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A-Section

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Abstract

Let $\{X_t\}$ be defined by $X_t = \theta X_{t-1} + U_t$ $(t = 1, 2, \ldots)$, where $\{U_t\}$ is a sequence of independently, identically and normally distributed random variables with mean 0 and variance 1 and X_0 is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each t X_0 is independent of U_t. We assume that $|\theta| < 1$ and consider the maximum likelihood estimator (MLE) of θ based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ in which X_T is missing. It is shown that the bias-adjusted MLE is second order asymptotically efficient. When in the above autoregressive process we assume that $X_0 = 0$, the asymptotic deficiency of the MLE is given.

1. Introduction.

In the first order autoregressive (AR) processes the first order and the second order asymptotic efficiency of the MLE was discussed by Akahira [1], [2], [3], [4]. The first order asymptotic efficiency was extended by Kabaila [9] to an autoregressive moving average (ARMA) process when the innovations are not necessarily Gaussian and the second order asymptotic efficiency was done by Taniguchi [11] to a Gaussian ARMA process.

In this paper we consider an AR process $\{X_t\}$ which is defined by $X_t = \theta X_{t-1} + U_t(t = 1, 2, \ldots)$, where $\{U_t\}$ is a sequence of independently, identically any normally distributed random variables with mean 0 and variance 1 and X_0 is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each t X_0 is independent of U_t. We assume $|\theta| < 1$. We consider the MLE based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ in which X_T is missing. We shall show that the bias-adjusted MLE is second order asymptotically efficient. Further we assume

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that $X_0 = 0$ in the AR process. We shall obtain the asymptotic deficiency of the MLE $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to the MLEs $\hat{\theta}^{T-1}_{ML}$, $\hat{\theta}^{T}_{ML}$ and $\hat{\theta}^{T+1}_{ML}$ based on the samples $(X_1, \dots, X_{T-1})$,$(X_1, \dots, X_T)$ and $(X_1, \dots, X_{T+1})$, respectively.

2. Definitions.

Let $(\mathfrak{X}, \mathfrak{B})$ be a sample space and Θ be a parameter space, which is assumed to be an open set in a Euclidean 1-space R^1. We shall denote by $(\mathfrak{X}^{(T)}, \mathfrak{B}^{(T)})$ the T-fold direct products of $(\mathfrak{X}, \mathfrak{B})$. For each $T = 1, 2, \dots$, the points of $\mathfrak{X}^{(T)}$ will be denoted by $\tilde{x}_T = (x_1, \dots, x_T)$. We consider a sequence of classes of probability measures $\{P_{T,\theta}: \theta \in \Theta\}$ $(T = 1, 2, \dots)$ each defined on $(\mathfrak{X}^{(T)}, \mathfrak{B}^{(T)})$ such that for each $T = 1, 2, \dots$ and each $\theta \in \Theta$ the following holds:

$$P_{T,\theta}(\mathrm{B}^{(T)}) = P_{T+1,\theta}(\mathrm{B}^{(T)} \times \chi)$$

for all $\mathrm{B}^{(T)} \in \mathfrak{B}^{(T)}$.

An estimator of θ is defined to be a sequence $\{\hat{\theta}_T\}$ of $\mathfrak{B}^{(T)}$-measurable functions $\hat{\theta}_T$. For simplicity we may denote an estimator $\hat{\theta}$ instead of $\{\hat{\theta}_T\}$. For an increasing sequence of positive numbers $\{c_T\}$ (c_T tending to infinity) an estimator $\hat{\theta}$ is called consistent with order $\{c_T\}$ (or $\{c_T\}$-consistent for short) if for every $\varepsilon > 0$ and every $\vartheta \in \Theta$ there exist a sufficiently small positive number δ and sufficiently large positive number L satisfying the following:

$$\lim_{T\to\infty} \sup_{\theta : |\theta - \vartheta| < \delta} P_{T,\theta}\{c_T |\hat{\theta}_T - \theta| \geq L\} < \varepsilon.$$

A $\{c_T\}$-consistent estimator $\hat{\theta}$ is k-th order asymptotically median unbiased (or k-th order *AMU*) if for any $\vartheta \in \Theta$ there exists a positive number δ such that

$$\lim_{T\to\infty} \sup_{\theta:\ |\theta - \vartheta| < \delta} c_T^{k-1} |P_{T,\theta}\{\hat{\theta} \leq \theta\} - \frac{1}{2}| = 0 ;$$

$$\lim_{T\to\infty} \sup_{\theta:\ |\theta - \vartheta| < \delta} c_T^{k-1} |P_{T,\theta}\{\hat{\theta} \geq \theta\} - \frac{1}{2}| = 0.$$

For each $k = 1, 2, \dots$ we denote by A_k the class of the all k-th order *AMU* estimators.

We have defined a first (second) order *AMU* estimator $\hat{\theta}^*$ to be first (second) order asymptotically efficient in the class A_1 (A_2) if for any first (second) order *AMU* estimator $\hat{\theta}$ and any $u > 0$

$$\varliminf_{T\to\infty} [P_{T,\theta}\{c_T |\hat{\theta}^* - \theta| < u\} - P_{T,\theta}\{c_T |\hat{\theta} - \theta| < u\}] \geq 0.$$

$$(\varliminf_{T\to\infty} c_T [P_{T,\theta}\{c_T |\hat{\theta}^* - \theta| < u\} - P_{T,\theta}\{c_T |\hat{\theta} - \theta| < u\}] \geq 0)$$

(e.g. see Akahira and Takeuchi [7]).

Let D be the class of estimators whose element $\hat{\theta}$ is best asymptotically normal and third order *AMU* and may be asymptotically expanded as

$$c_T(\hat{\theta} - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{1}{\sqrt{T}} Q(\theta) + o_p(\frac{1}{\sqrt{T}}),$$

where $I(\theta)$ is the Fisher information, $Z_1(\theta) = O_p(1)$, $Q(\theta) = O_p(1)$ and $E_\theta[Z_1(\theta)Q(\theta)^2] = o(1)$ with the notation $E_\theta[.]$ of the asymptotic expectation, and the distribution of $c_T(\hat{\theta} - \theta)$ admits an Edgeworth expansion. We have defined an estimator $\hat{\theta}^*$ in D to be third order asymptotically efficient in the class D if for any estimator $\hat{\theta}$ in D and any $u > 0$

$$\varliminf_{T\to\infty} c_T^2 [P_{T,\theta}\{c_T \mid \hat{\theta}^* - \theta \mid < u\} - P_{T,\theta}\{c_T \mid \hat{\theta} - \theta \mid < u\}] \geq 0.$$

In the subsequent discussion we shall deal with only the case when $c_T = \sqrt{T}$.

Let $k_T (T = 1, 2, ...)$ be positive numbers such that $d = \lim_{T\to\infty}(k_T - T)$ exists, and the estimators $\hat{\theta}_T$ and $\hat{\theta}^*_{k_T}$ in the class D based on the sample sizes T and k_T, respectively, are asymptotically equivalent in the sense that asymptotic distributions of $\sqrt{T}(\hat{\theta}_T - \theta)$ and $\sqrt{T}(\hat{\theta}^*_{k_T} - \theta)$ are equal up to the order T^{-1}. Then d is called the asymptotic deficiency of $\hat{\theta}^*_{k_T}$ with respect to $\hat{\theta}_T$ (See Hodges and Lehmann [8]). If we denote by Q and Q^* the terms of the order $T^{-1/2}$ in the stochastic expansions of $\sqrt{T}(\hat{\theta}_T - \theta)$ and $\sqrt{T}(\hat{\theta}^*_{k_T} - \theta)$, respectively, we see that the asymptotic deficiency d of $\hat{\theta}^*_{k_T}$ w.r.t. $\hat{\theta}_T$ is given by $I\{V_\theta(Q^*) - V_\theta(Q)\}$, where I is the Fisher information and V_θ designates the asymptotic variance (See Akahira [5], [6]).

3. Second order asymptotic efficiency.

Let X_t $(t = 1, 2, ...)$ be defined recursively by

(3.1) $X_t = \theta X_{t-1} + U_t$ $(t = 1, 2, ...)$,

where $\{U_t\}$ is a sequence of independently, identically and normally distributed random variables with mean 0 and variance 1 and X_0 is a normal random variable with mean 0 and variance $1/(1-\theta^2)$ and for each t X_0 is independent of U_t. We assume that $\Theta = (-1, 1)$, i.e., $|\theta| < 1$. Then it is easily seen that the process (3.1) is stationary.

Let $\hat{\theta}_{ML}$ be the *MLE* based on the sample $(X_0, X_1, \dots, X_T)$. Then it is known in Akahira [3] that the stochastic expansion of the *MLE* $\hat{\theta}_{ML}$ is given by

$$(3.2)\quad \sqrt{T}(\hat{\theta}_{ML}-\theta) = -\frac{2\theta}{\sqrt{T}} + \frac{2\theta(1-\theta^2)}{\sqrt{T}} X_0{}^2 + \frac{1-\theta^2}{\sqrt{T}} \sum_{t=1}^{T} U_t X_{t-1}$$

$$+\frac{(1-\theta^2)^2}{T\sqrt{T}}\{-\sum_{t=1}^{T}(X_{t-1}^2 - \frac{1}{1-\theta^2})\}\sum_{t=1}^{T} U_t X_{t-1}$$

$$+\frac{(1-\theta^2)^3\ 3J(\theta)+K(\theta)\}}{2T\sqrt{T}}(\sum_{t=1}^{T} U_t X_{t-1})^2 + o_p(\frac{1}{\sqrt{T}}),$$

where

$$J(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial^2}{\partial\theta^2}\log L(\theta)\}\{\frac{\partial}{\partial\theta}\log L(\theta)\}]\ ;$$

$$K(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial}{\partial\theta}\log L(\theta)\}^3]$$

with the likelihood function $L(\theta)$ of $(X_0, X_1, \ldots, X_T)$. It is also shown that the bias-adjusted *MLE* and the bias-adjusted least squares estimator are second order asymptotically efficient.

We consider the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ in which X_T is missing. The joint density of $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ is given by

$$L(\theta) = L(\theta : x_0, x_1, \ldots, x_{T-1}, x_{T+1})$$

$$= \frac{1}{(2\pi)^{(T+1)/2}} \cdot \sqrt{\frac{1-\theta^2}{1+\theta^2}} \exp[-\frac{1}{2}\{(1-\theta^2)x_0{}^2 + \sum_{t=1}^{T-1}(x_t - \theta x_{t-1})^2$$

$$+\frac{1}{1+\theta^2}(x_{T+1} - \theta^2 x_{T-1})^2\}].$$

We put

$$Z_1(\theta) = \frac{1}{\sqrt{T}}\frac{\partial}{\partial\theta}\log L(\theta)\ ;$$

$$Z_2(\theta) = \frac{1}{\sqrt{T}}\{\frac{\partial^2}{\partial\theta^2}\log L(\theta) - E_\theta[\frac{\partial^2}{\partial\theta^2}\log L(\theta)]\};$$

$$I(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial}{\partial\theta}\log L(\theta)\}^2];$$

$$J(\theta) = \frac{1}{T} E_\theta[\{\frac{\partial^2}{\partial\theta^2}\log L(\theta)\}\{\frac{\partial}{\partial\theta}\log L(\theta)\}]\ ;$$

$$K(\theta) = \frac{1}{T} E_\theta [\{\frac{\partial}{\partial\theta} \log L(\theta)\}^3].$$

Then it is known that the stochastic expansion of the *MLE* $\hat{\theta}_{ML}$ based on the sample $(X_0, X_1, \dots, X_{T-1}, X_{T+1})$ is given by

$$(3.3)\quad \sqrt{T}(\hat{\theta}_{ML} - \theta) = \frac{Z_1(\theta)}{I(\theta)} + \frac{Z_1(\theta)Z_2(\theta)}{I(\theta)^2\sqrt{T}} - \frac{3J(\theta) + K(\theta)}{2I(\theta)^3\sqrt{T}} Z_1(\theta)^2 + o_p(\frac{1}{\sqrt{T}})$$

(e.g. see Akahira [3] and Akahira and Takeuchi [7]).

Since

$$\frac{\partial \log L}{\partial\theta} = -\frac{\theta}{1-\theta^2} - \frac{\theta}{1+\theta^2} + \theta x_0^2 + \sum_{t=1}^{T-1} x_{t-1}(x_t - \theta x_{t-1})$$

$$+ \frac{\theta}{(1+\theta^2)^2}(x_{T+1}^2 + 2x_{T+1}x_{T-1}) - \frac{\theta^3(2+\theta^2)}{(1+\theta^2)^2} x_{T-1}^2 ;$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{1+\theta^2}{(1-\theta^2)^2} - \frac{1-\theta^2}{(1+\theta^2)^2} - \sum_{t=1}^{T-2} x_t^2 + \frac{1-3\theta^2}{(1+\theta^2)^3}(x_{T+1}^2 + 2x_{T+1}x_{T-1})$$

$$- \frac{\theta^2(\theta^4 + 3\theta^2 + 6)}{(1+\theta^2)^3} x_{T-1}^2 ;$$

$$E_\theta(\frac{\partial^2 \log L}{\partial\theta^2}) = -\frac{T-2}{1-\theta^2} + \frac{1 - 8\theta^2 - \theta^4}{(1+\theta^2)^2(1-\theta^2)} - \frac{1+\theta^2}{(1-\theta^2)^2} - \frac{1-\theta^2}{(1+\theta^2)^2},$$

We have

$$(3.4)\quad Z_1(\theta) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T-1} U_t X_{t-1} + \frac{1}{\sqrt{T}}(R_{T-a});$$

$$(3.5)\quad Z_2(\theta) = -\frac{1}{\sqrt{T}} \sum_{t=1}^{T-2} (X_t^2 - \frac{1}{1-\theta^2}) + o_p(\frac{1}{\sqrt{T}}),$$

where

$$R_T = \theta X_0^2 + \frac{\theta}{(1+\theta^2)^2}(X_{T+1}^2 + 2X_{T+1}X_{T-1}) - \frac{\theta^3(2+\theta^2)}{(1+\theta^2)^2} X_{T-1}^2 ;$$

$$a = \frac{\theta}{1-\theta^2} + \frac{\theta}{1+\theta^2} = \frac{2\theta}{1-\theta^4}.$$

Note that $E_\theta(R_T) = a$.

Since

$$X_{T+1} = \theta^2 X_{T-1} + \theta U_T + U_{T+1},$$

we obtain

$$(3.6) \quad R_T = \theta X_0{}^2 + \frac{\theta}{(1+\theta^2)^2}\{2(1+\theta^2)(\theta U_T + U_{T+1})X_{T-1} + (\theta U_T + U_{T+1})^2\}.$$

By (3.3), (3.4) and (3.5) we have

$$(3.7) \quad \sqrt{IT}(\hat{\theta}_{ML} - \theta) = \frac{1}{\sqrt{IT}}\sum_{t=1}^{T-1} U_t X_{t-1} + \frac{1}{\sqrt{IT}}(R_T - a) + \frac{1}{I^{3/2}\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}\right)$$

$$\cdot\left\{-\frac{1}{\sqrt{T}}\sum_{t=1}^{T-2}\left(X_t^2 - \frac{1}{1-\theta^2}\right)\right\} - \frac{3J+K}{2I^{5/2}\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}\right)^2 + o_p\left(\frac{1}{\sqrt{T}}\right),$$

where $I = I(\theta) = 1/(1-\theta^2) + o(1/\sqrt{T})$ and J and K denote $J(\theta)$ and $K(\theta)$. By (3.6) we obtain

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(R_T - a)\right]$$

$$= E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(R_T - \theta X_0^2)\right] + E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)(\theta X_0^2 - a)\right]$$

$$= 0;$$

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)\left\{-\sum_{t=1}^{T-2}\left(X_t^2 - \frac{1}{1-\theta^2}\right)\right\}(R_T - a)\right] = 0:$$

$$E_\theta\left[\left(\sum_{t=1}^{T-1} U_t X_{t-1}\right)^2 (R_T - a)\right] = 0.$$

Hence the stochastic expansion (3.7) of the *MLE* $\hat{\theta}_{ML}$ based on $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ can be essentially reduced to the case when the stochastic expansion of the *MLE* based on $(X_0, X_1, \ldots, X_T)$ is given by (3.2).

In a similar way as in Akahira [3] we have established the following:

Theorem 3.1. The bias-adjusted *MLE* $\hat{\theta}^*$ $(\in A_2)$ based on the sample $(X_0, X_1, \ldots, X_{T-1}, X_{T+1})$ is second order asymptotically efficient.

4. Asymptotic deficiency of the estimator.

In this section we deal with the case when $X_0 = 0$ in the AR process given by (3.1). Then we shall obtain the asymptotic deficiency of the MLE based on the sample $(X_1, \ldots, X_{T-1}, X_{T+1})$ with respect to the $MLEs$ based on the samples $(X_1, \ldots, X_{T-1})$, $(X_1, \ldots, X_T)$ and $(X_1, \ldots, X_{T+1})$, respectively. By (3.2) it is shown that the bias-adjusted MLE based on the sample $(X_1, \ldots, X_T)$ belongs to the class D. In a similar way as the independently and identically distributed sample case discussed in Akahira and Takeuchi [7] it is seen that the bias-adjusted MLE is third order asymptotically efficient in the class D.

Next we consider the MLE $\hat{\theta}_{ML}$ based on the sample $(X_1, \ldots, X_{T-1}, X_{T+1})$. By (3.7) we have

$$(4.1)\quad \sqrt{IT}(\hat{\theta}_{ML} - \theta) = \frac{Z_1}{\sqrt{I}} + \frac{1}{\sqrt{IT}}Q + o_p(\frac{1}{\sqrt{T}}),$$

where $Z_1 = Z_1{}' + \frac{1}{\sqrt{T}}(R_T{}' - a')$;

$$Q = \frac{1}{I}Z_1{}'Z_2 - \frac{3J+K}{2I^2}Z_1{}'^2,$$

with $Z_1{}' = \frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} U_t X_{t-1}$; $Z_2 = -\frac{1}{\sqrt{T}}\sum_{t=1}^{T-2}(X_t^2 - \frac{1}{1-\theta^2})$;

$$R_T{}' = \frac{\theta}{(1+\theta^2)^2}\{2(1+\theta^2)(\theta U_T + U_{T+1})X_{T-1} + (\theta U_T + U_{T+1})^2\};$$

$$a' = \frac{\theta}{1+\theta^2}$$

Note that $E_\theta(R_T{}') = a'$.

Since

$$E_\theta(Z_1{}'^3 Z_2{}^2) = O(\frac{1}{\sqrt{T}});\ E_\theta(Z'_1{}^4 Z_2) = O(\frac{1}{\sqrt{T}});\ E_\theta(Z_1{}'^5) = O(\frac{1}{\sqrt{T}});$$

$$E_\theta[Z_1{}'^2 Z_2{}^2(R_T{}' - a')] = O(\frac{1}{\sqrt{T}});\ E_\theta[Z_1{}'^3 Z_2(R_T{}' - a)] = O(\frac{1}{\sqrt{T}});$$

$$E_\theta[Z_1{}'^4(R_T{}' - a')] = O(\frac{1}{\sqrt{T}}),$$

it follows that

$$E_\theta[Z_1 Q^2] = O(\frac{1}{\sqrt{T}}).$$

Hence the bias-adjusted *MLE* $\hat{\theta}^*_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ belongs to the class D.

We consider the estimator $\hat{\theta}^{T-1}_{ML}$ based on the sample $(X_1, \dots, X_{T-1})$ which has the stochastic expansion

$$\text{(4.2)} \quad \sqrt{IT}(\hat{\theta}^{T-1}_{ML} - \theta) = \frac{Z'_1}{\sqrt{T}} + \frac{1}{\sqrt{IT}} Q_0 + o_p(\frac{1}{\sqrt{T}}),$$

where $Q_0 = \frac{1}{I} Z_1' Z_2 - \frac{3J+K}{2I^2} Z_1'^2$.

Let $\hat{\theta}^{T}_{ML}$ be the *MLE* based on the sample $(X_1, \dots, X_T)$ and $\hat{\theta}^{T*}_{ML}$ the bias-adjusted *MLE*. We put $S_T = \sqrt{IT}(\hat{\theta}^{T}_{ML} - \theta)$. Then $\hat{\theta}^{T}_{ML}$ has cumulants of the following form:

$$E_\theta(S_T) = \frac{\mu}{\sqrt{T}} + o(\frac{1}{T});$$

$$V_\theta(S_T) = 1 + \frac{\tau}{T} + o(\frac{1}{T});$$

$$E_\theta[\{S_T - E_\theta(S_T)\}^3] = \frac{\beta_3}{\sqrt{T}} + o(\frac{1}{T});$$

$$E_\theta[\{S_T - E_\theta(S_T)\}^4] - 3\{V_\theta(S_T)\}^2 = \frac{\beta_4}{T} + o(\frac{1}{T}).$$

It is noted by Akahira [5] that only the terms of the order of T^{-1} in the cumulants are essentially different between the estimators $\hat{\theta}^{T*}_{ML}$ and $\hat{\theta}^{*}_{ML}$ since they belong to the class D.

Also the Edgeworth expansion of the distribution of $\hat{\theta}^{T}_{ML}$ is given by

$$\text{(4.3)} \quad P_{T,\theta}\{\sqrt{IT}(\hat{\theta}^{T}_{ML} - \theta) \le u\} = \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2 - 1)\phi(u)$$

$$- \frac{\beta_4}{24T}(u^3 - 3u)\phi(u) - \frac{\beta_3^2}{72T}(u^5 - 10u^3 + 15u)\phi(u)$$

$$-\frac{\tau+\mu^2}{2T}u\,\phi(u)-\frac{\beta_3\mu}{6T}(u^3-3u)\,\phi(u)+o(\frac{1}{T}),$$

where $\Phi(u)=\int_{-\infty}^{u}\phi(x)dx$ with $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Since by (4.1)

$$E_\theta[Z_1'(R_T'-a')]=0\ ;$$

$$E_\theta[Q(R_T'-a')]=0,$$

it follows by (4.1), (4.2), (4.3) and a similar way as in Akahira ([5], page 71) that the Edgeworth expansion of the conditional distribution of $\hat{\theta}_{ML}$ based on the sample $(X_1,\dots,X_{T-1},X_{T+1})$ given R_T' is obtained by

(4.4) $P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}-\theta)\le u\mid R_T'\}$

$$=P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}^{T-1}_{ML}-\theta)\le u-\frac{1}{\sqrt{IT}}(R_T'-a')\mid R_T'\}$$

$$=\Phi(u-\frac{1}{\sqrt{IT}}(R_T'-a'))-\frac{\mu}{\sqrt{T}}\phi(u-\frac{1}{\sqrt{IT}}(R_T'-a'))$$

$$-\frac{\beta_3}{6\sqrt{T}}\{(u-\frac{1}{\sqrt{IT}}(R_T'-a'))^2-1\}\,\phi(u-\frac{1}{\sqrt{IT}}(R_T'-a'))$$

$$-\frac{\beta_4}{24T}(u^3-3u)\,\phi(u)-\frac{\beta_3^2}{72T}(u^5-10u^3+15u)\,\phi(u)$$

$$-\frac{\tau+u^2+1}{2T}u\,\phi(u)-\frac{\beta_3 u}{6T}(u^3-3u)\,\phi(u)+o(\frac{1}{T})$$

$$=\Phi(u)-\frac{\mu}{\sqrt{T}}\phi(u)-\frac{\beta_3}{6\sqrt{T}}(u^2-1)\,\phi(u)-\frac{\beta_4}{24T}(u^3-3u)\,\phi(u)$$

$$-\frac{\beta_3^2}{72T}(u^5-10u^3+15u)\,\phi(u)-\frac{\tau+\mu^2+1}{2T}u\,\phi(u)-\frac{\beta_3\mu}{6T}(u^3-3u)\,\phi(u)$$

$$-\frac{1}{\sqrt{IT}}(R_T'-a')\,\phi(u)+\frac{1}{2IT}(R_T'-a')^2u\,\phi(u)-\frac{\mu}{\sqrt{IT}}(R_T'-a')u\phi(u)$$

$$-\frac{\beta_3}{6\sqrt{I}\,T}(R_T'-a')\,u\,(u^2-1)\,\phi(u)+\frac{\beta_3}{3\sqrt{I}\,T}(R_T'-a')\,u\,\phi(u)+o(\frac{1}{T}).$$

By (4.4) we have

(4.5) $P_{T+1,\theta}\{\sqrt{IT}\,(\hat{\theta}_{ML}-\theta)\le u\}$

$$= E_\theta[P_{T+1,\theta}\{\sqrt{IT}\,(\hat{\theta}_{ML}-\theta)\le u \mid R_T'\}]$$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\,\phi(u) - \frac{\beta_4}{24T}(u^3-3u)\,\phi(u)$$

$$- \frac{\beta_3{}^2}{72T}(u^5-10u^3+15u)\,\phi(u) - \frac{\tau+\mu^2+1}{2T}\,u\,\phi(u) - \frac{\beta_3\mu}{6T}(u^3-3u)\,\phi(u)$$

$$+ \frac{1}{2IT}\{E_\theta\,(R_T'-a')^2\}\,u\,\phi(u) + o(\frac{1}{T}).$$

Since

$$E_\theta(X_t{}^2) = E_\theta[(\sum_{i=1}^{t}\theta^{t-i}\,U_i)^2] = \frac{1-\theta^{2\cdot t}}{1-\theta^2};$$

$$E_\theta[(\theta U_T + U_{T+1})^2] = 1+\theta^2\ ;$$

$$E_\theta[(\theta U_T + U_{T+1})^4] = 3(1+\theta^2)^2,$$

it follows that

$$E_\theta(R_T{}'^2) = E[\{\frac{\theta}{(1+\theta^2)^2}(\theta U_T + U_{T+1})(2(1+\theta^2)X_{T-1} + \theta U_T + U_{T+1})\}^2]$$

$$= \frac{\theta^2(\theta^2+7)}{(1+\theta^2)^2(1-\theta^2)}.$$

Hence the variance of $R_T{}'$ is given by

(4.6) $V_\theta(R_T') = E_\theta(R_T'-a')^2 = E_\theta(R_T{}'^2) - a'^2$

$$= \frac{2\theta^2(\theta^2+3)}{(1+\theta^2)^2(1-\theta^2)}\ .$$

In a similar way as in Akahira ([5], page 71) we have by (4.3)

(4.7) $P_{T-1,\theta}\{\sqrt{IT}\,(\hat{\theta}_{ML}^{T-1}-\theta)\le u\}$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\phi(u) - \frac{\beta_4}{24T}(u^3-3u)\phi(u)$$

$$-\frac{\beta_3^2}{72T}(u^5-10u^3+15u)\phi(u) - \frac{\tau_{-1}+\mu^2}{2T}u\phi(u)$$

$$-\frac{\beta_3\mu}{6T}(u^3-3u)\phi(u) + o(\frac{1}{T});$$

(4.8) $P_{T+1,\theta}\{\sqrt{IT}(\hat{\theta}_{ML}^{T+1}-\theta) \le u\}$

$$= \Phi(u) - \frac{\mu}{\sqrt{T}}\phi(u) - \frac{\beta_3}{6\sqrt{T}}(u^2-1)\phi(u) - \frac{\beta_4}{24T}(u^3-3u)\phi(u)$$

$$-\frac{\beta_3^2}{72T}(u^5-10u^3+15u)\phi(u) - \frac{\tau_1+\mu^2}{2T}u\,\phi(u)$$

$$-\frac{\beta_3\mu}{6T}(u^3-3u)\phi(u) + o(\frac{1}{T}),$$

where $\tau_{-1} = \tau + 1$ and $\tau_1 = \tau - 1$

with $V_\theta(\sqrt{I(T-1)}(\hat{\theta}_{ML}^{T-1}-\theta)) = 1 + \frac{\tau_{-1}}{T-1} + o(\frac{1}{T-1});$

$$V_\theta(\sqrt{I(T+1)}(\hat{\theta}_{ML}^{T+1}-\theta)) = 1 + \frac{\tau_1}{T+1} + o(\frac{1}{T+1}).$$

Note that the difference in the above (4.7) and (4.8) appears in the sixth terms of their right-hand sides. It is seen by (4.3),(4.7)and(4.8) that the asymptotic deficiencies of $\hat{\theta}_{ML}^{T-1}$ and $\hat{\theta}_{ML}^{T+1}$ with respect to $\hat{\theta}_{ML}^{T}$ are equal to 1 and −1, respectively, i.e., $\tau_{-1} - \tau = 1$ and $\tau_1 - \tau = -1$.

It follows by (4.3), (4.5), (4.7), (4.8) and Akahira [5] that the asymptotic deficiencies of $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ w.r.t. $\hat{\theta}_{ML}^{T-1}$, $\hat{\theta}_{ML}^{T}$ and $\hat{\theta}_{ML}^{T+1}$ are given by $-V_\theta(R_T')/I$, $1-\{V_\theta(R_T')/I\}$ and $2-\{V_\theta(R_T')/I\}$ with (4.6), respectively. Hence we have established the following.

Theorem 4.1. The asymptotic deficiencies of $\hat{\theta}_{ML}$ based on the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to $\hat{\theta}_{ML}^{T-1}$, $\hat{\theta}_{ML}^{T}$ and $\hat{\theta}_{ML}^{T+1}$ are given in the table below.

Estimator $\hat{\theta}$	Asymptotic deficiency of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}$
$\hat{\theta}_{ML}^{T-1} = \hat{\theta}_{ML}^{T-1}(X_1, \dots, X_{T-1})$	$-\dfrac{2\theta^2(\theta^2+3)}{(1+\theta^2)^2}$
$\hat{\theta}_{ML}^{T} = \hat{\theta}_{ML}^{T}(X_1, \dots, X_T)$	$\dfrac{5-(\theta^2+2)^2}{(1+\theta^2)^2}$
$\hat{\theta}_{ML}^{T+1} = \hat{\theta}_{ML}^{T+1}(X_1, \dots, X_{T+1})$	$\dfrac{2(1-\theta^2)}{(1+\theta^2)^2}$

Remark. It is seen that the loss of informations on θ from the sample $(X_1, \dots, X_{T-1}, X_{T+1})$ with respect to the samples $(X_1, \dots, X_{T-1})$, $(X_1, \dots, X_T)$ and $(X_1, \dots, X_{T+1})$ through the *MLE* are given by the asymptotic deficiencies depending on θ in the table in Theorem 4.1, respectively.

It is natural that the asymptotic deficiencies of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}_{ML}^{T-1}$ and $\hat{\theta}_{ML}^{T+1}$ are negative and positive, respectively since the based sample of $\hat{\theta}_{ML}$ includes that of $\hat{\theta}_{ML}^{T-1}$ and is done in that of $\hat{\theta}_{ML}^{T+1}$. It is also seen that the asymptotic deficiency of $\hat{\theta}_{ML}$ w.r.t. $\hat{\theta}_{ML}^{T}$ is positive if $|\theta| < \sqrt{\sqrt{5}-2} \doteqdot 0.486$ and negative if $\sqrt{\sqrt{5}-2} < |\theta| < 1$. The fact means that for the sample $(X_1, \dots, X_{T-1})$, X_T is more informative than X_{T+1} if $|\theta| < \sqrt{\sqrt{5}-2}$ and X_T is less informative than X_{T+1} if $\sqrt{\sqrt{5}-2} < |\theta| < 1$. It seems reasonable in the process (3.1) since it is better to contract the spacing of the observations if $|\theta|$ is small and expand it if $|\theta|$ is big. Further it may be extended to the problem on the optimum spacing of observations from a process ([10]).

In a similar way as the above discussion it may be possible to obtain the asymptotic deficiency of the *MLE* based on the sample $(X_1, \dots, X_i, X_{i+k}, \dots, X_{T+1})$ in which $X_{i+1}, \dots, X_{i+k-1}$ are missing, where $1 \leq i < i+k \leq T+1$ and that of the *MLE* based on the sample in which any observations except the extremes are missing.

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