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ASYMPTOTIC THEORY FOR ESTIMATION OF LOCATION IN NON-REGULAR CASES, II: BOUNDS OF ASYMPTOTIC DISTRIBUTIONS OF CONSISTENT ESTIMATORS*

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1. Introduction

A consistent estimator with order $\{c_n\}$ (or a $\{c_n\}$-consistent estimator) is defined in Akahira [1], where the necessary conditions for the existence of such an estimator are established and the bounds of the orders of convergence of consistent estimators are obtained for non-regular cases.

In the present paper the asymptotic accuracies of $\{c_n\}$-consistent estimators, that is, the bounds of their asymptotic distributions are discussed in similar cases as those studied in [1]. The approach is similar to Bahadur [2] dealing with the bound for asymptotic variances. We shall define an estimator to be uniformly asymptotically most accurate if the asymptotic distribution of it attains uniformly the bound of the asymptotic distributions of asymptotically median unbiased estimators. If the asymptotic distribution of an estimator $\{T_n\}$ attains the bound at one point, then $\{T_n\}$ is called to be asymptotically most accurate at the point. Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent identically distributed random variables having the density of the same location parameter case as that in [1]. If $\alpha=2$ and $\alpha>2$, then the maximum likelihood estimators with the asymptotic normal distributions $N(0, 1/I_1)$ and $N(0, 1/I_2)$ of order $\{(n \log n)^{1/2}\}$ and order $\{n^{1/2}\}$ are uniformly asymptotically most accurate, respectively, where $I_1=\frac{1}{2}\left(\frac{A'^2}{A'}+\frac{B'^2}{B'}\right)$ if $\beta=2$, $I_1=\frac{A'^2}{2A'}$ if $\beta>2$, and $I_2=E_\theta[\{(\partial/\partial\theta)\log f(x-\theta)\}^2]$. If $\alpha=\beta=1$ and $A'=B'$, then it is shown that certain estimators with the asymptotic Weibull distributions of order $\{n\}$ are asymptotically most accurate at some point. Furthermore we shall obtain the bounds of orders of convergence of consistent estimators and those of the orders of their asymptotic distributions in non-regular cases. Also some results in terms of the asymptotic distributions of estimators are given in Takeuchi [5].

2. Notations and Definitions

Let $(\mathfrak{X}, \mathfrak{B})$ be a sample space. We consider a family of probability measures on $\mathfrak{B}$, $\mathfrak{P}=\{P_\theta: \theta\in\Theta\}$, where the index set Θ is called a parameter space. We assume that Θ is an open set in a Euclidean 1-space R^1. Consider n-fold direct products $(\mathfrak{X}^{(n)}, \mathfrak{B}^{(n)})$ of $(\mathfrak{X}, \mathfrak{B})$ and the corresponding product measures $P_\theta^{(n)}$ of P_θ.

For each $n=1, 2, \cdots$, the points of $\mathfrak{X}^{(n)}$ will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$.

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An estimator of θ is defined to be a sequence $\{T_n: n=1, 2, \cdots\}$ of $\mathfrak{B}^{(n)}$-measurable function T_n on $\mathfrak{X}^{(n)}$ into Θ $(n=1, 2, \cdots)$. For an increasing sequence of positive numbers $\{c_n: n=1, 2, \cdots\}$ (c_n tending to infinity) an estimator $\{T_n: n=1, 2, \cdots\}$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every ϑ of Θ, there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\overline{\lim_{n\to\infty}} \sup_{\theta: |\theta-\vartheta|<\delta} P_\theta^{(n)}(\{c_n|T_n-\theta|\geq L\})<\varepsilon$$

(Akahira [1]).

Order $\{c_n'\}$ is called to be greater than order $\{c_n\}$ if $\lim_{n\to\infty} c_n/c_n'=0$.

Definition 2.1. A distribution function $F_{\theta,T}^{\mathfrak{C}}(.)$ is called to be the asymptotic distribution function of an estimator $T=\{T_n\}$ of orders $\mathfrak{C}=\{c_n\}$ if for each real number y, $F_{\theta,T}^{\mathfrak{C}}(y)$ is continuous in θ and for any $\vartheta\in\Theta$ there exists a positive number d such that at any continuity points y of $F_{\theta,T}^{\mathfrak{C}}(y)$,

$$\lim_{n\to\infty} \sup_{\theta: |\theta-\vartheta|<d} |P_\theta^{(n)}(\{c_n(T_n-\theta)\leq y\})-F_{\theta,T}^{\mathfrak{C}}(y)|=0.$$

Definition 2.2. $T=\{T_n\}$ is called asymptotically median unbiased if $F_{\theta,T}^{\mathfrak{C}}(0-)\leq 1/2$, $F_{\theta,T}^{\mathfrak{C}}(0+)\geq 1/2$. Specially if $F_{\theta,T}^{\mathfrak{C}}(0-)=F_{\theta,T}^{\mathfrak{C}}(0+)=1/2$, then T is called exactly asymptotically median unbiased.

Definition 2.3. Suppose that $T^*=\{T_n^*\}$ is (exactly) asymptotically median unbiased, it is called asymptotically most accurate at y if

$$F_{\theta,T^*}^{\mathfrak{C}}(y)=\begin{cases} \inf_{T\in\mathfrak{T}_M^{\mathfrak{C}}} F_{\theta,T}^{\mathfrak{C}}(y) & \text{for } y<0, \\ \sup_{T\in\mathfrak{T}_M^{\mathfrak{C}}} F_{\theta,T}^{\mathfrak{C}}(y) & \text{for } y>0, \end{cases}$$

where $\mathfrak{T}_M^{\mathfrak{C}}$ is the class of all (exactly) asymptotically median unbiased estimators with the same order $\mathfrak{C}=\{c_n\}$.

Definition 2.4. $T^*=\{T_n^*\}$ is called uniformly asymptotically most accurate if both (2.1) and (2.2) of the following hold:

$$F_{\theta,T^*}^{\mathfrak{C}}(y)=\inf_{T\in\mathfrak{T}_M^{\mathfrak{C}}} F_{\theta,T}^{\mathfrak{C}}(y) \quad \text{for all } y<0, \tag{2.1}$$

$$F_{\theta,T^*}^{\mathfrak{C}}(y)=\sup_{T\in\mathfrak{T}_M^{\mathfrak{C}}} F_{\theta,T}^{\mathfrak{C}}(y) \quad \text{for all } y>0. \tag{2.2}$$

Moreover, if either (2.1) or (2.2) hold, T^* is called one-sided most accurate.

3. Bounds of Asymptotic Distributions of Consistent Estimators

In this section we shall show that there exist the asymptotic distribution of the estimator of some order which is uniformly asymptotically most accurate or asymptotically most accurate at some point, that is, attains the bound of the asymptotic distributions of asymptotically median unbiased estimators uniformly or at some point.

Now we suppose that every $P_\theta(\cdot)(\theta\in\Theta)$ is absolutely continuous with respect to σ-finite measure μ. We denote the density $dP_\theta/d\mu$ by $f(\cdot:\theta)$ and by $A(\theta)$ the set of points in the space of $\mathfrak{X}$ for which $f(x:\theta)>0$.

For any two points θ and φ in Θ we put

$$Z(x:\varphi,\ \theta)=\chi_{A(\varphi)\cap A(\theta)}(x)\log\{f(x:\varphi)/f(x:\theta)\}$$

where $\chi_{A(\varphi)\cap A(\theta)}$ denotes the indicator (or the characteristic function) of $A(\varphi)\cap A(\theta)$. Then we shall consider the following cases.

Case I. For some sequence $\{c_n : n=1, 2, \cdots\}$ with $c_n \to \infty (n\to\infty)$,

$$\lim_{n\to\infty}\left\{\int_{A(\theta)} f(x: \theta - yc_n^{-1})dx\right\}^n = 1 \tag{3.1}$$

for $-\infty < y < \infty$ and the following hold:

$$0 < V_n(\theta) < \infty \quad (n=1, 2, \cdots);$$
$$0 < V_n^*(\theta) < \infty \quad (n=1, 2, \cdots);$$
$$\lim_{n\to\infty} V_n^*(\theta)/V_n(\theta) = \gamma (>0);$$
$$\lim_{n\to\infty}\frac{n(M_n^*(\theta)-M_n(\theta))}{\sqrt{nV_n(\theta)}} = \delta|y| \quad (\delta>0),$$

where

$$M_n(\theta) = \int_{\mathfrak{X}} x dQ_\theta(x|A(yc_n^{-1}: \theta));$$
$$M_n^*(\theta) = \int_{\mathfrak{X}} x dQ_{\theta - yc_n^{-1}}(x|A(yc_n^{-1}: \theta));$$
$$V_n(\theta) = \int_{\mathfrak{X}} \{x - M_n(\theta)\}^2 dQ_\theta(x|A(yc_n^{-1}: \theta));$$
$$V_n^*(\theta) = \int_{\mathfrak{X}} \{x - M_n^*(\theta)\}^2 dQ_{\theta - yc_n^{-1}}(x|A(yc_n^{-1}: \theta))$$

with $A(yc_n^{-1}: \theta) = A(\theta - yc_n^{-1}) \cap A(\theta)$, and the following relations hold: for any $\varepsilon > 0$

$$\frac{1}{V_n(\theta)}\int_{\{|x|>\varepsilon\sqrt{nV_n(\theta)}\}} x^2 dQ_\theta(x + M_n(\theta)|A(yc_n^{-1}: \theta)) \to 0 \text{ as } n\to\infty \tag{3.2}$$

uniformly in any compact subset of Θ, and for any $\varepsilon > 0$

$$\frac{1}{V_n^*(\theta)}\int_{\{|x|>\varepsilon\sqrt{nV_n^*(\theta)}\}} x^2 dQ_{\theta - yc_n^{-1}}(x + M_n^*(\theta)|A(yc_n^{-1}: \theta)) \to 0 \text{ as } n\to\infty \tag{3.3}$$

uniformly in any compact subset of Θ, where

$$Q_\theta(E|A(yc_n^{-1}: \theta)) = \frac{P_\theta(\{x: Z(x: \theta - yc_n^{-1}, \theta)\in E\}\cap A(yc_n^{-1}: \theta))}{P_\theta(A(yc_n^{-1}: \theta))}$$

for all $E \in \mathfrak{B}$ and

$$Q_{\theta - yc_n^{-1}}(F|A(yc_n^{-1}: \theta)) = \frac{P_{\theta - yc_n^{-1}}(\{x: Z(x: \theta - yc_n^{-1}, \theta)\in F\}\cap A(yc_n^{-1}: \theta))}{P_{\theta - yc_n^{-1}}(A(yc_n^{-1}: \theta))}$$

for all $F \in \mathfrak{B}$.

Case II. For some sequence $\{c_n' : n=1, 2, \cdots\}$ with $c_n' \to \infty (n\to\infty)$,

$$\lim_{n\to\infty}\left\{\int_{A(\theta - yc_n'^{-1})} f(x: \theta)d\mu\right\}^n = \begin{cases} e^{-C_\alpha^* y^\alpha} & \text{for } 0 \le y \le s_\alpha^*, \\ e^{-C_\alpha |y|^\alpha} & \text{for } -s_\alpha \le y \le 0, \end{cases} \tag{3.4}$$

$$\lim_{n\to\infty}\left\{\int_{A(\theta)} f(x: \theta - yc_n'^{-1})d\mu\right\}^n = \begin{cases} e^{-C_\alpha y^\alpha} & \text{for } 0 \le y \le s_\alpha^*, \\ e^{-C_\alpha^* |y|^\alpha} & \text{for } -s_\alpha \le y \le 0, \end{cases} \tag{3.5}$$

where α and C_α are positive numbers and $s_\alpha = \left(\frac{1}{C_\alpha}\log 2\right)$ and C_α^* is a nonnegative number and $s_\alpha^* = \left(\frac{1}{C_\alpha^*}\log 2\right)$ if $C_\alpha^* > 0$, $s_\alpha^* = \infty$ if $C_\alpha^* = 0$, and the following hold:

$$0 < V_n'(\theta) < \infty \quad (n=1, 2, \cdots);$$
$$0 < V_n^{*\prime}(\theta) < \infty \quad (n=1, 2, \cdots);$$

$$\lim_{n\to\infty} V_n^{*\prime}(\theta)/V_n'(\theta)=\gamma'(>0);$$

$$\lim_{n\to\infty}\frac{n(M_n^{*\prime}(\theta)-M_n'(\theta))}{\sqrt{nV_n'(\theta)}}=0,$$

where

$$M_n'(\theta)=\int_{\mathfrak{X}} x dQ_\theta(x|A(yc_n'^{-1}:\theta));$$

$$M_n^{*\prime}(\theta)=\int_{\mathfrak{X}} x dQ_{\theta-yc_n'^{-1}}(x|A(yc_n'^{-1}:\theta));$$

$$V_n'(\theta)=\int_{\mathfrak{X}}\{x-M_n'(\theta)\}^2 dQ_\theta(x|A(yc_n'^{-1}:\theta));$$

$$V_n^{*\prime}(\theta)=\int_{\mathfrak{X}}\{x-M_n^{*\prime}(\theta)\}^2 dQ_{\theta-yc_n'^{-1}}(x|A(yc_n'^{-1}:\theta)),$$

and for any $\varepsilon>0$

$$\frac{1}{V_n'(\theta)}\int_{\{|x|>\varepsilon\sqrt{nV_n'(\theta)}\}} x^2 dQ_\theta(x+M_n'(\theta)|A(yc_n'^{-1}:\theta))\to 0 \text{ as } n\to\infty \tag{3.6}$$

uniformly in any compact subset of Θ, and for any $\varepsilon>0$

$$\frac{1}{V_n^{*\prime}(\theta)}\int_{\{|x|>\varepsilon\sqrt{nV_n^{*\prime}(\theta)}\}} x^2 dQ_{\theta-yc_n'^{-1}}(x+M_n^{*\prime}(\theta)|A(yc_n'^{-1}:\theta))\to 0 \text{ as } n\to\infty \tag{3.7}$$

uniformly in any compact subset of Θ, where

$$Q_\theta(E|A(yc_n'^{-1}:\theta))=\frac{P_\theta(\{x: Z(x:\theta-yc_n'^{-1},\ \theta)\in E\}\cap A(yc_n'^{-1}:\theta))}{P_\theta(A(yc_n'^{-1}:\theta))}$$

for all $E\in\mathfrak{B}$ and

$$Q_{\theta-yc_n'^{-1}}(F|A(yc_n'^{-1}:\theta))=\frac{P_\theta I_{yc_n'^{-1}}(\{x: Z(x:\theta-yc_n'^{-1},\ \theta)\in F\}\cap A(yc_n'^{-1}:\theta))}{P_{\theta-yc_n'^{-1}}(A(yc_n'^{-1}:\theta))}$$

for all $F\in\mathfrak{B}$.

LEMMA 3.1. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent identically distributed (*i.i.d.*) random variables with a density function satisfying Case I. For testing $H:\theta=\theta_0$ against $A:\theta=\theta_0-yc_n^{-1}(y\neq 0)$, consider the sequence of the most powerful level α_n tests φ_n such that

$$\varphi_n(\tilde{x}_n)=\begin{cases}1, & \text{if } \tilde{x}_n\in D^n(yc_n^{-1}:\theta_0)\cup[\{\sum_{i=1}^n Z_{ni}(\theta_0)>k_n\}\cap A^n(yc_n^{-1}:\theta_0)],\\ 0, & \text{if } \tilde{x}_n\notin D^n(yc_n^{-1}:\theta_0)\cup[\{\sum_{i=1}^n Z_{ni}(\theta_0)>k_n\}\cap A^n(yc_n^{-1}:\theta_0)],\end{cases}$$

where $\alpha_n=E_{\theta_0}(\varphi_n)$, $Z_{ni}(\theta)=Z(x_i:\theta-yc_n^{-1},\ \theta)$, $k_n=-K|y|\sqrt{nV_n^*(\theta_0)}+M_n^*(\theta_0)$ $(K=\delta/\gamma)$, $D^n(yc_n^{-1}:\theta_0)=\mathop{\times}_{i=1}^n A(\theta_0-yc_n^{-1})-\mathop{\times}_{i=1}^n A(\theta_0)$ and $A^n(yc_n^{-1}:\theta_0)=\mathop{\times}_{i=1}^n A(yc_n^{-1}:\theta_0)$. Then the following hold:

$$\lim_{n\to\infty}\alpha_n=1/2; \tag{3.8}$$

$$\lim_{n\to\infty} E_{\theta_0-(y/K)c_n^{-1}}(\varphi_n)=\Phi(|y|) \quad \text{for} \quad -\infty<y<\infty, \tag{3.9}$$

where $\Phi(u)=\int_{-\infty}^{u}\frac{1}{\sqrt{2\pi}}e^{-(t^2/2)}dt$.

PROOF. From (3.1) we have

$$\begin{aligned}&\lim_{n\to\infty} P^{(n)}_{\theta_0-yc_n^{-1}}(D^n(yc_n^{-1}:\theta_0))\\ &=1-\lim_{n\to\infty}\left\{\int_{A(\theta_0)} f(x:\theta_0-yc_n^{-1})dx\right\}^n\\ &=0\end{aligned} \tag{3.10}$$

for $-\infty<y<\infty$;

$$\lim_{n\to\infty} P_{\theta_0}^{(n)}(D^n(yc_n^{-1}: \theta_0))=0 \tag{3.11}$$

for $-\infty<y<\infty$.

Since $Z_{n1}(\theta)$, $Z_{n2}(\theta)$, $\cdots$, are independent and (3.2) and (3.3) hold, it follows from the central limit theorem (Gnedenko and Kolmogorov [4]) that the distribution laws of $\{\sum_{i=1}^n Z_{ni}(\theta)-nM_n(\theta)\}/\sqrt{nV_n(\theta)}$ and $\{\sum_{i=1}^n Z_{ni}(\theta)-nM_n^*(\theta)\}/\sqrt{nV_n^*(\theta)}$ converge to the nomal law $N(0, 1)$ uniformly in any compact subset of Θ. Hence it follows from (3.11) that

$$\begin{aligned}
&\lim_{n\to\infty} E_{\theta_0}(\varphi_n)\\
&=\lim_{n\to\infty} P_{\theta_0}^{(n)}(\{\sum_{i=1}^n Z_{ni}(\theta_0)>k_n\}\cap A^n(yc_n^{-1}: \theta_0))\\
&=\lim_{n\to\infty} [P_{\theta_0}^{(n)}(\{(\sum_{i=1}^n Z_{ni}(\theta_0)-nM_n^*(\theta_0))/\sqrt{nV_n^*(\theta_0)}>-K|y|\}|A^n(yc_n^{-1}: \theta_0))\cdot P_{\theta_0}^{(n)}(A^n(yc_n^{-1}: \theta_0)]\\
&=\lim_{n\to\infty} \left[P_{\theta_0}^{(n)}\left(\left\{(\sum_{i=1}^n Z_{ni}(\theta_0)-nM_n(\theta_0))/\sqrt{nV_n(\theta_0)}>-K|y|\sqrt{\frac{V_n^*(\theta_0)}{V_n(\theta_0)}}+\frac{n}{\sqrt{nV_n(\theta_0)}}(M_n^*(\theta_0)-M_n(\theta_0))\right\}\right.\right.\\
&\qquad \left.\left.|A^n(yc_n^{-1}: \theta_0))\cdot P_{\theta_0}(A^n(yc_n^{-1}: \theta_0)\right)\right]\\
&=1-\Phi(0)\\
&=\frac{1}{2}.
\end{aligned}$$

Hence (3.8) holds.

Also it follows from (3.10) that

$$\begin{aligned}
&\lim_{n\to\infty} E_{\theta_0-yc_n^{-1}}(\varphi_n)\\
&=\lim_{n\to\infty} P^{(n)}{}_{\theta_0-yc_n^{-1}}(\{\sum_{i=1}^n Z_{ni}(\theta_0)>k_n\}\cap A^n(yc_n^{-1}: \theta_0))\\
&=\lim_{n\to\infty} [P^{(n)}{}_{\theta_0-yc_n^{-1}}(\{(\sum_{i=1}^n Z_{ni}(\theta_0)-nM_n^*(\theta_0))/\sqrt{nV_n^*(\theta_0)}>-K|y||A^n(yc_n^{-1}: \theta_0))\\
&\qquad \cdot P^{(n)}{}_{\theta_0-yc_n^{-1}}(A^n(yc_n^{-1}: \theta_0))]\\
&=1-\lim_{n\to\infty} P^{(n)}{}_{\theta_0-yc_n^{-1}}(\{(\sum_{i=1}^n Z_{ni}(\theta_0)-nM_n^*(\theta_0))/\sqrt{nV_n^*(\theta_0)}\leq -K|y||A^n(yc_n^{-1}: \theta_0))\\
&=1-\Phi(-K|y|)\\
&=\Phi(K|y|)
\end{aligned}$$

for $-\infty<y<\infty$.

Thus we complete the proof.

LEMMA 3.2. Let X_1, X_2, $\cdots$, X_n, $\cdots$ be a sequence of *i.i.d.* random variables with a density function satisfying Case II. For testing $H: \theta=\theta_0$ against $A^+: \theta=\theta_0-yc_n'^{-1}(y>0)$, consider the sequence of the most powerful level α_n^+ test φ_n^+ such that

$$\varphi_n^+(\tilde{x}_n)=\begin{cases}1, \text{ if } \tilde{x}_n\in D^n(yc_n'^{-1}: \theta_0)\cup[\{\sum_{i=1}^n Z_{ni}^+(\theta_0)>k_n^+\}\cap A^n(yc_n'^{-1}: \theta_0)],\\ 0, \text{ if } \tilde{x}_n\notin D^n(yc_n'^{-1}: \theta_0)\cup[\{\sum_{i=1}^n Z_{ni}^+(\theta_0)>k_n^+\}\cap A^n(yc_n'^{-1}: \theta_0)],\end{cases}$$

where $\alpha_n^+=E_{\theta_0}(\varphi_n^+)$, $Z_{ni}^+(\theta)=Z(x_i: \theta-yc_n'^{-1}, \theta)$, $k_n^+=-K_y^+y^\alpha\sqrt{nV_n^{*\prime}(\theta_0)}+M_n^{*\prime}(\theta_0)$ and $K_y^+=\gamma'^{-1}y^{-\alpha}\Phi^{-1}\left(\frac{1}{2}e^{C_\alpha^* y^\alpha}\right)$ if $C_\alpha^*>0$, $K_y^+=\gamma'^{-1}y^{-\alpha}\Phi^{-1}\left(\frac{1}{2}\right)$ if $C_\alpha^*=0$. For testing $H: \theta=\theta_0$ against $A^-: \theta=\theta_0-yc_n'^{-1}(y<0)$, consider the sequence of the most powerful level α_n^- tests φ_n^- such that

$$\varphi_n^-(\tilde{x}_n)=\begin{cases}1, \text{ if } \tilde{x}_n\in D^n(yc_n'^{-1}: \theta_0)\cup[\{\sum_{i=1}^n Z_{ni}^-(\theta_0)>k_n^-\}\cap A^n(yc_n'^{-1}: \theta_0)],\\ 0, \text{ if } \tilde{x}_n\notin D^n(yc_n'^{-1}: \theta_0)\cup[\{\sum_{i=1}^n Z_{ni}^-(\theta_0)>k_n^-\}\cap A^n(yc_n'^{-1}: \theta_0)],\end{cases}$$

where $\alpha_n^-=E_{\theta_0}(\varphi_n^-)$, $Z_{ni}^-(\theta)=Z(x_i: \theta-yc_n'^{-1}, \theta)$, $k_n=-K_y^-|y|^\alpha\sqrt{nV_n^*(\theta_0)}+M_n^{*\prime}(\theta_0)$ and $K_y^-=\gamma'^{-1}|y|^{-\alpha}\Phi^{-1}\left(\frac{1}{2}e^{C_\alpha|y|^\alpha}\right)$.

Then the following hold:

$$\lim_{n\to\infty}\alpha_n^+=\lim_{n\to\infty}\alpha_n^-=\frac{1}{2};$$

$$\lim_{n\to\infty}E_{\theta_0-yc_n'^{-1}}(\varphi_n^+)=1-e^{-C_\alpha y^\alpha}+\frac{1}{2}e^{-(C_\alpha-C_\alpha^*)y^\alpha} \text{ for } 0\leq y\leq s_\alpha^*; \tag{3.13}$$

$$\lim_{n\to\infty}E_{\theta_0-yc'_n{}^{-1}}(\varphi_n^-)=1-e^{-C_\alpha^*|y|^\alpha}+\frac{1}{2}e^{(C_\alpha-C_\alpha^*)|y|^\alpha} \text{ for } -s_\alpha\leq y\leq 0. \tag{3.14}$$

PROOF. From (3.4) and (3.5) we have

$$\begin{aligned}&\lim_{n\to\infty}P_{\theta_0}^{(n)}(A^n(yc_n'^{-1}: \theta_0))\\ &=\lim_{n\to\infty}\left\{\int_{A(\theta_0-yc_n'^{-1})}f(x: \theta_0)d\mu\right\}^n\\ &=\begin{cases}e^{-C_\alpha^*y^\alpha} & \text{for } 0\leq y\leq s_\alpha,\\ e^{-C_\alpha|y|^\alpha} & \text{for } -s_\alpha\leq y\leq 0,\end{cases}\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\lim_{n\to\infty}P^{(n)}{}_{\theta_0-yc_n'^{-1}}(A^n(yc_n'^{-1}: \theta_0))\\ &=\lim_{n\to\infty}\left\{\int_{A(\theta_0)}f(x: \theta_0-yc_n'^{-1})d\mu\right\}^n\\ &=\begin{cases}e^{-C_\alpha y^\alpha} & \text{for } 0\leq y\leq s_\alpha^*,\\ e^{-C_\alpha^*|y|^\alpha} & \text{for } -s_\alpha\leq y\leq 0,\end{cases}\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\lim_{n\to\infty}P^{(n)}{}_{\theta-yc_n'^{-1}}(D^n(yc_n'^{-1}: \theta_0))\\ &=1-\lim_{n\to\infty}\left\{\int_{A(\theta_0)}f(x: \theta_0-yc_n'^{-1})d\mu\right\}^n\\ &=\begin{cases}1-e^{-C_\alpha y^\alpha} & \text{for } 0\leq y\leq s_\alpha^*,\\ 1-e^{-C_\alpha^*|y|^\alpha} & \text{for } -s_\alpha\leq y\leq 0.\end{cases}\end{aligned} \tag{3.17}$$

Since $Z_{n1}^+(\theta)$, $Z_{n2}^+(\theta)$, $\cdots$ are independent and (3.6) and (3.7) hold, it follows from the central limit theorem (Gnedenko and Kolmogorov [4]) that the distribution laws of $\{\sum_{i=1}^n Z_{ni}^+(\theta)-nM_n'(\theta)\}/\sqrt{nV_n'(\theta)}$ and $\{\sum_{i=1}^n Z_{ni}^+(\theta)-nM_n^{*\prime}(\theta)\}/\sqrt{nV_n^{*\prime}(\theta)}$ converge to the normal law $N(0, 1)$ uniformly in any compact subset of Θ. Hence it follows from (3.15) that

$$\begin{aligned}&\lim_{n\to\infty}E_{\theta_0}(\varphi_n^+)\\ &=\lim_{n\to\infty}P_{\theta_0}^{(n)}(\{\sum_{i=1}^n Z_{ni}^+(\theta_0)>k_n^+\}\cap A^n(yc_n'^{-1}: \theta_0))\\ &=\lim_{n\to\infty}[P_{\theta_0}^{(n)}(\{(\sum_{i=1}^n Z_{ni}^+(\theta_0)-nM^{*\prime}(\theta_0))/\sqrt{nV_n^{*\prime}(\theta_0)}>-K_y^+y^\alpha\}|A^n(yc_n'^{-1}: \theta_0))\\ &\quad\cdot P_{\theta_0}^{(n)}(A^n(yc_n'^{-1}: \theta_0))]\\ &=\lim_{n\to\infty}\Bigg[P_{\theta_0}^{(n)}\Bigg(\Bigg\{(\sum_{i=1}^n Z_{ni}^+(\theta_0)-nM_n'(\theta_0)/\sqrt{nV_n'(\theta_0)}>-K_y^+y^\alpha\sqrt{\frac{V_n^{*\prime}(\theta_0)}{V_n'(\theta_0)}}\\ &\quad+\frac{n}{\sqrt{nV_n'(\theta_0)}}(M_n^{*\prime}(\theta_0)-M_n'(\theta_0)\Bigg\}|A^n(yc_n'^{-1}: \theta_0))\cdot P_{\theta_0}^{(n)}(A^n(yc_n'^{-1}: \theta_0))\Bigg]\end{aligned}$$

$$=\Phi(\gamma' K_y^+ y^\alpha)e^{-C_\alpha^* y^\alpha}$$
$$=1/2$$

for $0 \leq y \leq s_\alpha^*$.

Similary we have

$$\lim_{n\to\infty} E_{\theta_0}(\varphi_n^-)$$
$$=\Phi(\gamma' K_y^-|y|^\alpha)e^{-C_\alpha|y|^\alpha}$$
$$=1/2$$

for $-s_\alpha \leq y \leq 0$, where $\Phi(\gamma' K_y^-|y|^\alpha)=\frac{1}{2}e^{C_\alpha|y|^\alpha}$.

Hence (3.12) holds.

Also it follows from (3.16) that

$$\lim_{n\to\infty} P^{(n)}{}_{\theta_0-yc_n'^{-1}}(\{\sum_{i=1}^n Z_{ni}^+(\theta_0)>k_n^+\}\cap A^n(yc_n'^{-1}:\theta_0))$$
$$=\lim_{n\to\infty} [P^{(n)}{}_{\theta_0-yc_n'^{-1}}(\{(\sum_{i=1}^n Z_{ni}^+(\theta_0)-nM_n^{*\prime}(\theta))/\sqrt{nV_n^*(\theta_0)}>-K_y^+y^\alpha\}|A^n(yc_n'^{-1}:\theta_0))$$
$$\cdot P^{(n)}{}_{\theta_0-yc_n'^{-1}}(A^n(yc_n^{-1}:\theta_0))]$$
$$=\Phi(\gamma' K_y^+ y^\alpha)e^{-C_\alpha y^\alpha}$$
$$=\frac{1}{2}e^{-(C_\alpha-C_\alpha^*)y^\alpha} \tag{3.18}$$

for $0 \leq y \leq s_\alpha^*$.

Similary we have

$$\lim_{n\to\infty} P^{(n)}{}_{\theta_0-yc_n'^{-1}}(\{\sum_{i=1}^n Z_{ni}^-(\theta)>k_n^-\}\cap A^n(yc_n'^{-1}:\theta_0))$$
$$=\Phi(\gamma' K_y^-|y|^\alpha)e^{-C_\alpha^*|y|^\alpha}$$
$$=\frac{1}{2}e^{(C_\alpha-C_\alpha^*)|y|^\alpha} \tag{3.19}$$

for $-s_\alpha \leq y \leq 0$.

Hence it follows from (3.17), (3.18) and (3.19) that

$$\lim_{n\to\infty} E_{\theta-yc_n'^{-1}}(\varphi_n^+)=1-e^{-C_\alpha y^\alpha}+\frac{1}{2}e^{-(C_\alpha-C_\alpha^*)y^\alpha} \text{ for } 0\leq y\leq s_\alpha^*;$$
$$\lim_{n\to\infty} E_{\theta-yc_n'^{-1}}(\varphi_n^-)=1-e^{-C_\alpha^*|y|^\alpha}+\frac{1}{2}e^{(C_\alpha-C_\alpha^*)|y|^\alpha} \text{ for } -s_\alpha\leq y\leq 0.$$

Thus we complete the proof.

Let $\mathfrak{X}=R^1$. Now we suppose that every $P_\theta(\cdot)(\theta\in\Theta)$ is absolutely continuous with respect to a Lebesgue measure m. Then we denote that the density dP_θ/dm by $f(\cdot:\theta)$ and suppose $f(x:\theta)=f(x-\theta)$. Next we shall make the following assumptions. In [1] we discussed the order of convergence of consistent estimators in the same case.

Assumption (*A*). $f(x)>0$ for $a<x<b$,
$f(x)=0$ for $x\leq a$, $x\geq b$.

Assumption (*B*). $f(x)$ is twice continuously differentiable in the interval (a,b), and

$$\lim_{x\to a+0}(x-a)^{1-\alpha}f(x)=A'$$
$$\lim_{x\to b-0}(b-x)^{1-\beta}f(x)=B',$$

where both α and β are positive constants satisfying $\alpha\leq\beta<\infty$ and A' and B' are positive finite numbers.

Assumption (*C*). $A''=\lim_{x\to a+0}(x-a)^{2-\alpha}|f'(x)|$ and $B''=\lim_{x\to b-0}(b-x)^{2-\beta}|f'(x)|$ are finite. For $\alpha\geq 2$, $f''(x)$ is bounded.

For example it is easily seen that the beta distributions $Be(\alpha, \beta)$ $(0<\alpha\leq\beta\leq 2$ or $3<\alpha\leq\beta<\infty)$ satisfy Assumptions (A), (B) and (C).

Let u and v' be any real numbers satisfying $0<u<s_1=(1/C_1)\log 2$ and $0<v'<s_1$. We put $u'=(-1/C_1)\log\{1-e^{-C_1(s_1-u)}\}$, $v=(-1/C_1)\log\{1-e^{-C_1(s_1-v')}\}$, $T_n^*=\max\{T_n^+, T_n^-\}$ and $T_n^{**}=\min\{T_n'^+, T_n'^-\}$, where $T_n^+=\max_{1\leq i\leq n} X_i-b+un^{-1}$, $T_n^-=\min_{1\leq i\leq n} X_i-a-u'n^{-1}$, $T_n'^+=\max_{1\leq i\leq n} X_i-b+vn^{-1}$ and $T_n'^-=\min_{1\leq i\leq n} X_i-a-v'n^{-1}$. In the following theorem we shall show that there exist the asymptotic distributions with different orders according to α.

THEOREM 3.1. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of *i.i.d.* random variables with a density function satisfying Assumptions (A), (B) and (C). If $\alpha=2$, then the asymptotic distribution of a maximum likelihood estimator (*M.L.E.*) of order $\{(n\log n)^{1/2}\}$ is $N(0, 1/I_1)$, where $I_1=\frac{1}{2}\left(\frac{A''^2}{A'}+\frac{B''^2}{B'}\right)$ if $\beta=2$, $I_1=\frac{A''^2}{2A'}$ if $\beta>2$.

If $\alpha>2$, then the asymptotic distribution of an *M.L.E.* of order $\{n^{1/2}\}$ is $N(0, 1/I_2)$, where $I_2=E_\theta[\{(\partial/\partial\theta)\log f(x-\theta)\}^2]$. If $\alpha=\beta=1$ and $A'=B'=C_1$, then the asymptotic distributions of $T^*=\{T_n^*\}$ and $T^{**}=\{T_n^{**}\}$ of order $\{n\}$ are the Weibull distributions

$$F_{\theta,T^*}{}^{\{n\}}(y)=\begin{cases} e^{-C_1(u-y)}-e^{-C_1(u+u')} & \text{if } 0<y\leq u,\\ 1-e^{-C_1(u'+y)} & \text{if } u<y,\end{cases} \tag{3.20}$$

$$F_{\theta,T^{**}}{}^{\{n\}}(y)=\begin{cases} 1-e^{-C_1(v'+y)}+e^{-C_1(v'+y)} & \text{if } -v'\leq y<0,\\ e^{-C_1(v-y)} & \text{if } y<-v'.\end{cases} \tag{3.21}$$

PROOF. i) $\alpha=2$. Let $\{U_n\}$ be an *M.L.E.* of θ.
For any $\vartheta\in\Theta$ there exists a positive number d such that

$$\lim_{n\to\infty}\sup_{\theta:|\theta-\vartheta|<d}|P_\theta^{(n)}(\{(n\log n)^{1/2}(U_n-\theta)\leq y/\sqrt{I_1}\})-\Phi(y/\sqrt{I_1})|=0,$$

where $I_1=\frac{1}{2}\left(\frac{A''^2}{A'}+\frac{B''^2}{B'}\right)$ if $\beta=2$, $I_1=\frac{A''^2}{2A'}$ if $\beta>2$ (See the proof of Theorem 4.1 of [1]).

ii) $\alpha>2$. Let $\{U_n^*\}$ be an *M.L.E.* of θ. For any $\vartheta\in\Theta$ there exists a positive number d such that

$$\lim_{n\to\infty}\sup_{\theta:|\theta-\vartheta|<d}|P_\theta^{(n)}(\{n^{1/2}(U_n^*-\theta)\leq y/\sqrt{I_2}\})-\Phi(y/\sqrt{I_2})|=0,$$

where $I_2=E_\theta[\{\partial/\partial\theta)\log f(x-\theta)\}^2]$ (See the proof of Theorem 4.1 of [1]).

iii) $\alpha=\beta=1$. From Assumptions (A) and (B) we have for any $y>0$ and for sufficiently large n,

$$\begin{aligned}
&P_\theta^{(n)}(\{n(T_n^*-\theta)\leq y\})\\
&=P_\theta^{(n)}(\{n\cdot\max\{T_n^+-\theta,\ T_n^--\theta\}\leq y\})\\
&=P_\theta^{(n)}(\{\max_{1\leq i\leq n}x_i-\theta-b+un^{-1}\leq yn^{-1}\}\cap\{\min_{1\leq i\leq n}x_i-\theta-a-u'n^{-1}\leq yn^{-1}\})\\
&=P_\theta^{(n)}(\{\max_{1\leq i\leq n}x_i\leq\theta+b-(u-y)n^{-1}\})\\
&\quad-P_\theta^{(n)}(\{\max_{1\leq i\leq n}x_i\leq\theta+b-(u-y)n^{-1}\}\cap\{\min_{1\leq i\leq n}x_i>\theta+a+(u'+y)n^{-1}\})\\
&=\left\{1-\int_{b-(u-y)n^{-1}}^{b}f(x)dx\right\}^n-\left\{1-\int_a^{a+(u'+y)n^{-1}}f(x)dx-\int_{b-(u'-y)n^{-1}}^{b}f(x)dx\right\}^n.
\end{aligned}$$

Hence we obtain

$$F_{\theta,T^*}{}^{\{n\}}(y)=\begin{cases} e^{-C_1(u-y)}-e^{C_1(n+u')} & \text{if } 0<y\leq u,\\ 1-e^{-C_1(u'+y)} & \text{if } u<y.\end{cases}$$

Further from Assumptions (A) and (B) we have for each $y<0$ and for sufficiently large n,

$$\begin{aligned}
&P_\theta^{(n)}(\{n(T_n^{**}-\theta)\le y\}) \\
&=P_\theta^{(n)}(\{n\cdot \min\{T_n'^{+}-\theta,\ T_n'^{-}-\theta\}\le y\}) \\
&=1-P_\theta^{(n)}(\{n(T_n'^{+}-\theta)>y\}\cap\{n(T_n'^{-}-\theta)>y\}) \\
&=1-P_\theta^{(n)}(\{\max_{1\le i\le n} x_i-\theta-b+vn^{-1}>yn^{-1}\}\cap\{\min_{1\le i\le n} x_i-\theta-a-v'n^{-1}>yn^{-1}\}) \\
&=1-[P_\theta^{(n)}(\{\min_{1\le i\le n} x_i>\theta+a+(v'+y)n^{-1}\}) \\
&\quad -P_\theta^{(n)}(\{\min_{1\le i\le n} x_i>\theta+a+(v'+y)n^{-1}\}\cap\{\max_{1\le i\le n} x_i\le\theta+b-(v-y)n^{-1}\})] \\
&=1-\left\{1-\int_a^{a+(v'+y)n^{-1}} f(x)dx\right\}^n+\left\{1-\int_a^{a+(v+y)n^{-1}} f(x)dx-\int_{b-(v-y)n^{-1}}^{b} f(x)dx\right\}^n.
\end{aligned}$$

Hence we obtain

$$F_{\theta,T^{**}}{}^{(n)}(y)=\begin{cases}1-e^{-C_1(v'+y)}+e^{-C_1(v+v')} & \text{if } -v'\le y<0,\\ e^{-C_1(v-y)} & \text{if } y<-v'.\end{cases}$$

Thus we complete the proof.

REMARK: If $\alpha=2$ and $\alpha>2$, then the $M.L.E.$'s are a $\{(n\log n)^{1/2}\}$-consistent estimator and a $\{n^{1/2}\}$-consistent estimator, respectively (See Theorem 4.1 of [1]).

The following lemma is proved in a similar way as lemma 4.4 of [1].

LEMMA 3.3 If $\alpha\ge 2$, then the all density functions satisfying Assumptions (A), (B) and (C) fall in Case I.

If $\alpha=\beta=1$ and $A'=B'=C_1$, then the all density functions satisfying Assumptions (A) and (B) fall in Case II.

Throughout the subsequent discussions we denote by $\mathfrak{T}_M{}^{\mathfrak{C}}$ the class of all exactly asymptotically median unbiased estimators with the same order $\mathfrak{C}=\{c_n\}$.

THEOREM 3.2. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of $i.i.d.$ random variables with a density function satisfying Assumptions (A), (B) and (C). If $\alpha=2$ and $\alpha>2$, then the $M.L.E.$'s $\{U_n\}$ and $\{U_n^*\}$ with the asymptotic normal distributions $N(0, 1/I_1)$ and $N(0, 1/I_2)$ of order $\{(n\log n)^{1/2}\}$ and order $\{n^{1/2}\}$ are uniformly asymptotically most accurate, respectively, where I_1 and I_2 are defined in Theorem 3.1.

PROOF. It follows from lemma 3.3 that the case $\alpha\ge 2$ falls in Case I. We define an estimator $\hat\theta(=\{\hat\theta_n\})$ and order $\mathfrak{C}(=\{c_n\})$ and I as follows:

$$\hat\theta_n=\begin{cases}U_n & \text{if } \alpha=2,\\ U_n^* & \text{if } \alpha>2,\end{cases}$$

$$c_n=\begin{cases}(n\log n)^{1/2} & \text{if } \alpha=2,\\ n^{1/2} & \text{if } \alpha>2,\end{cases}$$

$$I=\begin{cases}I_1 & \text{if } \alpha=2,\\ I_2 & \text{if } \alpha>2.\end{cases}$$

We have for $T=\{T_n\}\in\mathfrak{T}_M{}^{\mathfrak{C}}$ and $-\infty<y<\infty$,

$$\begin{aligned}
&P^{(n)}{}_{\theta-(y/K\sqrt{I})c_n^{-1}}(\{c_n(T_n-\theta)\le 0\}) \\
&=P^{(n)}{}_{\theta-(y/K\sqrt{I})c_n^{-1}}\left(\left\{K\sqrt{I}\,c_n\left(T_n-\left(\theta-\frac{y}{K\sqrt{I}}c_n^{-1}\right)\right)\le y\right\}\right).
\end{aligned}$$

Put $A_T=\{\sqrt{I}\,c_n(T_n-\theta)\le 0\}$. Since for each y, $F_{\theta,T}{}^{\mathfrak{C}}(y)$ is continuous function of θ, we have

$$\lim_{\overline{n\to\infty}} |P_\theta^{(n)}(A_T)-F_{\theta,T}{}^{\mathfrak{C}}(0)|=0, \tag{3.22}$$

$$\lim_{\overline{n\to\infty}} |P^{(n)}{}_{\theta-(y/K\sqrt{I})c_n^{-1}}(A_T)-F_{\theta,T}{}^{\mathfrak{C}}(y)|=0. \tag{3.23}$$

Further it follows from (3.22) and (3.23) and lemma 3.1 that for each $y>0$ there exists a sequence $\{\varphi_n\}$ of tests such that

$$0=\lim_{n\to\infty}\Big|\sup_{T\in\mathfrak{T}_M\mathfrak{C}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})-\sup_{\phi\in\Psi_{\alpha_n}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\phi)\Big|$$
$$=\lim_{n\to\infty}\Big|\sup_{T\in\mathfrak{T}_M\mathfrak{C}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})-E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\varphi_n)\Big|$$

where $\Psi_{\alpha_n}=\{\text{test } \phi\colon E_\theta(\phi)=\alpha_n\}$.

Hence since for each $y>0$, $\lim_{n\to\infty} E_{\theta-(yK/\sqrt{I})c_n^{-1}}(\varphi_n)=\Phi(y/\sqrt{I})$, for each $y>0$ and for any $\varepsilon>0$ there exists a sufficiently large n_0 such that for all $n(\geq n_0)$

$$P^{(n)}{}_{\theta-(y/\sqrt{I})c_n^{-1}}(A_T)=E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})\leq\Phi(y/\sqrt{I})+\varepsilon \tag{3.24}$$

for all $T\in\mathfrak{T}_M\mathfrak{C}$. From (3.23) and (3.24) we have for all $y>0$ and for all $T\in\mathfrak{T}_M\mathfrak{C}$,

$$F_{\theta,T}{}^{\mathfrak{C}}(y)\leq\Phi(y/\sqrt{I}).$$

Since $F_{\theta,\hat\theta}{}^{\mathfrak{C}}(y)=\Phi(y/\sqrt{I})$, we have

$$\sup_{T\in\mathfrak{T}_M\mathfrak{C}} F_{\theta,T}{}^{\mathfrak{C}}(y)=F_{\theta,\hat\theta}{}^{\mathfrak{C}}(y) \quad \text{for all } y>0.$$

Furthermore from lemma 3.1 we have for each $y<0$,

$$0=\lim_{n\to\infty}\Big|\inf_{T\in\mathfrak{T}_M\mathfrak{C}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})-\inf_{\phi\in\Psi_{\alpha_n}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\phi)\Big|$$
$$=\lim_{n\to\infty}\Big|\inf_{T\in\mathfrak{T}_M\mathfrak{C}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})-\{1-\sup_{\phi\in\Psi_{\alpha_n}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\phi)\}\Big|$$
$$=\lim_{n\to\infty}\Big|\inf_{T\in\mathfrak{T}_M\mathfrak{C}} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\chi_{A_T})-\{1-E_{\theta-(yK/\sqrt{I})c_n^{-1}}(\varphi_n)\}\Big|.$$

Hence since $\lim_{n\to\infty} E_{\theta-(y/K\sqrt{I})c_n^{-1}}(\varphi_n)=\Phi(-y/\sqrt{I})$, it is shown by a similar way as the case $y>0$ that $F_{\theta,\hat\theta}{}^{\mathfrak{C}}(y)=\inf_{T\in\mathfrak{T}_M\mathfrak{C}} F_{\theta,T}{}^{\mathfrak{C}}(y)$ for all $y<0$.

Thus we complete the proof.

THEOREM 3.3. Let $X_1, X_2, \cdots, X_n$, be a sequence of *i.i.d.* random variables with a density function satisfying Assumptions (A), (B) and (C). If $\alpha=\beta=1$ and $A'=B'=C_1$, then $T^*=\{T_n{}^*\}$ and $T^{**}=\{T_n{}^{**}\}$ with the asymptotic distributions given in Theorem 3.1 of order $\{n\}$ are asymptotically most accurate at an arbitrary point u in $[0, s_1]$ and at an arbitrary point $-v'$ in $[-s_1, 0]$, respectively.

PROOF. It follows from lemma 3.3 that the case $\alpha=\beta=1$ and $A'=B'=C_1$ falls in Case II.

i) Case T^*. From lemma 3.2 we have for each $y>0$,

$$0=\lim_{n\to\infty}\Big|\sup_{T\in\mathfrak{T}_M\{n\}} E_{\theta-yn^{-1}}(\chi_{A_T})-\sup_{\phi\in\Psi_{\alpha_n+}} E_{\theta-yn^{-1}}(\phi)\Big|$$
$$=\lim_{n\to\infty}\Big|\sup_{T\in\mathfrak{T}_M\{n\}} E_{\theta-yn^{-1}}(\chi_{A_T})-E_{\theta-yn^{-1}}(\varphi_n{}^+)\Big|,$$

where $A_T=\{n(T_n-\theta)\leq 0\}$ and $\Psi_{\alpha_n}{}^+=\{\text{test } \phi\colon E_\theta(\phi)=\alpha_n{}^+\}$.

Furthermore since from (3.13) of lemma 3.2

$$\lim_{n\to\infty} E_{\theta-yn^{-1}}(\varphi_n{}^+)=\begin{cases}\dfrac{3}{2}-e^{-C_1 y} & \text{if } 0<y\leq s_1,\\ 1 & \text{if } s_1<y,\end{cases}$$

it is shown in a similar way as Theorem 3.2 that

$$\sup_{T\in\mathfrak{T}_M\{n\}} F_{\theta,T}{}^{\{n\}}(y)=\begin{cases}\dfrac{3}{2}-e^{-C_1 y} & \text{if } 0<y<s_1,\\ 1 & \text{if } s_1<y.\end{cases} \tag{3.25}$$

Hence it follows from (3.20) of Theorem 3.1 and (3.25) that T^* is asymptotically most accurate at an arbitrary point u in $[0, s_1]$.

ii) Case T^{**}. From lemma 3.2 we have for each $y<0$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big| \inf_{T\in\mathfrak{T}_M\{n\}} E_{\theta-y_n^{-1}}(\chi_{A_T}) - \inf_{\phi\in\Psi_{\alpha_n^-}} E_{\theta-y_n^{-1}}(\phi) \Big| \\ &= \lim_{n\to\infty} \Big| \inf_{T\in\mathfrak{T}_M\{n\}} E_{\theta-y_n^{-1}}(\chi_{A_T}) - \{1 - \sup_{\phi\in\Psi_{\alpha_n^-}} E_{\theta-y_n^{-1}}(\phi)\} \Big| \\ &= \lim_{n\to\infty} \Big| \inf_{T\in\mathfrak{T}_M\{n\}} E_{\theta-y_n^{-1}}(\chi_{A_T}) - \{1 - E_{\theta-y_n^{-1}}(\varphi_n^-)\} \Big| \end{aligned}$$

where $\Phi_{\alpha_n^-} = \{\text{test } \phi\colon E_\theta(\phi) = \alpha_n^-\}$.

Since from (3.14) of lemma 3.2

$$\lim_{n\to\infty} \{1 - E_{\theta-y_n^{-1}}(\varphi_n^-)\} = \begin{cases} -\dfrac{1}{2} + e^{c_1 y} & \text{if } -s_1 \leq y < 0, \\ 0 & \text{if } y < -s_1, \end{cases}$$

it is shown in a similar way as Theorem 3.2 that

$$\inf_{T\in\mathfrak{T}_M\{n\}} F_{\theta, T^{\{n\}}}(y) = \begin{cases} -\dfrac{1}{2} + e^{c_1 y} & \text{if } -s_1 \leq y < 0, \\ 0 & \text{if } y < -s_1. \end{cases} \tag{3.26}$$

Hence it follows from (3.21) of Theorem 3.1 and (3.26) that T^{**} is asymptotically most accurate at an arbitrary point $-v'$ in $[-s_1, 0]$. Thus we complete the proof.

4. Bounds of orders of asymptotic distributions of consistent estimators and others in non-regular cases.

Let $\mathfrak{X} = R^1$. We suppose that every $P_\theta(.)(\theta\in\Theta)$ is absolutely continuous with respect to a Lebesgue measure m. Then we denote the density dP_θ/dm by $f(x\colon\theta)$ and suppose that $f(x\colon\theta) = f(x-\theta)$ and $f(x)>0$ for all $x\in\mathfrak{X}$. For any points θ_1 and θ_2 in Θ, we define

$$d^{(n)}(\theta_1, \theta_2) = \int_{\mathfrak{X}^{(n)}} \Big| \prod_{i=1}^n f(x_i - \theta_1) - \prod_{i=1}^n f(x_i - \theta_2) \Big| \prod_{i=1}^n dx_2'.$$

Then $d^{(n)}$ is a metric on Θ.

In subsequent discussions + and − signs should be read consistently. If

$$\int_{\mathfrak{X}} \frac{\{f(x\pm\Delta) - f(x)\}^2}{f(x)} dx < \infty,$$

then it follows that

$$d^{(n)}(\theta\mp\Delta, \theta) \leq \Big[\Big\{ \int_{\mathfrak{X}} (f(x\pm\Delta) - f(x))^2 / f(x) dx + 1 \Big\}^n - 1 \Big]^{1/2}. \tag{4.1}$$

Further we shall assume that $f(x)$ is twice continuously differentiable for $x\in\mathfrak{X}$ except for $x=0$ and

$$\lim_{x\to-\infty} f'(x) = \lim_{x\to\infty} f'(x) = 0, \tag{4.2}$$

and there exists a positive number M_1 such that

$$f'(x) \geq 0 \text{ for } x < -M_1 \text{ and } f''(x) \geq 0 \text{ for } x < -M_1, \tag{4.3}$$

$$\int_{-\infty}^{-M_1} \{f'(x)\}^2 / f(x) dx < \infty, \tag{4.4}$$

and for any $\varepsilon>0$ there exists a positive number M_2 such that $M_2>\varepsilon$,

$$f'(x) \leq 0 \text{ for } x > M_2 - \varepsilon \text{ and } f''(x) \geq 0 \text{ for } x \geq M_2 - \varepsilon, \tag{4.5}$$

$$\int_{M_2}^{\infty} \{f'(x-\varepsilon)\}^2 / f(x) dx < \infty. \tag{4.6}$$

Here we consider the following Cases III, IV and V.

Case III. $a' = \lim_{x\to+0} f(x)$ and $b' = \lim_{x\to-0} f(x)$ are finite, where a' and b' are certain positive numbers satisfying $a' \neq b'$, and $f'(x)$ is bounded on $R^1 - \{0\}$.

Case IV. $\delta=\lim_{x\to\pm0}|x|^{-\alpha}(f(x)-\gamma)$ and $\eta=\overline{\lim}_{x\to\pm0}|x|^{1-\alpha}|f'(x)|$ are finite, where $0<\alpha\leq 1/2$ and δ, γ and η are positive numbers.

Case V. $\delta'=\lim_{x\to\pm0}|x|^{\alpha}f(x)$ and $\eta'=\overline{\lim}_{x\to\pm0}|x|^{1+\alpha}|f'(x)|$ are finite, where $0<\alpha<1$ and δ' and η' are positive numbers.

Let $\mathfrak{F}_{\text{III}}$, $\mathfrak{F}_{\text{IV}}$ and $\mathfrak{F}_{\text{V}}$ be classes of all density functions falling in Case III, Case IV and Case V, respectively.

REMARK: If $f\in\mathfrak{F}_{\text{IV}}$, then it is easily seen that Fisher's information is infinite:

$$\int_{\mathfrak{X}}\{f'(x)/f(x)\}^2 f(x)dx=\infty.$$

If $f\in\mathfrak{F}_{\text{III}}$, then there exist positive numbers K_1, K_2 and ε_1 such that

$$K_1\leq f(x)\leq K_2 \quad \text{for} \quad -\varepsilon_1<x<\varepsilon_1. \tag{4.7}$$

If $f\in\mathfrak{F}_{\text{IV}}$, then there exist positive numbers K_i $(i=3, 4, 5)$ and ε_2 such that

$$\gamma+K_3|x|^{\alpha}\leq f(x)\leq\gamma+K_4|x|^{\alpha} \quad \text{for} \quad -\varepsilon_2<x<\varepsilon_2, \tag{4.8}$$

$$|f'(x)|\leq K_5|x|^{\alpha-1} \quad \text{for} \quad -\varepsilon_2<x<\varepsilon_2. \tag{4.9}$$

If $f\in\mathfrak{F}_{\text{V}}$, then there exist positive numbers $K_i (i=6, 7, 8)$ and ε_3 such that

$$K_6|x|^{-\alpha}\leq f(x)\leq K_7|x|^{-\alpha} \quad \text{for} \quad -\varepsilon_3<x<\varepsilon_3, \tag{4.10}$$

$$|f'(x)|\leq K_8|x|^{-\alpha-1} \quad \text{for} \quad -\varepsilon_3<x<\varepsilon_3. \tag{4.11}$$

Putting $\varepsilon=\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, 1\}$, we see that (4.7), (4.8), (4.9), (4.10) and (4.11) hold for $-\varepsilon<x<\varepsilon$. Let $0<\Delta<\varepsilon/2$.

Now we define an information I

$$I=\int_{\mathfrak{X}}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx.$$

We divide I into six parts I_1, I_2, I_3, I_4, I_5 and I_6, that is, $I=\sum_{i=1}^{6}I_i$, where

$$I_1=\int_{-\infty}^{-\varepsilon},\ I_2=\int_{-\varepsilon}^{-2\Delta},\ I_3=\int_{-2\Delta}^{0},\ I_4=\int_{0}^{2\Delta},\ I_5=\int_{2\Delta}^{\varepsilon} \text{ and } I_6=\int_{\varepsilon}^{\infty}.$$

LEMMA 4.1. In each Case, the orders of I_1, I_2, I_3, I_4, I_5, I_6 and I are given by Table 1.

Table 1

<table>
<tr><th>Case</th><th>α</th><th>I_1</th><th>I_2</th><th>I_3</th><th>I_4</th><th>I_5</th><th>I_6</th><th>I</th></tr>
<tr><td>III</td><td></td><td rowspan="4">$O(\Delta^2)$</td><td>$O(\Delta^2)$</td><td>$O(\Delta)$</td><td>$O(\Delta)$</td><td>$O(\Delta^2)$</td><td rowspan="4">$O(\Delta^2)$</td><td>$O(\Delta)$</td></tr>
<tr><td rowspan="2">IV</td><td>$0<\alpha<\frac{1}{2}$</td><td>$O(\Delta^{2\alpha+1})$</td><td rowspan="2">$O(\Delta^{2\alpha+1})$</td><td rowspan="2">$O(\Delta^{2\alpha+1})$</td><td>$O(\Delta^{2\alpha+1})$</td><td>$O(\Delta^{2\alpha+1})$</td></tr>
<tr><td>$\alpha=\frac{1}{2}$</td><td>$O(\Delta^2|\log\Delta|)$</td><td>$O(\Delta^2|\log\Delta|)$</td><td>$O(\Delta^2|\log\Delta|)$</td></tr>
<tr><td>V</td><td>$0<\alpha<\frac{1}{2}$</td><td>$O(\Delta^{1-\alpha})$</td><td>$O(\Delta^{1-\alpha})$</td><td>$O(\Delta^{1-\alpha})$</td><td>$O(\Delta^{1-\alpha})$</td><td>$O(\Delta^{1-\alpha})$</td></tr>
</table>

PROOF. 1) *I_3 and I_4.* In Case *III*, it follows from (4.7) that

$$I_4=\int_0^{2\Delta}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx$$

$$\leq 2\frac{(K_2-K_1)^2}{K_1}\Delta. \tag{4.12}$$

In Case IV, it follows from (4.8) that

$$I_4=\int_0^{2\Delta}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx$$
$$\leq\frac{1}{\gamma}\int_0^{2\Delta}\{K_4|x-\Delta|^\alpha-K_3x^\alpha\}^2dx$$
$$=O(\Delta^{2\alpha+1}). \quad (4.13)$$

In Case V, it follows from (4.10) that

$$I_4=\int_0^{2\Delta}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx$$
$$\leq\int_0^{2\Delta}\frac{(K_7|x-\Delta|^{-\alpha}-K_6x^{-\alpha})^2}{K_6x^{-\alpha}}dx$$
$$\leq\frac{K_7^2}{K_6}\left\{\Delta^{1-\alpha}B(1+\alpha,\ 1-2\alpha)+(2\Delta)^\alpha\frac{\Delta^{1-2\alpha}}{1-2\alpha}\right\}-4\frac{K_7\Delta^{1-\alpha}}{1-\alpha}+K_6\frac{(2\Delta)^{1-\alpha}}{1-\alpha}$$
$$=O(\Delta^{1-\alpha}), \quad (4.14)$$

where $B(1+\alpha,\ 1-2\alpha)$ is the beta function and $0<\alpha<1/2$.
It follows from (4.12), (4.13) and (4.14) that

$$I_4=\begin{cases}O(\Delta) & \text{if } f\in\mathfrak{F}_{\text{III}},\\ O(\Delta^{2\alpha+1})\ (0<\alpha\leq1/2) & \text{if } f\in\mathfrak{F}_{\text{IV}},\\ O(\Delta^{1-\alpha})\ (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\text{V}},\end{cases}$$

Similarly it follows that the order of I_3 is consistent with that of I_4.

ii) I_2 *and* I_5. It follows by the mean value theorem that

$$I_5=\int_{2\Delta}^{\varepsilon}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx$$
$$=\int_{2\Delta}^{\varepsilon}\frac{\Delta^2\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx \quad (4.15)$$

where $\Delta<x-\Delta<\xi(x,\ \Delta)<x<\varepsilon$.

In Case III, since $f'(x)$ is bounded function on $R^1-\{0\}$, it follows from (4.7) and (4.15) that

$$I_5\leq\int_{2\Delta}^{\varepsilon}\Delta^2(1/K_1)\{f'(\xi(x,\ \Delta))\}^2dx$$
$$=K'_1\Delta^2(\varepsilon-2\Delta)$$
$$=O(\Delta^2), \quad (4.16)$$

where K_1' is some positive constant.

In Case IV, it follows from (4.8), (4.9) and (4.15) that

$$I_5\leq\int_{2\Delta}^{\varepsilon}(\Delta^2/\gamma)\{f'(\xi(x,\ \Delta))\}^2dx$$
$$\leq(\Delta^2/\gamma)\int_{2\Delta}^{\varepsilon}K_5^2\xi^{2\alpha-2}dx$$
$$\leq(K_5/\gamma)\Delta^2\int_{2\Delta}^{\varepsilon}(x-\Delta)^{2\alpha-2}dx. \quad (4.17)$$

If $0<\alpha<1/2$, then it follows from (4.17) that

$$I_5\leq(K_5^2\Delta^2/\gamma(2\alpha-1))\{(\varepsilon-\Delta)^{2\alpha-1}-\Delta^{2\alpha-1}\}$$
$$\leq(K_5^2/\gamma(2\alpha-1))\Delta^2(\varepsilon^{2\alpha-1}-\Delta^{2\alpha-1})$$
$$=(K_5^2/\gamma(1-2\alpha))\Delta^{2\alpha+1}. \quad (4.18)$$

If $\alpha=1/2$, then it follows from (4.17) that

$$I_5\leq(K_5^2/\gamma)\Delta^2\int_{2\Delta}^{\varepsilon}(x-\Delta)^{-1}dx$$
$$\leq(K_5^2/\gamma)\Delta^2(\log\varepsilon-\log\Delta)$$
$$=O(\Delta^2|\log\Delta|). \quad (4.19)$$

From (4.18) and (4.19) we have

$$I_5=\begin{cases}O(\Delta^{2\alpha+1}) & \text{if } 0<\alpha<1/2,\\ O(\Delta^2|\log\Delta|) & \text{if } \alpha=1/2.\end{cases} \tag{4.20}$$

In Case V, it follows from (4.10), (4.11) and (4.15) that

$$\begin{aligned}I_5&\leq\int_{2\Delta}^{\varepsilon}\Delta^2\frac{K_8\xi^{-2\alpha-2}}{K_6x^{-\alpha}}dx\\ &\leq(K_8/K_6)\Delta^2\int_{2\Delta}^{\varepsilon}x^{\alpha}(x-\Delta)^{-2\alpha-2}dx\\ &\leq O(\Delta^{1-\alpha}),\end{aligned} \tag{4.21}$$

where $0<\alpha<1/2$.
It follows from (4.16), (4.20) and (4.21) that

$$I_5=\begin{cases}O(\Delta^2) & & \text{if } f\in\mathfrak{F}_{\rm III},\\ O(\Delta^{2\alpha+1}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^2|\log\Delta|) & (\alpha=1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^{1-\alpha}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm V}.\end{cases}$$

Similarly we have

$$I_2=\begin{cases}O(\Delta^2) & & \text{if } f\in\mathfrak{F}_{\rm III},\\ O(\Delta^{2\alpha+1}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^2|\log\Delta|) & (\alpha=1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^{1-\alpha}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm V}.\end{cases}$$

iii) I_1 and I_6. It follows by the mean value theorem that

$$\begin{aligned}I_1&=\int_{-\infty}^{-\varepsilon}\frac{\{f(x-\Delta)-f(x)\}^2}{f(x)}dx\\ &=\int_{-\infty}^{-\varepsilon}\frac{\Delta^2\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx,\end{aligned} \tag{4.22}$$

where $-\infty<x-\Delta<\xi(x,\ \Delta)<x<-\varepsilon$.
Since $f(x)$ and $f'(x)$ are continuous functions on $(-\infty,\ -\varepsilon)$, it follows that

$$\int_{-M_1}^{-\varepsilon}\frac{\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx<\infty. \tag{4.23}$$

It also follows from (4.3) and (4.4) that

$$\int_{-\infty}^{-M_1}\frac{\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx\leq\int_{-\infty}^{-M_1}\frac{\{f'(x)\}^2}{f(x)}dx<\infty. \tag{4.24}$$

From (4.22), (4.23) and (4.24) we have

$$I_1=O(\Delta^2).$$

Since $f(x)$ and $f'(x)$ are continuous functions on $(\varepsilon,\ \infty)$, it also follows from (4.5) and (4.6) that

$$I_6=O(\Delta^2).$$

Since $I=\sum_{i=1}^{6}I_i$ and the orders of $I_i (i=1,\ 2,\ 3,\ 4,\ 5,\ 6)$ are obtained, it follows that

$$I=\begin{cases}O(\Delta) & & \text{if } f\in\mathfrak{F}_{\rm III},\\ O(\Delta^{2\alpha+1}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^2|\log\Delta|) & (\alpha=1/2) & \text{if } f\in\mathfrak{F}_{\rm IV},\\ O(\Delta^{1-\alpha}) & (0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\rm V}.\end{cases} \tag{4.25}$$

Thus we complete the proof.

REMARK: We also define another information I^* by

$$I^*=\int_{\mathfrak{X}}\frac{\{f(x+\Delta)-f(x)\}^2}{f(x)}dx.$$

In a similar way as the information I, we see that the order of I^* is consistent with that of I.

From (4.1) and lemma 4.1 we get the following lemma.

LEMMA 4.2.

$$d^{(n)}(\theta\mp\Delta,\ \theta)=\begin{cases}[\{1+O(\Delta)\}^n-1]^{1/2} & \text{if } f\in\mathfrak{F}_{\text{III}},\\ [\{1+O(\Delta^{2\alpha+1})\}^n-1]^{1/2}\quad(0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\text{IV}},\\ [\{1+O(\Delta^2|\log\Delta|)\}^n-1]^{1/2}\quad(\alpha=1/2) & \text{if } f\in\mathfrak{F}_{\text{IV}},\\ [\{1+O(\Delta^{1-\alpha})\}^n-1]^{1/2}\quad(0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\text{V}}.\end{cases}$$

LEMMA 4.3. In each Case let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of *i.i.d.* random variables with a density functon satisfying the Case. In each Case for a sequence $\{c_n\}$ given in Table 2 there exists a nonnegative function $H(\cdot)$ satisfying the following: for each $t>0$,

$$\overline{\lim_{n\to\infty}}\, d^{(n)}(\theta\mp tc_n^{-1},\ \theta)\leq H(C_t)<\infty, \tag{4.26}$$

where C_t is some positive number.

Furthermore in each Case, for any greater order c_n^* than values as given in Table 2,

$$\lim_{n\to\infty} d^{(n)}(\theta\mp tc^{*-1}_n,\ \theta)=0 \tag{4.27}$$

for all $t>0$.

Table 2

Case	α	c_n
III		n
IV	$0<\alpha<1/2$	$n^{1/(2\alpha+1)}$
	$\alpha=1/2$	$(n\log n)^{1/2}$
V	$0<\alpha<1/2$	$n^{1/(1-\alpha)}$

PROOF. It follows from lemma 4.2 that for sufficiently large n,

$$d^{(n)}(\theta\mp tc_n^{-1},\ \theta)=\begin{cases}[\{1+O(c_n^{-1})\}^n-1]^{1/2} & \text{if } f\in\mathfrak{F}_{\text{III}},\\ [\{1+O(c_n^{-2\alpha-1})\}^n-1]^{1/2}\quad(0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\text{IV}},\\ [\{1+O(c_n^{-2}\log c_n)\}^n-1]^{1/2}\quad(\alpha=1/2) & \text{if } f\in\mathfrak{F}_{\text{IV}},\\ [\{1+O(c_n^{\alpha-1})\}^n-1]^{1/2}\quad(0<\alpha<1/2) & \text{if } f\in\mathfrak{F}_{\text{V}}.\end{cases} \tag{4.28}$$

If c_n is chosen as Table 2 and $H(u)=(e^u-1)^{1/2}$ for all $u\geq 0$, then it follows from (4.28) that (4.26) holds. Further if c_n^* is greater than order c_n, then it follows from (4.28) that (4.27) holds. Thus we complete the proof.

THEOREM 4.1 Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of *i.i.d.* random variables with a density function satisfying Case V. Then for each α with $0<\alpha<1$ the asymptotic distribution of a sequence {Xmed.} of order $\{n^{1/(1-\alpha)}\}$ is $G(y\colon\alpha)$, where Xmed. is the median of $X_1, \cdots, X_n$ and $G(y\colon\alpha)$ is the *c.d.f.* of the random variable $|Y|^{1/(1-\alpha)}$ (sgn Y) and for large n, Y is asymptotically distributed according to the normal distribution $N(0,\ 1/4n)$.

PROOF. Let $F(x)$ be the *c.d.f.* of the random variable X. Put $U_i=F(X_i)$ $(i=1, 2, \cdots)$. Since $U_i (i=1, 2, \cdots)$ have the uniform distribution $U(0, 1)$, it follows that Umed.=F(Xmed.), where Umed. is the median of $U_1, \cdots, U_n$. Since $F(x)$ is continuous, the asymptotic distribution of (Umed.$-1/2$), for large n, is the normal distribution $N(0,\ 1/4n)$. On the other hand, since $\lim_{x\to\pm 0}|x|^\alpha f(x)=\delta'(>0)$ for $0<\alpha<1$, we have for sufficiently small $|x|>0$,

$$f(x) \propto |x|^{-\alpha}.$$

Hence we obtain for sufficiently small $|x|>0$,

$$(F(x)-1/2) \propto \begin{cases} x^{1-\alpha} & \text{for } x>0, \\ -(-x)^{1-\alpha} & \text{for } x<0. \end{cases}$$

Therefore we have

$$X \propto |\text{Umed.}-1/2|^{1/(1-\alpha)} \operatorname{sgn}(\text{Umed.}-1/2).$$

Thus putting $Y=\text{Umed.}-1/2$, we have the proof.

In follows from Theorem 4.1 that in Case V, {Xmed.} is a $\{n^{1/(1-\alpha)}\}$-consistent estimator and the asymptotic distribution of {Xmed.} has the largest order of consistency.

Next we define for sufficiently small $\varepsilon>0$,

$$f_\varepsilon(x)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x+t)dt. \tag{4.29}$$

Then we easily see that $f_\varepsilon(x)$ is a density function on $\mathfrak{X}$. An *M.L.E.* of θ with respect to the likelihood function $\prod_{i=1}^{n} f_\varepsilon(x_i-\theta)$ is said to be a ε-smooth *M.L.E.* of θ.

In the following theorems we shall show that the asymptotic distributions of the ε-smooth *M.L.E.*'s are certain normal distributions in Case III and IV.

THEOREM 4.2. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of *i.i.d.* random variables with a density function satisfying Case III. Then the asymptotic distribution of a ε-smooth *M.L.E.* of order $\{n^{1/2}\}$ is $N(0, 1/I_\varepsilon)$, where $I_\varepsilon=O(\varepsilon^{-1})$.

PROOF. It follows from (4.29) that for every $x(\neq 0)$

$$\lim_{\varepsilon\to 0} f_\varepsilon'(x)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\{f(x+\varepsilon)-f(x-\varepsilon)\}=f'(x), \tag{4.30}$$

$$\lim_{\varepsilon\to 0} f_\varepsilon''(x)=f''(x). \tag{4.31}$$

From (4.2)$\sim$(4.6), (4.30) and (4.31) we have

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty}\left\{\frac{\partial^2}{\partial\theta^2}\log f_\varepsilon(x-\theta)\right\}f(x-\theta)dx+\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log f_\varepsilon(x-\theta)\right\}^2 f(x-\theta)dx\right]=0. \tag{4.32}$$

Next we shall show that

$$I_\varepsilon \equiv \int_{-\infty}^{\infty}\{(\partial/\partial\theta)\log f_\varepsilon(x-\theta)\}^2 f(x-\theta)dx=O(\varepsilon^{-1}). \tag{4.33}$$

Indeed, it follows from (4.30) that

$$\begin{aligned} I_\varepsilon &= \int_{-\infty}^{\infty}\{(\partial/\partial\theta)\log f_\varepsilon(x-\theta)\}^2 f(x-\theta)dx \\ &\le K_1'\int_{-\infty}^{\infty}\left\{\frac{f_\varepsilon'(x)}{f_\varepsilon(x)}\right\}^2 f_\varepsilon(x)dx \\ &= K_1\int_{-\infty}^{\infty}\frac{\{f(x)-f(x-2\varepsilon)\}^2}{4\varepsilon^2 f(x)}dx \end{aligned}$$

where K_1 and K_1' are certain positive constants.

Hence it follows from I of lemma 4.1 that (4.33) holds. From (4.32) and (4.33) we obtain that the asymptotic distribution of a ε-smooth *M.L.E.* of order $\{n^{1/2}\}$ is $N(0, 1/I_\varepsilon)$ (Cramér [3]). Thus we complete the proof.

THEOREM 4.3. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of *i.i.d.* random variables with a density function satisfying Case IV. Then the asymptotic distribution of a ε-smooth *M.L.E.* of order $\{n^{1/2}\}$ is $N(0, 1/I_\varepsilon')$, where

$$I_\varepsilon'=\begin{cases} O(\varepsilon^{2\alpha-1}) & \text{if } 0<\alpha<1/2, \\ O(-\log\varepsilon) & \text{if } \alpha=1/2. \end{cases}$$

PROOF. From (4.2)~(4.6) and (4.29) we have

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty}\left\{\frac{\partial^2}{\partial\theta^2}\log f_\varepsilon(x-\theta)\right\}f(x-\theta)dx+\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log f_\varepsilon(x-\theta)\right\}^2 f(x-\theta)dx\right]=0. \quad (4.34)$$

Next we shall show that

$$I_\varepsilon'\equiv\int_{-\infty}^{\infty}\{(\partial/\partial\theta)\log f_\varepsilon(x-\theta)\}^2 f(x-\theta)dx=\begin{cases}O(\varepsilon^{2\alpha-1}) & \text{if } 0<\alpha<1/2,\\ O(-\log\varepsilon) & \text{if } \alpha=1/2.\end{cases} \quad (4.35)$$

Indeed, it follows that

$$\begin{aligned} I_\varepsilon' &= \int_{-\infty}^{\infty}\{f_\varepsilon'(x)/f_\varepsilon(x)\}^2 f(x)dx \\ &\leq K_2'\int_{-\infty}^{\infty}\{f_\varepsilon'(x)\}^2/f(x)dx \\ &\leq K_2\int_{-\infty}^{\infty}\frac{\{f(x)-f(x-2\varepsilon)\}^2}{4\varepsilon^2 f(x)}dx \end{aligned}$$

where K_2 and K_2' are certain positive constants.

Hence it follows from I of lemma 4.1 that (4.35) holds. From (4.34) and (4.35) we obtain that the asymptotic distribution of a ε-smooth $M.L.E.$ of order $\{n^{1/2}\}$ is $N(0, 1/I_\varepsilon')$ (Cramér [3]). Thus we complete the proof.

Using lemma 4.3 and the necessary condition for the existence of a $\{c_n\}$-consistent estimator (Theorem 3.3 of [1]), we get the following theorem.

THEOREM 4.4. In each Case III, IV and V, the order given by Table 2 of lemma 4.3 is the bound of the order of convergence of consistent estimators; that is, there does not exist a consistent estimator with the order greater than values given in Table 2.

REMARK: It follows from Theorem 4.4 that the bounds of orders of asymptotic distributions of consistent estimators in Case III, IV and V are given in Table 2 of lemma 4.3.

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