dx.doi.org/10.1186/1687-1847-2013-30

13. O. Martinez-Fuentes, F. Melendez-Vazquez, G. Fernandez-Anaya, J. Gomez-Aguilar, Analysis of fractional-order nonlinear dynamic systems with general analytic kernels: lyapunov stability and inequalities, *Mathematics*, **9** (2021), 2084. http://dx.doi.org/10.3390/math9172084

14. M. Matar, A. Lubbad, J. Alzabut, On p-Laplacian boundary value problems involving Caputo-Katugampula fractional derivatives, *Math. Method. Appl. Sci.*, in press. http://dx.doi.org/10.1002/mma.6534

15. M. Obloza, Hyers stability of the linear differential equation, *Rocznik NaukDydakt, Prace Mat.*, **13** (1993), 259–270.

16. K. Oldham, Fractional differential equations in electrochemistry, *Adv. Eng. Softw.*, **41** (2010), 9–12. http://dx.doi.org/10.1016/j.advengsoft.2008.12.012

17. A. Pratap, R. Raja, J. Cao, J. Alzabut, C. Huang, Finite-time synchronization criterion of graph theory perspective fractional order coupled discontinuous neural networks, *Adv. Differ. Equ.*, **2020** (2020), 97. http://dx.doi.org/10.1186/s13662-020-02551-x

18. U. Riaz, A. Zada, Analysis of (α,β)-order coupled implicit Caputo fractional differential equations using topological degree method, *Int. J. Nonlin. Sci. Num.*, **22** (2021), 897–915. http://dx.doi.org/10.1515/ijnsns-2020-0082

19. U. Riaz, A. Zada, Z. Ali, Y. Cui, J. Xu, Analysis of coupled systems of implicit impulsive fractional differential equations involving Hadamard derivatives, *Adv. Differ. Equ.*, **2019** (2019), 226. http://dx.doi.org/10.1186/s13662-019-2163-8

20. U. Riaz, A. Zada, Z. Ali, I. Popa, S. Rezapour, S. Etemad, On a Riemann-Liouville type implicit coupled system via generalized boundary conditions, *Mathematics*, **9** (2021), 1205. http://dx.doi.org/10.3390/math9111205

21. F. Rihan, Numerical modeling of fractional order biological systems, *Abstr. Appl. Anal.*, **2013** (2013), 816803. http://dx.doi.org/10.1155/2013/816803

22. J. Sabatier, O. Agrawal, J. Machado, *Advances in fractional calculus*, Dordrecht: Springer, 2007. http://dx.doi.org/10.1007/978-1-4020-6042-7

23. A. Seemab, M. Rehman, J. Alzabut, A. Hamdi, On the existence of positive solutions for generalized fractional boundary value problems, *Bound. Value Probl.*, **2019** (2019), 186. http://dx.doi.org/10.1186/s13661-019-01300-8

24. V. Tarasov, *Fractional dynamics: application of fractional calculus to dynamics of particles, fields and media*, Berlin: Springer, 2010. http://dx.doi.org/10.1007/978-3-642-14003-7

25. S. Ulam, *A collection of the mathematical problems*, New York: Interscience Publishers, 1960.

26. B. Vintagre, I. Podlybni, A. Hernandez, V. Feliu, Some approximations of fractional order operators used in control theory and applications, *Fract. Calc. Appl. Anal.*, **3** (2000), 231–248.

27. H. Waheed, A. Zada, R. Rizwan, I. Popa, Hyers-Ulam stability for a coupled system of fractional differential equation with p-Laplacian operator having integral boundary conditions, *Qual. Theory Dyn. Syst.*, **21** (2022), 92. http://dx.doi.org/10.1007/s12346-022-00624-8

28. H. Yepez-Martinez, J. Gomez-Aguilar, M. Inc, New modified Atangana-Baleanu fractional derivative applied to solve nonlinear fractional differential equations, *Phys. Scr.*, **98** (2023), 035202. http://dx.doi.org/10.1088/1402-4896/acb591

29. A. Zada, J. Alzabut, H. Waheed, I. Popa, Ulam-Hyers stability of impulsive integrodifferential equations with Riemann-Liouville boundary conditions, *Adv. Differ. Equ.*, **2020** (2020), 64. http://dx.doi.org/10.1186/s13662-020-2534-1

30. A. Zada, F. Khan, U. Riaz, T. Li, Hyers-Ulam stability of linear summation equations, *Punjab Univ. J. Math.*, **49** (2017), 19–24.

31. A. Zada, U. Riaz, F. Khan, Hyers-Ulam stability of impulsive integral equations, *Boll. Unione Mat. Ital.*, **12** (2019), 453–467. http://dx.doi.org/10.1007/s40574-018-0180-2

32. A. Zada, H. Waheed, J. Alzabut, X. Wang, Existence and stability of impulsive coupled system of fractional integrodifferential equations, *Demonstr. Math.*, **52** (2019), 296–335. http://dx.doi.org/10.1515/dema-2019-0035

33. L. Zhang, W. Zahag, X. Liu, M. Jia, Positive solutions of fractional p-Laplacian equations with integral boundary value and two parameters, *J. Inequal. Appl.*, **2020** (2020), 2. http://dx.doi.org/10.1186/s13660-019-2273-6

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