Comments on the height reducing property II

著者	Akiyama Shigeki, Thuswaldner Jorg M., Zaimi Toufik
journal or publication title	Indagationes mathematicae
volume	26
number	1
page range	28-39
year	2015-01
権利	(C) 2014 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. NOTICE: this is the author’s version of a work that was accepted for publication in Indagationes mathematicae. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Indagationes mathematicae, 26, 1, 2015, DOI:10.1016/j.indag.2014.07.002
URL	http://hdl.handle.net/2241/00123148

doi: 10.1016/j.indag.2014.07.002

COMMENTS ON THE HEIGHT REDUCING PROPERTY II

SHIGEKI AKIYAMA, JÖRG M. THUSWALDNER, AND TOUFIK ZAÏMI

Abstract. A complex number α is said to satisfy the height reducing property if there is a finite set $F \subset \mathbb{Z}$ such that $\mathbb{Z}[\alpha] = F[\alpha]$, where $\mathbb{Z}$ is the ring of the rational integers. It is easy to see that α is an algebraic number when it satisfies the height reducing property. We prove the relation $\mathrm{Card}(F) \geq \max\{2, |M_\alpha(0)|\}$, where M_α is the minimal polynomial of α over the field of the rational numbers, and discuss the related optimal cases, for some classes of algebraic numbers α. In addition, we show that there is an algorithm to determine the minimal height polynomial of a given algebraic number, provided it has no conjugate of modulus one.

1. Introduction

We continue, in this manuscript, the study of the numbers with height reducing property, in short HRP. Recall that a complex number α is said to satisfy HRP if there is a finite set $F \subset \mathbb{Z}$, such that each polynomial with coefficients in $\mathbb{Z}$, evaluated at α, belongs to the set

$$F[\alpha] := \left\{ \sum_{j=0}^{n} f_j \alpha^j \mid (f_0, \ldots, f_n) \in F^{n+1},\ n \in \mathbb{N} \right\},$$

(see *e.g.* [3, 6, 7]). In other words, α satisfies HRP when $\mathbb{Z}[\alpha]$ may be reduced to $F[\alpha]$. In [6] it is proved that α satisfies HRP if and only if α is an algebraic number whose conjugates (including α itself) are either all of modulus one, or all of modulus greater than one.

Throughout this paper, when we speak about conjugates, the minimal polynomial and the degree of an algebraic number α, this is meant over the field of the rational numbers $\mathbb{Q}$. The minimal polynomial M_α of α is supposed to be primitive, that is, the coefficients of M_α are integers whose greatest common divisor is one.

In fact the equality $\mathbb{Z}[\alpha] = S[\alpha]$, where S is a subset of the complex field $\mathbb{C}$, implies trivially the relations $S \subset \mathbb{Z}[\alpha]$ and $\mathbb{Z}[\alpha] = (-S)[\alpha]$. The following result shows that a complex number α satisfies HRP if and only if

$$\mathbb{Z}[\alpha] = S[\alpha] \text{ with } S \subset \mathbb{C} \text{ finite.} \tag{1}$$

Theorem 1. *If (1) holds for some pair (α, S), then there is a finite subset $F \subset \mathbb{Z}$, such that $\mathbb{Z}[\alpha] = F[\alpha]$ and*

$$\mathrm{Card}(F) \leq \mathrm{Card}(S)(\mathrm{Card}(S)^{s+1} - 1)/(\mathrm{Card}(S) - 1),$$

Date: March 26, 2014.

2010 *Mathematics Subject Classification.* 11R04, 12D10, 11R06.

Key words and phrases. Height of polynomials, Special algebraic numbers, Number systems.

The second author was supported by projects I1136 and W1230 funded by the Austrian Science Fund.

where s is the greatest exponent of α of some fixed choice of representations of the elements of S in $\mathbb{Z}[\alpha]$. Moreover, $\mathrm{Card}(S) \geq 2$ *and the set S contains at least $|M_\alpha(0)|$ elements of the form $P_j(\alpha)$, where $j \in \{0, \dots, |M_\alpha(0)| - 1\}$, $P_j \in \mathbb{Z}[x]$ and $P_j(0) \equiv j \bmod M_\alpha(0)$.*

For a given number α satisfying (1), we denote by S_α a fixed choice for $S \subset \mathbb{C}$, having the minimal number of elements. We also designate by $\mathcal{F}_N$ the set of those algebraic numbers α which satisfy $\mathrm{Card}(S_\alpha) = N$. Note that, using this notation, $\alpha \in \mathcal{F}_N$ for some N is equivalent to α satisfying HRP. It follows immediately from the second assertion in Theorem 1 that S_α contains a complete residue system, in short CRS, $\bmod\, \alpha$ in $\mathbb{Z}[\alpha]$, thus

$$\max\{2, |M_\alpha(0)|\} \leq \mathrm{Card}(S_\alpha) \tag{2}$$

and the index N in the notation $\mathcal{F}_N$ is at least 2. A result of Lagarias and Wang [19] implies that an expanding algebraic integer α, that is an algebraic integer whose conjugates are of modulus greater than one, satisfies (1) with $S = \{0, \pm 1, \dots, \pm(|M_\alpha(0)| - 1)\}$. It thus follows for expanding integers α that

$$|M_\alpha(0)| \leq \mathrm{Card}(S_\alpha) \leq 2\,|M_\alpha(0)| - 1, \tag{3}$$

and so $\alpha \in \mathcal{F}_N$ for some $N \in \{|M_\alpha(0)|, \dots, 2\,|M_\alpha(0)| - 1\}$.

It is interesting to determine the elements of the optimal set $\mathcal{F}_2$, and to characterize all algebraic numbers α satisfying $\alpha \in \mathcal{F}_{|M_\alpha(0)|}$. Theorem 2 below collects some partial answers to these questions.

It is easy to see that $\{0, 1, \dots, |\alpha| - 1)\}[\alpha] = \mathbb{Z}$, for $\alpha \in \mathbb{Z} \cap (-\infty, -2]$ and $\{-1, 0, \dots, \alpha - 2\}[\alpha] = \mathbb{Z}$, for $\alpha \in \mathbb{Z} \cap [3, \infty)$, and so, in both cases, $\alpha \in \mathcal{F}_{|M_\alpha(0)|}$. This fact is already proved by Grünwald [14]. The case where α is an expanding integer and $S_\alpha = \{0, 1, \dots, |M_\alpha(0)| - 1\}$, has been considered more than thirty years ago; such a pair (α, S_α) has been called a *canonical number system* (of the ring $\mathbb{Z}[\alpha]$). Many results about canonical number systems are known (see for instance the references and the results in [20]). For example, Kátai and Kovács [17] showed that a quadratic expanding integer α gives rise to a canonical number system if and only if $M_\alpha(x) = x^2 + a_1 x + a_2$, $a_2 \geq 2$ and $-1 \leq a_1 \leq a_2$. This was proved independently by Gilbert [13]. Kovács [18] showed that the conditions $M_\alpha(x) = x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d$, $a_d \geq 2$, $d \geq 2$ and $1 \leq a_1 \leq \cdots \leq a_{d-1} \leq a_d$ are sufficient to obtain a canonical number system, too. The problem to characterize canonical number systems of a given degree has later been embedded into the problem of determining shift radix systems, see *e.g.* [1, 2] for details.

Following the Hungarian tradition, we say that the pair (α, S_α) is a *number system* of the ring $\mathbb{Z}[\alpha]$, when $\mathrm{Card}(S_\alpha) = \max\{2, |M_\alpha(0)|\}$ and $0 \in S_\alpha$. It is worth noting that number systems have been defined in a more general context, and many related results are known. For instance, see [12, 23] for more recent developments. An important general result is due to Kátai [16], who showed that for any number field K there is an effectively computable constant $c(K) \geq 2$, such that if the conjugates of an element α of the ring of integers $\mathbb{Z}_K$ of K, are of modulus greater than $c(K)$, then α gives rise to a number system of $\mathbb{Z}_K$. The particular case where K is a real (resp., imaginary) quadratic field has been specified in [10] (resp., in [15, 22]). In fact by considering the companion matrix of the polynomial M_α it is easy to see by a theorem of Germán and Kovács [12] that we may choose $c(K) = 2$ for any K, without affecting the conclusion; this gives a complete answer to the above mentioned question when the conjugates of the algebraic integer α are all of modulus greater than 2.

Theorem 2. *Let α be an algebraic number with (primitive) minimal polynomial $M_\alpha \in \mathbb{Z}[x]$. Then the following assertions hold.*

(i) *The roots of unity belong to the set $\mathcal{F}_2$, and if $\alpha \in \mathcal{F}_2$ then α is an algebraic number whose conjugates are all of modulus 1 or is an expanding integer, with $|M_\alpha(0)| = 2$.*

(ii) *If an algebraic number α satisfies $|M_\alpha(1)| = 1$, then $\alpha \notin \mathcal{F}_{|M_\alpha(0)|}$.*

(iii) *Let α be an algebraic number whose conjugates are all of modulus 1. If $\mathrm{Card}(S_\alpha) = |M_\alpha(0)|$, then $S_\alpha \nsubseteq \mathbb{Z}$.*

(iv) *If α is an algebraic integer whose conjugates are all of modulus greater than 2, then $\alpha \in \mathcal{F}_{|M_\alpha(0)|}$.*

(v) *Let $\alpha = a/b$, where $a \in \mathbb{N}$, $b \in \mathbb{Z}$, $a > |b| \geq 1$ and $\gcd(a,b) = 1$. If $a \neq b+1$ (resp., $a = b+1$), then $\alpha \in \mathcal{F}_{|M_\alpha(0)|}$ (resp., $\alpha \in \mathcal{F}_{2|M_\alpha(0)|-1}$). Moreover, we can choose S_α in a way that $0 \in S_\alpha \subset \mathbb{Z}$.*

It follows, in particular, by Theorem 2 (iv), that for each algebraic integer α, there is a non-negative rational integer p such that $\alpha \pm k \in \mathcal{F}_{|M_{\alpha \pm k}(0)|}$, $\forall\ k \in \mathbb{N} \cap [p, \infty)$. This result may also be deduced from [20, Theorem 5], which uses the above mentioned result of Kovács [18]. Notice also, by Theorem 2 (v), that there is a number system for any rational number a/b, where $a \in \mathbb{N}$, $b \in \mathbb{Z}$, $a > |b| \geq 1$, $\gcd(a,b) = 1$ and $a \neq b+1$. See also [4] for an investigation of number systems in rational bases.

The height reducing problem is related to the multiplicity of representations of an element $z \in \mathbb{Z}[\alpha]$, *i.e.*, the number of equivalent representations of the same number as a polynomial in base α :

$$z = \sum_{i=0}^{\ell} a_i \alpha^i = \sum_{i=0}^{\ell} b_i \alpha^i$$

with $a_i, b_i \in \mathbb{Z}$. To find all these representations, we clearly need to study representations of 0 of the form

$$\sum_{i=0}^{\ell} (a_i - b_i)\alpha^i = 0.$$

If there is $H > 0$ such that $|a_i - b_i| < H$ for all $i \in \mathbb{N}$, then there is a finite automaton which recognizes representations of zero, under some assumption (for basics on automata theory we refer to [9]).

Theorem 3. *Let $H > 0$ and let α be an algebraic number without conjugates on the unit circle. Then, there is a finite automaton $Z(H)$ which recognizes words $d_m \dots d_0 \in \{-H, \dots, H\}^*$ such that $\sum_{i=0}^{m} d_i \alpha^i = 0$.*

This automaton tells the growth rate of the number of equivalent representations, and is used in the study of the boundary and the topology of fractal tilings, when α is an expanding algebraic integer (*cf. e.g.* [8, 11, 21]). We expect that it also has potential applications in the study of spectra of polynomials and related topics (see for instance [5, 24, 25]). In connection with the height reducing property, Theorem 3 enables one to determine *"the minimal height polynomial"* of a given algebraic number α, that is a non-zero polynomial $P \in \mathbb{Z}[x]$, satisfying $P(\alpha) = 0$

where the maximum $H(P)$ of the absolute values of the coefficients of P is as small as possible. Increasing one by one the value H until the automaton recognizes a non-empty word, this theorem gives an algorithm to determine $H(P)$. We do not have such an algorithm for α having a conjugate on the unit circle, *e.g.*, when α is a Salem number.

The proofs of our theorems are detailed in Section 3. In Section 2 we show auxiliary results, some of which are used to prove Theorem 2; these results are extensions of the corresponding ones of [16].

2. Some propositions

We will follow the same steps as in [16] to show some auxiliary results of independent interest. Note that our discussion is not restricted to expanding algebraic integers; it is valid for general algebraic numbers.

For a non-zero algebraic number α, the set $\{0,\dots,|M_\alpha(0)|-1)\}$ is a CRS mod α of the ring $\mathbb{Z}[\alpha]$, and so any CRS contains exactly $|M_\alpha(0)|$ elements. In other words, we identify $\mathbb{Z}[\alpha]$ with $\mathbb{Z}[x]/(M_\alpha)$ and consider its quotient ring by an ideal (x), that is isomorphic to $\mathbb{Z}[x]/(M_\alpha,x)\simeq\mathbb{Z}/M_\alpha(0)\mathbb{Z}$. A CRS may be identified with a set of representatives of the last quotient ring. Now fix a CRS, say R, then each element $\beta\in\mathbb{Z}[\alpha]$, can be written in a unique way $\beta=r+\alpha\beta'$, where $r\in R$ and $\beta'\in\mathbb{Z}[\alpha]$. Iterating the map

$$\begin{array}{rccc} J: & \mathbb{Z}[\alpha] & \to & \mathbb{Z}[\alpha], \\ & \beta & \mapsto & \dfrac{\beta-r}{\alpha}, \end{array} \tag{4}$$

where r is the unique element of R satisfying $\beta\equiv r \bmod \alpha$, we can associate to any $\beta\in\mathbb{Z}[\alpha]$, a sequence $(J^{(n)}(\beta))_{n\geq 0}$ of elements of $\mathbb{Z}[\alpha]$, where $J^{(0)}(\beta):=\beta$. In particular, if $J^{(n)}(\beta)=\beta$ for some $n\geq 1$, then β is said to be *periodic*; the set of periodic numbers is denoted by $\wp$. Setting $r_n=r_n(\beta):=J^{(n)}(\beta)-\alpha J^{(n+1)}(\beta)$, where $n\geq 0$ and $r_n\in R$, we have

$$\beta=r_0+\cdots+r_n\alpha^n+\alpha^{n+1}J^{(n+1)}(\beta), \tag{5}$$

and

$$J^{(n+1)}(\beta)=\frac{\beta}{\alpha^{n+1}}-\frac{r_0}{\alpha^{n+1}}-\cdots-\frac{r_n}{\alpha^1}. \tag{6}$$

The following result gives some necessary and sufficient conditions for S_α to contain exactly one representative of each element of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$.

Proposition 4. *Let α be a non-zero algebraic number and let R be a CRS of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$. Then the following assertions are equivalent.*

(i) $\mathbb{Z}[\alpha]=R[\alpha]$.

(ii) $\wp=\{J^{(n)}(0)\mid n\geq 0\}$, *and* $\forall\beta\in\mathbb{Z}[\alpha]$, $\exists\ s=s(\beta)\in\mathbb{N}$ *such that* $J^{(s+1)}(\beta)=0$.

(iii) $\wp=\{J^{(n)}(0)\mid n\geq 0\}$, *and* $\forall\beta\in\mathbb{Z}[\alpha]$ *the sequence* $(J^{(n)}(\beta))_{n\geq 0}$ *is eventually periodic.*

Proof. (i)⇒(ii). Suppose $\mathbb{Z}[\alpha]=R[\alpha]$ and let $e_0+\cdots+e_s\alpha^s$ be a representation in $R[\alpha]$ of an element $\beta\in\mathbb{Z}[\alpha]$, where $s=s(\beta)\in\mathbb{N}$. If $s=0$, then by (5) we have $\beta=r_0+\alpha J^{(1)}(\beta)=e_0+\alpha 0$ and so $J^{(1)}(\beta)=0$. Similarly, when $s\geq 1$ we have $\beta=r_0+\alpha(r_1+\cdots+r_s\alpha^{s-1}+\alpha^s J^{(s+1)}(\beta))=e_0+\alpha(e_1+\cdots+e_s\alpha^{s-1})$,

$r_0 = e_0$, $r_1 + \cdots + r_s\alpha^{s-1} + \alpha^s J^{(s+1)}(\beta) = e_1 + \cdots + e_s\alpha^{s-1}$, and by induction we obtain $J^{(s+1)}(\beta) = 0$. It follows in particular when $\beta = 0$ that there is a positive integer $p = s(0) + 1$ such that $J^{(p)}(0) = 0$; thus $0 \in \wp$. Moreover, if p designates the smallest integer satisfying the last equality, then

$$\{J^{(n)}(0) \mid n \geq 0\} = \{J^{(n)}(0) \mid 0 \leq n \leq p-1\} \subset \wp,$$

and for any $\beta \in \mathbb{Z}[\alpha]$, we have $\{J^{(n)}(\beta) \mid n \geq s(\beta)+1\} = \{J^{(n)}(0) \mid 0 \leq n \leq p-1\}$; so $\wp \subset \{J^{(n)}(0) \mid n \geq 0\}$.

(ii)⇒(iii) is trivial, since the relation $0 = J^{(s(0)+1)}(0) \in \wp$ gives that the sequence $(J^{(n)}(0))_{n\geq 0}$ is purely periodic, and so we have, by the hypothesis $J^{(s(\beta)+1)}(\beta) = 0$, where $\beta \in \mathbb{Z}[\alpha]$, that $(J^{(n)}(\beta))_{n\geq 0}$ is eventually periodic.

(iii)⇒(i). For each $\beta \in \mathbb{Z}[\alpha]$ there are two positive rational integers k and m such that $J^{(k)}(\beta) = J^{(k+m)}(\beta)$. Hence, $J^{(m)}(J^{(k)}(\beta)) = J^{(k)}(\beta)$, $J^{(k)}(\beta) \in \wp$ and so $J^{(k)}(\beta) = J^{(l)}(0)$ for some $l \in \{0, \ldots, p-1\}$, where p is a positive rational integer such that $0 = J^{(p)}(0)$; thus $J^{(k+p-l)}(\beta) = J^{(p)}(0) = 0$, and by (5) we see that $\beta \in R[\alpha]$. □

Corollary 5. *With the same assumption as in Proposition 4 we have the following equivalence: (α, R) is a number system $\iff \forall\beta \in \mathbb{Z}[\alpha]$, the sequence $(J^{(n)}(\beta))_{n\geq 0}$ is eventually periodic, and $\wp = \{0\}$.*

Proof. The result is an immediate consequence of Proposition 4. Indeed, if (α, R) is a number system, then $0 \in R$, $0 = 0 + \alpha 0$, $J^{(1)}(0) = 0$ and so $\wp = \{0\}$. Conversely, if $\wp = \{0\}$, then $0 \in \wp$, $J^{(1)}(0) = 0$ and by the relation (5) (with $n = 0$), we have that $0 \in R$. □

Proposition 6. *With the same hypothesis as in Proposition 4, for each $\beta \in \mathbb{Z}[\alpha]$ there is a constant $c = c(\alpha, \beta) \in \mathbb{N}$ and a positive integer $L = L(\alpha, R)$ such that $LJ^{(n)}(\beta)$ is an algebraic integer, $\forall\, n \geq c$.*

Proof. Clearly for any element $\gamma \in \mathbb{Z}[\alpha]$, there is a positive integer $c = c(\alpha, \gamma)$ such that $\gamma/\alpha^c \in \mathbb{Z}[1/\alpha]$. Put $\ell = \max\{c(\alpha, r) \mid r \in R\}$. Then by (5) and (6), for every $n \geq c(\alpha, \beta)$ we have $\alpha^{-\ell}J^{(n)}(\beta) \in \mathbb{Z}[1/\alpha] \cap \alpha^{-\ell}\mathbb{Z}[\alpha]$, i.e., $J^{(n)}(\beta) \in \alpha^{\ell}\mathbb{Z}[1/\alpha] \cap \mathbb{Z}[\alpha]$. Letting L be the absolute norm of the denominator of the fractional ideal (α^{ℓ}) in $\mathbb{Q}(\alpha)$, we obtain $LJ^{(n)}(\beta) \in L\alpha^{\ell}\mathbb{Z}[1/\alpha] \cap \mathbb{Z}[\alpha]$, and we can deduce the result similarly to the proof of [7, Lemma 3]. □

For an algebraic number α we designate by $E(\alpha)$ the set of the distinct embeddings of the field $\mathbb{Q}(\alpha)$ into $\mathbb{C}$. The following assertion may be easily deduced from [16, Lemma 1].

Proposition 7. *Let α be an expanding algebraic number and let R be a CRS. Then, there is a constant $c = c(\alpha, R)$ with the following property: for each $\beta \in \mathbb{Z}[\alpha]$ there is $n_0 \in \mathbb{N}$ such that $\left|\sigma(J^{(n)}(\beta))\right| \leq c$ for all $n \geq n_0$ and $\sigma \in E(\alpha)$.*

Proof. For each element $\sigma \in E(\alpha)$, set $K_\sigma := \max\{|\sigma(r)| \mid r \in R\}$. Then, by (6), we have $\sigma(J^{(n+1)}(\beta)) = \frac{\sigma(\beta)}{(\sigma(\alpha))^{n+1}} - \frac{\sigma(r_0)}{(\sigma(\alpha))^{n+1}} - \cdots - \frac{\sigma(r_n)}{(\sigma(\alpha))^{1}}$, $\left|\sigma(J^{(n+1)}(\beta))\right| \leq \frac{|\sigma(\beta)|}{|\sigma(\alpha)|^{n+1}} + \frac{K_\sigma}{|\sigma(\alpha)|-1}$, and the result follows immediately by setting (for example) $c(\alpha, R)$ the greatest value of the quantities $1 + \frac{K_\sigma}{|\sigma(\alpha)|-1}$, when σ runs through $E(\alpha)$. □

The first, second and last assertions of the corollary below, have been mentioned in [16], when α is an expanding integer.

Corollary 8. *Under the assumptions of Proposition 7 the following assertions hold.*

(i) $\wp$ *is a finite set.*
(ii) $\forall \beta \in \mathbb{Z}[\alpha]$, *the sequence* $(J^{(n)}(\beta))_{n\geq 0}$ *is eventually periodic.*
(iii) $\mathbb{Z}[\alpha] = R[\alpha] \Leftrightarrow \wp = \{J^{(n)}(0) \mid n \geq 0\}$.
(iv) (α, R) *is a number system* $\Leftrightarrow \wp = \{0\}$.

Proof. We see that (i) and (ii) are consequences of the Propositions 6 and 7, and from this we deduce, by Proposition 4, the last two assertions. □

3. Proofs of theorems

Proof of Theorem 1. Let (α, S) be a pair satisfying (1), and fix for each element of S a representation, say $\sum_{k=0}^{s_j} a_{k,j}\alpha^k$, where $j \in \{1, \dots, \mathrm{Card}(S)\}$, $s_j \in \mathbb{N}$ and $a_{k,j} \in \mathbb{Z}$. Padding with zeros the last sums may also be written

$$\sum_{k=0}^{s} a_{k,j}\alpha^k, \tag{7}$$

where $s := \max\{s_j \mid 1 \leq j \leq \mathrm{Card}(S)\}$. If $\beta = \sum_{j=0}^{n} \varepsilon_j \alpha^j$, where $n \in \mathbb{N}$ and $(\varepsilon_0, \dots, \varepsilon_n) \in S^{n+1}$, then we see, by (7), that $\beta = R(\alpha)$ for some $R(x) := \sum_{k=0}^{D} A_k x^k \in \mathbb{Z}[x]$ and $D \geq s$. Moreover, a short computation shows that the values of A_k, are among the numbers $a_{0,j_0} + a_{1,j_1} + \cdots + a_{\min\{k,s\}, j_{\min\{k,s\}}}$, where $(j_0, \dots, j_{\min\{k,s\}}) \in \{1, \dots, \mathrm{Card}(S)\}^{\min\{k,s\}+1}$. Hence, the number of possible values of the coefficients of R is given by $\sum_{k=1}^{s+1} \mathrm{Card}(S)^k < \infty$, and so α satisfies HRP. It follows immediately from [7, Theorem 1], that α is an algebraic number. Now, we show that $\mathrm{Card}(S) \geq 2$. Clearly $S[\alpha] = \{0\} \neq \mathbb{Z}[\alpha]$ when $S = \{0\}$. The representation $\sum_{j=0}^{n} s\alpha^j$ of 0, must exist when $\{s\} = S \neq \{0\}$. However this implies that α is a root of unity not equal to 1, and so $S[\alpha]$ is a bounded subset of $\mathbb{C}$. This means that $S[\alpha] \neq \mathbb{Z}[\alpha]$. Hence, $\mathrm{Card}(S) \geq 2$, and the first inequality in Theorem 1 is true, as

$$\sum_{k=1}^{s+1} \mathrm{Card}(S)^k = \mathrm{Card}(S)(\mathrm{Card}(S)^{s+1} - 1)/(\mathrm{Card}(S) - 1).$$

To end the proof of Theorem 1 assume without loss of generality that $|M_\alpha(0)| \geq 2$. For each $\beta \in \mathbb{Z}[\alpha]$ the representation $\beta = a_0 + a_1\alpha + \cdots + a_L\alpha^L \in S_\alpha[\alpha]$ has $a_0 \equiv \beta \bmod \alpha$. Thus, S_α contains a complete system of coset representatives of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$. Thus by Gauss' Lemma for each $j \in \mathbb{Z}$ there is $P(\alpha) \in S_\alpha$ such that $P(0) \equiv j \bmod M_\alpha(0)$. □

Proof of Theorem 2 (i). It is clear that $\mathbb{Z}[1] = \mathbb{Z} = \{-1, 1\}[1]$. Let $\alpha \neq 1$ be a root of unity, and let $m \in \mathbb{N} \cap [2, \infty)$ satisfying $\alpha^m = 1$. Then, using the fact that $\alpha^{jm} = 1$, where $j \in \mathbb{N}$, a simple induction shows that $\mathbb{N} \subset \{0, 1\}[\alpha]$. Similarly, by the equation $-1 = \sum_{j=1}^{m-1} \alpha^j$,we obtain that every negative rational integer belongs to the set $\{0,1\}[\alpha]$. After this, a simple induction on the degree of the representations of the elements of $\mathbb{Z}[\alpha]$, leads immediately to the equation $\mathbb{Z}[\alpha] = \{0,1\}[\alpha]$.

Now, consider $\alpha \in \mathcal{F}_2$ which is not a root of unity. Then, the relation (2) gives that $|M_\alpha(0)| \leq 2$, and so by [7, Theorem 1], we obtain that α is an expanding integer, or is an algebraic number whose conjugates are of modulus 1, with $|M_\alpha(0)| = 2$. Indeed, if α is an expanding number which is not an expanding integer, then the leading coefficient, say c, of M_α, satisfies $|c| \geq 2$, and so

$$\left| \frac{M_\alpha(0)}{c} \right| \leq \frac{2}{2};$$

this last inequality leads to a contradiction, because the absolute value of the product of the conjugates of α is greater than 1. □

Proof of Theorem 2 (ii). Suppose that $\alpha \in \mathcal{F}_{|M_\alpha(0)|}$. Then, α satisfies HRP. By Theorem 1, we see that $|M_\alpha(0)| \geq 2$, and any corresponding set S_α, satisfies $\mathrm{Card}(S_\alpha) = |M_\alpha(0)|$; thus S_α is a complete residue system $\bmod\, \alpha$ in $\mathbb{Z}[\alpha]$. Set $M_\alpha(x) = A_0 + A_1 x + \cdots + A_d x^d$, and assume on the contrary that $M_\alpha(1)^2 = 1$. We shall obtain a contradiction by considering the non-zero number

$$\beta_0 := \frac{M_\alpha(1)}{(1-\alpha)}.$$

Indeed, a simple computation shows that

$$\beta_0 = \sum_{j=0}^{d-1} \alpha^j \sum_{k=j+1}^{d} A_k,$$

and so $\beta_0 \in \mathbb{Z}[\alpha]$. Moreover, if we fix a non-zero element $s \in S_\alpha$, and we set

$$\beta := s M_\alpha(1) \beta_0,$$

then $\beta \in \mathbb{Z}[\alpha]$, $\beta = s + \alpha\beta$, and so $J^{(1)}(\beta) = \beta$; thus $J^{(n)}(\beta) = \beta$ for all $n \geq 0$, and by Proposition 4 we obtain a contradiction, since $\beta \neq 0$. □

Remark 9. It follows immediately by Theorem 2 (ii), that $2 \in \mathcal{F}_3$, since $M_2(1) = 1 - 2 \Rightarrow 2 \notin \mathcal{F}_2$, and $\{-1,0,1\}[2] = \mathbb{Z} \Rightarrow \mathrm{Card}(S_2) \leq 3$ (this relation may also be deduced from Theorem 2 (v)), and so, by Theorem 2 (i), we have $\mathbb{Q} \cap \mathcal{F}_2 = \{-2,-1,1\}$. Concerning the quadratic case the results of Gilbert [13] and Kátai and Kovács [17] imply that there are at least 8 quadratic (non-real) expanding integers in $\mathcal{F}_2$. Also, a short computation shows that if α is an expanding real quadratic integer satisfying $|M_\alpha(0)| = 2$, then $M_\alpha(x) = x^2 - 2$, and so $\alpha \notin \mathcal{F}_2$, as $|M_\alpha(1)| = 1$. For higher degrees consider for example the $p-$Eisenstein polynomials $x^d + px^k + \cdots + p$, where p is a prime, $d \geq 2$, and k runs through $\{0,1,\ldots,d-1\}$. We see, by the above mentioned result of Kovács [18], that each set $\mathcal{F}_p$ contains at least d^2 expanding integers with degree d. On the other hand, if $d \equiv 0 \bmod 2$ and α is a root of $M_\alpha(x) = x^d + px^{d-1} + \cdots + px + p$, then $-\alpha$ is of degree d and satisfies $-\alpha \notin \mathcal{F}_p$ as $M_{-\alpha}(1) = 1$.

Remark 10. It is easy to see when $\alpha \in \mathcal{F}_2$ and α is not an algebraic integer, then $0 \notin S_\alpha$ (that is S_α is not a number system). Indeed, if $0 \in S_\alpha$ and b designates the other (non-zero) element of S_α, then the representation of $-b$ in $S_\alpha[\alpha]$ is of the form

$b + b\alpha^{n_1} + \cdots + b\alpha^{n_s}$, where $1 \leq n_1 < \ldots < n_s$. Thus α is a root of the polynomial $2 + x^{n_1} + \cdots + x^{n_s}$, contradicting the fact that α is not an algebraic integer. Notice that Theorem 2 (ii) can also be applied in the non-integral case. For example, if $M_\alpha(x) = 2x^2 - 3x + 2$, then α is a quadratic algebraic number whose conjugates are of modulus 1 (α satisfies HRP by [7, Theorem 2]) and $\alpha \notin \mathcal{F}_2$, since $M_\alpha(1) = 1$.

Proof of Theorem 2 (iii). Let α be an algebraic number whose conjugates are all of modulus 1, such that $\mathrm{Card}(S_\alpha) = |M_\alpha(0)|$. Then, α is not an algebraic integer, since otherwise α is a root of unity (by Kronecker's theorem), and Theorem 2 (i) gives, in this case, that $\mathrm{Card}(S_\alpha) = 2 > 1 = |M_\alpha(0)|$. Notice also that $M_\alpha(x) = x^d M_\alpha(1/x)$, where d is the degree of M_α, and so the leading coefficient of M_α, say c, satisfies $M_\alpha(0) = c \geq 2$. Assume on the contrary, that there is a set $S_\alpha \subset \mathbb{Z}$ satisfying $\mathrm{Card}(S_\alpha) = c$. Then, by Theorem 1, the set S_α is a CRS $\bmod\, \alpha$ in $\mathbb{Z}[\alpha]$, and is also a CRS $\bmod\, c$ in $\mathbb{Z}$. Now, we claim that the map J, which was defined on the ring $\mathbb{Z}[\alpha]$ in (4), restricted to the set

$$U := \mathbb{Z}[\alpha] \cap \frac{1}{\alpha}\mathbb{Z}\left[\frac{1}{\alpha}\right],$$

is a bijection of U. Indeed, if $\beta \in U$ and $r = r(\beta)$ is the unique element of S_α such that

$$J(\beta) = (\beta - r)/\alpha,$$

then $(\beta - r) \in \mathbb{Z}[1/\alpha]$, $(\beta - r)/\alpha \in (1/\alpha)\mathbb{Z}[1/\alpha]$, and $J(\beta) \in U$; thus $J(U) \subset U$. Moreover, if $\sum_{j=1}^{s} a_j/\alpha^j$ and $\sum_{j=1}^{t} b_j/\alpha^j$ designate, respectively, some representations in $(1/\alpha)\mathbb{Z}[1/\alpha]$ of two elements β and γ of U, then the equation $J(\beta) = J(\gamma)$ gives immediately that α is a root of a polynomial with integer coefficients, whose leading coefficient is $(r(\gamma) - r(\beta))$. It follows by Gauss' Lemma that $r(\gamma) \equiv r(\beta) \bmod c$ and so $r(\gamma) = r(\beta)$; thus $\beta = \gamma$ and J is injective. To complete the proof of the claim, fix again a representation $\sum_{j=1}^{v} c_j/\alpha^j$, in $(1/\alpha)\mathbb{Z}[1/\alpha]$, of an element y of U. Then, the relation $\alpha y - c_1 \in (1/\alpha)\mathbb{Z}[1/\alpha]$, together with the equality $-c_1 = r(-c_1) - ck$, where $k \in \mathbb{Z}$, yield $\alpha y + r(-c_1) \in ck + (1/\alpha)\mathbb{Z}[1/\alpha]$, and so

$$\alpha y + r(-c_1) \in (1/\alpha)\mathbb{Z}[1/\alpha], \tag{8}$$

as $M_\alpha(\alpha) = 0 \Rightarrow c \in (1/\alpha)\mathbb{Z}[1/\alpha]$ and

$$c\mathbb{Z} \subset (1/\alpha)\mathbb{Z}[1/\alpha]. \tag{9}$$

Since $\alpha y + r(-c_1) \in \mathbb{Z}[\alpha]$, $r(-c_1) \in S_\alpha$ and $J(\alpha y + r(-c_1)) = y$, we see by (8) that J is a surjective, and so J is a bijection of U. Notice also that U and S_α have only one common element (which is the unique element in $c\mathbb{Z} \cap S_\alpha$). It follows immediately that $J^{(n)}(0) \in U$, for all $n \in \mathbb{N}$, and so $\wp = \{J^{(n)}(0) \mid n \geq 0\} \subset U$. Recall, by Proposition 4, that $\wp$ is finite and for each $\beta \in U$, there exists $s \geq 1$ such that $J^{(s)}(\beta) = 0$. Moreover, as $\wp \subset U$ is finite and J is bijective on U there is $t \in \mathbb{N}$ such that $J^{(t)}(0) = 0$. Thus, again by bijectivity of J the number β has to occur somewhere in the cycle (each arrow indicates an application of J)

$$0 \xrightarrow{J} y_1 \xrightarrow{J} y_2 \xrightarrow{J} \cdots \xrightarrow{J} y_{t-1} \xrightarrow{J} 0$$

and, hence, $J^{(t-s)}(0) = \beta$ which implies that $\beta \in U$. Thus $U \subset \wp$, a contradiction, because by (9) we have that $c\mathbb{Z} \subset U$ and so U cannot be finite. $\square$

Proof of Theorem 2 (iv). Since the eigenvalues of the companion matrix of the polynomial M_α are all of modulus greater than 2, the result follows immediately from [12, Theorem 4]. □

Proof of Theorem 2 (v). It is clear that $M_\alpha(x) = bx - a$, and if R is a CRS $\bmod a$ in $\mathbb{Z}$, then R is also a CRS $\bmod \alpha$ in $\mathbb{Z}[\alpha]$, as $a = b\alpha$. Suppose first that $a \neq b+1$. We shall show that there is a CRS $\bmod a$ in $\mathbb{Z}$, say S, such that every element $\beta \in \mathbb{Z}[\alpha]$ may be written $\beta = \varepsilon_0 + \cdots + \varepsilon_n \alpha^n$, for some $n \in \mathbb{N}$ and $(\varepsilon_0, \ldots, \varepsilon_n) \in S^{n+1}$. For this purpose it is enough to prove the inclusion

$$\mathbb{Z} \subset S[\alpha]. \tag{10}$$

Indeed, assume that all elements of $\mathbb{Z}[x]$ with degree at most $d-1$, where $d \geq 1$, evaluated at α belong to the set $S[\alpha]$, and let $P(x) = a_0 + a_1 x + \cdots + a_d x^d \in \mathbb{Z}[x]$. Since the constant term a_0 may be written

$$a_0 = \varepsilon + a a_1' = \varepsilon + b a_1' \alpha,$$

for some $\varepsilon \in S$, and $a_1' \in \mathbb{Z}$, we see that

$$P(\alpha) = \varepsilon + (a_1 + b a_1')\alpha + \cdots + a_d \alpha^d = \varepsilon + \alpha Q(\alpha),$$

where $Q(x) = (a_1 + b a_1') + \cdots + a_d x^{d-1} \in \mathbb{Z}[x]$, and by the induction hypothesis we obtain the result. Furthermore, to show the inclusion (10), it suffices to prove that $ka \in \alpha S[\alpha]$, $\forall\ k \in \mathbb{Z}$, or equivalently

$$kb \in S[\alpha], \quad \forall k \in \mathbb{Z}, \tag{11}$$

as any rational integer may be written $\varepsilon + ka$ for some $\varepsilon \in S$ and $k \in \mathbb{Z}$.

Assume first that $0 < -b < a$ and choose $S_\alpha = \{0, \ldots, a-1\}$. As S_α is a CRS modulo a the mapping J in (4) is well-defined and as in (5) we can be used to attach a formal sum

$$kb = \sum_{i=0}^{\infty} d_i \alpha^i$$

to each kb with $k \in \mathbb{Z}$. We denote this by $kb = (\ldots, d_1, d_0)_\alpha$. To prove that $b\mathbb{Z} \subset S_\alpha[\alpha]$ we need to show that for each $k \in \mathbb{Z}$ there is $\ell \in \mathbb{N}$ such that $a_i = 0$ for each $i \geq \ell$ (in this case we say that kb has a *finite expansion*). This will be done by an induction involving a so-called *transducer* automaton (see *e.g.* [9] for the definition of these objects).

As $0 = (\ldots, 0, 0)_\alpha$ it is clear that 0 has a finite expansion. Now assume that kb has a finite expansion for some given k. To conclude the induction proof we have to show that the same is true for $(k \pm 1)b$. To this matter we study the effect of "addition and subtraction of b" on the expansion of a number.

We first deal with the addition of b. Let $kb = (\ldots, d_2, d_1, d_0)_\alpha$. If $d_0 + b \in S_\alpha$, then $(k+1)b = (\ldots d_2, d_1, d_0 + b)_\alpha$ and we are done. If, however, $d_0 + b \notin S_\alpha$, then certainly $d_0 + b + a \in S_\alpha$, and, observing that $b\alpha - a = 0$, we get that $(k+1)b = (\ldots, d_2, d_1 - b, d_0 + b + a)_\alpha$. In this case, again two things can happen: either $d_1 - b \in S_\alpha$, in which case we are done, or $d_1 - b - a \in S_\alpha$, in which case we gain $(k+1)b = (\ldots, d_2 + b, d_1 - b - a, d_0 + b + a)_\alpha$ and have to go on again. Subtraction of b is treated analogously. As in Akiyama *et al.* [4] we use a transducer to model this process (see Figure 1).

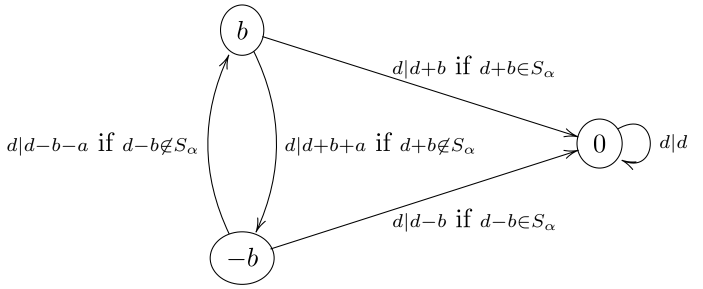
b
$d|d+b$ if $d+b\in S_\alpha$
$d|d-b-a$ if $d-b\notin S_\alpha$
$d|d+b+a$ if $d+b\notin S_\alpha$
0
$d|d$
$d|d-b$ if $d-b\in S_\alpha$
$-b$

FIGURE 1. The transducer for $0<-b<a$. If we use b as a starting state, this transducer reads the digits of kb from right to left and writes out the digits of $(k+1)b$. If we use $-b$ as starting state, it writes out the digits of $(k-1)b$. The label $d_1|d_2$ means that reading d_1 the transducer writes out d_2. As soon as we arrive at the state 0, the remaining digits are just copied.

Let $kb=(\dots,d_1,d_0)_\alpha$ be given. Feeding the digits of this expansion in the transducer depicted in Figure 1 from right to left starting at the state $\pm b$ the transducer will write out the digits of the expansion of $(k\pm 1)b$. Since, by the induction assumption, kb has a finite expansion, eventually we read only the digit 0. However, as it is easily seen that two zeros in a row make sure that the transducer arrives in the "accepting state" 0, we conclude that the length of the expansion of $(k\pm 1)b$ can be at most by two (nonzero) digits longer than the expansion of kb. This proves that the expansion of $(k\pm 1)b$ is finite and the induction proof is finished.

Let now $0<b<a-1$. In this case the choice of the digit set is a bit more subtle; the multiples of $a-b$ play a special role here and need to be shifted to the negative by a. Indeed, set $R=\{0,\dots,a-1\}$ and $B=\{k(a-b),k(a-b)-a \mid 1\le k\le \frac{a-1}{a-b}\}$. Then a convenient digit set is given by the symmetric difference $S_\alpha=R\triangle B$. The following assertions are easily checked.

(A) S_α is a CRS modulo a.
(B) If $d\in S_\alpha$ then either $d+b\in S_\alpha$ or $d+b-a\in S_\alpha$.
(C) If $d\in S_\alpha$ then either $d-b\in S_\alpha$ or $d-b-a\in S_\alpha$.
(D) $\{-b,b\}\subset S_\alpha$ (here we have to use that $b\neq a-1$).

The mapping J is well defined by (A). Moreover, (B) and (C) make sure that the transducer in Figure 2 can process all digit strings $(\dots,d_2,d_1,d_0)\in\{0,1,\dots,a-1\}^{\mathbb{N}}$ and produces a well-defined unique output. Indeed, direct calculations similar to the ones presented in the case $0<-b<a$ show that this transducer performs the addition of $\pm b$ to expansions $kb=(\dots,d_2,d_1,d_0)_\alpha$. Again we can now use induction to show that kb has finite expansion for all $k\in\mathbb{Z}$. Now (D) implies that two zeros in a row make sure that the transducer arrives in the "accepting state" 0 and we conclude again that the length of the expansion of $(k\pm 1)b$ can be at most by two (nonzero) digits longer than the expansion of kb.

It remains to deal with the case $0<b=a-1$. Let S_α be given and suppose that $d\in S_\alpha$ is nonzero. Then $J(-bd)=\frac{-bd-d}{\alpha}=-bd$. Thus, to get a finite expansion of $-bd$ we need another element of S_α that lies in the same residue class modulo a. Therefore, for each nonzero residue class we need at least two representatives to be contained in S_α in order to guarantee that $b\mathbb{Z}\subset S_\alpha[\alpha]$ and,

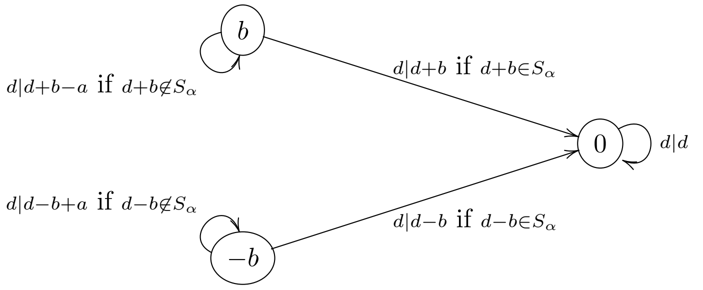
b
$d|d+b-a$ if $d+b\notin S_\alpha$
$d|d+b$ if $d+b\in S_\alpha$
0
$d|d$
$d|d-b+a$ if $d-b\notin S_\alpha$
$d|d-b$ if $d-b\in S_\alpha$
$-b$

Figure 2. The transducer for $0 < b < a-1$.

hence, $|S_\alpha| > 2|M_\alpha(0)|-1$. As it follows immediately from the results on the case $0 < -b < a$ that $S_\alpha = \{-a+1,\dots,a-1\}$ satisfies $S_\alpha[\alpha] = \mathbb{Z}[\alpha]$, we conclude that $\alpha \in \mathcal{F}_{2|M_\alpha(0)|-1}$. □

Proof of Theorem 3. Let $\alpha = \alpha_1, \alpha_2, \dots, \alpha_d$ be the conjugates of α, arranged so that $|\alpha_k| > 1$ for $k = 1,\dots,n$ and $|\alpha_k| < 1$ for $k = n+1,\dots,d$. For each k we denote by x_k the corresponding conjugate of any element x in $\mathbb{Z}[\alpha]$. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ designate the prime ideals which appear in the denominator of the prime ideal decomposition of (α) in $\mathbb{Q}(\alpha)$. Set

$$S(H) := \left\{ x \in \mathbb{Z}[\alpha] \;\middle|\; |x_j| \le \frac{H}{|1-|\alpha_j||},\ (j=1,\dots,d),\ \ \mu_j(x) \ge 0 \quad (j=1,\dots,s) \right\},$$

where $\mu_j(x)$ is the discrete valuation of x lying over $\mathfrak{p}_j$. By definition $S(H)$ is a finite set, because it is a subset of $\mathbb{Z}[\alpha]$ whose elements have bounded denominators and conjugates. Clearly $S(H) \subset S(H+1)$. Let $S_1 = \{0\}$ and we inductively define

$$S_{j+1} = \left\{ \alpha y + d \;\middle|\; \begin{array}{l} d \in \{-H,\dots,H\},\ y \in S_j, \\ |\alpha_k y_k + d| \le \frac{H}{|1-|\alpha_k||} \text{ for } k \le n, \\ \mu_j(\alpha y + d) \ge 0 \text{ for } 1 \le j \le s \end{array} \right\}$$

Then we easily see that

$$S_\infty = \bigcup_{j=1}^{\infty} S_j$$

is a subset of $S(H)$. Construct an automaton $Z(H)$ having states S_∞, transitions $\delta(y,d) = \alpha y + d$ are defined if there exists a j with $y \in S_j$ and $\alpha y + d \in S_{j+1}$, and 0 is both an initial and a final state. We claim that this automaton has the required property. In fact, assume that $\sum_{j=0}^{m} d_j \alpha^j = 0$ with $d_j \in \{-H,\dots,H\}$. It is obvious that

$$\left| \sum_{j=u}^{m} d_j \alpha_k^{j-u} \right| \le \frac{H}{1-|\alpha_k|}$$

holds for $k \ge n+1$ and $u = 0,\dots,m$. For $k \le n$, note that if $|x_k| > H/(|\alpha_k|-1)$ and $x_k \in \mathbb{Q}(\alpha_k)$, then $|\alpha_k x_k - d| > H/(|\alpha_j|-1)$ for $|d| \le H$. In plain words, this means that once x_k becomes larger than $H/(|\alpha_k|-1)$, then there is no way to come back to zero. Thus we see that

$$\left| \sum_{j=u}^{m} d_j \alpha_k^{j-u} \right| \le \frac{H}{|1-|\alpha_k||}$$

is valid for all $u = 0, \dots, m$ and $k = 1, \dots, d$. Similarly since $\mu_k(x) < 0$ implies the relation $\mu_k(\alpha x - d) = \mu_k(\alpha x) < \mu_k(x) < 0$, we see that

$$\mu_k \left(\sum_{j=u}^{m} d_j \alpha_k^{j-u} \right) \geq 0$$

for all u and k. Therefore the sequence of states $\sum_{j=u}^{m} d_j \alpha_k^{j-u}$ with $u = m, m-1, \dots, 0$ gives a successful path of $Z(H)$ whose output is $d_m d_{m-1} \dots d_0$. □

Remark 11. By standard operations in automata, we can recognize the set of the mirrored words $d_0 d_1 \dots d_m$. The automaton constructed in the proof of Theorem 3 is not *trim*, i.e., 0 may not be reachable from some states, but it is easy to make it to a trim automaton.

Acknowledgment. We thank the referee for careful reading of this paper.

References

[1] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner. Generalized radix representations and dynamical systems. I. *Acta Math. Hungar.*, 108(3):207–238, 2005.

[2] S. Akiyama, H. Brunotte, A. Pethő, and J. M. Thuswaldner. Generalized radix representations and dynamical systems. II. *Acta Arith.*, 121(1):21–61, 2006.

[3] S. Akiyama, P. Drungilas, and J. Jankauskas. Height reducing problem on algebraic integers. *Funct. Approx. Comment. Math.*, 47(part 1):105–119, 2012.

[4] S. Akiyama, C. Frougny, and J. Sakarovitch. Powers of rationals modulo 1 and rational base number systems. *Israel J. Math.*, 168:53–91, 2008.

[5] S. Akiyama and V. Komornik. Discrete spectra and Pisot numbers. *J. Number Theory*, 133(2):375–390, 2013.

[6] S. Akiyama, J. M. Thuswaldner, and T. Zaïmi. Characterization of the numbers which satisfy the height reducing property. *Indag. Math.*, to appear, 2014.

[7] S. Akiyama and T. Zaïmi. Comments on the height reducing property. *Cent. Eur. J. Math.*, 11(9):1616–1627, 2013.

[8] V. Berthé and A. Siegel. Tilings associated with beta-numeration and substitutions. *Integers*, 5(3):A2, 46, 2005.

[9] S. Eilenberg. *Automata, languages, and machines. Vol. A.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974. Pure and Applied Mathematics, Vol. 58.

[10] G. Farkas. Number systems in real quadratic fields. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 18:47–59, 1999.

[11] C. Fuchs and R. Tijdeman. Substitutions, abstract number systems and the space filling property. *Ann. Inst. Fourier (Grenoble)*, 56(7):2345–2389, 2006. Numération, pavages, substitutions.

[12] L. Germán and A. Kovács. On number system constructions. *Acta Math. Hungar.*, 115(1-2):155–167, 2007.

[13] W. J. Gilbert. Radix representations of quadratic fields. *J. Math. Anal. Appl.*, 83(1):264–274, 1981.

[14] V. Grünwald. Intorno all'aritmetica dei sistemi numerici a base negativa con particolare riguardo al sistema numerico a base negativo-decimale per lo studio delle sue analogie coll'aritmetica ordinaria (decimale). *Giornale di matematiche di Battaglini*, 23:203–221, 367, 1885.

[15] I. Kátai. Number systems in imaginary quadratic fields. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 14:91–103, 1994. Festschrift for the 50th birthday of Karl-Heinz Indlekofer.

[16] I. Kátai. Construction of number systems in algebraic number fields. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 18:103–107, 1999.

[17] I. Kátai and B. Kovács. Canonical number systems in imaginary quadratic fields. *Acta Math. Acad. Sci. Hungar.*, 37(1-3):159–164, 1981.

[18] B. Kovács. Canonical number systems in algebraic number fields. *Acta Math. Acad. Sci. Hungar.*, 37(4):405–407, 1981.

[19] J. C. Lagarias and Y. Wang. Integral self-affine tiles in $\mathbf{R}^n$. II. Lattice tilings. *J. Fourier Anal. Appl.*, 3(1):83–102, 1997.

[20] A. Pethő. Connections between power integral bases and radix representations in algebraic number fields. In *Proceedings of the 2003 Nagoya Conference "Yokoi-Chowla Conjecture and Related Problems"*, pages 115–125, Saga, 2004. Saga Univ.

[21] A. Siegel and J. M. Thuswaldner. Topological properties of Rauzy fractals. *Mém. Soc. Math. Fr. (N.S.)*, (118):140, 2009.

[22] G. Steidl. On symmetric radix representation of Gaussian integers. *BIT*, 29(3):563–571, 1989.

[23] C. van de Woestijne. Noncanonical number systems in the integers. *J. Number Theory*, 128(11):2914–2938, 2008.

[24] T. Zaïmi. Approximation by polynomials with bounded coefficients. *J. Number Theory*, 127(1):103–117, 2007.

[25] T. Zaïmi. Commentaires sur quelques résultats sur les nombres de Pisot. *J. Théor. Nombres Bordeaux*, 22(2):513–524, 2010.

Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, 350-0006 Japan

E-mail address: akiyama@math.tsukuba.ac.jp

Department of mathematics and statistics, Leoben University, Franz-Josef-Strasse 18, A-8700, Leoben, Austria

E-mail address: joerg.thuswaldner@unileoben.ac.at

Department of mathematics and informatics, Larbi Ben M'hidi University, Oum El Bouaghi 04000, Algeria

E-mail address: toufikzaimi@yahoo.com