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THE LICHNEROWICZ THEOREM ON CR MANIFOLDS

By

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Abstract. For any compact strictly pseudoconvex CR manifold M endowed with a contact form θ we obtain the Bochner type formula $\frac{1}{2}\Delta_b(|\nabla^H f|^2) = |\pi_H \nabla^2 f|^2 + (\nabla^H f)(\Delta_b f) + \rho(\nabla^H f, \nabla^H f) + 2Lf$ (involving the sublaplacian Δ_b and the pseudohermitian Ricci curvature ρ). When M is compact of CR dimension n and $\rho(X,X) + 2A(X,JX) \geq kG_\theta(X,X)$, $X \in H(M)$, we derive the estimate $-\lambda \geq 2nk/(2n-1)$ on each nonzero eigenvalue λ of Δ_b satisfying $\mathrm{Eigen}(\Delta_b;\lambda) \cap \mathrm{Ker}(T) \neq (0)$ where T is the characteristic direction of $d\theta$.

1. Introduction

By a well known result by A. Lichnerowicz, [18], and M. Obata, [21], on any m-dimensional compact Riemannian manifold (M,g) with $\mathrm{Ric} \geq kg$ the first eigenvalue of the Laplacian satisfies the estimate

$$\lambda_1 \geq mk/(m-1), \tag{1}$$

with equality if and only if M is isometric to the standard sphere S^m. The proof of (1) relies on the Bochner formula (cf. e.g. [3], p. 131)

$$-\frac{1}{2}\Delta(|df|^2) = |\mathrm{Hess}(f)|^2 - (df, d\Delta f) + \mathrm{Ric}((df)^\#, (df)^\#), \tag{2}$$

for any $f \in C^\infty(M)$. On the other hand, given a compact strictly pseudoconvex CR manifold M, with any fixed contact form θ one may associate a natural second order differential operator Δ_b (the *sublaplacian*) which is similar in many

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respects to the Laplacian of a Riemannian manifold. Indeed, Δ_b is hypoelliptic and (by a result of [20]) has a discrete spectrum

$$0 < -\lambda_1 < -\lambda_2 < \cdots < -\lambda_k < \cdots \uparrow +\infty.$$

Also (M, θ) carries a natural linear connection ∇ (the *Tanaka-Webster connection*, cf. [24]–[25]) preserving the Levi form and the maximally complex distribution, and resembling to both the Levi-Civita connection and the Chern connection (in Hermitian geometry). Moreover the Ricci tensor ρ of ∇ is likely to play the role of the Ricci curvature in Riemannian geometry. To give an example, by a result of J. M. Lee, [15], if $\rho(Z, \bar{Z}) > 0$ for any $Z \in T_{1,0}(M)$, $Z \neq 0$, then the first Kohn-Rossi cohomology group $H^{0,1}(M, \bar{\partial}_b)$ vanishes (as a CR counterpart of the classical result in [5]). It is a natural question whether we may estimate the spectrum of Δ_b from below, under appropriate geometric assumptions (on ρ). The first attempt to bring (1) to CR geometry belongs to A. Greenleaf, [12]. His result is that on any compact strictly pseudoconvex CR manifold M, of CR dimension $n \geq 3$, one has

$$-\lambda_1 \geq nC/(n+1) \tag{3}$$

provided that

$$R_{\alpha\bar{\beta}} Z^\alpha \bar{Z}^\beta + i(A_{\bar{\alpha}\bar{\beta}} \bar{Z}^\alpha \bar{Z}^\beta - A_{\alpha\beta} Z^\alpha Z^\beta) \geq 2C g_{\alpha\bar{\beta}} Z^\alpha \bar{Z}^\beta, \tag{4}$$

for some constant $C > 0$. Here $R_{\alpha\bar{\beta}} = \rho(T_\alpha, T_{\bar{\beta}})$ is the *pseudohermitian Ricci tensor* while $A_{\alpha\beta}$ is the *pseudohermitian torsion* (cf. e.g. [7], p. 102) and $\{T_\alpha : 1 \leq \alpha \leq n\}$ is a local frame of the CR structure. The proof of (3) relies on the rather involved Bochner like formula

$$\begin{aligned} \Delta_b(|\nabla^{1,0} f|^2) = {} & 2\sum_{\alpha,\beta}(f_{\alpha\bar{\beta}} f_{\bar{\alpha}\beta} + f_{\alpha\beta} f_{\bar{\alpha}\bar{\beta}}) + 4i\sum_{\alpha}(f_{\bar{\alpha}} f_{0\alpha} - f_\alpha f_{0\bar{\alpha}}) \\ & + 2\sum_{\alpha,\beta} R_{\alpha\bar{\beta}} f_{\bar{\alpha}} f_\beta + 2in\sum_{\alpha,\beta}(A_{\alpha\beta} f_{\bar{\alpha}} f_{\bar{\beta}} - A_{\bar{\alpha}\bar{\beta}} f_\alpha f_\beta) \\ & + \sum_{\alpha}\{f_{\bar{\alpha}}(\Delta_b f)_\alpha + f_\alpha(\Delta_b f)_{\bar{\alpha}}\} \end{aligned} \tag{5}$$

where $\nabla^{1,0} f = f^\alpha T_\alpha$. Cf. also Chapter 9 in [11]. Recently, a large number of results were obtained within CR and pseudohermitian geometry, mainly by analogy to similar findings in Riemannian geometry (cf. e.g. S. Dragomir et al., [8]–[10]). On this line of thought, one scope of this paper is to establish the Bochner like formula[1]

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[1] Under the conventions in the present paper the sublaplacian of [12] is $-\Delta_b$.

$$\frac{1}{2}\Delta_b(|\nabla^H f|^2) = |\pi_H \nabla^2 f|^2 + (\nabla^H f)(\Delta_b f) + \rho(\nabla^H f, \nabla^H f) + 2Lf, \tag{6}$$

for any $f \in C^\infty(M)$, where the differential operator L given by

$$Lf \equiv (J\nabla^H f)(Tf) - (J\nabla_T \nabla^H f)(f). \tag{7}$$

As an application we shall prove

Theorem 1. *Let M be a compact strictly pseudoconvex CR manifold, of CR dimension n. Let θ be a contact form on M such that the Levi form G_θ is positive definite. Let λ be a nonzero eigenvalue of the sublaplacian Δ_b. Suppose that there is a constant $k > 0$ such that* i)

$$\rho(X, X) + 2A(X, JX) \geq kG_\theta(X, X), \quad X \in H(M), \tag{8}$$

and ii) *there is an eigenfunction $f \in \mathrm{Eigen}(\Delta_b; \lambda)$ such that $T(f) = 0$. Then λ satisfies the estimate*

$$-\lambda \geq 2nk/(2n-1). \tag{9}$$

Another lower bound on $-\lambda_1$ (in terms of the diameter of (M, g_θ), where g_θ is the Webster metric) was found in [1] (by using estimates of the horizontal gradient at a point, rather than L^2 methods) as an extension of the work by Z. Jiaqing & Y. Hongcang, [13], in Riemannian geometry. Although under more restrictive assumptions our estimate (9) is sharper than (3). When (M, θ) is Sasakian (i.e. $A_{\alpha\beta} = 0$) A. Greenleaf's assumption (4) coincides with our (8).

The Bochner type formula (6) (as compared to Greenleaf's (5)) presents a closer resemblance to (2) in Riemannian geometry, perhaps enabling one to look for an analogue to the result by M. Obata, [21], as well. Restated in the CR category, the problem is whether equality in (9) implies that M is CR isomorphic to the sphere S^{2n+1}. As it turns out when $M = S^{2n+1}$ the assumptions in our Theorem 2 (see below) are satisfied if and only if $n = 1$. We conjecture that any strictly pseudoconvex CR manifold M carrying a contact form θ satisfying (8) for some $k > 0$ and such that i) $-2nk/(2n-1) \in \mathrm{Spec}(\Delta_b)$, and ii) $\mathrm{Eigen}(\Delta_b; -2nk/(2n-1)) \cap \mathrm{Ker}(T) \neq (0)$, is CR isomorphic to S^3.

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2. A Reminder of CR Geometry

Let $(M, T_{1,0}(M))$ be an oriented CR manifold, of CR dimension n. For a review of the main notions of CR and pseudohermitian geometry one may see [7]. Let $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the maximally complex distribution and $J(Z + \bar{Z}) = i(Z - \bar{Z})$, $Z \in T_{1,0}(M)$, its complex structure. Let θ be a *pseudohermitian structure* on M, i.e. θ is a differential 1-form such that $\mathrm{Ker}(\theta) = H(M)$. The *Levi form* is given by $G_\theta(X, Y) = (d\theta)(X, JY)$, $X, Y \in H(M)$. The given CR manifold is *nondegenerate* (respectively *strictly pseudoconvex*) if G_θ is nondegenerate (respectively positive definite). From now on, let us assume that M is nondegenerate. Then each pseudohermitian structure θ is a contact form i.e. $\Psi = \theta \wedge (d\theta)^n$ is a volume form on M. Let T be the *characteristic direction* of $d\theta$ i.e. the unique globally defined nowhere zero tangent vector field T on M determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. Let g_θ be the *Webster metric* i.e.

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. (M, g_θ) is a semi-Riemannian manifold. If M is strictly pseudoconvex and θ is chosen such that G_θ is positive definite (note that $G_{-\theta}$ is negative definite) then (M, g_θ) is a Riemannian manifold (whose canonical Riemannian volume form is $c_n\Psi$, where $c_n = 2^{-n}/n!$).

Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that the Levi form G_θ is positive definite. The *sublaplacian* is

$$\Delta_b f = \mathrm{div}(\nabla^H f), \quad f \in C^2(M),$$

where $\mathrm{div}(X)$ is the divergence of the vector field X (with respect to the Riemannian metric g_θ) and $\nabla^H f = \pi_H \nabla f$ is the *horizontal gradient*. Precisely ∇f is the ordinary gradient (i.e. $g_\theta(\nabla f, X) = X(f)$ for any $X \in T(M)$) and $\pi_H : T(M) \to H(M)$ is the projection associated to the direct sum decomposition $T(M) = H(M) \oplus \boldsymbol{R}T$. Let ∇ be the Tanaka-Webster connection of (M, θ) i.e. the unique linear connection on M obeying to i) $H(M)$ is ∇-parallel, ii) $\nabla g_\theta = 0$, $\nabla J = 0$, iii) the torsion T_∇ of ∇ satisfies

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \bar{W}) = 2iG_\theta(Z, \bar{W})T, \quad Z, W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0,$$

where $\tau(X) = T_\nabla(T, X)$, $X \in T(M)$. A strictly pseudoconvex CR manifold M is a Sasakian manifold (in the sense of [4], p. 73) if and only if $\tau = 0$. Given two CR manifolds M and N a *CR map* is a C^∞ map $f : M \to N$ such that $(d_x f)T_{1,0}(M)_x \subseteq T_{1,0}(N)_{f(x)}$ for any $x \in M$. A *CR isomorphism* is a C^∞ diffeomorphism and a CR map. By a recent result of G. Marinescu et al., [19], any Sasakian manifold is CR isomorphic to a real submanifold of C^N, for some $N \geq 2$, carrying the induced CR structure.

3. The Bochner Formula

Let $\{X_1, \dots, X_{2n}\}$ be a local orthonormal (i.e. $G_\theta(X_j, X_k) = \delta_{jk}$) frame of $H(M)$, defined on the open subset $U \subseteq M$. Then

$$\Delta_b f = \sum_{j=1}^{2n} \{X_j^2 f - (\nabla_{X_j} X_j) f\} \tag{10}$$

on U. Let $x_0 \in M$ be an arbitrary point. As well known $H(M)$ and g_θ are parallel with respect to ∇. Therefore, by parallel displacement of a given orthonormal frame $\{v_1, \dots, v_{2n}\} \subset H(M)_{x_0}$ with $v_{\alpha+n} = J_x v_\alpha$, $1 \leq \alpha \leq n$, along the geodesics of ∇ issuing at x_0 we may build a local orthonormal frame $\{X_j\}$ of $H(M)$, defined on an open neighborhood of x_0, such that

$$(\nabla_{X_j} X_k)(x_0) = 0, \quad 1 \leq j, k \leq 2n. \tag{11}$$

Also $X_{\alpha+n} = JX_\alpha$ (as a consequence of $\nabla J = 0$). Then (by (10) and $\nabla g_\theta = 0$)

$$\begin{aligned}
\Delta_b(|\nabla^H f|^2)(x_0) &= \sum_j X_j^2(|\nabla^H f|^2)(x_0) \\
&= 2\sum_j X_j(g_\theta(\nabla_{X_j} \nabla^H f, \nabla^H f))_{x_0} \\
&= 2\sum_j \{g_\theta(\nabla_{X_j} \nabla_{X_j} \nabla^H f, \nabla^H f) + g_\theta(\nabla_{X_j} \nabla^H f, \nabla_{X_j} \nabla^H f)\}_{x_0}.
\end{aligned}$$

As $\{X_j\}$ is orthonormal, the first term in the above sum is

$$\sum_{j,k} g_\theta(\nabla_{X_j} \nabla_{X_j} \nabla^H f, X_k) X_k(f).$$

Moreover (by (11))

$$\begin{aligned}
g_\theta(\nabla_{X_j} \nabla_{X_j} \nabla^H f, X_k)_{x_0} &= \{X_j(g_\theta(\nabla_{X_j} \nabla^H f, X_k)) - g_\theta(\nabla_{X_j} \nabla^H f, \nabla_{X_j} X_k)\}_{x_0} \\
&= X_j(X_j(g_\theta(\nabla^H f, X_k)) - g_\theta(\nabla^H f, \nabla_{X_j} X_k))_{x_0} \\
&= X_j(X_j X_k f - (\nabla_{X_j} X_k) f)_{x_0} = X_j((\nabla^2 f)(X_j, X_k))_{x_0}
\end{aligned}$$

where the Hessian is defined with respect to the Tanaka-Webster connection

$$(\nabla^2 f)(X, Y) = (\nabla_X df)Y = X(Y(f)) - (\nabla_X Y)f, \quad X, Y \in T(M).$$

Unlike the Hessian in Riemannian geometry $\nabla^2 f$ is never symmetric

$$(\nabla^2 f)(X, Y) = (\nabla^2 f)(Y, X) - T_\nabla(X, Y)(f), \tag{12}$$

where T_∇ is the torsion of ∇. On the other hand T_∇ is pure (cf. [7], p. 102) hence

$$T_\nabla(X, Y) = -2\Omega(X, Y)T, \quad X, Y \in H(M). \tag{13}$$

Here $\Omega(X, Y) = g_\theta(X, JY)$ (so that $\Omega = -d\theta$). Then (by (12)–(13))

$$\begin{aligned} g_\theta(\nabla_{X_j}\nabla_{X_j}\nabla^H f, X_k)_{x_0} &= X_j((\nabla^2 f)(X_j, X_k))_{x_0} \\ &= X_j((\nabla^2 f)(X_k, X_j) + 2\Omega(X_j, X_k)Tf)_{x_0} \\ &= g_\theta(\nabla_{X_j}\nabla_{X_k}\nabla^H f, X_j)_{x_0} + 2\Omega(X_j, X_k)_{x_0} X_j(Tf)_{x_0} \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{2}\Delta_b(|\nabla^H f|^2)(x_0) = \sum_j |\nabla_{X_j}\nabla^H f|^2_{x_0} + \sum_{j,k} \{ g_\theta(\nabla_{X_j}\nabla_{X_k}\nabla^H f, X_j) \\ + 2\Omega(X_j, X_k)X_j(Tf)\}_{x_0} X_k(f)_{x_0}. \end{aligned} \tag{14}$$

If B is a bilinear form on $T(M)$ we denote by $\pi_H B$ its restriction to $H(M)$. The norm of $\pi_H B$ is given by $|\pi_H B|^2 = \sum_{j,k} B(X_j, X_k)^2$. Then

$$\begin{aligned} |\pi_H \nabla^2 f|^2 &= \sum_{j,k} (\nabla^2 f)(X_j, X_k)^2 = \sum_{j,k} (X_j X_k f - (\nabla_{X_j} X_k) f)^2 \\ &= \sum_{j,k} g_\theta(\nabla_{X_j}\nabla^H f, X_k)^2 = \sum_j g_\theta(\nabla_{X_j}\nabla^H f, \nabla_{X_j}\nabla^H f) \end{aligned}$$

so that

$$|\pi_H \nabla^2 f|^2 = \sum_j |\nabla_{X_j}\nabla^H f|^2. \tag{15}$$

Next $[X_j, X_k] = \nabla_{X_j} X_k - \nabla_{X_k} X_j - T_\nabla(X_j, X_k)$ hence (by applying (11) and (13))

$$[X_j, X_k]_{x_0} = 2\Omega(X_j, X_k)_{x_0} T_{x_0}$$

and taking into account

$$\nabla_X \nabla_Y = \nabla_Y \nabla_X + R(X, Y) + \nabla_{[X,Y]}$$

(where R is the curvature tensor field of ∇) we obtain

$$\nabla_{X_j}\nabla_{X_k}\nabla^H f = \nabla_{X_k}\nabla_{X_j}\nabla^H f + R(X_j, X_k)\nabla^H f + 2\Omega(X_j, X_k)\nabla_T\nabla^H f \tag{16}$$

at x_0. Moreover

$$\begin{aligned} g_\theta(\nabla_{X_k}\nabla_{X_j}\nabla^H f, X_j)_{x_0} &= \{X_k(g_\theta(\nabla_{X_j}\nabla^H f, X_j)) - g_\theta(\nabla_{X_j}\nabla^H f, \nabla_{X_k}X_j)\}_{x_0} \\ &= X_k(X_j^2 f - (\nabla_{X_j}X_j)f)_{x_0} \end{aligned}$$

that is

$$\sum_j g_\theta(\nabla_{X_k}\nabla_{X_j}\nabla^H f, X_j)_{x_0} = X_k(\Delta_b f)_{x_0}. \tag{17}$$

Therefore (by (16)–(17))

$$\begin{aligned} &\sum_{j,k} g_\theta(\nabla_{X_j}\nabla_{X_k}\nabla^H f, X_j)_{x_0} X_k(f)_{x_0} \\ &\quad = \sum_k \{X_k(\Delta_b f)X_k f\}_{x_0} + \sum_{j,k}\{g_\theta(R(X_j, X_k)\nabla^H f, X_j)X_k f \\ &\qquad + 2\Omega(X_j, X_k) g_\theta(\nabla_T\nabla^H f, X_j)X_k f\}_{x_0} \\ &\quad = (\nabla^H f)(\Delta_b f)_{x_0} + \sum_j \{g_\theta(R(X_j, \nabla^H f)\nabla^H f, X_j) \\ &\qquad + 2g_\theta(X_j, J\nabla^H f) g_\theta(\nabla_T\nabla^H f, X_j)\}_{x_0} \\ &\quad = (\nabla^H f)(\Delta_b f)_{x_0} + \rho(\nabla^H f, \nabla^H f)_{x_0} + 2g_\theta(\nabla_T\nabla^H f, J\nabla^H f)_{x_0} \end{aligned}$$

where $\rho(X, Y) = \operatorname{trace}\{Z \mapsto R(Z, Y)X\}$. Then (by (15)) the identity (14) becomes

$$\begin{aligned} \frac{1}{2}\Delta_b(|\nabla^H f|^2) &= |\pi_H\nabla^2 f|^2 + (\nabla^H f)(\Delta_b f) + \rho(\nabla^H f, \nabla^H f) \\ &\quad + 2g_\theta(\nabla_T\nabla^H f, J\nabla^H f) + 2g_\theta(\nabla^H Tf, J\nabla^H f) \end{aligned}$$

which yields (6).

4. A Lower Bound on $-\lambda$ for $\lambda \in \operatorname{Spec}(\Delta_b)$ with $\operatorname{Eigen}(\Delta_b; \lambda) \cap \operatorname{Ker}(T) \neq (0)$

Let M be a compact strictly pseudoconvex CR manifold and θ a contact form on M with G_θ positive definite. Let $(u, v) = \int_M uv\Psi$ be the L^2 inner product on M and $\|u\| = (u, u)^{1/2}$ the L^2 norm. For any $f \in C^\infty(M)$ let $f_0 = T(f)$. We shall need the following two lemmas.

LEMMA 1.

$$\operatorname{div}(J\nabla^H f) = 2nf_0. \tag{18}$$

PROOF. Let $\{T_\alpha : 1 \le \alpha \le n\}$ be a local frame of $T_{1,0}(M)$, defined on $U \subseteq M$. Then $\nabla^H f = f^\alpha T_\alpha + f^{\bar\alpha} T_{\bar\alpha}$ on U, where $f^\alpha = g^{\alpha\bar\beta} f_{\bar\beta}$, $f_{\bar\beta} = T_{\bar\beta}(f)$ and $T_{\bar\beta} = \overline{T}_\beta$, hence

$$J\nabla^H f = i(f^\alpha T_\alpha - f^{\bar\alpha} T_{\bar\alpha}). \tag{19}$$

We wish to compute the divergence of the vector field (19). As Ψ is parallel with respect to ∇

$$\operatorname{div}(J\nabla^H f) = \operatorname{trace}\{T_A \mapsto \nabla_{T_A} J\nabla^H f\}$$

where $A \in \{0, 1, \dots, n, \bar 1, \dots, \bar n\}$ (with the convention $T_0 = T$). We set $f_{AB} = (\nabla^2 f)(T_A, T_B)$. Then

$$\nabla_{T_\beta} J\nabla^H f = i(f_\beta{}^\alpha T_\alpha - f_\beta{}^{\bar\alpha} T_{\bar\alpha})$$

where $f_\beta^\alpha = g^{\alpha\bar\gamma} f_{\beta\bar\gamma}$, etc., so that

$$\operatorname{div}(J\nabla^H f) = i(f_\alpha{}^\alpha - f_{\bar\alpha}{}^{\bar\alpha}). \tag{20}$$

The identities (12)–(13) furnish the commutation formula $f_{\alpha\bar\beta} = f_{\bar\beta\alpha} - 2ig_{\alpha\bar\beta} f_0$. In particular

$$f_{\bar\alpha}{}^{\bar\alpha} = f_\alpha{}^\alpha + 2inf_0 \tag{21}$$

hence (20) yields (18). Q.e.d.

LEMMA 2.

$$\int_M Lf\Psi = -4n\|f_0\|^2 + \int_M A(\nabla^H f, J\nabla^H f)\Psi, \tag{22}$$

Here $A(X, Y) = g_\theta(\tau X, Y)$ is the pseudohermitian torsion of (M, θ) and L is given by (7).

PROOF. By the very definition of Lf

$$\int_M Lf\Psi = J_1 - J_2$$

where

$$J_1 = \int_M (J\nabla^H f)(f_0)\Psi, \quad J_2 = \int_M (J\nabla_T \nabla^H f)(f)\Psi.$$

By Green's lemma and (18)

$$J_1 = \int_M \{\mathrm{div}(f_0 J\nabla^H f) - f_0\,\mathrm{div}(J\nabla^H f)\}\Psi$$
$$= -\int_M f_0\,\mathrm{div}(J\nabla^H f)\Psi = -2n\|f_0\|^2,$$
$$J_2 = \int_M \mathrm{div}(fJ\nabla_T\nabla^H f)\Psi - \int_M f\,\mathrm{div}(J\nabla_T\nabla^H f)\Psi$$
$$= -\int_M f\,\mathrm{div}(J\nabla_T\nabla^H f)\Psi.$$

Let us compute in local coordinates $\mathrm{div}(J\nabla_T\nabla^H f)$. According to the notations used in the proof of lemma 1, set $f_{ABC} = (\nabla^3 f)(T_A, T_B, T_C)$ where

$$(\nabla^3 f)(X, Y, Z) = (\nabla_X \nabla^2 f)(Y, Z)$$
$$= X((\nabla^2 f)(Y, Z)) - (\nabla^2 f)(\nabla_X Y, Z) - (\nabla^2 f)(Y, \nabla_X Z),$$

for any $X, Y, Z \in T(M)$. Then

$$J\nabla_T\nabla^H f = i(f_0{}^\alpha T_\alpha - f_0{}^{\bar\alpha} T_{\bar\alpha})$$

yields

$$\nabla_{T_\alpha}(J\nabla_T\nabla^H f) = i(f_{\alpha 0}{}^\beta T_\beta - f_{\alpha 0}{}^{\bar\beta} T_{\bar\beta})$$

so that (by $\nabla T = 0$)

$$\mathrm{div}(J\nabla_T\nabla^H f) = \mathrm{trace}\{T_A \mapsto \nabla_{T_A} J\nabla_T\nabla^H f\} = i(f_{\alpha 0}{}^\alpha - f_{\bar\alpha 0}{}^{\bar\alpha}), \tag{23}$$

where $f_{\alpha 0}{}^\beta = g^{\beta\bar\gamma} f_{\alpha 0\bar\gamma}$, etc. We need the third order commutation formula

$$f_{\bar\beta 0\alpha} = f_{\alpha 0\bar\beta} + 2ig_{\alpha\bar\beta} f_{00} + A^\gamma_{\bar\beta} f_{\alpha\gamma} - A^{\bar\gamma}_{\alpha} f_{\bar\beta\bar\gamma} + A^\gamma_{\bar\beta,\alpha} f_\gamma - A^{\bar\gamma}_{\alpha,\bar\beta} f_{\bar\gamma} \tag{24}$$

where $f_{00} = (\nabla^2 f)(T, T) = T^2(f)$. This follows from

$$(\nabla^3 f)(X, T, Y) - (\nabla^3 f)(Y, T, X)$$
$$= 2\Omega(X, Y) f_{00} - X(\tau(Y)f) + Y(\tau(X)f) + \tau([X, Y])f$$

i.e.

$$(\nabla^3 f)(X, T, Y) = (\nabla^3 f)(Y, T, X) + 2\Omega(X, Y) f_{00}$$
$$+ (\nabla^2 f)(Y, \tau(X)) - (\nabla^2 f)(X, \tau(Y)) - S(X, Y) f,$$

for any $X, Y \in H(M)$, where $S(X, Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$. Indeed we may set $X = T_\alpha$ and $Y = T_{\bar\beta}$ in the previous identity and observe that $S(T_\alpha, T_{\bar\beta}) = A^{\gamma}_{\bar\beta,\alpha} T_\gamma - A^{\bar\gamma}_{\alpha,\bar\beta} T_{\bar\gamma}$ where $\tau(T_\alpha) = A^{\bar\beta}_{\alpha} T_{\bar\beta}$ and the covariant derivatives $A^{\gamma}_{\bar\beta,\alpha}$ are given by $(\nabla_{T_\alpha}\tau)T_{\bar\beta} = A^{\gamma}_{\bar\beta,\alpha} T_\gamma$. The identity (24) leads to

$$f_{\bar\alpha 0}{}^{\bar\alpha} = f_{\alpha 0}{}^{\alpha} + 2inf_{00} + A^{\alpha\beta} f_{\alpha\beta} - A^{\bar\alpha\bar\beta} f_{\bar\alpha\bar\beta} + A^{\alpha\beta}{}_{,\alpha} f_\beta - A^{\bar\alpha\bar\beta}{}_{,\bar\alpha} f_{\bar\beta}$$

hence (23) becomes

$$\operatorname{div}(J\nabla_T \nabla^H f) = 2nf_{00} - i(A^{\alpha\beta} f_{\alpha\beta} - A^{\bar\alpha\bar\beta} f_{\bar\alpha\bar\beta} + A^{\alpha\beta}{}_{,\alpha} f_\beta - A^{\bar\alpha\bar\beta}{}_{,\bar\alpha} f_{\bar\beta}).$$

Therefore

$$J_2 = -2n \int ff_{00}\Psi + i \int f(A^{\alpha\beta} f_{\alpha\beta} - A^{\bar\alpha\bar\beta} f_{\bar\alpha\bar\beta} + A^{\alpha\beta}{}_{,\alpha} f_\beta - A^{\bar\alpha\bar\beta}{}_{,\bar\alpha} f_{\bar\beta})\Psi$$

where

$$\begin{aligned} \int_M ff_{00} &= \int_M fT(f_0)\Psi = \int_M \{T(ff_0) - f_0^2\}\Psi \\ &= \int_M \{\operatorname{div}(ff_0 T) - ff_0 \operatorname{div}(T)\}\Psi - \|f_0\|^2 \end{aligned}$$

hence (by $\operatorname{div}(T) = 0$)

$$\int_M ff_{00}\Psi = -\|f_0\|^2.$$

On the other hand $\operatorname{div}(Z^\alpha T_\alpha) = Z^\alpha{}_{,\alpha}$ hence (by Green's lemma)

$$\begin{aligned} \int_M fA^{\alpha\beta}{}_{,\alpha} f_\beta &= \int_M \{(ff_\beta A^{\alpha\beta})_{,\alpha} - A^{\alpha\beta} f_\alpha f_\beta - fA^{\alpha\beta} f_{\beta,\alpha}\}\Psi \\ &= -\int_M (A^{\alpha\beta} f_\alpha f_\beta + fA^{\alpha\beta} f_{\beta,\alpha})\Psi \end{aligned}$$

where $f_{\beta,\alpha} = (\nabla_{T_\alpha} df)T_\beta = f_{\alpha\beta}$. Hence

$$J_2 = 2n\|f_0\|^2 + i \int_M (A^{\bar\alpha\bar\beta} f_{\bar\alpha} f_{\bar\beta} - A^{\alpha\beta} f_\alpha f_\beta)\Psi$$

and then (by $A(\nabla^H f, J\nabla^H f) = i(A_{\alpha\beta} f^\alpha f^\beta - A_{\bar\alpha\bar\beta} f^{\bar\alpha} f^{\bar\beta})$) we may conclude that

$$J_2 = 2n\|f_0\|^2 - \int_M A(\nabla^H f, J\nabla^H f)\Psi$$

so Lemma 2 is proved.

Let us prove Theorem 1. Note that $(\nabla^H f)(f) = |\nabla^H f|^2$. Let λ be an eigenvalue of Δ_b and f an eigenfunction corresponding to λ such that $T(f) = 0$. Then (6) becomes

$$\frac{1}{2}\Delta_b(|\nabla^H f|^2) = |\pi_H \nabla^2 f|^2 + \lambda |\nabla^H f|^2 + \rho(\nabla^H f, \nabla^H f) + 2Lf.$$

Let us integrate over M and use Green's lemma, Lemma 2 and the assumptions (i)–(ii) in Theorem 1 to get

$$\begin{aligned} 0 &= \|\pi_H \nabla^2 f\|^2 + \lambda \|\nabla^H f\|^2 + \int_M \{\rho(\nabla^H f, \nabla^H f) + 2A(\nabla^H f, J\nabla^H f)\}\Psi \\ &\geq \|\pi_H \nabla^2 f\|^2 + (\lambda + k)\|\nabla^H f\|^2 \end{aligned}$$

that is

$$0 \geq \|\pi_H \nabla^2 f\|^2 + (\lambda + k)\|\nabla^H f\|^2. \tag{25}$$

Once again, as f is an eigenfunction

$$\begin{aligned} \|\Delta_b f\|^2 &= \int_M |\Delta_b f|^2 \Psi = \lambda \int_M f \Delta_b f \Psi = \lambda \int_M f \operatorname{div}(\nabla^H f)\Psi \\ &= \lambda \int_M \{\operatorname{div}(f\nabla^H f) - (\nabla^H f)(f)\}\Psi \end{aligned}$$

that is

$$\|\Delta_b f\|^2 = -\lambda \|\nabla^H f\|^2. \tag{26}$$

Next (with the notations in Section 2) we set

$$v_j = ((\nabla^2 f)(X_j, X_1), \ldots, (\nabla^2 f)(X_j, X_{2n})), \quad 1 \leq j \leq 2n,$$

so that

$$|\pi_H \nabla^2 f|^2 = \sum_{j,k} (\nabla^2 f)(X_j, X_k)^2 = \sum_j |v_j|^2 = |w|^2$$

where $w = (|v_1|, \ldots, |v_{2n}|)$ (and $|v_j|$, $|w|$ are the Euclidean norm of v_j, w). By the Cauchy-Schwarz inequality

$$\begin{aligned} |\pi_H \nabla^2 f|^2 = |w|^2 &\geq \frac{1}{2n}|w \cdot (1, \ldots, 1)|^2 \\ &= \frac{1}{2n}\left(\sum_j |v_j|\right)^2 \geq \frac{1}{2n}\left(\sum_j |(\nabla^2 f)(X_j, X_j)|\right)^2 \end{aligned}$$

hence

$$|\pi_H \nabla^2 f|^2 \geq \frac{1}{2n}(\Delta_b f)^2. \tag{27}$$

Finally, by (26)–(27) the inequality (25) becomes (whenever $\|\Delta_b f\| \neq 0$, that is $\lambda \neq 0$)

$$0 \geq \left(\frac{1}{2n} - \frac{\lambda + k}{\lambda}\right)\|\Delta_b f\|^2$$

to conclude that $-\lambda \geq 2nk/(2n-1)$. Q.e.d.

We close the section with the following remark on assumption (ii) in Theorem 1. The problem whether $\mathrm{Eigen}(\Delta_b; \lambda) \cap \mathrm{Ker}(T) \neq \emptyset$ is in general open. Nevertheless if $M = S^{2n+1}$ is the standard sphere then eigenfunctions $f \in \mathrm{Eigen}(\Delta_b; -4(n+1))$ with $T(f) = 0$ may be easily produced (here $-4(n+1)$ is the second nonzero eigenvalue of the ordinary Laplacian on S^{2n+1}). Indeed let Δ be the Laplace-Beltrami operator of (S^{2n+1}, g_{θ_0}). As well known (cf. e.g. [3]) $\Delta v = -\ell(\ell + 2n)v$, where v is the restriction to S^{2n+1} of a harmonic polynomial $H \in \mathscr{H}_\ell$ (here $\mathscr{H}_\ell$ is the space of harmonic, i.e. $\Delta_{\boldsymbol{R}^{2n+2}} H = 0$, polynomials $H : \boldsymbol{R}^{2n+2} \to \boldsymbol{R}$ which are homogeneous of degree ℓ) and the whole spectrum of Δ on S^{2n+1} may be obtained this way. Note that $\mathscr{H}_2$ consists of all $H = \sum_{i,j}(a_{ij}x^i x^j + b_{ij}x^i y^j + c_{ij}y^i y^j)$ with $\sum_i (a_{ii} + c_{ii}) = 0$. For the sphere $(d\imath)T = T_0$ where $\imath : S^{2n+1} \to \boldsymbol{C}^{n+1}$ is the inclusion while $T_0 = x^j \partial/\partial y^j - y^j \partial/\partial x^j$ and (x^j, y^j) are the natural coordinates on $\boldsymbol{C}^{n+1} \approx \boldsymbol{R}^{2n+2}$, hence

$$\mathscr{H}_2 \cap \mathrm{Ker}(T_0) = \left\{ H = \sum_{i,j} a_{ij}(x^i x^j + y^i y^j) : \sum_i a_{ii} = 0 \right\}. \tag{28}$$

Finally, by a formula of A. Greenleaf (cf. *op. cit.*)

$$\Delta_b = \Delta - T^2 \tag{29}$$

hence $-4(n+1) \in \mathrm{Spec}(\Delta_b)$ and $(0) \neq \mathrm{Eigen}(\Delta; -4(n+1)) \cap \mathrm{Ker}(T) \subseteq \mathrm{Eigen}(\Delta_b; -4(n+1))$. On the other hand note that $\mathscr{H}_1 \cap \mathrm{Ker}(T_0) = (0)$. So the eigenfunctions of Δ_b we consider (cf. (28) above) are spherical harmonics of degree 2. However $4(n+1)$ is greater equal than minus the third eigenvalue of Δ_b (cf. Proposition 3 below). See also our Appendix A for a short proof of (29).

5. Consequences of $-2nk/(2n-1) \in \mathrm{Spec}(\Delta_b)$

Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite. We recall a few concepts from *sub-Riemannian* *geometry* (cf. e.g. R. S. Strichartz, [23]) on a strictly pseudoconvex CR manifold. Let $x \in M$ and $g(x) : T_x^*(M) \to H(M)_x$ determined by

$$G_{\theta,x}(v, g(x)\xi) = \xi(v), \quad v \in H(M)_x, \quad \xi \in T_x^*(M).$$

Note that the kernel of g is precisely the conormal bundle

$$H(M)_x^{\perp} = \{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\}, \quad x \in M.$$

That is G_θ is a *sub-Riemannian metric* on $H(M)$ and g its alternative description (cf. (2.1) in [23], p. 225).

Let $\gamma : I \to M$ be a piecewise C^1 curve (where $I \subseteq \boldsymbol{R}$ is an interval). Then γ is a *lengthy curve* if $\dot\gamma(t) \in H(M)_{\gamma(t)}$ for every $t \in I$ such that $\dot\gamma(t)$ is defined. A piecewise C^1 curve $\xi : I \to T^*(M)$ is a *cotangent lift* of γ if $\xi(t) \in T_{\gamma(t)}^*(M)$ and $g(\gamma(t))\xi(t) = \dot\gamma(t)$ for every t (where defined). The *length* of a lengthy curve $\gamma : I \to M$ in sub-Riemannian geometry

$$L(\gamma) = \int_I \{\xi(t)[g(\gamma(t))\xi(t)]\}^{1/2}\, dt = \int_I G_{\theta,\gamma(t)}(\dot\gamma(t), \dot\gamma(t))^{1/2}$$

coincides with the Riemannian length of γ as a curve in (M, g_θ). The *Carnot-Carathéodory distance* $\rho(x, y)$ among $x, y \in M$ is the infimum of the lengths of all lengthy curves joining x and y. By a well known theorem of W. L. Chow, [6], any two points $x, y \in M$ may be joined by a lengthy curve (and one may easily check that ρ is a distance function on M).

Let g_θ be the Webster metric of (M, θ). Then g_θ is a *contraction* of the sub-Riemannian metric G_θ (G_θ is an *expansion* of g_θ) i.e.

$$d(x, y) \le \rho(x, y), \quad x, y \in M. \tag{30}$$

(cf. [23], p. 230) where d is the distance function corresponding to the Webster metric. Although ρ and d are inequivalent distance functions, they determine the same topology. A first step towards recovering M. Obata's arguments (cf. [21]) is the following

THEOREM 2. *Let (M, θ) be a compact strictly pseudoconvex CR manifold of CR dimension n, such that $\rho(X, X) + 2A(X, JX) \ge kG_\theta(X, X)$ for some $k > 0$ and any $X \in H(M)$. Assume that $\lambda \equiv -2nk/(2n-1) \in \mathrm{Spec}(\Delta_b)$ and $\mathscr{H} \equiv \mathrm{Eigen}(\Delta_b; \lambda) \cap \mathrm{Ker}(T) \neq (0)$. Then any eigenfunction $f \in \mathscr{H}$ is given by*

$$f(\gamma(s)) = \alpha \cos(s\sqrt{c}), \quad s \in \boldsymbol{R},\ c = k/(2n-1), \tag{31}$$

along each lengthy geodesic $\gamma : \boldsymbol{R} \to M$ *of the Tanaka-Webster connection* ∇ *such that* $|\dot{\gamma}(s)| = 1$ *and* $\gamma(0) = x_0$, *where* $x_0 \in M$ *is a point such that* $f(x_0) = \sup_{x \in M} f(x) \equiv \alpha$.

Assume additionally that (M, θ) *is Sasakian* ($\tau = 0$). *If any two points of* M *can be joined by a Carnot-Carathéodory minimizing lengthy geodesic then* $f(x) = \alpha \cos(r(x)\sqrt{c})$, $x \in M$, *where* $r(x) = \rho(x_0, x)$ *is the Carnot-Carathéodory distance from* x_0. *If* $y_0 \in M$ *is a point such that* $f(y_0) = \inf_{x \in M} f(x)$ *then* $f(y_0) = -\alpha$. *Consequently* $M_{\pi/\sqrt{c}}$ *consists solely of critical points of* f *and each* $x \in M_{\pi/\sqrt{c}}$ *is degenerate.*

Here, for a given $s \in \boldsymbol{R}$ we let M_s consist of all points $x \in M$ such that there is a lengthy geodesic $\gamma : \boldsymbol{R} \to M$ of ∇, parametrized by arc length, such that $\gamma(0) = x_0$ and $\gamma(s) = x$. The assumptions in Theorem 2 are rather restrictive and, among all odd dimensional spheres, are satisfied only on S^3 (thus motivating the conjecture in the Introduction). Precisely

PROPOSITION 1. *Let* $M = S^{2n+1}$ *with the standard contact form* $\theta = \frac{i}{2}(\bar{\partial} - \partial)|z|^2$. *If* i) *the inequality* (8) *is satisfied for some* $k > 0$, ii) $-2nk/(2n-1) \in \mathrm{Spec}(\Delta_b)$, *and* iii) $\mathrm{Eigen}(\Delta_b; -2nk/(2n-1)) \cap \mathrm{Ker}(T) \neq (0)$, *then* $k = 4$ *and* $n = 1$. *Conversely the statements* i)–iii) *hold on* S^3. *Moreover if* $M = S^3$ *and*

$$H = a(x_1^2 + y_1^2 - x_2^2 - y_2^2) + 2b(x_1 x_2 + y_1 y_2), \quad f = H|_{S^3}, \quad b \neq 0,$$

(*a spherical harmonic of degree* 2 *on* S^3 *such that* $T(f) = 0$) *then* $\alpha = \sup_{x \in S^3} f(x) = \sqrt{a^2 + b^2}$ *and* $f(\gamma(s)) = \alpha \cos(2s)$ *for any lengthy geodesic* $\gamma : \boldsymbol{R} \to S^3$ *of* ∇ (*the Tanaka-Webster connection of* S^3) *parametrized by arc length and such that* $\gamma(0)$ *is a maximum point of* f. *Moreover*

$$M_{\pi/2} = \left\{ \left(\lambda, \mu, -\frac{b\lambda}{\alpha - a}, -\frac{b\mu}{\alpha - a} \right) : \lambda^2 + \mu^2 = \frac{\alpha - a}{2\alpha}, \lambda, \mu \in \boldsymbol{R} \right\} \tag{32}$$

consists solely of degenerate critical points of f.

The proof of Proposition 1 is relegated to Appendix A.

PROOF OF THEOREM 2. Assume that $\lambda = -2nk/(2n-1)$ is an eigenvalue of Δ_b and let $f \in \mathscr{H}$ be an eigenfunction of Δ_b corresponding to λ such that $T(f) = 0$. By the Bochner type formula (6) one has

$$\frac{1}{2}\Delta_b(|\nabla^H f|^2) = |\pi_H \nabla^2 f|^2 + \lambda |\nabla^H f|^2 + \rho(\nabla^H f, \nabla^H f) + 2Lf.$$

Once again we integrate and use Lemma 2 and the assumption (8). We get

$$0 \geq \|\pi_H \nabla^2 f\|^2 - \left(1 + \frac{k}{\lambda}\right) \|\Delta_b f\|^2 = \|\pi_H \nabla^2 f\|^2 - \frac{1}{2n} \|\Delta_b f\|^2 \geq 0$$

(the last inequality is a consequence of (27)) hence

$$\int_M \left\{ |\pi_H \nabla^2 f|^2 - \frac{1}{2n} (\Delta_b f)^2 \right\} \Psi = 0$$

so that (again by (27))

$$|\pi_H \nabla^2 f|^2 = \frac{1}{2n} (\Delta_b f)^2. \tag{33}$$

The following lemma of linear algebra is well known. If $A \in \boldsymbol{R}^{m^2}$ satisfies $m|A|^2 = \mathrm{trace}(A)^2$ then $A = (1/m)\,\mathrm{trace}(A) I_m$, where I_m is the unit matrix of order m. Therefore (by (33))

$$\pi_H \nabla^2 f = \frac{1}{2n} (\Delta_b f) G_\theta.$$

In particular the identities (12)–(13) are consistent with our assumption that $f_0 = 0$. Using again $\Delta_b f = \lambda_1 f$ we may conclude that

$$\pi_H \nabla^2 f = -c f G_\theta, \tag{34}$$

where $c = k/(2n-1)$.

M. Obata's proof (cf. *op. cit.*) of the fact that equality in (1) yields $M^m \approx S^m$ (an isometry) is an indication that we should evaluate (34) along a lengthy geodesic of the Tanaka-Webster connection, and integrate the resulting ODE. Let us recall briefly the needed material on geodesics (as developed in [2]). Let $(U, x^1, \ldots, x^{2n+1})$ be a system of local coordinates on M and let us set $g\, dx^i = g^{ij} \partial_j$, where $\partial_i = \partial / \partial x^i$. A *sub-Riemannian geodesic* is a C^1 curve $\gamma(t)$ in M satisfying the Hamilton-Jacobi equations associated to the Hamiltonian function $H(x, \xi) = \frac{1}{2} g^{ij}(x) \xi_i \xi_j$ that is

$$\frac{dx^i}{dt} = g^{ij}(\gamma(t)) \xi_j(t), \tag{35}$$

$$\frac{d\xi_k}{dt} = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} (\gamma(t)) \xi_i(t) \xi_j(t), \tag{36}$$

for some cotangent lift $\xi(t) \in T^*(M)$ of $\gamma(t)$. Let $\gamma(t) \in M$ be a sub-Riemannian geodesic and $s = \phi(t)$ a C^1 diffeomorphism. As shown in [2], if $\gamma(t) = \bar{\gamma}(\phi(t))$ then

$\bar{\gamma}(s)$ is a sub-Riemannian geodesic if and only if ϕ is affine, i.e. $\phi(t) = \alpha t + \beta$, for some $\alpha, \beta \in \boldsymbol{R}$. In particular, every sub-Riemannian geodesic may be reparametrized by arc length $\phi(t) = \int_0^t |\dot{\gamma}(u)|\, du$. In [2] we introduced a *canonical* cotangent lift of a given lengthy curve $\gamma : I \to M$ by setting

$$\xi : I \to T^*(M), \quad \xi(t)T_{\gamma(t)} = 1, \quad \xi(t)X = g_\theta(\dot{\gamma}, X),$$

for any $X \in H(M)_{\gamma(t)}$, and showed that

THEOREM 3. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite. A C^1 curve $\gamma(t) \in M$, $|t| < \varepsilon$, is a sub-Riemannian geodesic of $(M, H(M), G_\theta)$ if and only if $\gamma(t)$ is a solution to*

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -2b(t)J\dot{\gamma}, \quad b'(t) = A(\dot{\gamma}, \dot{\gamma}), \quad |t| < \varepsilon, \tag{37}$$

with $\dot{\gamma}(0) \in H(M)_{\gamma(0)}$, for some C^1 function $b : (-\varepsilon, \varepsilon) \to \boldsymbol{R}$.

R. S. Strichartz's paper [23] manifestly doesn't involve any elements of connection theory or curvature. As argued by R. S. Strichartz (cf. *op. cit.*) curvature is a measurement of the deviation of the given Riemannian manifold from its Euclidean model (and sub-Riemannian manifolds exhibit no approximate Euclidean behavior). Nevertheless, in view of Theorem 2 when (M, θ) is a Sasakian manifold (i.e. $\tau = 0$) the lengthy geodesics of ∇ are among the sub-Riemannian geodesics and it is likely that a variational theory of the geodesics of ∇ (as started in [2]) is the key step towards bringing the results of [21] to CR geometry.

Our approach (based on ∇) is not in contradiction with the arguments in [23]: indeed the curvature of ∇ is related to the pseudoconvexity properties of M (as understood in complex analysis in several variables) rather than to its intrinsic shape. To emphasize the impact of connection theory within our approach we may prove the following elementary regularity result. Note that a sub-Riemannian geodesic is required to be of class C^2 (cf. [23], p. 233) and no higher regularity is expected *a priori*. In turn, any C^1 geodesic of ∇ is automatically of class C^∞ [as a projection on M of an integral curve of some standard horizontal vector field (cf. Prop. 6.3 in [14], Vol. I, p. 139) having C^∞ coefficients].

Let $\gamma(t) \in M$ be a lengthy geodesic of the Tanaka-Webster connection, parametrized by arc-length ($|\dot{\gamma}(t)| = 1$). Then (by (34))

$$\frac{d^2(f \circ \gamma)}{dt^2} = -cf \circ \gamma$$

hence $f(\gamma(t)) = A\cos(t\sqrt{c}) + B\sin(t\sqrt{c})$. As M is compact there is $x_0 \in M$ such that $f(x_0) = \sup_{x \in M} f(x) =: \alpha$. Let $\gamma(t)$ be a lengthy geodesic of ∇ such that

$\gamma(0) = x_0$. Then $A = \alpha$ and $\{d(f \circ \gamma)/dt\}_{t=0} = 0$ yields $B = 0$ so that $f(\gamma(t)) = \alpha \cos(t\sqrt{c})$, which is (31).

Again by compactness (M, ρ) is a complete metric space, hence (cf. Theorem 7.1 in [23], p. 244) any sub-Riemannian geodesic can be extended indefinitely. Since $\tau = 0$ the statements about sub-Riemannian geodesics in [23] apply to the lengthy geodesics of ∇ as well. Let $\gamma : \boldsymbol{R} \to M$ be a lengthy geodesic of ∇ such that $|\dot{\gamma}(s)| = 1$, $\gamma(0) = x_0$ and $\gamma(s_{\min}) = y_0$. By (31)

$$0 = \frac{d}{ds}\{f \circ \gamma\}_{s=s_{\min}} = -\alpha\sqrt{c}\,\sin(\sqrt{c}s_{\min})$$

hence $s_{\min} = m\pi/\sqrt{c}$ for some $m \in \mathbf{Z}$. Then $\alpha > f(y_0) = (-1)^m \alpha$ implies that m is odd. Again by (31), $M_{\pi/\sqrt{c}} \subset f^{-1}(-\alpha)$. Finally, let $x \in M_{\pi/\sqrt{c}}$ and (U, x^i) a local coordinate system on M such that $x \in U$. As $T(f) = 0$

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(x) T^i(x) T^j(x) = (\nabla^2 f)(T, T)_x = T(f_0)_x = 0$$

hence x is a degenerate critical point. Therefore, the points of $M_{\pi/\sqrt{c}}$ may fail to be isolated. Nevertheless

Proposition 2. *Let (M, θ) be a compact Sasakian manifold. If for any $x \in B(x_0, \pi/\sqrt{c})$ there is a length minimizing (with respect to the Carnot-Carathéodory distance) lengthy geodesic joining x_0 and x then the exponential map $\exp_{x_0} : N(x_0, \pi/\sqrt{c}) \to B(x_0, \pi/\sqrt{c})$ (with respect to the Tanaka-Webster connection) is a surjection.*

Here $B(x_0, R) = \{x \in M : \rho(x_0, x) < R\}$ is the Carnot-Carathéodory ball of center x_0 and radius $R > 0$. Also $N(x_0, R) = \{w \in H(M)_{x_0} : |w| < R\}$.

Proof of Proposition 2. To see that the restriction of $\exp_{x_0}$ to $N(x_0, \pi/\sqrt{c})$ is indeed $B(x_0, \pi/\sqrt{c})$-valued let $w \in N(x_0, \pi/\sqrt{c})$, $w \neq 0$, and $t = |w|$. Let us set $v = (1/t)w$ and consider the geodesic $\gamma : \boldsymbol{R} \to M$ of ∇ with the initial data $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v$, so that

$$\exp_{x_0}(w) = \exp_{x_0}(tv) = \gamma(t).$$

Then

$$\rho(x_0, \gamma(t)) \leq \int_0^t |\dot{\gamma}(s)|\, ds = t < \pi/\sqrt{c}$$

i.e. $\gamma(t) \in B(x_0, \pi/\sqrt{c})$.

To see that $\exp_{x_0} : N(x_0, \pi/\sqrt{c}) \to B(x_0, \pi/\sqrt{c})$ is on-to let $x \in B(x_0, \pi/\sqrt{c})$ and let $\gamma : \boldsymbol{R} \to M$ be a length minimizing geodesic joining x_0 and x and such that $\dot{\gamma}(0) = v \in H(M)_{x_0}$, with $|v| = 1$. Then $\gamma(t) = x$ for some $t \in \boldsymbol{R}\backslash\{0\}$, so that $\exp_{x_0}(tv) = x$. Next

$$g_{\theta, x_0}(tv, tv) = t^2|v|^2 = r(x)^2 < \pi^2/c$$

i.e. $tv \in N(x_0, \pi/\sqrt{c})$. Q.e.d.

A generalization of the Lichnerowicz-Obata theorem ([18], [21]) to the case of Riemannian foliations was obtained by J. M. Lee & K. Richardson, [17] (see also [16]). The leaf space of a Riemannian foliation is often an orbifold (for instance if all leaves are compact) so that (in light of [9]) one expects analogs to Theorems 1 and 2 on a CR orbifold (see also E. Stanhope, [22]). This matter will be addressed in a further paper.

Appendix A. On the Spectrum of the Sublaplacian on the Standard Sphere

Let M be a strictly pseudoconvex CR manifold and θ a contact form on M with G_θ positive definite. Let ∇^θ be the Levi-Civita connection of the semi-Riemannian manifold (M, g_θ). Then (cf. e.g. [11], Chapter 1)

$$\nabla^\theta = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \odot J \tag{38}$$

where $\odot$ is the symmetric tensor product. Then

$$\nabla^\theta_X X = \nabla_X X - A(X, X)T, \quad X \in H(M). \tag{39}$$

Given a local G_θ-orthonormal frame $\{X_a : 1 \leq a \leq 2n\}$ of $H(M)$ one has (by (39) and $\mathrm{trace}(\tau) = 0$)

$$\Delta f = \sum_{j=0}^{2n}\{X_j(X_j f) - (\nabla^\theta_{X_j} X_j)(f)\} = T(T(f)) + \Delta_b f$$

for any $f \in C^\infty(M)$, where $X_0 = T$, proving Greenleaf's formula (29). Let $\mathscr{P}_\ell$ be the set of all homogeneous polynomials $H : \boldsymbol{R}^{2n+2} \to \boldsymbol{R}$ of degree $\deg(H) = \ell$ and $\mathscr{H}_\ell = \mathscr{P}_\ell \cap \mathrm{Ker}(\Delta_{\boldsymbol{R}^{2n+2}})$. To compute eigenvalues of Δ_b starting from $\mathrm{Spec}(\Delta)$ we consider the equation

$$\Delta_b f + T^2(f) = -\ell(2n + \ell)f \tag{40}$$

with $f = H|_{S^{2n+1}}$ and $H \in \mathscr{H}_\ell$. For example if $\ell = 1$ and $H \in \mathscr{H}_1 = \mathscr{P}_1$ then $T_0^2(H) = -H$ hence $-2n \in \mathrm{Spec}(\Delta_b)$. In general

PROPOSITION 3. *If there is* $\lambda \in \boldsymbol{R}$ *and* $\mathscr{H}_\ell \cap \mathrm{Ker}(T_0^2 + \lambda I) \neq (0)$ *then* $\lambda - \ell(2n+\ell) \in \mathrm{Spec}(\Delta_b)$. *For instance one may produce the eigenvalues* $\{-2n, -4n, -6n-8, -6n\} \subset \mathrm{Spec}(\Delta_b)$ *and* $\mathrm{Eigen}(\Delta_b; -2n) \cap \mathscr{P}_1 \neq (0)$, $\mathrm{Eigen}(\Delta_b; -4n) \cap \mathscr{P}_2 \neq (0)$ *and* $\mathrm{Eigen}(\Delta_b; \lambda) \cap \mathscr{P}_3 \neq (0)$ *for each* $\lambda \in \{-6n-8, -6n\}$.

If $H = \sum_{i,j=1}^{n+1} (a_{ij}x^i x^j + b_{ij} x^i y^j + c_{ij} y^i y^j) \in \mathscr{P}_2$ (with $a_{ij}, b_{ij} \in \boldsymbol{R}$, $a_{ji} = a_{ij}$, $c_{ji} = c_{ij}$) then $T_0^2 H = -\lambda H$ if and only if $2(c_{ij} - a_{ij}) = -\lambda a_{ij}$, $2(b_{ij} + b_{ji}) = \lambda b_{ij}$ and $2(c_{ij} - a_{ij}) = \lambda c_{ij}$. Hence $\mathrm{Ker}(T_0^2 + \lambda I) \cap \mathscr{P}_2 = (0)$ for any $\lambda \in \boldsymbol{R} \backslash \{4\}$ and

$$\mathrm{Ker}(T_0^2 + 4I) \cap \mathscr{P}_2 = \{a_{ij}(x^i x^j - y^i y^j) + b_{ij} x^i y^j : a_{ij}, b_{ij} \in \boldsymbol{R}, a_{ij} = a_{ji}\} \subset \mathscr{H}_2.$$

Similarly $\mathrm{Ker}(T_0^2 + \lambda I) \cap \mathscr{P}_3 = (0)$ for any $\lambda \in \boldsymbol{R} \backslash \{1, 9\}$ and

$$\mathrm{Ker}(T_0^2 + I) \cap \mathscr{H}_3 = \left\{ (a_{ijk} x^i + b_{ijk} y^i)(x^j x^k + y^j y^k) : a_{ijk}, b_{ijk} \in \boldsymbol{R} \text{ symmetric}, \right.$$
$$\left. \sum_j a_{ijj} = \sum_j b_{ijj} = 0, 1 \leq i \leq n+1 \right\},$$

$$\mathrm{Ker}(T_0^2 + 9I) \cap \mathscr{H}_3 = \left\{ a_{ijk} x^i (x^j x^k - 3 y^j y^k) + b_{ijk}(y^i y^j - 3 x^i x^j) y^k : a_{ijk}, b_{ijk} \in \boldsymbol{R} \right.$$
$$\left. \text{symmetric}, \sum_j a_{ijj} = \sum_j b_{ijj} = 0, 1 \leq i \leq n+1 \right\}.$$

Proposition 3 is proved. The calculation of the full $\mathrm{Spec}(\Delta_b)$ on S^{2n+1} is an open problem.

PROOF OF PROPOSITION 1. Let R be the curvature of the Tanaka-Webster connection. Then (cf. Chapter 1 in [11])

$$\begin{aligned} R(X,Y)Z &= G_\theta(Y,Z)X - G_\theta(X,Z)Y \\ &\quad + G_\theta(JY,Z)JX - G_\theta(JX,Z)JY - 2G_\theta(JX,Y)JZ \end{aligned} \tag{41}$$

for any $X, Y, Z \in H(S^{2n+1})$. Taking the trace in (41) we obtain

$$\rho(X,X) = 2(n+1) G_\theta(X,X). \tag{42}$$

The assumptions i)–ii) imply that $-2nk/(2n-1)$ is an eigenvalue of the ordinary Laplacian on S^{2n+1}. On the other hand $\mathscr{H}_1 \cap \mathrm{Ker}(T_0) = (0)$ hence $2nk/(2n-1)$ is greater equal than $4(n+1)$. Finally (by (42)) $k \leq 2(n+1)$ hence $n = 1$ and $k = 4$.

Let ∇^θ be the Levi-Civita connection of S^3. As S^3 is a Sasakian manifold $\nabla^\theta_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + 2\theta(\dot\gamma)J\dot\gamma$ (by (38)) for any C^1 curve $\gamma(t)$ in S^3. In particular any lengthy geodesic γ of ∇ is a geodesic of S^3 as well. Moreover any geodesic γ of ∇ with $\dot{\gamma}(0) \in H(S^3)_{\gamma(0)}$ is lengthy. Indeed (as $\nabla T = 0$)

$$\frac{d}{dt}\{\theta_{\gamma(t)}(\dot{\gamma}(t))\} = g_\theta(\dot{\gamma}, \nabla_{\dot{\gamma}} T)_{\gamma(t)} = 0$$

hence $\theta(\dot{\gamma})_{\gamma(t)} = \theta_{\gamma(0)}(\dot{\gamma}(0)) = 0$. Let $x_0 \in S^3$ such that $\alpha = f(x_0)$. Let γ be a lengthy geodesic of ∇, parametrized by arc length, such that $\gamma(0) = x_0$. Then $\gamma(s) = x_0 \cos s + x \sin s$, $s \in \boldsymbol{R}$, for some $x \in \boldsymbol{R}^4$ such that $\|x\| = 1$ and $\langle x_0, x\rangle = 0$. If $U = S^3 \setminus \{x_2 = y_2 = 0\}$ the Levi distribution $H(S^3)|_U$ is spanned by

$$X = \frac{\partial}{\partial x_1} - F\frac{\partial}{\partial x_2} - G\frac{\partial}{\partial y_2}, \quad Y = \frac{\partial}{\partial y_1} + G\frac{\partial}{\partial x_2} - F\frac{\partial}{\partial y_2},$$

$$F(x, y) = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \quad G(x, y) = \frac{x_1 y_2 - y_1 x_2}{x_2^2 + y_2^2},$$

hence the condition that γ is lengthy reads

$$Q^j \frac{\partial}{\partial x^j}\bigg|_{x_0} + R^j \frac{\partial}{\partial y^j}\bigg|_{x_0} = \lambda X_{x_0} + \mu Y_{x_0}$$

for some $\lambda, \mu \in \boldsymbol{R}$, where $Q^j = x^j(x)$ and $R^j = y^j(x)$, $j \in \{1, 2\}$, or

$$Q^1 = \lambda, \quad Q^2 = \mu G(x_0) - \lambda F(x_0), \tag{43}$$

$$R^1 = \mu, \quad R^2 = -\mu F(x_0) - \lambda G(x_0). \tag{44}$$

Let us set $P^j = x^j(x_0)$ and $S^j = y^j(x_0)$. The solution to the constrained extreme value problem $\alpha = \sup_{x \in S^3} f(x)$ is $\alpha = \sqrt{a^2 + b^2}$ and

$$P^1 = \xi, \quad S^2 = \eta, \quad P^2 = A\xi, \quad S^2 = A\eta, \quad \xi^2 + \eta^2 = \frac{\alpha - a}{2\alpha},$$

where $A = (\alpha - a)/b$, hence $F(x_0) = b/(\alpha - a)$ and $G(x_0) = 0$. Finally $\|x\| = 1$ may be written $\lambda^2 + \mu^2 = (\alpha - a)/(2\alpha)$ hence (43)–(44) yield (32) in Proposition 1.

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