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ON BOUNDEDNESS OF THE GENERALIZED RIESZ POTENTIAL IN LOCAL MORREY-TYPE SPACES

V. I. Burenkov[1,2,3] · **M. A. Senouci**[1,2]

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Abstract
For all admissible values of the numerical parameters sharp sufficient conditions on the functional parameters are obtained ensuring the boundedness of the generalized Riesz potential from one general local Morrey-type space to another one, which, for a certain range of the numerical parameters, coincide with the necessary ones.

Keywords Generalized Riesz potential operator · Boundedness · Local Morrey-type spaces

Introduction

In this paper, we study the boundedness from one general local Morrey-type space to another one of the generalized Riesz potential

$$\left(I_{\rho(\cdot)}f\right)(x) = \int_{\mathbb{R}^n} \rho(|x-y|)f(y)dy,\ x \in \mathbb{R}^n,$$

under certain assumptions on the kernel ρ. Our aim is to generalize the results obtained in [6] for the case of the classical Riesz potential I_α, in which $\rho(t) = t^{\alpha-n}$, $t > 0$, $0 < \alpha < n$. Some of the presented results were stated without proofs in [7]. Let $F, G : A \times B \longrightarrow [0, \infty]$. Throughout this paper we say that F is dominated by G uniformly in $x \in A$ and write

$$F \lesssim G \text{ uniformly in } x \in A$$

if there exists $c(B) > 0$ such that

$$F(x, y) \le c(B)G(x, y) \text{ for all } x \in A.$$

✉ V. I. Burenkov
burenkov@cardiff.ac.uk

M. A. Senouci
senoucim@yandex.ru

1 S.M. Nikol'skii Mathematical Institute, RUDN University, 6 Miklukho Maklay St, Moscow 117198, Russian Federation

2 V.A. Steklov Mathematical Institute, Russian Academy of Sciences, 8 Gubkin St, Moscow 117966, Russian Federation

3 Cardiff School of Mathematics, Cardiff University, Senghennydd Rd, Cardiff CF24 4AG, United Kingdom

(So $c(B)$ is independent of $x \in A$, but may depend on $y \in B$.) Respectively, we say that F dominates G uniformly in $x \in A$ and write

$$F \gtrsim G \text{ uniformly in } x \in A$$

if there exists $c(B) > 0$ such that

$$F(x, y) \geq c(B)G(x, y) \text{ for all } x \in A.$$

We also say that F is equivalent to G uniformly in $x \in A$ and write

$$F \approx G \text{ uniformly in } x \in A$$

if F and G dominate each other uniformly in $x \in A$.

Definitions and basic properties of general Morrey-type spaces

In this section we recall basic facts of the theory of general Morrey-type spaces.
Let, for a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$, $\mathfrak{M}(\Omega)$ denote the space of all functions $f : \Omega \longrightarrow \mathbb{C}$ Lebesgue measurable on Ω, and $\mathfrak{M}^+(\Omega)$ denote the subset of $\mathfrak{M}(\Omega)$ of all non-negative functions.

Definition 2.1 Let $0 < p, \theta \leq \infty$ and let $w \in \mathfrak{M}^+\big((0,\infty)\big)$ be not equivalent to 0. We denote by $LM_{p\theta,w(\cdot)}$ the local Morrey-type space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,w(\cdot)}} \equiv \|f\|_{LM_{p\theta,w(\cdot)}(\mathbb{R}^n)} = \|w(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)}.$$

Definition 2.2 Let $0 < p, \theta \leq \infty$. We denote by Ω_θ the set of all functions $w \in \mathfrak{M}^+((0,\infty))$ which are not equivalent to 0 and such that

$$\|w\|_{L_\theta(t,\infty)} < \infty \tag{2.1}$$

for some $t > 0$.

Lemma 2.1 [3–5]. *Let $0 < p, \theta \leq \infty$ and let w be a non-negative Lebesgue measurable function on $(0,\infty)$, which is not equivalent to* 0.

Then the space $LM_{p\theta,w(\cdot)}$ is nontrivial if and only if $w \in \Omega_\theta$.

In the sequel, keeping in mind Lemma 2.1, we always assume that $w \in \Omega_\theta$ for the local Morrey-type spaces $LM_{p\theta,w(\cdot)}$.
It is well known that the spaces $LM_{p\theta,w(\cdot)}$ are Banach spaces if $1 \leq p, \theta \leq \infty$ and are quasi-Banach spaces if $0 < p < 1$, or $0 < \theta < 1$, or both $0 < p, \theta < 1$.

For further properties of general Morrey-type spaces, operators acting in such spaces and applications see, for example, survey papers [1, 2, 12–16] and references therein.

L_p and WL_p estimates of generalized Riesz potentials over balls

Let $n \in \mathbb{N}, 0 < p < \infty, \Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set. Recall that the weak L_p-space $WL_p(\Omega)$ is the space of all Lebesgue measurable functions $f : \Omega \to \mathbb{C}$ for which

$$\|f\|_{WL_p(\Omega)} = \sup_{t \geq 0} t|\{y \in \Omega : |f(y)| > t\}|^{\frac{1}{p}} < \infty.$$

Here $|G|$ denotes the Lebesgue measure of a set $G \subset \mathbb{R}^n$.

Remark 3.1 Let $M > 0$. It follows directly from the above definition that, if f is equivalent to M on Ω, then

$$\|M\|_{WL_p(\Omega)} = \|M\|_{L_p(\Omega)} = M|\Omega|^{\frac{1}{p}}$$

and if $f : \Omega \to \mathbb{R}$ is such that $f \geqslant M$ almost everywhere on Ω, then

$$\|f\|_{WL_p(\Omega)} \geq M|\Omega|^{\frac{1}{p}}.$$

Definition 3.1 Let $n \in \mathbb{N}$. We say that $\rho \in S_n$ if $\rho \in \mathfrak{M}^+((0,\infty))$ and

1) $\int_0^r \rho(t)t^{n-1}dt < \infty$ for all $r > 0$,
2) for some $c_1, c_2 > 0$,

$$c_1\rho(t) \leq \rho(s) \leq c_2\rho(t)$$

for all $s, t > 0$ satisfying the inequality $\frac{t}{2} \leq s \leq 2t$.

Remark 3.2 Let $x \in \mathbb{R}^n$, $r > 0$, $y \in B(x,r)$, $z \in {}^cB(x,2r)$. Then

$$|y-z| \leq |y-x| + |x-z| \leq r + |x-z| \leq \frac{1}{2}|x-z| + |x-z| < 2|x-z|$$

and

$$|y-z| \geq |x-z| - |y-x| \geq |x-z| - r \geq |x-z| - \frac{1}{2}|x-z| = \frac{1}{2}|x-z|.$$

Let $s = |y-z|$ and $t = |x-z|$. Then by Condition 2) of Definition 3.1 we have
3) $c_1\rho(|x-z|) \leq \rho(|y-z|) \leq c_2\rho(|x-z|)$
for all $x \in \mathbb{R}^n$, $r > 0$, $y \in B(x,r)$, $z \in {}^cB(x,2r)$.

Remark 3.3 If functions $\rho_1, \rho_2 : (0,\infty) \to (0,\infty)$ satisfy Condition 2) of Definition 3.1 with c_{11}, $c_{12} > 0$, c_{21}, $c_{22} > 0$, respectively, then the product $\rho_1\rho_2$ satisfies Condition 2) with $c_1 = c_{11}c_{21}$ and $c_2 = c_{12}c_{22}$. Indeed, it suffices to multiply the inequalities

$$c_{i1}\ \rho_i(t) \leq \rho_i(s) \leq c_{i2}\rho_i(t),$$

where $s, t > 0, \frac{t}{2} \leq s \leq 2t$ and $i = 1, 2$.

Remark 3.4 If a function $\rho_1 : (0,1] \to (0,\infty)$ satisfies Condition 2) of Definition 3.1 with c_{11}, $c_{12} > 0$ and a function $\rho_2 : (1,\infty) \to (0,\infty)$ satisfies Condition 2) of Definition 3.1 with $c_{21}, c_{22} > 0$, then the function

$$\rho(t) = \begin{cases} \rho_1(t), & 0 < t \leq 1, \\ \rho_2(t), & 1 < t < \infty \end{cases}$$

satisfies Condition 2) of Definition 3.1 with certain $c_1, c_2 > 0$.

Indeed, let $s, t > 0, \frac{t}{2} \leq s \leq 2t$. If $t \leq \frac{1}{2}$, then $s \leq 1$, $\rho(s) = \rho_1(s)$, $\rho(t) = \rho_1(t)$ and Condition 2) of Definition 3.1 is satisfied with $c_{11}, c_{12} > 0$. Respectively, if $t > 2$, then $s > 1$, $\rho(s) = \rho_2(s)$, $\rho(t) = \rho_2(t)$ and Condition 2) of Definition 3.1 is satisfied with $c_{21}, c_{22} > 0$. Let $\frac{1}{2} < t \leq 2$, then $\frac{1}{4} < s \leq 4$,

$$\min\{c_{11}, c_{21}\} \leq \begin{cases} c_{11}, & \frac{1}{4} < s \leq 1, \\ c_{21}, & 1 < s < 4 \end{cases} \leq \rho(s) \leq \begin{cases} c_{12}, & \frac{1}{4} < s \leq 1, \\ c_{22}, & 1 < s < 4 \end{cases} \leq \max\{c_{12}, c_{22}\},$$

$$\min\{c_{11}, c_{21}\} \leq \begin{cases} c_{11}, & \frac{1}{2} < t \leq 1, \\ c_{21}, & 1 < t \leq 2 \end{cases} \leq \rho(t) \leq \begin{cases} c_{12}, & \frac{1}{2} < t \leq 1, \\ c_{22}, & 1 < t \leq 2 \end{cases} \leq \max\{c_{12}, c_{22}\},$$

hence

$$\min\{c_{11}, c_{21}\}(\max\{c_{12}, c_{22}\})^{-1}\rho(t) \le \rho(s) \le \max\{c_{12}, c_{22}\}(\min\{c_{11}, c_{21}\})^{-1}\rho(t).$$

So, for $\frac{1}{2} < t \le 2, \frac{t}{2} \le s \le 2t$, Condition 2) of Definition 3.1 is satisfied with

$$c_1 = \min\{c_{11}, c_{21}\}\max\{c_{12}, c_{22}\}^{-1},\ c_2 = \max\{c_{12}, c_{22}\}\min\{c_{11}, c_{21}\}^{-1}.$$

The above inequality is satisfied for all $s, t > 0, \frac{t}{2} \le s \le 2t$, because $c_{11} \le 1 \le c_{12}, c_{21} \le 1 \le c_{22}$, hence $c_1 \le c_{11}, c_1 \le c_{21}$, $c_{12} \le c_2, c_{22} \le c_2$.

Remark 3.5 If a function $\rho : (0,\infty) \to (0,\infty)$ satisfies Condition 2) of Definition 3.1, then for any $\gamma > 0$ $c_{1\gamma}, c_{2\gamma} > 0$ such that for any $t > 0$

$$c_{1\gamma}\rho(t) \le \rho(\gamma t) \le c_{2\gamma}\rho(t).$$

Indeed, if $\frac{1}{2} \le \gamma \le 2$, then this inequality follows by Condition 2) with $s = \gamma t$. Let $2 \le \gamma < 4$, then by Condition 2) with $t = 2\tau, \tau > 0$ and $s = \gamma\tau$ and the above inequality with $t = \tau$ and $\gamma = 2$

$$c_1^2\rho(\tau) \le c_1\rho(2\tau) \le \rho(\gamma\tau) \le c_2\rho(2\tau) \le c_2^2\rho(\tau),\ \tau > 0,$$

because $\frac{1}{2}(2\tau) \le \gamma\tau \le 2(2\tau)$.

Next, let $4 \le \gamma < 8$ by Condition 2) with $t = 4\tau, \tau > 0, s = \gamma\tau$,

$$\frac{1}{2}(4\tau) \le \gamma\tau \le 2(4\tau),$$

then

$$c_1^3\rho(\tau) \le c_1^2\rho(2\tau) \le c_1\rho(4\tau) \le \rho(\gamma\tau) \le c_2\rho(4\tau) \le c_2^2\rho(2\tau) \le c_2^3\rho(\tau),\ \tau > 0.$$

Furthermore, if $2^{k-1} \le \gamma < 2^k, k \in \mathbb{N}, k > 3 \Leftrightarrow k = \left[\frac{\ln\gamma}{\ln 2}\right] + 1$, then

$$c_1^k\rho(\tau) \le c_1^{k-1}\rho(2\tau) \le \dots \le \rho(\gamma\tau) \le c_2^2\rho(2\tau) \le \dots \le c_2^k\rho(\tau).$$

The argument for the case $0 < \gamma < \frac{1}{2}$ is similar.

Example 1 Let $\rho(t) = t^{\alpha-n}, t > 0, 0 < \alpha < n$. Then $\rho \in S_n$ because

$$\int_0^r t^{\alpha-n+n-1}dt < \infty \ \text{ for all }\ 0 < r < \infty\ \Leftrightarrow \alpha > 0$$

and, for any $s, t > 0$ for which $\frac{t}{2} \le s \le 2t$,

$$2^{\alpha-n}\rho(s) = 2^{\alpha-n}s^{\alpha-n} = (2s)^{\alpha-n}$$

$$\le \rho(t) = t^{\alpha-n} \le \left(\frac{s}{2}\right)^{\alpha-n} = 2^{n-\alpha}s^{\alpha-n} = 2^{n-\alpha}\rho(s).$$

Example 2 Let $0 < \alpha < n, -\infty < \beta_1, \beta_2 < \infty$,

$$\Psi_{\beta_1,\beta_2}(t) = \begin{cases} (1 + |\ln t|)^{\beta_1},\ 0 < t \le 1, \\ (1 + |\ln t|)^{\beta_2},\ 1 < t < \infty \end{cases}$$

and $\rho(t) = t^{\alpha-n}\Psi_{\beta_1\beta_2}(t), t > 0$. Then $\rho \in S_n$.

Since the function ρ is continuous on $(0,\infty)$ it suffices to prove Condition 1) of Definition 3.1 for $r=1$. Recall that for any $\varepsilon>0$ there exists $C_\varepsilon \ge 1$ such that

$$1+|\ln t| \le C_\varepsilon t^{-\varepsilon}$$

for all $t\in(0,1]$. If $\beta_1 \le 0$, then

$$\int_0^1 \rho(t)t^{n-1}dt = \int_0^1 t^{\alpha-1}(1+|\ln t|)^{\beta_1}dt \le \int_0^1 t^{\alpha-1}dt < \infty.$$

If $\beta_1>0$, then

$$\int_0^1 \rho(t)t^{n-1}dt = \int_0^1 t^{\alpha-1}(1+|\ln t|)^{\beta_1}dt \le C_\varepsilon^{\beta_1}\int_0^1 t^{\alpha-\varepsilon\beta_1-1}dt < \infty$$

if we choose $\varepsilon>0$ to be such that $\varepsilon\beta_1<\alpha$.

As for Condition 2) of Definition 3.1, by Remark 3.3 and Example 1, it suffices to prove that the function $\varphi_{\beta_1,\beta_2}$ satisfies this condition.

Let $s,t>0$ and $\frac{t}{2}\le s\le 2t$. If $s>1$, hence $t>\frac{1}{2}$, then

$$1+|\ln s| \le 1+\ln 2t = 1+\ln 2+\ln t \le 2(1+|\ln t|)$$

and

$$1+|\ln t| \le 1+\ln 2 < 2(1+|\ln s|),$$

hence

$$\frac{1}{2}(1+|\ln t|) \le 1+|\ln s| \le 2(1+|\ln t|).$$

For similar reasons this inequality also holds if $s\le 1$, hence $t\le 2$. Therefore, for $s\le 1$

$$2^{-|\beta_1|}(1+|\ln t|)^{\beta_1} \le (1+|\ln s|)^{\beta_1} \le 2^{|\beta_1|}(1+|\ln t|)^{\beta_1}.$$

If $t\le 1$, this means that

$$2^{-|\beta_1|}\Psi_{\beta_1,\beta_2}(t) \le \Psi_{\beta_1,\beta_2}(s) \le 2^{|\beta_1|}\Psi_{\beta_1,\beta_2}(t).$$

If $1<t\le 2$, then

$$2^{-|\beta_1|-|\beta_2|} \le (1+|\ln t|)^{\beta_1}(1+|\ln t|)^{-\beta_2} \le 2^{|\beta_1|+|\beta_2|},$$

hence

$$2^{-2|\beta_1|-|\beta_2|}(1+|\ln t|)^{\beta_2} \le (1+|\ln s|)^{\beta_1} \le 2^{2|\beta_1|+|\beta_2|}(1+|\ln t|)^{\beta_2}$$

which means that

$$2^{-2|\beta_1|-|\beta_2|}\Psi_{\beta_1,\beta_2}(t) \le \Psi_{\beta_1,\beta_2}(s) \le 2^{2|\beta_1|+|\beta_2|}\Psi_{\beta_1,\beta_2}(t).$$

So, this inequality holds for all $s\le 1$ and all t satisfying the inequality $\frac{t}{2}\le s\le 2t$.

For similar reasons, for all $s>1$ and all t satisfying the inequality $\frac{t}{2}\le s\le 2t$

$$2^{-|\beta_1|-2|\beta_2|}\Psi_{\beta_1,\beta_2}(t) \le \Psi_{\beta_1,\beta_2}(s) \le 2^{|\beta_1|+2|\beta_2|}\Psi_{\beta_1,\beta_2}(t).$$

Example 3 Let $0<\alpha<n, -\infty<\beta<\infty$ and

$$\rho(t) = \begin{cases} t^{\alpha-n}, & 0 < t \le 1, \\ t^{\beta}, & 1 < t < \infty. \end{cases}$$

Then $\rho \in S_n$. Indeed, Condition 1) is clearly satisfied. As for Condition 2), it is also satisfied by Remark 3.4, because $\rho_1(t) = t^{\alpha-n}$ satisfies Condition 2) on (0, 1] by Example 1 and $\rho_2(t) = t^{\beta}$ satisfies Condition 2) on $(1, \infty)$ by an argument similar to that of the proof of Example 1.

Note that t^{β} can be replaced by any function $\rho(t)$ satisfying Condition 2) on $(1, \infty)$.

Definition 3.2 For $\rho \in S_n$ and $f \in \mathfrak{M}(\mathbb{R}^n)$

$$I_{\rho(\cdot)}f(x) = \int_{\mathbb{R}^n} \rho(|x-y|)f(y)dy,$$

$$\overline{I}_{\rho(\cdot),r}f(x) = \int_{^cB(x,r)} \rho(|x-y|)f(y)dy$$

and

$$\underline{I}_{\rho(\cdot),r}f(x) = \int_{B(x,r)} \rho(|x-y|)f(y)dy.$$

Lemma 3.1 *Let $n \in \mathbb{N}, \rho \in S_n, 0 < p \le \infty, x \in \mathbb{R}^n, r > 0, f \in \mathfrak{M}(\mathbb{R}^n)$. Then*[1] $G = r^{\frac{n}{p}}\left(\overline{I}_{\rho(\cdot),2r}|f|\right)(x)$.

$$\|I_{\rho(.)}|f|\|_{WL_p(B(x,r))} \gtrsim r^{\frac{n}{p}}\left(\overline{I}_{\rho(\cdot),2r}|f|\right)(x) \tag{3.1}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

Proof By Property 3) of Remark 3.2 for any $y \in B(x, r)$

$$I_{\rho(.)}|f|(y) \ge \int_{^cB(x,2r)} \rho(|y-z|)|f(z)|dz$$

$$\ge c1 \int_{^cB(x,2r)} \rho(|x-z|)|f(z)|dz,$$

hence, by Remark 3.1, we get

$$\|I_{\rho(\cdot)}|f|\|_{WL_p(B(x,r))} \ge c_1\left(v_n r^n\right)^{\frac{1}{p}} \int_{^cB(x,2r)} \rho(|x-z|)|f(z)|dz$$

$$= c_1 v_n^{\frac{1}{p}} r^{\frac{n}{p}} \overline{I}_{\rho(\cdot),2r}|f|(x)$$

[1] According to the definitions introduced in Introduction, this means that for any $n \in \mathbb{N}, \rho \in S_n, 0 < p \le \infty$ there exists $c(n, \rho, p) > 0$ such that

$$\|I_{\rho(.)}|f|\|_{WL_p(B(x,r))} \gtrsim c(n,\rho,p) r^{\frac{n}{p}}\left(\overline{I}_{\rho(\cdot),2r}|f|\right)(x)$$

for all $x \in \mathbb{R}^n, r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

In this case $A = \{\mathbb{R}^n, (0,\infty), \mathfrak{M}(\mathbb{R}^n)\}$, $B = \left\{(n,\rho) : n \in \mathbb{N}, \rho \in S_n; [0,\infty\right]$, $F = \|I_{\rho(\cdot)}|f|\|_{WL_p(B(x,r))}$, for all $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$. Here v_n is the volume of the unit ball in $\mathbb{R}^n$.

Lemma 3.2 *Let $n \in \mathbb{N}$, $\rho \in S_n$, $0 < p \le \infty$, $x \in \mathbb{R}^n$, $r > 0$, $f \in \mathfrak{M}(\mathbb{R}^n)$. Then*

$$\|I_{\rho(\cdot)}|f|\|_{L_p(B(x,r))} \approx \|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + r^{\frac{n}{p}}\left(\bar{I}_{\rho(\cdot),2r}|f|\right)(x) \tag{3.2}$$

and

$$\|I_{\rho(\cdot)}|f|\|_{WL_p(B(x,r))} \approx \|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))} + r^{\frac{n}{p}}\left(\bar{I}_{\rho(\cdot),2r}|f|\right)(x) \tag{3.3}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

Proof According to the properties[2] of the spaces $L_p(\Omega)$ and $WL_p(\Omega)$

$$\|I_{\rho(\cdot)}|f|\|_{L_p(B(x,r))} \le 2^{(\frac{1}{p}-1)_+}\left(\|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + \|I_{\rho(\cdot)}(|f|\chi_{{}^cB(x,2r)})\|_{L_p(B(x,r))}\right), \tag{3.4}$$

$$\|I_{\rho(\cdot)}|f|\|_{WL_p(B(x,r))} \le 2^{\frac{1}{p}}\left(\|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))} + \|I_{\rho(\cdot)}(|f|\chi_{{}^cB(x,2r)})\|_{WL_p(B(x,r))}\right). \tag{3.5}$$

$$\|I_{\rho(\cdot)}|f|\|_{L_p(B(x,r))} \ge \frac{1}{2}\left(\|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + \|I_{\rho(\cdot)}(|f|\chi_{{}^cB(x,2r)})\|_{L_p(B(x,r))}\right), \tag{3.6}$$

and

$$\|I_{\rho(\cdot)}|f|\|_{L_p(B(x,r))}\|I_{\rho(\cdot)}|f|\|_{WL_p(B(x,r))} \ \ge\ \|I_{\rho(\cdot)}(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))}. \tag{3.7}$$

Next,

$$\|I_{\rho(\cdot)}(|f|\chi_{{}^cB(x,2r)})\|_{WL_p(B(x,r))} \ \le\ \|I_{\rho(\cdot)}(|f|\chi_{{}^cB(x,2r)})\|_{L_p(B(x,r))}$$

$$= \left[\int_{B(x,r)}\left(\int_{{}^cB(x,2r)}\rho(|y-z|)|f(z)|dz\right)^p dy\right]^{\frac{1}{p}} = J.$$

By Condition 3) of Remark 3.2

$$J \le_{C2} \left[\int_{B(x,r)}\left(\int_{{}^cB(x,2r)}\rho(|x-z|)|f(z)|dz\right)^p dy\right]^{\frac{1}{p}} =_{C2} v_n^{\frac{1}{p}} r^{\frac{n}{p}}\left(\bar{I}_{\rho(.),2r}|f|\right)(x).$$

Thus, by (3.4) and (3.5)

$$\|I_{\rho(.)}|f|\|_{L_p(B(x,r))} \lesssim \|I_{\rho(.)}(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + r^{\frac{n}{p}}\left(\bar{I}_{\rho(.),2r}|f|\right)(x),$$

and

$$\|I_{\rho(.)}|f|\|_{WL_p(B(x,r))} \lesssim \|I_{\rho(.)}(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))} + r^{\frac{n}{p}}\left(\bar{I}_{\rho(.),2r}|f|\right)(x),$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

[2] Below $a_+ = a$ if $a \ge 0$ and $a_+ = 0$ if $a < 0$.

Also, by Condition 3) of Remark 3.2

$$J \geq_{C1} \left[\int_{B(x,r)} \left(\int_{{}^cB(x,2r)} \rho(|x-z|)|f(z)|dz\right)^p dy\right]^{\frac{1}{p}} =_{C1}^{-1} v_n^{\frac{1}{p}} r^{\frac{n}{p}} \left(\bar{I}_{\rho(.),2r}|f|\right)(x).$$

Hence, by (3.6)

$$\|I_{\rho(.)}|f|\|_{L_p(B(x,r))} \gtrsim \|I_{\rho(.)}(|f|\chi_{B(x,2r)})\|_{L_p(B(x,r))} + r^{\frac{n}{p}} \left(\bar{I}_{\rho(.),2r}|f|\right)(x),$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

Finally, by adding inequalities (3.1) and (3.7), we get that

$$\|I_{\rho(.)}|f|\|_{WL_p(B(x,r))} \gtrsim \|I_{\rho(.)}(|f|\chi_{B(x,2r)})\|_{WL_p(B(x,r))} + r^{\frac{n}{p}} \left(\bar{I}_{\rho(.),2r}|f|\right)(x),$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n)$.

Lemma 3.3 *Let, for $n \in \mathbb{N}$, $\rho \in S_n$, $1 \leq p_1 < p_2 < \infty$, and*

$$\varphi_{n,\rho,p_1,p_2}(t) = \rho(t) t^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)}, \ t > 0, \tag{3.8}$$

where p_1' is the conjugate number to p_1.

1. *Assume that ρ is such that the function φ_{n,ρ,p_1,p_2} is almost non-decreasing on $(0, \infty)$, that is for some $c > 0$*

$$\varphi_{n,\rho,p_1,p_2}(t_1) \leq c\varphi_{n,\rho,p_1,p_2}(t_2) \text{ for all } 0 < t_1 < t_2 < \infty. \tag{3.9}$$

If $1 < p_1 < p_2 < \infty$, then

$$\|I_{\rho(\cdot)}\left(|f|\chi_{B(x,2r)}\right)\|_{L_{p_2}(B(x,r))} \lesssim \varphi_{n,\rho,p_1,p_2}(r)\|f\|_{L_{p_1}(B(x,2r))} \tag{3.10}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in L_{p_1}(B(x, 2r))$.

Also, if $p_1 = 1, 1 < p_2 < \infty$, then

$$\|I_{\rho(\cdot)}\left(|f|\chi_{B(x,2r)}\right)\|_{WL_{p_2}(B(x,r))} \lesssim \varphi_{n,\rho,p_1,p_2}(r)\|f\|_{L_{p_1}(B(x,2r))} \tag{3.11}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in L_1(B(x, 2r))$.

2. *Inequality (3.10) also holds for $p_1 = 1$ and $1 \leq p_2 \leq \infty$ under the assumption*

$$\|\rho(t)t^{\frac{n-1}{p_2}}\|_{L_{p_2}(0,r)} \lesssim \rho(r)r^{\frac{n}{p_2}} \tag{3.12}$$

uniformly in $r > 0$, replacing assumption (3.9).

3. *Inequality (3.10) and hence inequality (3.11) also holds for $1 \leq p_1 < \infty$ and $0 < p_2 \leq p_1$ under the assumption*

$$\|\rho(t)t^{n-1}\|_{L_1(0,r)} \lesssim \rho(r)r^n \tag{3.13}$$

uniformly in $r > 0$, replacing assumption (3.9).

4. *Condition* (3.13) *is necessary for the validity of inequalities* (3.10) *and* (3.11) *for all* $0 < p_1, p_2 \le \infty$.

5. *Condition* (3.13) *is necessary and sufficient for the validity of inequalities* (3.10) *and* (3.11) *for all* $1 \le p_1 < \infty$ *and* $0 < p_2 \le p_1$.

Proof 1. Let $1 \le p_1 < p_2 < \infty$, $x \in \mathbb{R}^n$, $r > 0$, $z \in B(x,r)$ and $f \in L_{p_1}(B(x,2r))$. Then

$$\Big(I_{\rho(\cdot)}\Big(|f|\chi_{B(x,2r)}\Big)\Big)(z) = \int_{\mathbb{R}^n} \rho(|z-y|)|f(y)|\chi_{B(x,2r)}(y)dy$$

$$= \int_{B(x,2r)} \rho(|z-y|)|z-y|^{n\left(1-\frac{1}{p_1}+\frac{1}{p_2}\right)} \frac{|f(y)|\chi_{B(x,2r)}(y)}{|z-y|^{n-n(\frac{1}{p_1}-\frac{1}{p_2})}}dy$$

$$\le \Big(\sup_{\substack{z\in B(x,r)\\ y\in B(x,2r)}} \varphi_{n,\rho,p_1,p_2}(|z-y|)\Big)\Big(I_{n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\Big(|f|\chi_{B(x,2r)}\Big)\Big)(z).$$

Since the function φ_{n,ρ,p_1,p_2} is almost non-decreasing on $(0,\infty)$ and, for $z \in B(x,r)$, $y \in B(x,2r)$,

$$|z-y| \le |z-x| + |x-y| \le r + 2r = 3r,$$

we have by (3.9)

$$\sup_{\substack{z\in B(x,r)\\ y\in B(x,2r)}} \varphi_{n,\rho,p_1,p_2}(|z-y|) \le c\varphi_{n,\rho,p_1,p_2}(3r) = c3^{n(1-\frac{1}{p_1}+\frac{1}{p_2})}\rho(3r)r^{n(1-\frac{1}{p_1}+\frac{1}{p_2})}.$$

By Remark 3.5 with $\gamma = 3$ it follows that

$$\rho(3r) \le c_{23}\rho(r), \ \ r > 0.$$

Hence,

$$\sup_{\substack{z\in B(x,r)\\ y\in B(x,2r)}} \varphi_{n,\rho,p_1,p_2}(|z-y|) \le c\, c_{23}\, 3^{n(1-\frac{1}{p_1}+\frac{1}{p_2})}\varphi_{n,\rho,p_1,p_2}(r)$$

and

$$\Big(I_{\rho(\cdot)}\Big(|f|\chi_{B(x,2r)}\Big)\Big)(z) \lesssim \varphi_{n,\rho,p_1,p_2}(r)\Big(I_{n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\Big(|f|\chi_{B(x,2r)}\Big)\Big)(z) \tag{3.14}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$, $z \in B(x,r)$ and $f \in L_{p_1}(B(x,2r))$.

Next, we apply the well-known inequalities for the Riesz potential. If $1 < p_1 < p_2 < \infty$, then

$$\|I_{n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}|f|\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_{p_1}(\mathbb{R}^n)} \tag{3.15}$$

uniformly in $f \in L_{p_1}(\mathbb{R}^n)$. Also, if $1 < p_2 < \infty$, then

$$\|I_{n\left(1-\frac{1}{p_2}\right)}|f|\|_{WL_q(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)} \tag{3.16}$$

uniformly in $L_1(\mathbb{R}^n)$.

If $1 < p_1 < p_2 < \infty$, then, by (3.14) and (3.15), it follows that

$$\|I_{\rho(\cdot)}\Big(|f|\chi_{B(x,2r)}\Big)\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \varphi_{n,\rho,p_1,p_2}(r) \parallel |f|\chi_{B(x,2r)}\|_{L_{p_1}(\mathbb{R}^n)}$$

$$= \varphi_{n,\rho,p_1,p_2}(r)\|f\|_{L_{p_1}(B(x,2r))}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in L_{p_1}(B(x, 2r))$, which is inequality (3.10).

Respectively, if $1 < p_2 < \infty$, then by (3.14) and (3.16) there follows inequality (3.11).

2. If $p_1 = 1$ and $1 \le p_2 \le \infty$, then, by applying Young's inequality for truncated convolutions, we get

$$\| I_{\rho(.)}(|f|\chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))}= \left\| \int_{B(x,2r)} \rho(|. - y|)|f(y)|dy \right\|_{L_{p_2}(B(x,r))}$$

$$\le\| \rho \|_{L_{p_2}(B(x,r)-B(x,2r))}\| f \|_{L_1(B(x,2r))}$$

$$=\| \rho \|_{L_{p_2}(B(0,3r))}\| f \|_{L_1(B(x,2r))}$$

$$= \sigma_n^{\frac{1}{p_2}} \| \rho(t)t^{\frac{n-1}{p_2}} \|_{L_{p_2}(0,3r)}\| f \|_{L_1(B(x,2r))},$$

where $\sigma_n = nv_n$ is the surface area of the unit ball in $\mathbb{R}^n$.

By inequality (3.12) and Remark 3.5 with $\gamma = 3$ it follows that

$$\| \rho(t)t^{\frac{n-1}{p_2}} \|_{L_{p_2}(0,3r)}\lesssim \rho(3r)r^{\frac{n}{p_2}} \lesssim \rho(r)r^{\frac{n}{p_2}} = \varphi_{n,\rho,1,p_2}(r),$$

which implies inequality (3.10).

3. If $1 \le p_2 = p_1 < \infty$, then similarly to Step 2

$$\| I_{\rho(.)}(|f|\chi_{B(x,2r)}) \|_{L_{p_1}(B(x,r))}\le\| \rho \|_{L_1(B(x,r)-B(x,2r))}\| f \|_{L_{p_1}(B(x,2r))}$$

$$= \sigma_n \| \rho(t)t^{n-1} \|_{L_1(0,3r)}\| f \|_{L_{p_1}(B(x,2r))} .$$

By inequality (3.13) and Remark 3.5 with $\gamma = 3$ it follows that

$$\| \rho(t)t^{n-1} \|_{L_1(0,3r)}\lesssim \rho(3r)r^n \lesssim \rho(r)r^n = \varphi_{n,\rho,p_1,p_1}(r),$$

which implies inequality (3.10).

If $0 < p_2 < p_1$, $1 \le p_1 < \infty$, then, by applying Hölder's inequality and inequality (3.10) with $p_2 = p_1$ we get

$$\| I_{\rho(.)}(|f|\chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))}\le (v_nr^n)^{\frac{1}{p_2}-\frac{1}{p_1}} \| I_{\rho(.)}(|f|\chi_{B(x,2r)}) \|_{L_{p_1}(B(x,r))}$$

$$\lesssim \rho(r)r^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)} \| f \|_{L_{p_1}(B(x,2r))}= \varphi_{n,\rho,p_1,p_2}(r) \| f \|_{L_{p_1}(B(x,2r))},$$

which is inequality (3.10).

4. Assume that for some $0 < p_1, p_2 \le \infty$ inequality (3.10) and (3.11) is satisfied. If inequality (3.10) is satified, then inequality (3.11) is also satisfied. Take in this inequality $x = 0$ and $f \equiv 1$. Then for any $y \in B(0, r)$ we have $B(y, 2r) \supset B(0, r)$ and

$$I_{\rho(.)}\left(\chi_{B(0,2r)}\right)(y) = \int_{B(0,2r)} \rho(y-z)dz = \int_{B(y,2r)} \rho(u)du$$

$$\geq \int_{B(0,r)} \rho(u)du = \sigma_n \int_0^r \rho(t)t^{n-1}dt.$$

By Remark 3.1

$$\left\| I_{\rho(.)}\left(\chi_{B(0,2r)}\right)\right\|_{WL_{p_2}(B(0,r))} \gtrsim r^{\frac{n}{p_2}} \int_0^r \rho(t)t^{n-1}dt$$

and $\left\|\chi_{B(0,2r)}\right\|_{L_{p_2}(B(0,r))} = (\sigma_n r^n)^{\frac{1}{p_1}}$, hence by (3.11)

$$r^{\frac{n}{p_2}} \int_0^r \rho(t)t^{n-1}dt \lesssim \rho(r) r^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)} r^{\frac{n}{p_1}}$$

uniformly in $r > 0$, which implies inequality (3.13).

5. The last statement of Lemma 3.3 follows by Steps 3 and 4.

Remark 3.6 If $1 \leq p_1 < p_2 < \infty$ and

$$\lim_{t\to 0^+} \varphi_{n,\rho,p_1,p_2}(t) = 0, \tag{3.17}$$

then inequality (3.10) for $1 < p_1 < p_2 < \infty$ and inequality (3.11) for $1 < p_2 < \infty$ cannot hold for any $f \in L_{p_1}(\mathbb{R}^n)$, $f \in L_1(\mathbb{R}^n)$, respectively, not equivalent to 0 on $\mathbb{R}^n$. Indeed, by passing to the limit as $r \to \infty$, in (3.10) and (3.11), it follows that $\| I_{\rho(.)}|f| \|_{L_{p_2}(\mathbb{R}^n)} = 0$, $\| I_{\rho(.)}|f| \|_{WL_{p_2}(\mathbb{R}^n)} = 0$, hence, f is equivalent to 0 on $\mathbb{R}^n$.

If $\rho(t) = t^{\alpha-n}$, $0 < \alpha < n$ and $1 \leq p_1 < p_2$, then Condition (3.9) reduces to the condition $\alpha \geq n(\frac{1}{p_1} - \frac{1}{p_2})$, Condition (3.12) reduces to the condition $\alpha > n(1 - 1/p_2)$ and Condition (3.13) is satisfied since $\alpha > 0$.

Corollary 3.1 *Let the assumptions of Lemma 3.3 for n, ρ, p_1, p_2 and the function φ_{n,ρ,p_1,p_2} be satisfied. Then, for $1 < p_1 < p_2 < \infty$, for $p_1 = 1, 1 < p_2 \leq \infty$, and for $1 \leq p_1 < \infty, 0 < p_2 \leq p_1$*

$$\| I_{\rho(.)}|f| \|_{L_{p_2}(B(x,r))} \lesssim \varphi_{n,\rho,p_1,p_2}(r) \| f \|_{L_{p_1}(B(x,2r))} + r^{\frac{n}{p_2}} \left(\bar{I}_{\rho(.),2r}|f|\right)(x) \tag{3.18}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n) \bigcap L_{p_1}(B(x, 2r))$ and, for $1 < p_2 < \infty$

$$\| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))} \lesssim \varphi_{n,\rho,1,p_2}(r) \| f \|_{L_1(B(x,2r))} + r^{\frac{n}{p_2}} \left(\bar{I}_{\rho(.),2r}|f|\right)(x) \tag{3.19}$$

uniformly in $x \in \mathbb{R}^n$, $r > 0$ and $f \in \mathfrak{M}(\mathbb{R}^n) \bigcap L_1(B(x, 2r))$.

Lemma 3.4 *Let $f \in \mathfrak{M}(\mathbb{R}^n)$ be a non-negative function and ρ be a non-negative, non-increasing, continuously differentiable function on $(0, \infty)$ such that $\lim_{t\to+\infty} \rho(t) = 0$. Then for any $x \in \mathbb{R}^n$ and $r > 0$*

$$\int_{{}^c B(x,r)} \rho(|x-y|)f(y)dy = \int_r^\infty \left(\int_{B(x,t)\setminus B(x,r)} f(y)dy\right)|\rho'(t)|dt. \tag{3.20}$$

For $\rho(t) = t^{-\beta}$, $t > 0$, $\beta > 0$ this lemma was proved in [3] (Lemma 3).

Proof By applying the Fubini theorem we get

$$\begin{aligned}\int_{^{c}B(x,r)} \rho(|x-y|)f(y)dy &= \int_{y:|x-y|>r} \rho(|x-y|)f(y)dy \\ &= \int_{y:|x-y|>r} \left(\int_{|x-y|}^{\infty} |\rho'(t)|dt \right) f(y)dy \\ &= \int_r^{\infty} \left(\int_{y:|x-y|>r, |x-y|<t} |\rho'(t)|f(y)dy \right) dt \\ &= \int_r^{\infty} \left(\int_{B(x,t)\setminus B(x,r)} f(y)dy \right) |\rho'(t)|dt.\end{aligned}$$

Definition 3.3 Let $n \in \mathbb{N}, 1 \le p_1 < \infty, 0 < p_2 \le \infty$. Then $\rho \in S_{n,p_1,p_2}$ if $\rho \in \mathfrak{M}^+((0,\infty))$ and there exists a positive non-increasing continuously differentiable function $\tilde{\rho} : (0,\infty) \to (0,\infty)$ such that

1) $\tilde{\rho}(t) \approx \rho(t)$ uniformly in $t > 0$,

2) $\lim\limits_{t\to\infty} \tilde{\rho}(t) = 0$,

3) $\tilde{\rho} \in S_n$,

4) $\int\limits_0^1 \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} dt = \infty, \int\limits_1^{\infty} \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} dt < \infty$,

5) $|\tilde{\rho}'(t)| t \gtrsim \tilde{\rho}(t)$ uniformly in $t > 0$,

6) if $0 < p_2 \le p_1$ and $1 \le p_1 < \infty$, then

$$\int_0^r \tilde{\rho}(t) t^{n-1} dt \lesssim \tilde{\rho}(r) r^n$$

uniformly in $r > 0$,

7) if $1 \le p_1 < p_2 < \infty$, then the function $\phi_{n,\tilde{\rho},p_1,p_2}(t) = \tilde{\rho}(t) t^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)}$ is almost non-decreasing on $(0,\infty)$.

Moreover, if $p_1 = 1$ and $1 < p_2 < \infty$, then $\rho \in S_{n,1,p_2}$ if there exists a positive non-increasing continuously differentiable function $\tilde{\rho} : (0,\infty) \to (0,\infty)$ such that Conditions 1)–3) and 5) are satisfied, Conditions 4) and 6) are satisfied for $p_1 = 1$ and instead of Condition 7) the following condition is satisfied

8)

$$\left\| \tilde{\rho}(t) t^{\frac{n-1}{p_2}} \right\|_{L_{p_2}(0,r)} \lesssim \tilde{\rho}(r) r^{\frac{n}{p_2}}$$

uniformly in $r > 0$.

Remark 3.7 For the Riesz potential $\rho(t) = t^{\alpha-n}, 0 < \alpha < n, \tilde{\rho} = \rho$. Condition 2) is satisfied because $\alpha < n$, Condition 3) is satisfied because $\alpha > 0$, Condition 4) is satisfied if $p_1 < \infty$ and $\alpha < \frac{n}{p_1}$, Condition 5) is obviously satisfied, Condition 6) is satisfied because $\alpha > 0$, Condition 7) is satisfied if $\alpha \ge n(1/p_1 - 1/p_2$, Condition 8) is satisfied if $\alpha > n(1 - 1/p_2)$.

Remark 3.8 Clearly, due to Condition 1), Conditions 2)–4), 6)–8) are equivalent to the conditions, obtained from them by replacing $\tilde{\rho}$ by ρ

Theorem 3.1 *Let $n \in \mathbb{N}, 1 \le p_1 < \infty, 0 < p_2 \le \infty$.*

1. *If $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1,p_2}$, then*

$$\| I_{\rho(.)}(|f|) \|_{L_{p_2}(B(x,r))} \lesssim r^{\frac{n}{p_2}} \int_r^{\infty} \rho(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} dt \tag{3.21}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)$.

2. *If* $p_1 = 1$ *and* $0 < p_2 < \infty$, *and* $\rho \in \tilde{S}_{n,1,p_2}$, *then*

$$\| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))} \approx \| I_{\rho(.)}|f| \|_{L_{p_2}(B(x,r))} \approx r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt \tag{3.22}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ *and* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

3. *If* $p_1 = 1$ *and* $1 < p_2 < \infty$, *and* $\rho \in S_{n,1,p_2}$, *then*

$$\| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))} \approx r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt \tag{3.23}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ *and* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

For $\rho(t) = t^{\alpha-n}, 0 < \alpha < n$, hence for the Riesz potential I_α, this theorem is proved in [6] (Lemma 3.6 and Theorem 3.9).

Remark 3.9 If $p_1 = 1$ and $0 < p_2 < \infty$ and $\rho \in \tilde{S}_{n,1,p_2}$, equivalence (3.22) holds if Condition 6) of Definition 3.3 for $0 < p_2 \leq 1$ and Condition 7) of Definition 3.3 for $1 < p_2 < \infty$ are satisfied. If $\rho(t) = t^{\alpha-n}, 0 < \alpha < n$, then this means that $n(1 - \frac{1}{p_2})_+ < \alpha < n$.

If $p_1 = 1$ and $1 < p_2 < \infty$ and $\rho \in S_{n,1,p_2}$, equivalence (3.23) holds if Condition 8) of Definition 3.3 is satisfied. If $\rho(t) = t^{\alpha-n}, 0 < \alpha < n$, then this means that $n(1 - \frac{1}{p_2}) \leq \alpha < n$.

In particular, for $\alpha = n(1 - 1/p_2)$ Statement 3 of Theorem 3.1 holds, while, as proved in Lemma 3.3 of [6], Statement 2 does not hold.

Proof Step 1 (proof of (3.21)). We apply equivalence (3.2).

1.1a. To the first summand in the right-hand side of (3.2) we apply Condition 2) of Definition 3.1 with ρ replaced by $\tilde{\rho}$, s by r, t by $2r$ and inequality (3.10) and get that

$$\begin{aligned} & r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} \, dt \\ & \gtrsim r^{\frac{n}{p_2}} \int_r^\infty \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} \, dt \\ & \geq r^{\frac{n}{p_2}} \int_r^{2r} \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} \, dt \\ & \geq r^{\frac{n}{p_2}} \tilde{\rho}(2r) \left(\int_r^{2r} t^{\frac{n}{p_1'}-1} dt \right) \| f \|_{L_{p_1}(B(x,r))} \end{aligned}$$

$$\gtrsim \tilde{\rho}(r) r^{\frac{n}{p_1'}+\frac{n}{p_2}} \| f \|_{L_{p_1}(B(x,r))} = \varphi_{n,\tilde{\rho},p_1,p_2}(r) \| f \|_{L_{p_1}(B(x,r))}$$

$$\gtrsim \| I_{\tilde{\rho}(.)}(|f| \chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))} \gtrsim \| I_{\rho(.)}(|f| \chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))} \tag{3.24}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

1.1b (second proof of inequality (3.24)). To the first summand in the right-hand side of (3.2) we apply inequality (3.10) and Condition 2) of Definition 3.1 with ρ replaced by $\tilde{\rho}$, firstly, s replaced by r and t by $2r$, secondly, s replaced by $\frac{\tau}{2}$ and t by τ, and we get that

$$\| I_{\rho(.)}(|f| \chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))} \lesssim \| I_{\tilde{\rho}(.)}(|f| \chi_{B(x,2r)}) \|_{L_{p_2}(B(x,r))}$$

$$
\begin{aligned}
&\lesssim \tilde{\rho}(r) r^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)} \| f \|_{L_{p_1}(B(x,2r))} \lesssim \tilde{\rho}(2r) r^{n\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)} \| f \|_{L_{p_1}(B(x,2r))} \\
&\lesssim r^{\frac{n}{p_2}} \left(\int_{2r}^{4r} \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} dt \right) \| f \|_{L_{p_1}(B(x,2r))} \lesssim r^{\frac{n}{p_2}} \int_{2r}^{\infty} \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,2t))} \, dt \\
&= 2^{-\frac{n}{p_1'}} r^{\frac{n}{p_2}} \int_r^{\infty} \tilde{\rho}\left(\frac{\tau}{2}\right) \tau^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,\tau))} \, d\tau \lesssim r^{\frac{n}{p_2}} \int_r^{\infty} \tilde{\rho}(\tau) \tau^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,\tau))} \, d\tau \\
&\lesssim r^{\frac{n}{p_2}} \int_r^{\infty} \rho(\tau)^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,\tau))} \, d\tau
\end{aligned}
$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$.

1.2a. In order to estimate the second summand in the right-hand side of (3.2) we obtain an estimate for $(\bar{I}_{\rho(.),r}|f|)(x)$. First we note that

$$
\int_{2r}^{\infty} \tilde{\rho}(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt = \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1} r} \tilde{\rho}(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt
$$

$$
\geq \sum_{k=1}^{\infty} \tilde{\rho}(2^k r) \| f \|_{L_1(B(x,2^k r))} \int_{2^k r}^{2^{k+1} r} \frac{dt}{t} = \ln 2 \sum_{k=1}^{\infty} \tilde{\rho}(2^k r) \| f \|_{L_1(B(x,2^k r))}
$$

$$
\geq \ln 2 \sum_{k=1}^{\infty} \tilde{\rho}(2^k r) \| f \|_{L_1(B(x,2^k r) \setminus B(x,2^{k-1} r))}
$$

Since by Condition 2) of Definition 3.1 $\tilde{\rho}(2^k r) \geq c_1 \tilde{\rho}(2^{k-1} r)$ and $|x-y| \geq 2^{k-1} r$ for all $y \in B(x, 2^k r) \setminus B(x, 2^{k-1} r)$, it follows that

$$
\int_{2r}^{\infty} \tilde{\rho}(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt \gtrsim \sum_{k=1}^{\infty} \int_{B(x,2^k r) \setminus B(x,2^{k-1} r)} \tilde{\rho}(2^{k-1} r) |f(y)| dy
$$

$$
\geq \sum_{k=1}^{\infty} \int_{B(x,2^k r) \setminus B(x,2^{k-1} r)} \tilde{\rho}(|x-y|) |f(y)| dy = \int_{{}^c B(x,r)} \tilde{\rho}(|x-y|) |f(y)| dy = (\bar{I}_{\tilde{\rho}(.),r} |f|)(x).
$$

So,

$$
(\bar{I}_{\tilde{\rho}(.),r} |f|)(x) \lesssim \int_{2r}^{\infty} \tilde{\rho}(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt \tag{3.25}
$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$.

Hence, by applying Hölder's inequality, we get that

$$
r^{\frac{n}{p_2}} \left(\bar{I}_{\rho(.),2r} |f| \right)(x) \lesssim r^{\frac{n}{p_2}} \left(\bar{I}_{\tilde{\rho}(.),2r} |f| \right)(x) \lesssim r^{\frac{n}{p_2}} \int_{4r}^{\infty} \tilde{\rho}(t) t^{-1} \| f \|_{L_1(B(x,t))} \, dt
$$

$$
\lesssim r^{\frac{n}{p_2}} \int_{4r}^{\infty} \tilde{\rho}(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} \, dt \lesssim r^{\frac{n}{p_2}} \int_r^{\infty} \rho(t) t^{\frac{n}{p_1'}-1} \| f \|_{L_{p_1}(B(x,t))} \, dt \tag{3.26}
$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$.

Step 2 (proof of equivalences (3.22) and (3.23) for $p_1 = 1$). We apply equivalences (3.2) and (3.3).

2.1a. Under the assumption $\tilde{\rho} \in \tilde{S}_{n,1,p_2}$ inequality (3.12) holds for $p_1 = 1$ and $1 \le p_2 \le \infty$ and inequality (3.13) holds for $p_1 = 1$ and $0 < p_2 \le 1$, therefore, Lemma 3.3 ensures the validity of inequality (3.10) with $p_1 = 1$ and we have

$$r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt \gtrsim r^{\frac{n}{p_2}} \int_r^\infty \tilde{\rho}(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt$$

$$\gtrsim \varphi_{n,\tilde{\rho},1,p_2}(r) \parallel f \parallel_{L_1(B(x,r))}$$

$$\gtrsim \parallel I_{\rho(.)}(|f|\chi_{B(x,2r)}) \parallel_{L_{p_2}(B(x,r))} \ge \parallel I_{\rho(.)}(|f|\chi_{B(x,2r)}) \parallel_{WL_{p_2}(B(x,r))} \tag{3.27}$$

uniformly in $x \in \mathbb{R}^n,\ r > 0$ and $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and we get the estimate above for the first summand in (3.22).

2.1b. If $\rho \in S_{n,1,p_2}$, then condition (3.9) holds for $p_1 = 1$ and $1 < p_2 < \infty$, therefore, Lemma 3.3 ensures the validity of inequality (3.11) and we, respectively, have

$$r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt$$

$$\gtrsim r^{\frac{n}{p_2}} \int_r^\infty \tilde{\rho}(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt \ge \varphi_{n,\tilde{\rho},1,p_2}(r) \parallel f \parallel_{L_1(B(x,r))}$$

$$\gtrsim \parallel I_{\rho(.)}(|f|\chi_{B(x,2r)}) \parallel_{WL_{p_2}(B(x,r))} \tag{3.28}$$

uniformly in $x \in \mathbb{R}^n,\ r > 0$ and $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and we get the estimate above for the first summand in (3.23).

2.2a. If $\tilde{\rho} \in \tilde{S}_{n,1,p_2}$ and $1 \le p_2 \le \infty$, to the second summand in the right-hand side of (3.2) we apply inequality (3.25). Equivalence (3.2) and inequalities (3.25) and (3.27) imply that

$$\parallel I_{\rho(.)}|f| \parallel_{WL_{p_2}(B(x,r))} \le \parallel I_{\rho(.)}|f| \parallel_{L_{p_2}(B(x,r))} \lesssim r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt \tag{3.29}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

2.2b. Accordingly, if $\tilde{\rho} \in S_{n,1,p_2}$, $p_1 = 1$ and $1 < p_2 < \infty$, then equivalence (3.3) and inequalities (3.25) and (3.28) imply that

$$\parallel I_{\rho(.)}|f| \parallel_{WL_{p_2}(B(x,r))} \lesssim r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{-1} \parallel f \parallel_{L_1(B(x,t))} dt \tag{3.30}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

2.3. For all $y \in B(x,r)$ we have $|y-z| \le 2r$ if $z \in B(x,r)$ and $|y-z| \le 2|x-z|$ if $z \in^c B(x,r)$ (since $|y-z| \le |y-x| + |x-z| \le r + |x-z| \le 2|x-z|$). Therefore, since $\tilde{\rho}$ is non-increasing

$$I_{\tilde{\rho}}(.)(|f|)(y) = \int_{B(x,r)} \tilde{\rho}(|y-z|)|f(z)|dz + \int_{^cB(x,r)} \tilde{\rho}(|y-z|)|f(z)|dz$$

$$\ge \tilde{\rho}(2r) \int_{B(x,r)} |f(z)|dz + \int_{^cB(x,r)} \tilde{\rho}(2|x-z|)|f(z)|dz.$$

By Condition 2) of Definition 3.1 with $s = 2t$ we have $\tilde{\rho}(2t) \ge c_1 \tilde{\rho}(z)$ for any $t > 0$, hence, by equality (3.18) uniformly in $x \in \mathbb{R}^n, r > 0,\ y \in B(x,r)$ and $f \in L_1^{\rm loc}(\mathbb{R}^n)$

$$I_{\rho(.)}(|f|)(y) \gtrsim I_{\tilde{\rho}(.)}(|f)(y) \ge c_1 \left(\tilde{\rho}(r) \int_{B(x,r)} |f(z)|dz + \int_{^cB(x,r)} \tilde{\rho}(|x-z|)|f(z)|dz \right)$$

$$= c_1\left(\tilde{\rho}(r)\int_{B(x,r)}|f(z)|dz + \int_r^\infty \left(\int_{B(x,t)\setminus B(x,r)}|f(z)|dz\right)|(\tilde{\rho})'(t)|dt\right)$$

$$= c_1\left(\tilde{\rho}(r)\int_{B(x,r)}|f(z)|dz + \int_r^\infty \left(\int_{B(x,t)}|f(z)|dz\right)|(\tilde{\rho})'(t)|dt - \left(\int_r^\infty |(\tilde{\rho})'(t)|dt\right)\int_{B(x,r)}|f(z)|dz\right)$$

$$= c_1\int_r^\infty \left(\int_{B(x,t)}|f(z)|dz\right)|(\tilde{\rho})'(t)|dt.$$

Consequently, by Remark 3.1 and by Condition 5) of Definition 3.3

$$\| I_{\rho(.)}|f| \|_{L_{p_2}(B(x,r))} \geq \| I_{\rho(.)}|f| \|_{WL_{p_2}(B(x,r))} \gtrsim \| I_{\tilde{\rho}}|f| \|_{WL_{p_2}(B(x,r))}$$

$$\geq c_1 v_n^{\frac{n}{p_2}} r^{\frac{n}{p_2}} \int_r^\infty \left(\int_{B(x,t)}|f(z)|dz\right)|(\tilde{\rho})'(t)|dt$$

$$\gtrsim r^{\frac{n}{p_2}} \int_r^\infty \tilde{\rho}(t)t^{-1} \| f \|_{L_1(B(x,t))}\, dt \gtrsim r^{\frac{n}{p_2}} \int_r^\infty \rho(t)t^{-1} \| f \|_{L_1(B(x,t))}\, dt \tag{3.31}$$

uniformly in $x \in \mathbb{R}^n, r > 0$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Step 3. Statement 1 of Theorem 3.1 follows by inequalities (3.24) and (3.26). Statement 2 follows by inequalities (3.25), (3.27) and (3.31). Statement 3 follows by inequalities (3.25), (3.28) and (3.31).

Generalized Riesz potential and Hardy operator

Let $\mathfrak{M}^+((0,\infty))$ be the subset of $\mathfrak{M}((0,\infty))$ consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+((0,\infty);\downarrow)$ the cone of all functions in $\mathfrak{M}^+((0,\infty))$ which are non-increasing on $(0,\infty)$ and we set

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+((0,\infty);\downarrow) : \lim_{t\to\infty}\varphi(t) = 0\right\}.$$

Let H be the Hardy operator

$$(Hg)(t) := \int_0^t g(r)dr,\ 0 < t < \infty$$

Moreover, let, for $0 < p \leq \infty$ and a Lebesgue measurable function $v : (0,\infty) \to [0,\infty)$, $L_{p_1 v(\cdot)}(0,\infty)$ denote the space of all Lebesgue measurable functions $f : (0,\infty) \to \mathbb{C}$ for which

$$\|f\|_{L_{p,v(\cdot)}(0,\infty)} = \|fv\|_{L_p(0,\infty)} < \infty.$$

Theorem 4.1 *Let $n \in \mathbb{N}, 1 \leq p_1 < \infty, 0 < p_2 \leq \infty, 0 < \theta_2 \leq \infty, w_2 \in \Omega_{\theta_2}$, ρ be a positive continuous function on $(0,\infty)$ and*

$$\mu_{n,\rho,p_1}(r) = \frac{\int_r^\infty \rho(t)t^{\frac{n}{p_1'}-1}dt}{\int_1^\infty \rho(t)t^{\frac{n}{p_1'}-1}dt},\ r > 0, \tag{4.1}$$

$$v_2(r) = w_2\Big(\mu^{(-1)}_{n,\rho,p_1}(r)\Big)\Big(\mu^{(-1)}_{n,\rho,p_1}(r)\Big)^{\frac{n}{p_2}} |(\mu^{(-1)}_{n,\rho,p_1}(r))'|^{\frac{1}{\theta_2}},\ r > 0, \tag{4.2}$$

$$g_{n,\rho,p_1}(t) = \|f\|_{L_{p_1}\left(B(0,\mu^{(-1)}_{n,\rho,p_1}(t))\right)},\ t > 0. \tag{4.3}$$

1. *If* $1 < p_1 < p_2 < \infty$ *or* $1 \le p_1 < \infty$ *and* $0 < p_2 \le p_1$, *and* $\rho \in S_{n,p_1,p_2}$, *then*

$$\|I_{\rho(\cdot)}f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \lesssim \|Hg_{n,\rho,p_1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \tag{4.4}$$

uniformly in $f \in \mathfrak{M}(\mathbb{R}^n)$.

2. *If* $p_1 = 1$ *and* $0 < p_2 < \infty$, *and* $\rho \in \tilde{S}_{n,1,p_2}$, *then*

$$\|I_{\rho(\cdot)}f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \approx \|Hg_{n,\rho,1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \tag{4.5}$$

uniformly in all non-negative functions $f \in \mathfrak{M}(\mathbb{R}^n)$.

3. *If* $p_1 = 1$ *and* $1 < p_2 < \infty$, *and* $\rho \in S_{n,1,p_2}$, *then*

$$\|I_{\rho(\cdot)}f\|_{WLM_{p_2\theta_2,w_2(\cdot)}} \approx \|Hg_{n,\rho,1}\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \tag{4.6}$$

uniformly in all non-negative functions $f \in \mathfrak{M}(\mathbb{R}^n)$.

Remark 4.1 If $\rho(t) = t^{\alpha-n}$, $1 \le p_1 < \infty$, $0 < p_2 < \infty$, $0 < \alpha < \frac{n}{p_1}$, then $\mu_{n,\rho,p_1}(r) = r^{-\sigma}$, where $\sigma = \frac{n}{p_1} - \alpha$, $\mu^{(-1)}_{n,\rho,p_1}(r) = r^{-\frac{1}{\sigma}}$, $v_2(r) = \sigma^{-\frac{1}{\theta_2}} w_2(r^{-\frac{1}{\sigma}}) r^{-\frac{n}{\sigma p_2} - \frac{1}{\sigma\theta_2} - \frac{1}{\theta_2}}$, $g_{n,\rho,p_1}(t) = \| f \|_{L_{p_1}(B(0,t^{-\frac{1}{\sigma}}))}$ and Lemma 4.1 takes the form of Lemma 4.1 in [6].

Remark 4.2 Due to Condition 4) of Definition 3.3

$$\lim_{r\to 0^+} \mu_{n,\rho,p_1}(r) = \lim_{r\to 0^+} \mu_{n,\tilde{\rho},p_1}(r) = \infty,\ \lim_{r\to\infty} \mu_{n,\rho,p_1}(r) = \lim_{r\to\infty} \mu_{n,\tilde{\rho},p_1}(r) = 0 \tag{4.7}$$

and μ_{n,ρ,p_1} is a strictly decreasing continuously differentiable function on $(0,\infty)$. Moreover,

$$\lim_{r\to 0^+} \mu^{(-1)}_{n,\rho,p_1}(r) = \infty,\ \lim_{r\to\infty} \mu^{(-1)}_{n,\rho,p_1}(r) = 0. \tag{4.8}$$

Proof 1. Let $1 < p_1 < p_2 < \infty$ or $1 \le p_1 \le \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1,p_2}$. By inequality (3.21) we have $\left(c = \left(\int_1^\infty \rho(t) t^{\frac{n}{p_1'}-1} dt\right)^{-1}\right)$

$$\|I_{\rho(\cdot)}f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \lesssim \left\| w_2(r) r^{\frac{n}{p_2}} \int_r^\infty \rho(t) t^{\frac{n}{p_1'}-1} \|f\|_{L_{p_1}(B(0,t))} dt \right\|_{L_{\theta_2}(0,\infty)}$$

$$= c\left\| w_2(r) r^{\frac{n}{p_2}} \left(-\int_r^\infty \|f\|_{L_{p_1}(B(0,t))} d\mu_{n,\rho,p_1}(t) \right) \right\|_{L_{\theta_2}(0,\infty)}$$

$$= c\Big(\mu_{n,\rho,p_1}(t) = \tau,\ t = \mu^{(-1)}_{n,\rho,p_1}(\tau),\ \lim_{r\to\infty} \mu_{n,\rho,p_1}(t) = 0\Big)$$

$$= c\left\| w_2(r) r^{\frac{n}{p_2}} \int_0^{\mu_{n,\rho,p_1}(r)} \|f\|_{L_{p_1}\left(B(0,\mu^{(-1)}_{n,\rho,p_1}(\tau))\right)} d\tau \right\|_{L_{\theta_2}(0,\infty)}$$

$$= c\Big(\mu_{n,\rho,p_1}(r) = u,\ r = \mu^{(-1)}_{n,\rho,p_1}(u),\ \lim_{r\to 0}\mu_{n,\rho,p_1}(r) = \infty,\ \lim_{r\to\infty}\mu_{n,\rho,p_1}(r) = 0\Big)$$

$$= c\Big\|w_2\Big(\mu^{(-1)}_{n,\rho,p_1}(u)\Big)\Big(\mu^{(-1)}_{n,\rho,p_1}(u)\Big)^{\frac{n}{p_2}}\Big|\Big(\mu^{(-1)}_{n,\rho,p_1}(u)\Big)'\Big|^{\frac{1}{\theta_2}}\int_0^u \|f\|_{L_{p_1}\left(B(0,\mu^{(-1)}_{n,\rho,p_1}(\tau))\right)}d\tau\Big\|_{L_{\theta_2}(0,\infty)}$$

$$= c\Big\|v_2(u)\big(Hg_{n,\rho,p_1}\big)(u)\Big\|_{L_{\theta_2}(0,\infty)} = c\|Hg_{n,\rho,p_1}\|_{L_{\theta_2,v_2(\cdot)},(0,\infty)}$$

uniformly in $f \in \mathfrak{M}(\mathbb{R}^n)$.

2. Let $p_1 = 1$ and $0 < p_2 \le 1$, and $\rho \in \tilde{S}_{n,p_1,p_2}$. By inequality (3.22) we have similarly to Step 1

$$\Big\|I_{\rho(\cdot)}f\|_{LM_{p_2\theta_2,w_2(\cdot)}} \approx \|w_2(r)r^{\frac{n}{p_2}}\int_r^\infty \rho(t)t^{-1}\|f\|_{L_1(B(0,t))}\Big\|_{L_{\theta_2}(0,\infty)}$$

$$= c\Big\|w_2(r)r^{\frac{n}{p_2}}\Big(-\int_r^\infty \|f\|_{L_1(B(0,t))}d\mu_{n,\rho,1}(t)\Big)\Big\|_{L_{\theta_2}(0,\infty)}$$

$$= c\Big\|v_2(u)\big(Hg_{n,1}\big)(u)\Big\|_{L_{\theta_2}(0,\infty)} = c\|Hg_{n,\rho,1}\|_{L_{\theta_2,v_2(\cdot)},(0,\infty)}$$

uniformly in all non-negative functions $f \in \mathfrak{M}(\mathbb{R}^n)$.

3. Let $p_1 = 1$ and $1 < p_2 < \infty$, and $\rho \in S_{n,1,p_2}$. Equivalence (4.6) follows similarly to Step 2 from equivalence (3.23). □

Theorem 4.2 *Assume that $n \in \mathbb{N}$, $1 \le p_1 < \infty$, $0 < p_2 \le \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$, μ_{n,ρ,p_1} is defined by formula* (4.1),

$$v_1(r) = w_1\Big(\mu^{(-1)}_{n,\rho,p_1}(r)\Big)\Big|\Big(\mu^{(-1)}_{n,\rho,p_1}(r)\Big)'\Big|^{\frac{1}{\theta_1}},\ r > 0, \tag{4.9}$$

and v_2 is defined by formula (4.2).

1. *Let $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1,p_2}$. If the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0,\infty)$ to $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$, that is*

$$\|Hg\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,v_1(\cdot)}(0,\infty)} \tag{4.10}$$

uniformly in $g \in \mathbb{A}$, then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2. *Let $p_1 = 1$, $0 < p_2 < \infty$ and $\rho \in \tilde{S}_{n,1,p_2}$. Then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$ if and only if the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0,\infty)$ to $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$.*

3. *Let $p_1 = 1$, $1 < p_2 < \infty$ and $\rho \in S_{n,1,p_2}$. Then the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $WLM_{p_2\theta_2,w_2(\cdot)}$ if and only if the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0,\infty)$ to $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$.*

Remark 4.3 If we put $r = \mu_{n_1,\rho,p_1}(t)$ in (4.9), then, taking into account that $\big(\mu^{(-1)}_{n_1,\rho,p_1}(r)\big)' = \Big(\mu'_{n_1,\rho,p_1}\big(\mu^{(-1)}_{n_1,\rho,p_1}(r)\big)\Big)^{-1}$, $r > 0$, we get

$$v_1\left(\mu_{n_1,\rho,p_1}(t)\right) = w_1(t)\left|\mu'_{n_1,\rho,p_1}(t)\right|^{-\frac{1}{\theta_1}},\ t > 0 \tag{4.11}$$

and

$$v_2\left(\mu_{n_1,\rho,p_1}(t)\right) = w_2(t)t^{\frac{n}{p_2}}\left|\mu'_{n,\rho,p_1}(t)\right|^{-\frac{1}{\theta_1}},\ t > 0, \tag{4.12}$$

Lemma 4.1 *Assume that* $0 < p_1, p_2, \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, ρ *is a positive continuous function on* $(0, \infty)$ *such that*

$$\int_0^1 \rho(t)t^{\frac{n}{p_1'}-1}dt = \infty,\ \int_1^\infty \rho(t)t^{\frac{n}{p_1'}-1}dt < \infty, \tag{4.13}$$

and the functions μ_{n,ρ,p_1}, g_{n,ρ,p_1}, v_1, v_2 *are defined by formulas* (4.1), (4.3), (4.9), (4.2), *respectively. Then*

$$\|g_{n,\rho,p_1}\|_{L_{\theta_1,v_1(\cdot)}} = \|f\|_{LM_{p_1\theta_1,w_1(\cdot)}} \tag{4.14}$$

for all $f \in LM_{p_1\theta_1,w_1(\cdot)}$, *for any measurable function* $\phi_1 : (0, \infty) \to (0, \infty)$ *and* $t > 0$

$$\|v_1\phi_1\|_{L_{\theta_1}(0,t)} = \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_1}(\mu^{(-1)}_{n,\rho,p_1}(t),\infty)}, \tag{4.15}$$

$$\|v_1\phi_1\|_{L_{\theta_1}(t,\infty)} = \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_1}(0,\mu^{(-1)}_{n,\rho,p_1}(t))}, \tag{4.16}$$

$$\|v_1\phi_1\|_{L_{\theta_1}(0,\infty)} = \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_1}(0,\infty)}, \tag{4.16'}$$

$$\|v_1\|_{L_{\theta_1}(0,\infty)} = \left\|w_1\right\|_{L_{\theta_1}(0,\infty)}, \tag{4.17}$$

and for any measurable function $\phi_2 : (0, \infty) \to (0, \infty)$ *and* $t > 0$

$$\|v_2\phi_2\|_{L_{\theta_2}(0,t)} = \|w_2(r)r^{\frac{n}{p_2}}\phi_2(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_2}(\mu^{(-1)}_{n,\rho,p_1}(t),\infty)}, \tag{4.18}$$

$$\|v_2\phi_2\|_{L_{\theta_2}(t,\infty)} = \|w_2(r)r^{\frac{n}{p_2}}\phi_2(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_2}(0,\mu^{(-1)}_{n,\rho,p_1}(t))}, \tag{4.19}$$

$$\|v_2\phi_2\|_{L_{\theta_2}(0,\infty)} = \|w_2(r)r^{\frac{n}{p_2}}\phi_2(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_2}(0,\infty)}, \tag{4.19'}$$

$$\|v_2\|_{L_{\theta_2}(0,\infty)} = \left\|w_2(r)r^{\frac{n}{p_2}}\right\|_{L_{\theta_2}(0,\infty)}. \tag{4.20}$$

Proof 1. Indeed,

$$\|g_{n,\rho,p_1}\|_{L_{\theta_1,v_1(\cdot)}} = \left\|v_1(t)\|f\|_{L_{p_1}(B(0,\mu^{(-1)}_{n,\rho,p_1}(t)))}\right\|_{L_{\theta_1}(0,\infty)}$$

$$= \left(\mu^{(-1)}_{n,\rho,p_1}(t) = r, t = \mu_{n,\rho,p_1}(r),\ \text{by (4.13)}\ \lim_{r\to 0^+}\mu^{(-1)}_{n,\rho,p_1}(t) = \infty, \lim_{n\to\infty}\mu^{(-1)}_{n,\rho,p_1}(t) = 0\right)$$

$$= \left\|v_1\left(\mu_{n,\rho,p_1}(r)\right)|\mu'_{n,\rho,p_1}(r)|^{\frac{1}{\theta_1}}\|f\|_{L_{p_1}(B(0,r))}\right\|_{L_{\theta_1}(0,\infty)}$$

$$= \left\| w_1(r) \|f\|_{L_{p_1}(B(0,r))} \right\|_{L_{\theta_1}(0,\infty)} = \|f\|_{LM_{p_1\theta_1,w_1(\cdot)}}.$$

(We have used equality (4.11)).

2. Note that, by (3.10) and (4.7), for $0 < \theta_1 < \infty$, $t > 0$

$$\|v_1\phi_1\|_{L_{\theta_1}(0,t)} = \left(-\int_0^t w_1\left(\mu^{(-1)}_{n,\rho,p_1}(s)\right)^{\theta_1} \phi_1(s)^{\theta_1} \left(\mu^{(-1)}_{n,\rho,p_1}(s)\right)' ds \right)^{\frac{1}{\theta_1}}$$

$$= \left(\mu^{(-1)}_{n,\rho,p_1}(s) = r\right) = \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(t)}^{\infty} w_1(r)^{\theta_1} \phi_1(\mu_{n,\rho,p_1}(r))^{\theta_1} dr \right)^{\frac{1}{\theta_1}}$$

$$= \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_1}(\mu^{(-1)}_{n,\rho,p_1}(t),\infty)}$$

and, similarly

$$\|v_1\|_{L_{\theta_1}(t,\infty)} = \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_{\theta_1}(0,\mu^{(-1)}_{n,\rho,p_1}(t))}.$$

These equalities also hold for $\theta_1 = \infty$, because, for example, by (4.9)

$$\|v_1\phi_1\|_{L_\infty(0,t)} = \left\| w_1\left(\mu^{(-1)}_{n,\rho,p_1}(s)\right)\phi_1(s) \right\|_{L_\infty(0,t)}$$

$$= \operatorname*{ess\,sup}_{0<s<t} \left| w_1\left(\mu^{(-1)}_{n,\rho,p_1}(s)\right)\phi_1(s) \right|$$

$$= \left(\mu^{(-1)}_{n,\rho,p_1}(s) = r, \mu^{(-1)}_{n,\rho,p_1}((0,t)) = \left(\mu^{(-1)}_{n,\rho,p_1}, \infty\right) \right)$$

$$= \operatorname*{ess\,sup}_{\mu^{(-1)}_{n,\rho,p_1}(t)<r<\infty} \left| w_1(r)\phi_1\left(\mu_{n,\rho,p_1}(r)\right) \right| = \|w_1(r)\phi_1(\mu_{n,\rho,p_1}(r))\|_{L_\infty\left(\mu^{(-1)}_{n,\rho,p_1}(t),\infty\right)}.$$

Similarly, by (4.2), (4.7) and (4.12) for $0 < \theta_2 \le \infty$, $t > 0$

$$\|v_2\|_{L_{\theta_2}(0,t)} = \left\| w_2(r) r^{\frac{n}{p_2}} \right\|_{L_{\theta_2}\left(\mu^{(-1)}_{n,\rho,p_1}(t),\infty\right)}$$

and

$$\|v_2\|_{L_{\theta_2}(t,\infty)} = \left\| w_2(r) r^{\frac{n}{p_2}} \right\|_{L_{\theta_2}\left(0,\mu^{(-1)}_{n,\rho,p_1}(t)\right)}.$$

The proved equalities imply equalities (4.17) and (4.20) by passing to the limit as $t \to 0^+$ or $t \to +\infty$. □

Proof of Theorem 4.2 1. Let $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$ and $0 < p_2 \le p_1$, and $\rho \in S_{n,p_1,p_2}$. Assume that the operator H is bounded from $L_{\theta_1,v_1(\cdot)}(0,\infty)$ to $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$. Since $g_{n,\rho,p_1} \in \mathbb{A}$, by Statement 1 of Theorem 4.1 and formula (4.14) we have

$$\left\| I_{\rho(\cdot)} f \right\|_{LM_{p_2\theta_2,w_2(\cdot)}} \lesssim \left\| H g_{n,\rho,p_1} \right\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \lesssim \left\| g_{n,\rho,p_1} \right\|_{L_{\theta_1,v_1(\cdot)}(0,\infty)} = \|f\|_{LM_{p_1\theta_1,w_1(\cdot)}} \tag{4.21}$$

uniformly in $f \in LM_{p_1\theta_1,w_1(\cdot)}$, hence the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2.1. Let $p_1 = 1, 0 < p_2 < \infty$ and $\rho \in \tilde{S}_{n,1,p_2}$. If the operator H is bounded from $L_{\theta_1,v_1(\cdot)(0,\infty)}$ to $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$, then by Statement 2 of Theorem 4.1 as in Step 1 it follows that the operator $I_{\rho(\cdot)}$ is bounded from $LM_{p_1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$.

2.2. Assume that $I_{\rho(\cdot)}$ is bounded from $LM_{1\theta_1,w_1(\cdot)}$ to $LM_{p_2\theta_2,w_2(\cdot)}$. Then by Statement 2 of Theorem 4.1 and formula (4.14) with $p_1 = 1$

$$\left\|H_{g_{n,\rho,1}}\right\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \approx \left\|I_{\rho(\cdot)}f\right\|_{LM_{p_2\theta_2,w_2(\cdot)}} \lesssim \|f\|_{LM_{1\theta_1,w_1(\cdot)}} = \left\|g_{n,\rho,1}\right\|_{L_{\theta_1,v_1(\cdot)}} \tag{4.22}$$

uniformly in all non-negative functions $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Let $g \in \mathbb{A}$ be continuously differentiable on $(0,\infty)$. Consider the positive Lebesgue measurable function h on $(0,\infty)$ defined uniquely up to equivalence by the equality

$$g(t) = \|h(|\cdot|)\|_{L_1(B(0,\mu^{(-1)}_{n,\rho,1}(t)))} = \sigma_n \int_0^{\mu^{(-1)}_{n,\rho,1}(t)} h(\tau)\tau^{n-1}d\tau,\ t > 0$$

$$\Leftrightarrow g'(t) = \sigma_n \left(\mu^{(-1)}_{n,\rho,1}(t)\right)' h\left(\mu^{(-1)}_{n,\rho,1}(t)\right)\left(\mu^{(-1)}_{n,\rho,1}(t)\right)^{n-1}, t > 0$$

$$\Leftrightarrow h(r) = \sigma_n^{-1} g'(\mu_{n,\rho,1}(r))\mu'_{n,\rho,1}(r)r^{1-n}, r > 0.$$

If we take in (4.22) $f(x) = h(|x|)$, then by (4.3) $\|f\|_{L_1(B(0,\mu^{(-1)}_{n,\rho,1}(t)))} = g(t)$ and by (4.21)

$$\|Hg\|_{L_{\theta_2,v_2(\cdot)}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,v_1(\cdot)}} \tag{4.23}$$

uniformly in all $g \in \mathbb{A}$ which are continuously differentiable on $(0,\infty)$.

Finally, if g is an arbitrary function in $\mathbb{A}$, then there exist functions $g_k \in \mathbb{A}$, $k \in \mathbb{N}$, which are continuously differentiable on $(0,\infty)$ and such $g_k \leq g_{k+1}$, $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} g_k = g$ on $(0,\infty)$. Therefore, by passing to the limit in (4.23), with g replaced by g_k, as $k \to \infty$, it follows that (4.23) holds for all $g \in \mathbb{A}$.

3. Let $p_1 = 1, 1 < p_2 < \infty$ and $\rho \in S_{n,1,p_2}$. In this case the argument is similar to that of Step 2, only Statement 3 of Theorem 4.1 should be used and the space $LM_{p_2\theta_2,w_2(\cdot)}$ should be replaced by the space $WLM_{p_2\theta_2,w_2(\cdot)}$. □

Hardy's inequality on the cone of monotonic functions

Necessary and sufficient conditions for the boundedness of the operator H from one weighted Lebesgue space $L_{\theta_1,v_1(\cdot)}(0,\infty)$ to another one $L_{\theta_2,v_2(\cdot)}(0,\infty)$ on the cone $\mathbb{A}$ are knows for all values of $0 < \theta_1, \theta_2 \leq \infty$. We present below these results which are corollaries of more general statements contained in the survey by A. Gogatishvili and V.D. Stepanov [10] (Theorems 2.5, 3.15 and 3.16). See also the survey by M.L. Goldman [11].

Theorem 5.1 *Let* $0 < \theta_1, \theta_2 \leq \infty$, v_1, v_2 *be non-negative Lebesgue measurable functions on* $(0,\infty)$. *Then the operator* H *is bounded from* $L_{\theta_1,v_1(\cdot)}(0,\infty)$ *to* $L_{\theta_2,v_2(\cdot)}(0,\infty)$ *on the cone* $\mathbb{A}$ *if and only if*

(a) if $1 < \theta_1 \leq \theta_2 < \infty$, *then*

$$A_{11} := \sup_{t>0} \left(\int_0^t s^{\theta_2} v_2^{\theta_2}(s)ds\right)^{\frac{1}{\theta_2}} \left(\int_0^t v_1^{\theta_1}(s)ds\right)^{-\frac{1}{\theta_1}} < \infty,$$

and

$$A_{12} := \sup_{t>0}\left(\int_t^\infty v_2^{\theta_2}(\tau)d\tau\right)^{\frac{1}{\theta_2}}\left(\int_0^t z^{\theta_1'}\left(\int_0^z v_1^{\theta_1}(s)ds\right)^{-\theta_1'} v_1^{\theta_1}(z)dz\right)^{\frac{1}{\theta_1'}} < \infty,$$

(b) if $0<\theta_1\le 1, \theta_1\le\theta_2<\infty$, *then*

$$A_2 := \sup_{t>0}\left(\int_0^t v_1^{\theta_1}(s)ds\right)^{-\frac{1}{\theta_1}}\left(\int_0^\infty \min\{s,t\}^{\theta_2} v_2^{\theta_2}(s)ds\right)^{\frac{1}{\theta_2}} < \infty,$$

(c) if $1<\theta_1<\infty, 0<\theta_2<\theta_1<\infty$, *then*

$$A_{31} := \left(\int_0^\infty \left(\int_0^t v_1^{\theta_1}(\tau)d\tau\right)^{-\frac{r}{\theta_1}}\left(\int_0^t s^{\theta_2} v_2^{\theta_2}(s)ds\right)^{\frac{r}{\theta_1}} t^{\theta_2} v_2^{\theta_2}(t)dt\right)^{\frac{1}{r}} < \infty,$$

and

$$A_{32} := \left(\int_0^\infty \left(\int_t^\infty v_2^{\theta_2}(s)ds\right)^{\frac{r}{\theta_1}}\left(\int_0^t s^{\theta_1'}\left(\int_0^s v_1^{\theta_1}(\tau)d\tau\right)^{-\theta_1'} v_1^{\theta_1}(s)ds\right)^{\frac{r}{\theta_1'}} v_2^{\theta_2}(t)dt\right)^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r}=\frac{1}{\theta_2}-\frac{1}{\theta_1}$,

(d) if $0<\theta_2<\theta_1\le 1$, *then* $A_{41} := A_{31}<\infty$ *and*

$$A_{42} := \left(\int_0^\infty \left(\sup_{0<s\le t} s^{\theta_1}\left(\int_0^s v_1^{\theta_1}(\tau)d\tau\right)^{-1}\right)^{\frac{r}{\theta_1}}\left(\int_t^\infty v_2^{\theta_2}(z)dz\right)^{\frac{r}{\theta_1}} v_2^{\theta_2}(t)dt\right)^{\frac{1}{r}} < \infty,$$

(e) if $0<\theta_1\le 1, \theta_2=\infty$, *then*

$$A_5 := \sup_{t>0}\left(\int_0^t v_1^{\theta_1}(s)ds\right)^{-\frac{1}{\theta_1}}\left(\operatorname*{ess\,sup}_{y>0} \min(t,y)v_2(y)\right) < \infty,$$

(f) if $1<\theta_1<\infty, \theta_2=\infty$, *then*

$$A_6 := \operatorname*{ess\,sup}_{s>0} v_2(s)\left(\int_0^s\left(\int_t^s\left(\int_0^y v_1^{\theta_1}(z)dz\right)^{-1}dy\right)^{\theta_1'} v_1^{\theta_1}(t)dt\right)^{\frac{1}{\theta_1'}} < \infty,$$

(g) if $\theta_1=\infty, 0<\theta_2<\infty$, *then*

$$A_7 := \left(\int_0^\infty\left(v_2(x)\int_0^x \frac{dy}{\operatorname*{ess\,sup}_{0<z<y} v_1(z)}\right)^{\theta_2} dx\right)^{\frac{1}{\theta_2}} < \infty,$$

(h) if $\theta_1=\theta_2=\infty$, *then*

$$A_8 := \operatorname*{ess\,sup}_{x>0}\left(v_2(x)\int_0^x \frac{dy}{\operatorname*{ess\,sup}_{0<z<y} v_1(z)}\right) < \infty.$$

Conditions, ensuring boundedness of the generalized Riesz potential

In order to obtain sufficient conditions and necessary and sufficient conditions for $p_1 = 1$ on the weight functions w_1, w_2 ensuring the boundedness of $I_{\rho(.)}$ from $LM_{p_1,\theta_1,w_1(.)}$ to $LM_{p_2,\theta_2,w_2(.)}$ for $1 < p_1 < p_2 < \infty$ or $1 \le p_1 < \infty$, it, clearly, suffices to apply Theorems 4.2 and 5.1.
In order to make these conditions have a simpler form we shall carry out certain changes of variables and apply equalities (4.15)–(4.20) proved in Lemma 4.1.

Theorem 6.1 *Assume that* $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.

1. *Let* $1 < p_1 < p_2 < \infty$ *or* $1 \le p_1 < \infty$ *and* $0 < p_2 \le \infty$, *and* $\rho \in S_{n,p_1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{p_1,\theta_1,w_1(\cdot)}$ *to* $LM_{p_2,\theta_2,w_2(\cdot)}$ *if the following conditions are satistied.*

(*a*) *If* $1 < \theta_1 \le \theta_2 < \infty$, *then*

$$B_{11} := \sup_{t>0} \left(\int_t^\infty w_{2,n,p_2}^{\theta_2}(x) \mu_{n,\rho,p_1}^{\theta_2}(x)dx \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(x)dx \right)^{-\frac{1}{\theta_1}} < \infty,$$

and

$$B_{12} := \sup_{t>0} \left(\int_0^t w_{2,n,p_2}^{\theta_2}(x)dx \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(x)\mu_{n,\rho,p_1}^{\theta_1'}(x)}{(\int_x^\infty w_1^{\theta_1}(s)ds)^{\theta_1'}} dx \right)^{\frac{1}{\theta_1'}} < \infty.$$

where $w_{2,n,p_2}(x) = w_2(x)x^{\frac{n}{p_2}}$.

(*b*) *If* $0 < \theta_1 \le 1, \theta_1 \le \theta_2 < \infty$, *then*

$$B_2 := \sup_{t>0} \left(\int_t^\infty w_1^{\theta_1}(x)dx \right)^{-\frac{1}{\theta_1}} \left(\int_0^\infty w_{2,n,p_2}^{\theta_2}(x) \min\{\mu_{n,\rho,p_1}(t), \mu_{n,\rho,p_1}(x)\}^{\theta_2} dx \right)^{\frac{1}{\theta_2}} < \infty.$$

(*c*) *If* $1 < \theta_1 < \infty, 0 < \theta_2 < \theta_1 < \infty$, *then*

$$B_{31} := \left(\int_0^\infty \left(\frac{\int_t^\infty w_{2,n,p_2}^{\theta_2}(x)\mu_{n,\rho,p_1}^{\theta_2}(x)dx}{\int_t^\infty w_1^{\theta_1}(x)dx} \right)^{\frac{r}{\theta_1}} w_{2,n,p_2}^{\theta_2}(t)\mu_{n,\rho,p_1}^{\theta_2}(t)dt \right)^{\frac{1}{r}} < \infty,$$

and

$$B_{32} := \left(\int_0^\infty \left(\int_z^\infty w_{2,n,p_2}^{\theta_2}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int_z^\infty \frac{\mu_{n,\rho,p_1}^{\theta_1'}(\tau)w_1^{\theta_1}(\tau)}{\left(\int_\tau^\infty w_1^{\theta_1}(u)du\right)^{\theta_1'}} d\tau \right)^{\frac{r}{\theta_1'}} w_{2,n,p_2}^{\theta_2}(z)dz \right)^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{\theta_2} - \frac{1}{\theta_1}$.

(*d*) *If* $0 < \theta_2 < \theta_1 \le 1$, *then* $B_{41} := B_{31} < \infty$ *and*

$$B_{42} := \left(\int_0^\infty \sup_{y<z<\infty} \mu_{n,\rho,p_1}(z)^r \left(\int_z^\infty w_1^{\theta_1}(s)ds \right)^{-\frac{r}{\theta_1}} \left(\int_0^y w_{2,n,p_2}^{\theta_2}(x)dx \right)^{\frac{r}{\theta_1}} w_{2,n,p_2}^{\theta_2}(y)dy \right)^{\frac{1}{r}} < \infty.$$

(e) If $0 < \theta_1 \le 1, \theta_2 = \infty$, *then*

$$B_5 := \sup_{t>0}\left(\int_t^\infty w_1^{\theta_1}(x)dx\right)^{-\frac{1}{\theta_1}} \operatorname*{ess\,sup}_{x>0} w_{2,n,p_2}(x)\min\{\mu_{n,\rho,p_1}(t), \mu_{n,\rho,p_1}(x)\} < \infty.$$

(f) If $1 < \theta_1 < \infty, \theta_2 = \infty$, *then*

$$B_6 := \operatorname*{ess\,sup}_{z>0} w_{2,n,p_2}(z)\left(\int_z^\infty\left(\int_x^z\left(\int_u^\infty w_1^{\theta_1}(s)ds\right)^{-1}\rho(u)u^{\frac{n}{p_1'}-1}du\right)^{\theta_1'} w_1^{\theta_1}(x)dx\right)^{\frac{1}{\theta_1'}} < \infty.$$

(g) If $\theta_1 = \infty, 0 < \theta_2 < \infty$, *then*

$$B_7 := \left(\int_0^\infty\left(w_{2,n,p_2}(\tau)\int_\tau^\infty \frac{\rho(z)z^{\frac{n}{p_1'}-1}dz}{\operatorname*{ess\,sup}_{z<s<\infty} w_1(s)}\right)^{\theta_2}d\tau\right)^{\frac{1}{\theta_2}} < \infty.$$

(h) If $\theta_1 = \theta_2 = \infty$, *then*

$$B_8 := \operatorname*{ess\,sup}_{\tau>0}\left(\int_\tau^\infty \frac{\rho(z)z^{\frac{n}{p_1'}-1}dz}{\operatorname*{ess\,sup}_{z<s<\infty} w_1(s)}\right)w_{2,n,p_2}(\tau) < \infty.$$

2. *Let* $p_1 = 1, 0 < p_2 < \infty$ *and* $\rho \in \tilde{S}_{n,1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{1,\theta_1,w_1(.)}$ *to* $LM_{p_2,\theta_2,w_2(.)}$ *if and only if conditions* (a) - (h) *with* $p_1 = 1$ *are satisfied.*

3. *Let* $p_1 = 1, 1 < p_2 < \infty$ *and* $\rho \in S_{n,1,p_2}$. *Then the operator* $I_{\rho(.)}$ *is bounded from* $LM_{1,\theta_1,w_1(.)}$ *to* $WLM_{p_2,\theta_2,w_2(.)}$ *if and only if conditions* (a) - (h) *with* $p_1 = 1$ *are satisfied.*

Proof It sufficies to prove by using Lemma 4.1 and the appropriate changes of variables that, for μ_{n,ρ,p_1}, v_1 and v_2 defined by formulas (4.1), (4.9), (4.2), respectively, $A_{11} = B_{11}$. $A_{12} = B_{12}$, $A_2 = B_2$, $A_{31} = B_{31}$, $A_{32} = B_{32}$, $A_{41} = B_{41}$, $A_5 = B_5$ and $A_6 = cB_6$, $A_7 = cB_7$, $A_8 = cB_8$, where $c = \left(\int_1^\infty \rho(t)t^{\frac{n}{p_1'}-1}dt\right)^{-1}$.

(a) If $1 < \theta_1 \le \theta_2 < \infty$, then according to Theorem 5.1 and by (4.15) with $\phi_1 \equiv 1$ and (4.18) with $\phi_2(s) \equiv s$, we get

$$A_{11} = \sup_{t>0}\left(\int_{\mu^{(-1)}_{n,\rho,p_1}(t)}^\infty w_{2,n,p_2}^{\theta_2}(r)\mu_{n,\rho,p_1}^{\theta_2}(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_{\mu^{(-1)}_{n,\rho,p_1}(t)}^\infty w_1^{\theta_1}(r)dr\right)^{-\frac{1}{\theta_1}}$$

$$\left(\mu^{(-1)}_{n,\rho,p_1}(t) = \tau\right)$$

$$= \sup_{\tau>0}\left(\int_\tau^\infty w_{2,n,p_2}^{\theta_2}(r)\mu_{n,\rho,p_1}^{\theta_2}(r)dr\right)^{\frac{1}{\theta_2}}\left(\int_\tau^\infty w_1^{\theta_1}(r)dr\right)^{-\frac{1}{\theta_1}} = B_{11}$$

and

$$A_{12} = \sup_{t>0}\left(\int_0^{\mu^{(-1)}_{n,\rho,p_1}(t)} w_{2,n,p_2}^{\theta_2}(x)dx\right)^{\frac{1}{\theta_2}}\left(\int_0^t z^{\theta_1'}\left(\int_{\mu^{(-1)}_{n,\rho,p_1}(z)}^\infty w_1^{\theta_1}(r)dr\right)^{-\theta_1'} w_1^{\theta_1}(z)dz\right)^{\frac{1}{\theta_1'}}.$$

Next, by (4.16) with

$$\phi_1(z) = \left(z^{\theta_1'} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(z)}^{\infty} w_1^{\theta_1}(r)dr \right)^{-\theta_1'} \right)^{\frac{1}{\theta_1}},$$

we get

$$A_{12} = \sup_{t>0} \left(\int\limits_0^{\mu^{(-1)}_{n,\rho,p_1}(t)} w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{1}{\theta_2}} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(t)}^{\infty} \mu^{\theta_1'}_{n,\rho,p_1}(x) \left(\int\limits_x^{\infty} w_1^{\theta_1}(r)dr \right)^{-\theta_1'} w_1^{\theta_1}(x)dx \right)^{\frac{1}{\theta_1'}}$$

$$\left(\mu^{(-1)}_{n,\rho,p_1}(t) = \tau \right)$$

$$= \sup_{\tau>0} \left(\int\limits_0^{\tau} w_2^{\theta_2}(x) x^{\frac{n\theta_2}{p_2}} dx \right)^{\frac{1}{\theta_2}} \left(\int\limits_{\tau}^{\infty} \mu^{\theta_1'}_{n,\rho,p_1}(x) \left(\int\limits_x^{\infty} w_1^{\theta_1}(r)dr \right)^{-\theta_1'} w_1^{\theta_1}(x)dx \right)^{\frac{1}{\theta_1'}} = B_{12}.$$

(b) If $0 < \theta_1 \leq 1, \theta_1 \leq \theta_2 < \infty$, then by (4.15) with $\phi_1 \equiv 1$ and by (4.19′) with $\phi_2(s) = \min\{t, s\}, s > 0$, we get

$$A_2 = \sup_{t>0} \left(\int_{\mu^{-1}_{n,\rho,p_1}(t)}^{\infty} w_1^{\theta_1}(r)dr \right)^{-\frac{1}{\theta_1}} \left(\int_0^{\infty} w^{\theta_2}_{2,n,p_2}(r) \min\{t, \mu_{n,\rho,p_1}(r)\}^{\theta_2} dr \right)^{\frac{1}{\theta_2}}$$

$$(\mu^{-1}_{n,\rho,p_1}(t) = \tau)$$

$$= \sup_{\tau>0} \left(\int_{\tau}^{\infty} w_1^{\theta_1}(r)dr \right)^{-\frac{1}{\theta_1}} \left(\int_0^{\infty} w^{\theta_2}_{2,n,p_2}(r) \min\{\mu_{n,\rho,p_1}(\tau), \mu_{n,\rho,p_1}(r)\}^{\theta_2} dr \right)^{\frac{1}{\theta_2}} = B_2.$$

(c) If $1 < \theta_1 < \infty, 0 < \theta_2 < \theta_1 < \infty$, then by (4.15) with $\phi_1 \equiv 1$, (4.18) with $\phi_s = s$ and by (4.19') we get

$$A_{31} = \left(\int\limits_0^{\infty} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1(t)}}^{\infty} w_1^{\theta_1}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1(t)}}^{\infty} w^{\theta_2}_{2,n,p_2}(x) \mu^{\theta_2}_{n,\rho,p_1}(x)dx \right)^{\frac{r}{\theta_1}} t^{\theta_2} v_2^{\theta_2}(t)dt \right)^{\frac{1}{r}}$$

$$= \left(\int\limits_0^{\infty} \left(\int\limits_z^{\infty} w_1^{\theta_1}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int\limits_z^{\infty} w^{\theta_2}_{2,n,p_2}(x) \mu^{\theta_2}_{n,\rho,p_1}(x)dx \right)^{\frac{r}{\theta_1}} \mu^{\theta_2}_{n,\rho,p_1}(z) w^{\theta_2}_{2,n,p_2}(z)dz \right)^{\frac{1}{r}} = B_{31}$$

and

$$A_{32} = \left(\int\limits_0^{\infty} \left(\int\limits_0^{\mu^{(-1)}_{n,\rho,p_1}(t)} w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int\limits_0^{t} \frac{s^{\theta_1'}}{\left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(s)}^{\infty} w_1^{\theta_1}(u)du \right)^{\theta_1'}} v_1^{\theta_1}(s)ds \right)^{\frac{r}{\theta_1'}} v_2^{\theta_2}(t)dt \right)^{\frac{1}{r}}$$

$$= \left(\int_0^\infty \left(\int_0^{\mu^{(-1)}_{n,\rho,p_1}(t)} w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(t)}^\infty \frac{\mu^{\theta_1'}_{n,\rho,p_1}(\tau) w_1^{\theta_1}(\tau)}{\left(\int_\tau^\infty w_1^{\theta_1}(u)du \right)^{\theta_1'}} d\tau \right)^{\frac{r}{\theta_1'}} v_2^{\theta_2}(t)dt \right)^{\frac{1}{r}}$$

$$= \left(\int_0^\infty \left(\int_z^\infty w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} \left(\int_z^\infty \frac{\mu^{\theta_1'}_{n,\rho,p_1}(\tau) w_1^{\theta_1}(\tau)}{\left(\int_\tau^\infty w_1^{\theta_1}(u)du \right)^{\theta_1'}} d\tau \right)^{\frac{r}{\theta_1'}} w^{\theta_2}_{2,n,p_2}(z)dz \right)^{\frac{1}{r}} = B_{32}.$$

(d) If $0 < \theta_2 < \theta_1 \le 1$, then by (4.15) with $\phi_1 \equiv 1$, by (4.19) with $\phi_2 \equiv 1$, we get

$$A_{41} = \left(\int_0^\infty \sup_{0<\tau<t} \tau^r \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(\tau)}^\infty w_1^{\theta_1}(u)du \right)^{\frac{r}{\theta_1}} \left(\int_0^{\mu^{(-1)}_{n,\rho,p_1}(t)} w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} v_2^{\theta_2}(t)dt \right)^{\frac{1}{r}}$$

$$= \left(\int_0^\infty \sup_{0<\tau<\mu_{n,\rho,p_1}(y)} \tau^r \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(\tau)}^\infty w_1^{\theta_1}(s)ds \right)^{\frac{r}{\theta_1}} \left(\int_0^y w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} w^{\theta_2}_{2,n,p_2}(y)dy \right)^{\frac{1}{r}}.$$

$$\left(\tau = \mu^{(-1)}_{n,\rho,p_1}(z) \right)$$

$$= \left(\int_0^\infty \operatorname*{ess\,sup}_{y<z<\infty} \mu_{n,\rho,p_1}(z)^r \left(\int_z^\infty w_1^{\theta_1}(s)ds \right)^{\frac{r}{\theta_1}} \left(\int_0^y w^{\theta_2}_{2,n,p_2}(x)dx \right)^{\frac{r}{\theta_1}} \mu_{n,\rho,p_1}(x)^{\theta_2}(y)dy \right)^{\frac{1}{r}} = B_{42}.$$

(e) If $0 < \theta_1 \le 1$, $\theta_2 = \infty$, then by (4.15) with $\phi_1 \equiv 1$ and (4.19') with $\phi(y) = \min\{t, y\}$, $y > 0$, we get

$$A_5 = \sup_{t>0} \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(t)}^\infty w_1^{\theta_1}(x)dx \right)^{-\frac{1}{\theta_1}} \operatorname*{ess\,sup}_{x>0} w_{2,n,p_2}(x) \min\{t, \mu_{n,\rho,p_1}(x)\}$$

$$(\mu^{(-1)}_{n,\rho,p_1}(t) = \tau)$$

$$= B_{42}. = \sup_{\tau>0} \left(\int_\tau^\infty w_1^{\theta_1}(x)dx \right)^{-\frac{1}{\theta_1}} \operatorname*{ess\,sup}_{x>0} w_{2,n,p_2}(x) \min\{\mu_{n,\rho,p_1}(\tau), \mu_{n,\rho,p_1}(x)\} = B_5.$$

(f) If $1 < \theta_1 < \infty$, $\theta_2 = \infty$, then by (4.15) with $\phi_1 \equiv 1$ and by the same formula with

$$\phi_1(t) = \left(\int_t^s \left(\int_{\mu^{(-1)}_{n,\rho,p_1}(y)}^\infty w_1^{\theta_1}(s)ds \right)^{-1} dy \right)^{\frac{\theta_1'}{\theta_1}}$$

and, finally, by (4.19') with $\theta_2 = \infty$ and

$$\phi_2(s) = \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(s)}^{\infty} \left(\int\limits_{\mu_{n,\rho,p_1}(r)}^{s} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(y)}^{\infty} w_1^{\theta_1}(s)ds \right)^{-1} dy \right)^{\theta_1'} w_1^{\theta_1}(r)dr \right)^{\frac{1}{\theta_1'}}$$

we get

$$A_6 = \operatorname*{ess\,sup}_{s>0} v_2(s) \left(\int\limits_0^s \left(\int\limits_t^s \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(y)}^{\infty} w_1^{\theta_1}(s)ds \right)^{-1} dy \right)^{\theta_1'} v_1^{\theta_1}(t)dt \right)^{\frac{1}{\theta_1'}}$$

$$= \operatorname*{ess\,sup}_{s>0} v_2(s) \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(s)}^{\infty} \left(\int\limits_{\mu_{n,\rho,p_1}(r)}^{s} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(y)}^{\infty} w_1^{\theta_1}(s)ds \right)^{-1} dy \right)^{\theta_1'} w_1^{\theta_1}(r)dr \right)^{\frac{1}{\theta_1'}}$$

$$= \operatorname*{ess\,sup}_{z>0} w_2(z) z^{\frac{n}{p_2}} \left(\int\limits_z^{\infty} \left(\int\limits_{\mu_{n,\rho,p_1}(r)}^{\mu_{n,\rho,p_1}(z)} \left(\int\limits_{\mu^{(-1)}_{n,\rho,p_1}(y)}^{\infty} w_1^{\theta_1}(s)ds \right)^{-1} dy \right)^{\theta_1'} w_1^{\theta_1}(r)dr \right)^{\frac{1}{\theta_1'}}$$

$$(y = \mu_{n,\rho,p_1}(u))$$

$$= cB_6, = \operatorname*{ess\,sup}_{z>0} \; w_{2,n,p_2}(z) \left(\int\limits_z^{\infty} \left(\int\limits_r^{z} \left(\int\limits_u^{\infty} w_1(s)^{\theta_1} ds \right)^{-1} |(\mu_{n,\rho,p_1}(u))'| du \right)^{\theta_1'} w_1^{\theta_1}(r)dr \right)^{\frac{1}{\theta_1'}}$$

where

$$c = \left(\int\limits_1^{\infty} \rho(t) t^{\frac{n}{p_1'}-1} dt \right)^{-1}.$$

(g) If $\theta_1 = \infty, 0 < \theta_2 < \infty$, then by using formula (4.15) with $\phi_1 \equiv 1$ and formula (4.19') with $\phi_2(x) = \int\limits_0^x \frac{dy}{\operatorname*{ess\,sup}_{r>\mu^{(-1)}_{n,\rho,p_1}(y)} w_1(r)}$

we get

$$A_7 = \left(\int\limits_0^{\infty} \left(v_2(x) \int\limits_0^x \frac{dy}{\operatorname*{ess\,sup}_{r>\mu^{(-1)}_{n,\rho,p_1}(y)} w_1(r)} \right)^{\theta_2} dx \right)^{\frac{1}{\theta_2}}$$

$$= \left(\int\limits_0^{\infty} \left(w_{2,n,p_2}(\tau) \int\limits_0^{\mu_{n,\rho,p_1}(\tau)} \frac{dy}{\operatorname*{ess\,sup}_{r>\mu^{(-1)}_{n,\rho,p_1}(y)} w_1(r)} \right)^{\theta_2} d\tau \right)^{\frac{1}{\theta_2}}$$

$$\left(\mu^{(-1)}_{n,\rho,p_1}(y) = z \right)$$

$$= c\left(\int\limits_0^\infty \left(w_{2,n,p_2}(\tau)\int\limits_\tau^\infty \frac{\rho(z)z^{\frac{n}{p_1'}-1}}{\operatorname*{ess\ sup}\limits_{z<r<\infty} w_1(r)}dz\right)^{\theta_2} d\tau\right)^{\frac{1}{\theta_2}} = cB_7.$$

(h) If $\theta_1 = \theta_2 = \infty$, then in A_8 according to (4.10) and (4.2)

$$v_1(x) = w_1\left(\mu^{(-1)}_{n,\rho,p_1}(x)\right),\ v_2(x) = w_2\left(\mu^{(-1)}_{n,\rho,p_1}(x)\right)\left(\mu^{(-1)}_{n,\rho,p_1}(x)\right)^{\frac{n}{p_2}}, x > 0.$$

By using formula (4.15) with $\psi_1 \equiv 1$ and by carrying out the following changes of variables: $y = \mu_{n,\rho,p_1}(z)$, $z > 0$, and, finally, $t = \mu_{n,\rho,p_1}(\tau)$, $\tau > 0$, we get

$$A_8 = \operatorname*{ess\ sup}_{t>0}\left(\int_0^t \frac{dy}{\operatorname*{ess\ sup}\limits_{s>\mu^{(-1)}_{n,\rho,p_1}(y)} w_1(s)}\right) w_2\left(\mu^{(-1)}_{n,\rho,p_1}(t)\right)\left(\mu^{(-1)}_{n,\rho,p_1}(t)\right)^{\frac{n}{p_2}}$$

$$= c\operatorname*{ess\ sup}_{t>0}\left(\int_{\mu^{(-1)}_{n,\rho,p_2}(t)}^\infty \frac{\rho(z)z^{\frac{n}{p_1'}-1}}{\operatorname*{ess\ sup}\limits_{s>z} w_1(s)}dz\right) w_2\left(\mu^{(-1)}_{n,\rho,p_2}(t)\right)\left(\mu^{(-1)}_{n,\rho,p_2}(t)\right)^{\frac{n}{p_2}}$$

$$= c\operatorname*{ess\ sup}_{\tau>0}\left(\int_\tau^\infty \frac{\rho(z)z^{\frac{n}{p_1'}-1}}{\operatorname*{ess\ sup}\limits_{z<s<\infty} w_1(s)}dz\right) w_{2,n,p_2}(\tau) = cB_8.$$

□

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Declarations

Conflict of interest The authors declare no competing interests.

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